

Joint influence of leakage delays and proportional delays on almost periodic solutions for FCNNs

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Abstract

This paper deals with fuzzy cellular neural networks (FCNNs) with leakage delays and proportional delays. Applying the differential inequality strategy, fixed point theorem and almost periodic function principle, some sufficient criteria which ensure the existence and global attractivity of a unique almost periodic solution for fuzzy cellular neural networks with leakage delays and proportional delays are derived. Computer simulations are carried out to illustrate our theoretical findings. Our results are new and complement some previous published ones.

Keywords: Fuzzy cellular neural networks, almost periodic solution, exponential stability, leakage delay, time-varying delay, proportional delay.

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1 Introduction

During the past decades, cellular neural networks have been extensively studied due to their potential applications in various fields such as psychophysics, perception, robotics, vision, adaptive pattern recognition and image processing and so on [22,28,32,34-36]. In order to make better use of cellular neural networks, many scholars pay much attention to the dynamical nature of cellular neural networks such as the existence and stability of the equilibrium point, periodic and almost periodic solutions (see [1,4]). In recent decades, a lot of achievement on cellular neural networks have been available. For example, Balasubramaniam et al. [3] studied the existence and global asymptotic stability of the equilibrium point for fuzzy cellular neural networks with time delay in the leakage term and unbounded distributed delays, Qin et al. [25] analyzed the convergence and attractivity of memristor-based cellular neural networks with time delays, Song et al. [30] discussed the exponential stability of delayed and impulsive cellular neural networks with partially Lipschitz continuous activation functions. For more related works on these aspects, we refer the readers to [2-3,6-8,13-14,17-19,25,33,37-39,41-45,49] and the references cited therein.

Fuzzy cellular neural networks have fuzzy logic between their template input and/or output besides the sum of product operation. Many authors think that fuzzy neural networks are very useful for image processing problems and pattern recognition [48]. The leakage delay often appear in the negative feedback term of the neural networks and has a great impact on the dynamics of neural networks [20,26-27,29,40,44]. For example, the leakage delay can make the system be unstable [21], Balasubramaniam et al. [5] mentioned that the leakage delay has no effect on the existence and uniqueness of the equilibrium state. In addition, time delay appearing in neural networks can be proportional delay with the form $\xi(s) = s - \varsigma s$ where $s > 0$ and $\varsigma \in (0, 1)$ is a constant. In real world, proportional delay plays a key role in many areas (including web quality, collection of current [24], nonlinear dynamics [10,12,31] and probability

theory [9]. Here we would like to point out that almost periodicity is more suitable to describe the object world than periodicity. Therefore, the research on the existence and stability of almost periodic solutions of fuzzy cellular neural networks with leakage delays and proportional delays has important theoretical and practical value. However, so far, there is no published reports about the almost periodic solution of fuzzy cellular neural networks with leakage delays and proportional delays.

Inspired by the above discussions, it is necessary for us to study the existence and global attractivity of almost periodic solutions for fuzzy cellular neural networks with leakage delays and proportional delays. All the above works in [1-5,9-10,12-13,19-22,24-32,34-45,48-49] are only concerned with the leakage delays or proportional delays, they can only reveal the effect of leakage delay or proportional delays. They do not involve both type delays: leakage delays or proportional delays. A natural problem arises: what is the effect of both delays (leakage delays and proportional delays) on the almost periodic solution?

The main aim of this article is to establish some sufficient conditions for the existence and global attractivity of almost periodic solutions of (1). The main advantages of this paper lie in three aspects: (i) The study on the existence and global attractivity of almost periodic solutions for fuzzy cellular neural networks with leakage delays and proportional delays is firstly put forward; (ii) The fuzzy cellular neural networks with leakage delays and proportional delays in this paper are more general than those of numerous previous studies; (iii) The derived results of this article are more general and it is valid for some other similar fuzzy neural networks.

The architecture of this manuscript is planned as follows. In Section 2, the considered model is established and some notations and basic results are prepared. In Section 3, a sufficient criterion to ensure the existence of almost periodic solution for fuzzy cellular neural networks with leakage delays and proportional delays is obtained. In Section 4, a sufficient condition to guarantee the global attractivity of almost periodic solution for fuzzy cellular neural networks with leakage delays and proportional delays is established. In Section 5, software simulations are given to confirm the rationality of the analysis results. We end this manuscript in Section 6.

2 Preliminaries

In this manuscript, we will consider the following fuzzy cellular neural networks

$$\left\{ \begin{array}{l} \dot{x}_i(t) = -d_i(t)x_i(t - \delta_i(t)) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t)u_j(t) + I_i(t) \\ \quad + \bigwedge_{j=1}^n \alpha_{ij}(t)f_j(x_j(q_{ij}t)) + \bigvee_{j=1}^n \beta_{ij}(t)f_j(x_j(q_{ij}t)) \\ \quad + \bigwedge_{j=1}^n T_{ij}(t)u_j(t) + \bigvee_{j=1}^n H_{ij}(t)u_j(t), t \geq t_0 \geq 0, i \in \Lambda = \{1, 2, \dots, n\}, \end{array} \right. \quad (1)$$

where $\alpha_{ij}(t), \beta_{ij}(t), T_{ij}(t), H_{ij}(t)$ are the elements of the fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feedforward MIN template and fuzzy feedforward MAX template, respectively; $a_{ij}(t)$ is the element of feedback template and $c_{ij}(t)$ is the element of the feedforward template; \bigwedge, \bigvee denote the fuzzy AND and fuzzy OR operation, respectively; $x_i(t), u_i(t)$ and $I_i(t)$ denote the state, input and bias of the i th neuron, respectively; $d_i(t)$ is a diagonal matrix and it represents the rate coefficient; $f(\cdot)$ is the activation function. $\delta_i(t)$ is the transmission delay. $q_{ij}, i, j \in \Lambda$ are proportional delay factors and satisfy $0 < q_{ij} \leq 1$, and $q_{ij}t = t - (1 - q_{ij})t$, in which $\tau_{ij}(t) = (1 - q_{ij})t$ is the transmission delay function, and $(1 - q_{ij})t \rightarrow \infty$ as $q_{ij} \neq 1, t \rightarrow \infty$.

The initial conditions of system (1) are given by

$$x_i(s) = \varphi_i(s), s \in [t_0 - \varrho, t_0], i \in \Lambda, \quad (2)$$

where $\varrho = \max_{i \in \Lambda} \{\delta_i^+, q^+ t_0\}$, $\delta_i^+ = \sup_{t \in \mathbb{R}} \delta_i(t)$, $q^+ = \sup_{t \in \mathbb{R}} q_{ij}$ and $\varphi_i(\cdot)$ denotes a real-valued continuous function defined on $[t_0 - \varrho, t_0]$.

The following assumptions are given:

(P1) $d_i, \delta_i, a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij}, H_{ij}, T_{ij}, I_i, u_i : \mathbb{R} \rightarrow \mathbb{R}$ are almost periodic functions, $i, j \in \Lambda$.

(P2) $\forall i \in \Lambda$, $M[d_i] = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_t^{t+T} d_i(\theta) d\theta > 0$ and there exist a bounded continuous function: $d_i^* : \mathbb{R} \rightarrow (0, +\infty)$ and a positive constant κ_i such that $e^{-\int_s^t d_i(\theta) d\theta} \leq \kappa_i e^{-\int_s^t d_i^*(\theta) d\theta} \forall t, s \in \mathbb{R}$ and $t - s \geq 0$.

(P3) $\forall i \in \Lambda$, there exists a constant $L_j \geq 0$ such that $|f_j(u) - f_j(v)| \leq L_j|u - v|, \forall u, v \in \mathbb{R}$.

(P4) $\forall i \in \Lambda, \exists \varrho_1 > 0, \varrho_2 > 0, \dots, \varrho_n > 0$ and $\eta_i > 0$ such that

$$\sup_{t \in \mathbb{R}} \{-d_i^*(t) + d_i^+ \delta_i^+ + \kappa_i \left[\varrho_i^{-1} \sum_{j=1}^n |a_{ij}(t)| L_j \varrho_j + \varrho_i^{-1} \sum_{j=1}^n |\alpha_{ij}(t)| L_j \varrho_j + \varrho_i^{-1} \sum_{j=1}^n |\beta_{ij}(t)| L_j \varrho_j \right]\} < -\eta_i < 0.$$

Remark 2.1. In (P1)-(P4), all the parameters and functions have actual implications on neural networks. For given the initial value (2), the hypothesis (P1) and (P4) are the need of theoretical derivation to obtain the existence and global attractivity of almost periodic solutions. The hypotheses (P2) and (P3) can ensure the existence and uniqueness of solution for model (1).

Firstly, we give some definitions, lemmas and notations.

We denote by $\mathbb{R}^n (\mathbb{R} = \mathbb{R}^1)$ the set of all n -dimensional real vector. A matrix or vector $M \geq 0$ means that all entries of M are greater than or equal to zero. $M > 0$ can be defined in a similar way. For matrices or vectors M_1 and M_2 , $M_1 \geq M_2$ (resp. $M_1 > M_2$) means that $M_1 - M_2 \geq 0$ (resp. $M_1 - M_2 > 0$). $BC(\mathbb{R}, \mathbb{R}^n)$ denotes the set of bounded and continuous functions from \mathbb{R} to \mathbb{R}^n . $AP(\mathbb{R}, \mathbb{R}^n)$ denotes the set of all almost periodic functions from \mathbb{R} to \mathbb{R}^n . Denote

$$h^+ = \sup_{t \in \mathbb{R}} |h(t)|, h^- = \inf_{t \in \mathbb{R}} |h(t)|,$$

where $h : \mathbb{R} \rightarrow \mathbb{R}$ is an almost periodic function.

If $X = \{\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T | \varphi_i \in C^1(\mathbb{R}, \mathbb{R}), \varphi_i \text{ is the almost periodic function on } \mathbb{R}, i \in \Lambda\}$ with the norm $\|\varphi\| = \max\{|\varphi|_0, |\dot{\varphi}|_0\}, |\varphi|_0 = \max_{1 \leq i \leq n} \varphi_i^+, |\dot{\varphi}|_0 = \max_{1 \leq i \leq n} (\dot{\varphi}_i)^+, C^1(\mathbb{R}, \mathbb{R})$ is the set of continuous functions with continuous derivatives on \mathbb{R} , then X is a Banach space.

Lemma 2.2. [46] Suppose that x_j and y_j are two states of system (1), then

$$\left| \bigwedge \alpha_{ij}(t) f_j(x_j) - \bigwedge \alpha_{ij}(t) f_j(y_j) \right| \leq \sum_{j=1}^n |\alpha_{ij}(t)| |f_j(x_j) - f_j(y_j)|$$

and

$$\left| \bigvee \beta_{ij}(t) f_j(x_j) - \bigvee \beta_{ij}(t) f_j(y_j) \right| \leq \sum_{j=1}^n |\beta_{ij}(t)| |f_j(x_j) - f_j(y_j)|.$$

Lemma 2.3. $\forall i \in \Lambda$, if $\delta_i(t), x_j(t) \in AP(\mathbb{R}, \mathbb{R})$ and $q \in \mathbb{R} (q > 0)$, then $x_j(qt), d_i(t) \int_{t-\delta_i(t)}^t \dot{x}_j(s) ds \in AP(\mathbb{R}, \mathbb{R})$.

Proof. Let $y_j(t) = x_j(qt), j \in \Lambda$. Since $x_j(t)$ is almost periodic, then $\forall \epsilon > 0, \exists \iota = \iota(\epsilon) > 0, \forall$ interval with length $\iota, \exists \sigma$ in this interval such that

$$|x_j(t + \sigma) - x_j(t)| < \epsilon, t \in \mathbb{R}. \quad (3)$$

For the given ϵ above, we choose $\frac{\epsilon}{q} > 0$, then $[c, c + \frac{\epsilon}{q}] > 0 (c \in \mathbb{R})$ is an arbitrary interval with length $\frac{\epsilon}{q}$. Then $\exists \rho \in [qc, qc + \iota]$ such that (2.1) holds. Clearly, $\frac{\rho}{q} \in [c, c + \frac{\epsilon}{q}]$. Thus one has

$$\left| y_j \left(t + \frac{\rho}{q} \right) - y_j(t) \right| = \left| x_j \left(q \left(t + \frac{\rho}{q} \right) \right) - x_j(qt) \right| = |x_j(qt + \rho) - x_j(qt)| < \epsilon, t \in \mathbb{R},$$

which implies that $x_j(qt) \in AP(\mathbb{R}, \mathbb{R})$. In a similar way, we can also prove that $d_i(t) \int_{t-\delta_i(t)}^t \dot{x}_j(s) ds \in AP(\mathbb{R}, \mathbb{R})$. \square

Lemma 2.4. $\forall i, j \in \Lambda, q_{ij} \in \mathbb{R}$, if $x_j(t), \delta_i(t), \alpha_{ij}(t), \beta_{ij}(t), H_{ij}(t), T_{ij}(t) \in AP(\mathbb{R}, \mathbb{R})$ and (P3) holds, then

$$\bigwedge_{j=1}^n \alpha_{ij}(t) f_j(x_j(q_{ij}t)), \bigvee_{j=1}^n \beta_{ij}(t) f_j(x_j(q_{ij}t)), \bigwedge_{j=1}^n T_{ij}(t) u_j(t), \bigvee_{j=1}^n H_{ij}(t) u_j(t) \in AP(\mathbb{R}, \mathbb{R}).$$

Proof. It follows from (P3) that $f_j (j \in \Lambda)$ is uniformly continuous on \mathbb{R} . In view of Lemma 2.3 and [47, Theorem 1.9], one has

$$f_j(x_j(q_{ij}t)) \in AP(\mathbb{R}, \mathbb{R}), i, j \in \Lambda.$$

Denote $A^f = \max_{i,j \in \Lambda} \{\sup_{t \in \mathbb{R}} |f_j(x_j(q_{ij}t))|\}$, $A^\alpha = \max_{i,j \in \Lambda} \{\sup_{t \in \mathbb{R}} |\alpha_{ij}(t)|\}$. $\forall \epsilon > 0$, [11, Corollary 2.3], $\exists \iota = \iota(\epsilon) > 0$ with the property that \forall interval length ι , $\exists \varsigma = \varsigma(\epsilon)$ in this interval such that

$$|\alpha_{ij}(t + \varsigma) - \alpha_{ij}(t)| < \frac{\epsilon}{2n(A^f + A^\alpha)}, i, j \in \Lambda, \quad (4)$$

and

$$|f_j(x_j(q_{ij}(t + \varsigma))) - f_j(x_j(q_{ij}(t)))| < \frac{\epsilon}{2n(A^f + A^\alpha)}, i, j \in \Lambda, \quad (5)$$

It follows from (3), (4) and Lemma 2.2 that

$$\begin{aligned} & \left| \bigwedge_{j=1}^n \alpha_{ij}(t + \varsigma) f_j(x_j(q_{ij}(t + \varsigma))) - \bigwedge_{j=1}^n \alpha_{ij}(t) f_j(x_j(q_{ij}t)) \right| \leq \left| \bigwedge_{j=1}^n \alpha_{ij}(t + \varsigma) f_j(x_j(q_{ij}(t + \varsigma))) - \bigwedge_{j=1}^n \alpha_{ij}(t) f_j(x_j(q_{ij}(t + \varsigma))) \right| \\ & + \left| \bigwedge_{j=1}^n \alpha_{ij}(t) f_j(x_j(q_{ij}(t + \varsigma))) - \bigwedge_{j=1}^n \alpha_{ij}(t) f_j(x_j(q_{ij}t)) \right| \leq \sum_{j=1}^n |f_j(x_j(q_{ij}(t + \varsigma)))| |\alpha_{ij}(t + \varsigma) - \alpha_{ij}(t)| \\ & + \sum_{j=1}^n |\alpha_{ij}(t)| |f_j(x_j(q_{ij}(t + \varsigma))) - f_j(x_j(q_{ij}t))| \leq A^f \sum_{j=1}^n |\alpha_{ij}(t + \varsigma) - \alpha_{ij}(t)| + A^\alpha \sum_{j=1}^n |f_j(x_j(q_{ij}(t + \varsigma))) - f_j(x_j(q_{ij}t))| \\ & \leq A^f \sum_{j=1}^n \frac{\epsilon}{2n(A^f + A^\alpha)} + A^\alpha \sum_{j=1}^n \frac{\epsilon}{2n(A^f + A^\alpha)} < \epsilon, \end{aligned}$$

then $\bigwedge_{j=1}^n \alpha_{ij}(t) f_j(x_j(q_{ij}t)) \in AP(\mathbb{R}, \mathbb{R})$. Similarly we can prove that

$$\bigvee_{j=1}^n \beta_{ij}(t) f_j(x_j(q_{ij}t)), \bigwedge_{j=1}^n T_{ij}(t) u_j(t), \bigvee_{j=1}^n H_{ij}(t) u_j(t) \in AP(\mathbb{R}, \mathbb{R}).$$

□

3 Existence of almost periodic solutions

In this section, we consider the existence of almost periodic solutions of system (1).

Theorem 3.1. *If (P1)-(P4) hold, then system (1) has a unique almost periodic solution.*

Proof. It is easy to see that system (1) can be rewritten as the following form:

$$\left\{ \begin{aligned} \dot{x}_i(t) &= -d_i(t)x_i(t) + d_i(t) \int_{t-\delta_i(t)}^t \dot{x}_i(s) ds + \sum_{j=1}^n a_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t) u_j(t) \\ &+ \bigwedge_{j=1}^n \alpha_{ij}(t) f_j(x_j(q_{ij}t)) + \bigvee_{j=1}^n \beta_{ij}(t) f_j(x_j(q_{ij}t)) \\ &+ \bigwedge_{j=1}^n T_{ij}(t) u_j(t) + \bigvee_{j=1}^n H_{ij}(t) u_j(t) + I_i(t), i \in \Lambda. \end{aligned} \right. \quad (6)$$

Let $x_i^*(t) = \varrho_i^{-1} x_i(t)$, $i \in \Lambda$, then (8) becomes:

$$\left\{ \begin{aligned} \dot{x}_i^*(t) &= -d_i(t)x_i^*(t) + d_i(t) \int_{t-\delta_i(t)}^t \dot{x}_i^*(s) ds + \varrho_i^{-1} \sum_{j=1}^n a_{ij}(t) f_j(\varrho_j x_j(t)) + \varrho_i^{-1} \sum_{j=1}^n b_{ij}(t) u_j(t) \\ &+ \varrho_i^{-1} \bigwedge_{j=1}^n \alpha_{ij}(t) f_j(\varrho_j x_j(q_{ij}t)) + \varrho_i^{-1} \bigvee_{j=1}^n \beta_{ij}(t) f_j(\varrho_j x_j(q_{ij}t)) \\ &+ \varrho_i^{-1} \bigwedge_{j=1}^n T_{ij}(t) u_j(t) + \varrho_i^{-1} \bigvee_{j=1}^n H_{ij}(t) u_j(t) + \varrho_i^{-1} I_i(t), i \in \Lambda, \end{aligned} \right. \quad (7)$$

Let $\varphi \in AP(\mathbb{R}, \mathbb{R})$. According to Lemmas 2.3 and ??, we have

$$\begin{aligned}
 & d_i(t) \int_{t-\delta_i(t)}^t \dot{\varphi}_i^*(s) ds + \varrho_i^{-1} \sum_{j=1}^n a_{ij}(t) f_j(\varrho_j \varphi_j(t)) + \varrho_i^{-1} \sum_{j=1}^n b_{ij}(t) u_j(t) \\
 & + \varrho_i^{-1} \bigwedge_{j=1}^n \alpha_{ij}(t) f_j(\varrho_j \varphi_j(q_{ij}t)) + \varrho_i^{-1} \bigvee_{j=1}^n \beta_{ij}(t) f_j(\varrho_j \varphi_j(q_{ij}t)) \\
 & + \varrho_i^{-1} \bigwedge_{j=1}^n T_{ij}(t) u_j(t) + \varrho_i^{-1} \bigvee_{j=1}^n H_{ij}(t) u_j(t) + \varrho_i^{-1} I_i(t) \in AP(\mathbb{R}, \mathbb{R}), i \in \Lambda,
 \end{aligned} \tag{8}$$

Since $M[c_i] > 0 (i \in \Lambda)$, by (8) and Lemma 2.1 of Wu and Chen [40], we can conclude that the following equation

$$\left\{ \begin{aligned}
 \dot{x}_i^*(t) &= -d_i(t)x_i^*(t) + d_i(t) \int_{t-\delta_i(t)}^t \dot{\varphi}_i^*(s) ds + \varrho_i^{-1} \sum_{j=1}^n a_{ij}(t) f_j(\varrho_j \varphi_j(t)) + \varrho_i^{-1} \sum_{j=1}^n b_{ij}(t) u_j(t) \\
 &+ \varrho_i^{-1} \bigwedge_{j=1}^n \alpha_{ij}(t) f_j(\varrho_j \varphi_j(q_{ij}t)) + \varrho_i^{-1} \bigvee_{j=1}^n \beta_{ij}(t) f_j(\varrho_j \varphi_j(q_{ij}t)) \\
 &+ \varrho_i^{-1} \bigwedge_{j=1}^n T_{ij}(t) u_j(t) + \varrho_i^{-1} \bigvee_{j=1}^n H_{ij}(t) u_j(t) + \varrho_i^{-1} I_i(t), i \in \Lambda,
 \end{aligned} \right. \tag{9}$$

has exactly one almost periodic solution:

$$\begin{aligned}
 x^\varphi(t) = \{x_i^\varphi(t)\} &= \left\{ \int_{-\infty}^t e^{-\int_s^t d_i(\theta) d\theta} \left[d_i(t) \int_{s-\delta_i(s)}^s \dot{\varphi}_i^*(\theta) d\theta + \varrho_i^{-1} \sum_{j=1}^n a_{ij}(s) f_j(\varrho_j \varphi_j(s)) \right. \right. \\
 &+ \varrho_i^{-1} \sum_{j=1}^n b_{ij}(s) u_j(s) + \varrho_i^{-1} \bigwedge_{j=1}^n \alpha_{ij}(s) f_j(\varrho_j \varphi_j(q_{ij}s)) + \varrho_i^{-1} \bigvee_{j=1}^n \beta_{ij}(s) f_j(\varrho_j \varphi_j(q_{ij}s)) \\
 &\left. \left. + \varrho_i^{-1} \bigwedge_{j=1}^n T_{ij}(s) u_j(s) + \varrho_i^{-1} \bigvee_{j=1}^n H_{ij}(s) u_j(s) + \varrho_i^{-1} I_i(s) \right] ds \right\}, i \in \Lambda.
 \end{aligned} \tag{10}$$

Define a mapping: $\Gamma : AP(\mathbb{R}, \mathbb{R}^n) \rightarrow AP(\mathbb{R}, \mathbb{R}^n)$ as follows:

$$(\Gamma\varphi)(t) = x^\varphi(t), \text{ for all } \varphi \in AP(\mathbb{R}, \mathbb{R}^n). \tag{11}$$

Next we will prove that Γ is a contraction mapping of $AP(\mathbb{R}, \mathbb{R}^n)$. In fact, by (10), (P2)-(P4), for $\varphi, \phi \in AP(\mathbb{R}, \mathbb{R}^n)$, one has

$$\begin{aligned}
 |(\Gamma\varphi)(t) - (\Gamma\phi)(t)| &= \{ |((\Gamma\varphi)(t) - (\Gamma\phi)(t))_i| \} \\
 &= \left\{ \left| \int_{-\infty}^t e^{-\int_s^t d_i(\theta) d\theta} \left[d_i(t) \int_{s-\delta_i(s)}^s (\dot{\varphi}_i^*(\theta) - \dot{\phi}_i^*(\theta)) d\theta \right. \right. \right. \\
 &\quad + \varrho_i^{-1} \sum_{j=1}^n a_{ij}(s) (f_j(\varrho_j \varphi_j(s)) - f_j(\varrho_j \phi_j(s))) \\
 &\quad + \varrho_i^{-1} \bigwedge_{j=1}^n \alpha_{ij}(s) (f_j(\varrho_j \varphi_j(q_{ij}s)) - f_j(\varrho_j \phi_j(q_{ij}s))) \\
 &\quad \left. \left. + \varrho_i^{-1} \bigvee_{j=1}^n \beta_{ij}(s) (f_j(\varrho_j \varphi_j(q_{ij}s)) - f_j(\varrho_j \phi_j(q_{ij}s))) \right] ds \right\} \\
 &\leq \left\{ \kappa_i e^{-\int_s^t d_i^*(\theta) d\theta} \left[d_i^+ \delta_i^+ |\dot{\varphi}_i^* - \dot{\phi}_i^*|_0 \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + \varrho_i^{-1} \sum_{j=1}^n |a_{ij}(s)| L_j \varrho_j |\varphi_j(s) - \phi_j(s)| \\
& + \varrho_i^{-1} \sum_{j=1}^n |\alpha_{ij}(s)| L_j \varrho_j |\varphi_j(q_{ij}s) - \phi_j(q_{ij}s)| \\
& + \varrho_i^{-1} \sum_{j=1}^n |\beta_{ij}(s)| L_j \varrho_j |\varphi_j(q_{ij}s) - \phi_j(q_{ij}s)| \Big] ds \Bigg\}.
\end{aligned}$$

In view of the definition of norm, one has

$$\begin{aligned}
|(\Gamma\varphi)(t) - (\Gamma\phi)(t)| & \leq \left\{ \kappa_i e^{-\int_s^t d_i^*(\theta) d\theta} \left[d_i^+ \delta_i^+ + \varrho_i^{-1} \sum_{j=1}^n |a_{ij}(s)| L_j \varrho_j \right. \right. \\
& \left. \left. + \varrho_i^{-1} \sum_{j=1}^n |\alpha_{ij}(s)| L_j \varrho_j + \varrho_i^{-1} \sum_{j=1}^n |\beta_{ij}(s)| L_j \varrho_j \right] ds \|\varphi - \phi\| \right\} \\
& \leq \left\{ \int_{-\infty}^t e^{-\int_s^t d_i^*(\theta) d\theta} [d_i^*(s) - \eta_i] ds \|\varphi - \phi\| \right\} \\
& \leq \left\{ \int_{-\infty}^t e^{-\int_s^t d_i^*(\theta) d\theta} \left[1 - \frac{\eta_i}{(d_i^*)^+} \right] d_i^*(s) ds \|\varphi - \phi\| \right\}.
\end{aligned}$$

Then

$$|(\Gamma\varphi)(t) - (\Gamma\phi)(t)| \leq \left\{ \left[1 - \frac{\eta_i}{(d_i^*)^+} \right] \|\varphi - \phi\| \right\}. \quad (12)$$

Thus

$$\|\Gamma\varphi - \Gamma\phi\| \leq \max_{i \in \Lambda} \left[1 - \frac{\eta_i}{(d_i^*)^+} \right] \|\varphi - \phi\|. \quad (13)$$

It follows from (13) and (P4) that Γ is a contraction mapping. Thus Γ has a unique fixed point

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \in AP(\mathbb{R}, \mathbb{R}^n), \Gamma \bar{x} = \bar{x}$$

and \bar{x} satisfies (9). Therefore, (1) has a unique almost periodic solution $\tilde{x} = (\varrho_1 \bar{x}_1, \varrho_2 \bar{x}_2, \dots, \varrho_n \bar{x}_n)$. \square

4 Global attractivity of almost periodic solutions

In this section, we will focus on the global attractivity of almost periodic solution of (1).

Theorem 4.1. *If (P1)-(P4) hold, system (1) has a unique almost periodic solution \tilde{x} which is global attractivity, namely, for arbitrary solution $x(t)$ of system (1) with initial value (2), there exist constants $\lambda > 0$ and $K > 0$ such that*

$$|x_j(t) - \tilde{x}_j(t)| \leq K^* \frac{\max_{i \in \Lambda} \{\max_{t \in [t_0 - \varrho, t_0]} |\varphi_i(t) - \tilde{x}_i(t)|\}}{(1+t)^\mu}, \text{ for all } t > t_0, j \in \Lambda.$$

Proof. From Theorem 3.1, we know that system (1) has a unique almost periodic solution $\tilde{x}(t) = \{\tilde{x}_i(t)\}$. Assume that $x(t) = \{x_i(t)\}$ is an arbitrary solution of system (1) with initial condition $\varphi(t) = \{\varphi_i(t)\}$ satisfying (2). Let $u(t) = \{u_i(t)\} = \{(x_i(t) - \tilde{x}_i(t)), i \in \Lambda, t \geq t_0 - \varrho$, and $\|u\|_\varrho = \{\max_{t \in [t_0 - \varrho, t_0]} |\varphi_i(t) - \tilde{x}_i(t)|\}$. Then one has

$$\left\{ \begin{aligned}
& \dot{u}_i(t) = -d_i(t)u_i(t) + d_i(t) \int_{t-\delta_i(t)}^t \dot{u}_i(s) ds \\
& \quad + \sum_{j=1}^n a_{ij}(t)[f_j(x_j(t)) - f_j(\tilde{x}_j(t))] \\
& \quad + \bigwedge_{j=1}^n \alpha_{ij}(t)[f_j(x_j(q_{ij}t)) - f_j(\tilde{x}_j(q_{ij}t))] \\
& \quad + \bigvee_{j=1}^n \beta_{ij}(t)[f_j(x_j(q_{ij}t)) - f_j(\tilde{x}_j(q_{ij}t))], i \in \Lambda.
\end{aligned} \right. \quad (14)$$

Define a continuous function $\Psi_i(\nu)$ as follows:

$$\Psi_i(\nu) = \sup_{t \geq t_0} \left\{ \mu \varrho_i - d_i^*(t) \varrho_i + d_i^+ \delta_i^+ \varrho_i + \kappa_i \left[\sum_{j=1}^n |a_{ij}(t)| L_j \varrho_j + \sum_{j=1}^n |\alpha_{ij}(t)| L_j \varrho_j e^{\nu \ln \frac{1}{q_{ij}}} \varrho_j + \sum_{j=1}^n |\beta_{ij}(t)| L_j \varrho_j e^{\nu \ln \frac{1}{q_{ij}}} \right] \right\},$$

where $\mu \in [0, \min_{i \in \Lambda} \inf_{t \geq t_0, i \in \Lambda} d_i^*(t)]$. By (A4), one has

$$\begin{aligned} \Psi_i(0) = \sup_{t \geq t_0} \left\{ -d_i^*(t) \varrho_i + d_i^+ \delta_i^+ \varrho_i + \kappa_i \left[\sum_{j=1}^n |a_{ij}(t)| L_j \varrho_j \right. \right. \\ \left. \left. + \sum_{j=1}^n |\alpha_{ij}(t)| L_j \varrho_j + \sum_{j=1}^n |\beta_{ij}(t)| L_j \varrho_j \right] \right\} < 0, i \in \Lambda. \end{aligned} \quad (15)$$

In view of the continuity of $\Theta_i(\mu)$ and note that

$$\frac{\mu \varrho_i}{1+t} \leq \mu \varrho_i, \ln \left(\frac{1+t}{1+q_{ij}t} \right) \leq \ln \frac{1}{q_{ij}}$$

for all $\mu \geq 0, t \geq 0$, then we can choose a constant $\mu \in (0, \min_{t \in \Lambda} \inf_{i \geq 0} d_i^*(t))$ such that

$$\begin{aligned} & \sup_{t \geq t_0} \left\{ \frac{\mu \varrho_i}{1+t} - d_i^*(t) \varrho_i + d_i^+ \delta_i^+ \varrho_i + \kappa_i \left[\sum_{j=1}^n |a_{ij}(t)| L_j \varrho_j \right. \right. \\ & \left. \left. + \sum_{j=1}^n |\alpha_{ij}(t)| L_j \varrho_j e^{\mu \ln \left(\frac{1+t}{1+q_{ij}t} \right)} \varrho_j + \sum_{j=1}^n |\beta_{ij}(t)| L_j \varrho_j e^{\mu \ln \left(\frac{1+t}{1+q_{ij}t} \right)} \right] \right\} \\ & \leq \sup_{t \geq t_0} \left\{ \mu \varrho_i - d_i^*(t) \varrho_i + d_i^+ \delta_i^+ \varrho_i + \kappa_i \left[\sum_{j=1}^n |a_{ij}(t)| L_j \varrho_j \right. \right. \\ & \left. \left. + \sum_{j=1}^n |\alpha_{ij}(t)| L_j \varrho_j e^{\mu \ln \frac{1}{q_{ij}}} \varrho_j + \sum_{j=1}^n |\beta_{ij}(t)| L_j \varrho_j e^{\mu \ln \frac{1}{q_{ij}}} \right] \right\} = \Psi_i(\mu) < 0. \end{aligned} \quad (16)$$

$\forall \epsilon > 0$, we define functions $V_i(t) (i \in \Lambda)$ as follows:

$$V_i(t) = K \varrho_*^{-1} (\|u\|_\varrho + \epsilon) \varrho_i e^{-\mu \ln \frac{1+t}{1+t_0}}, \quad (17)$$

where $K \geq \max\{\max_{i \in \Lambda} \kappa_i, 1\}$, $\varrho_* = \min_{i \in \Lambda} \varrho_i, t \geq 0$. Then

$$\begin{aligned} V_l(q_{lj}t) &= K \varrho_*^{-1} (\|u\|_\varrho + \epsilon) \varrho_l e^{-\mu \ln \frac{1+q_{lj}t}{1+t_0}} \\ &= K \varrho_*^{-1} (\|u\|_\varrho + \epsilon) \varrho_l e^{-\mu \ln \frac{1+t}{1+t_0}} e^{\mu \ln \frac{1+t}{1+q_{lj}t}} \\ &\leq V_l(q_{lj}t) e^{\mu \ln \frac{1}{1+q_{lj}}} \end{aligned} \quad (18)$$

$\forall t \geq t_0, l, j \in \Lambda$. Thus

$$|u_i(t_0)| = \varrho_*^{-1} (\|u\|_\varrho + \epsilon) \varrho_i \leq K \varrho_*^{-1} (\|u\|_\varrho + \epsilon) \varrho_i = V_i(t_0), i \in \Lambda. \quad (19)$$

Now we will show that

$$|u_i(t)| < V_i(t), \text{ for all } t > t_0, i \in \Lambda. \quad (20)$$

If (21) does not hold, there must exist $i \in \Lambda$ and $\vartheta > t_0$ such that

$$V_i(\vartheta) = K \varrho_* (\|u\|_\varrho + \epsilon) \varrho_i e^{-\mu \ln \frac{1+\vartheta}{1+t_0}} \quad (21)$$

and

$$|u_j(t)| < V_j(t) \text{ for all } t \in [t_0, \vartheta], j \in \Lambda. \quad (22)$$

Notice that

$$\left\{ \begin{aligned} \dot{u}_i(s) + d_i(s)u_i(s) &= d_i(s) \int_{s-\delta_i(s)}^s \dot{u}_i(\theta) d\theta + \sum_{j=1}^n a_{ij}(s)[f_j(x_j(s)) - f_j(\varrho_j \tilde{x}_j(s))] \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}(s)[f_j(x_j(q_{ij}s)) - f_j(\tilde{x}_j(q_{ij}s))] \\ &\quad + \bigvee_{j=1}^n \beta_{ij}(s)[f_j(x_j(q_{ij}s)) - f_j(\tilde{x}_j(q_{ij}s))], \end{aligned} \right. \quad (23)$$

where $s \in [t_0, t]$, $t \in [t_0, \vartheta]$, $i \in \Lambda$. (24) $\times e^{\int_{t_0}^s d_i(\theta) d\theta}$ and then integrating it on $[t_0, t]$, one has

$$\left\{ \begin{aligned} \dot{u}_i(t) &= d_i(t_0)e^{-\int_{t_0}^t d_i(\theta) d\theta} + \int_{t_0}^t e^{-\int_s^t d_i(\theta) d\theta} \left\{ d_i(s) \int_{s-\delta_i(s)}^s \dot{u}_i(\theta) d\theta \right. \\ &\quad + \sum_{j=1}^n a_{ij}(s)[f_j(\varrho_j x_j(s)) - f_j(\varrho_j \tilde{x}_j(s))] \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}(s)[f_j(x_j(q_{ij}s)) - f_j(\tilde{x}_j(q_{ij}s))] \\ &\quad \left. + \bigvee_{j=1}^n \beta_{ij}(s)[f_j(x_j(q_{ij}s)) - f_j(\tilde{x}_j(q_{ij}s))] \right\} ds, \end{aligned} \right. \quad (24)$$

According to the Lemma 2.1, (17),(19),(20) and (23), we have

$$\begin{aligned} |\dot{u}_i(\vartheta)| &= \left| d_i(t_0)e^{-\int_{t_0}^{\vartheta} d_i(\theta) d\theta} + \int_{t_0}^{\vartheta} e^{-\int_s^{\vartheta} d_i(\theta) d\theta} \left\{ d_i(s) \int_{s-\delta_i(s)}^s \dot{u}_i(\theta) d\theta \right. \right. \\ &\quad + \sum_{j=1}^n a_{ij}(s)[f_j(x_j(s)) - f_j(\tilde{x}_j(s))] \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}(s)[f_j(x_j(q_{ij}s)) - f_j(\varrho_j \tilde{x}_j(q_{ij}s))] \\ &\quad \left. \left. + \bigvee_{j=1}^n \beta_{ij}(s)[f_j(x_j(q_{ij}s)) - f_j(\tilde{x}_j(q_{ij}s))] \right\} ds \right| \\ &\leq \varrho_*^{-1}(\|u\|_{\varrho} + \epsilon) \varrho_i \kappa_i e^{-\int_{t_0}^{\vartheta} d_i^*(\theta) d\theta} + \int_{t_0}^{\vartheta} e^{-\int_s^{\vartheta} d_i^*(\theta) d\theta} \kappa_i \left[d_i^+ \delta_i^+ + \sum_{j=1}^n |a_{ij}(s)| L_j |u_j(s)| \right. \\ &\quad \left. + \sum_{j=1}^n |\alpha_{ij}(s)| L_j |u_j(q_{ij}s)| + \sum_{j=1}^n |\beta_{ij}(s)| L_j |u_j(q_{ij}s)| \right] ds. \end{aligned}$$

By (18), one has

$$\begin{aligned} |\dot{u}_i(\vartheta)| &\leq \varrho_*^{-1}(\|u\|_{\varrho} + \epsilon) \varrho_i \kappa_i e^{-\int_{t_0}^{\vartheta} d_i^*(\theta) d\theta} + \int_{t_0}^{\vartheta} e^{-\int_s^{\vartheta} d_i^*(\theta) d\theta} \kappa_i \left[d_i^+ \delta_i^+ + \sum_{j=1}^n |a_{ij}(s)| L_j \right. \\ &\quad \left. + \sum_{j=1}^n |\alpha_{ij}(s)| L_j e^{-\mu \ln \frac{1+s}{1+q_{ij}s}} + \sum_{j=1}^n |\beta_{ij}(s)| L_j e^{-\mu \ln \frac{1+s}{1+q_{ij}s}} \right] \\ &\quad \times K \varrho_*^{-1}(\|u\|_{\varrho} + \epsilon) \varrho_j e^{-\mu \ln \frac{1+s}{1+t_0}} ds \\ &\leq K \varrho_*^{-1}(\|u\|_{\varrho} + \epsilon) \varrho_i e^{-\mu \ln \frac{1+\vartheta}{1+t_0}} \left\{ \frac{\kappa_i}{K} e^{-\int_{t_0}^{\vartheta} (d_i^*(\theta) - \frac{\mu}{1+\vartheta}) d\theta} \right\} \end{aligned}$$

$$\begin{aligned}
 & + \int_{t_0}^{\vartheta} e^{-\int_s^{\vartheta} (d_i^*(\theta) - \frac{\mu}{1+\theta}) d\theta} \varrho_i^{-1} \kappa_i \left[d_i^+ \delta_i^+ + \sum_{j=1}^n |a_{ij}(s)| L_j \varrho_j \right. \\
 & \left. + \sum_{j=1}^n |\alpha_{ij}(s)| L_j \varrho_j e^{\mu \ln \frac{1+s}{1+q_{ij}s}} + \sum_{j=1}^n |\beta_{ij}(s)| L_j \varrho_j e^{-\mu \ln \frac{1+s}{1+q_{ij}s}} \right] ds \Big\} \\
 = & K \varrho_*^{-1} (\|u\|_{\varrho} + \epsilon) \varrho_i e^{-\mu \ln \frac{1+\vartheta}{1+t_0}} \left\{ \frac{\kappa_i}{K} e^{-\int_{t_0}^{\vartheta} (d_i^*(\theta) - \frac{\mu}{1+\theta}) d\theta} \right. \\
 & \left. + \int_{t_0}^{\vartheta} e^{-\int_s^{\vartheta} (d_i^*(\theta) - \frac{\mu}{1+\theta}) d\theta} \left(d_i^*(s) - \frac{\mu}{1+s} \right) ds \right\} \\
 \leq & K \varrho_*^{-1} (\|u\|_{\varrho} + \epsilon) \varrho_i e^{-\mu \ln \frac{1+\vartheta}{1+t_0}}. \tag{25}
 \end{aligned}$$

In view of (22) and (26), we know that (21) holds. Let $\epsilon \rightarrow 0^+$, then

$$\|x(t) - \tilde{x}(t)\| = \max_{i \in \Lambda} |u_i(t)| \leq K^* \frac{\max_{i \in \Lambda} \{ \max_{t \in [t_0 - \varrho, t_0]} |\varphi_i(t) - \tilde{x}_i(t)| \}}{(1+t)^\mu},$$

for all $t \geq t_0, j \in \Lambda$, where $K^* = K \varrho_*^{-1} \max_{i \in \Lambda} \varrho_i (1+t_0)^\mu$. □

Remark 4.2. Huang [16] investigated the almost periodic solutions for fuzzy cellular neural networks with time-varying delays, Huang [15] studied the almost periodic solutions for fuzzy cellular neural networks with proportional delays, the models in [15-16] does not involves leakage delay. In this paper, we studies the the existence and global attractivity of almost periodic solutions of fuzzy cellular neural networks with leakage delays and proportional delays. All the obtained results in [15-16] can not be applicable to model (1) of this paper to obtain the existence and global attractivity of almost periodic solutions. Up to now, there are no results on existence and global attractivity of almost periodic solutions for fuzzy neural networks with leakage delays and proportional delays. From the viewpoint, our results on the almost periodic solutions for fuzzy cellular neural networks are essentially new and complement previously known results to some extent.

5 Examples

Considering the following model

$$\left\{ \begin{aligned}
 \dot{x}_1(t) &= -d_1(t)x_1(t - \delta_1(t)) + \sum_{j=1}^2 a_{1j}(t)f_j(x_j(t)) + \sum_{j=1}^2 b_{ij}(t)u_j(t) \\
 &+ \bigwedge_{j=1}^2 \alpha_{1j}(t)f_j(x_j(q_{1j}t)) + \bigvee_{j=1}^2 \beta_{1j}(t)f_j(x_j(q_{1j}t)) \\
 &+ \bigwedge_{j=1}^2 T_{1j}(t)u_j(t) + \bigvee_{j=1}^2 H_{1j}(t)u_j(t) + I_1(t), \\
 \dot{x}_2(t) &= -d_2(t)x_2(t - \delta_2(t)) + \sum_{j=1}^2 a_{2j}(t)f_j(x_j(t)) + \sum_{j=1}^2 b_{2j}(t)u_j(t) \\
 &+ \bigwedge_{j=1}^2 \alpha_{2j}(t)f_j(x_j(q_{2j}t)) + \bigvee_{j=1}^2 \beta_{2j}(t)f_j(x_j(q_{2j}t)) \\
 &+ \bigwedge_{j=1}^2 T_{2j}(t)u_j(t) + \bigvee_{j=1}^2 H_{2j}(t)u_j(t) + I_2(t),
 \end{aligned} \right. \tag{26}$$

where $f_1(u) = f_2(u) = 0.5(|u+1| - |u-1|)$ and

$$\begin{aligned} \begin{bmatrix} d_1(t) & \delta_1(t) \\ d_2(t) & \delta_2(t) \end{bmatrix} &= \begin{bmatrix} 2 + 0.8 \sin 10t & 0.01 |\sin(10\pi t)| \\ 2 + 0.8 \cos 10t & 0.02 |\sin(10\pi t)| \end{bmatrix}, \\ \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} &= \begin{bmatrix} 0.01 |\cos(10\pi t)| & 0.02 |\cos(10\pi t)| \\ 0.01 |\sin(18\pi t)| & 0.01 |\sin(18\pi t)| \end{bmatrix}, \\ \begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix} &= \begin{bmatrix} 0.01 |\sin(15\pi t)| & 0.01 |\cos(12\pi t)| \\ 0.01 |\sin(15\pi t)| & 0.01 |\cos(12\pi t)| \end{bmatrix}, \\ \begin{bmatrix} \alpha_{11}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) \end{bmatrix} &= \begin{bmatrix} 0.02 |\sin(15\pi t)| & 0.02 |\sin(15\pi t)| \\ 0.03 |\cos(15\pi t)| & 0.03 |\sin(15\pi t)| \end{bmatrix}, \\ \\ \begin{bmatrix} \beta_{11}(t) & \beta_{12}(t) \\ \beta_{21}(t) & \beta_{22}(t) \end{bmatrix} &= \begin{bmatrix} 0.02 |\cos(10\pi t)| & 0.01 |\cos(12\pi t)| \\ 0.03 |\sin(10\pi t)| & 0.02 |\sin(12\pi t)| \end{bmatrix}, \\ \begin{bmatrix} T_{11}(t) & T_{12}(t) \\ T_{21}(t) & T_{22}(t) \end{bmatrix} &= \begin{bmatrix} 0.05 |\sin(20\pi t)| & 0.05 |\sin(12\pi t)| \\ 0.07 |\cos(20\pi t)| & 0.05 |\cos(12\pi t)| \end{bmatrix}, \\ \begin{bmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{bmatrix} &= \begin{bmatrix} 0.02 |\sin(12\pi t)| & 0.03 |\sin(12\pi t)| \\ 0.01 |\sin(12\pi t)| & 0.02 |\sin(12\pi t)| \end{bmatrix}, \\ \begin{bmatrix} u_1(t) & I_1(t) \\ u_2(t) & I_2(t) \end{bmatrix} &= \begin{bmatrix} 0.26 |\cos(15\pi t)| & 0.36 |\sin(15\pi t)| \\ 0.34 |\cos(15\pi t)| & 0.42 |\sin(15\pi t)| \end{bmatrix}, \\ \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} &= \begin{bmatrix} 0.02 & 0.02 \\ 0.02 & 0.01 \end{bmatrix}. \end{aligned}$$

Hence $L_1 = L_2 = 1$ and

$$\begin{aligned} \begin{bmatrix} d_1^+ & \delta_1^+ \\ d_2^+ & \delta_2^+ \end{bmatrix} &= \begin{bmatrix} 2.8 & 0.01 \\ 2.8 & 0.02 \end{bmatrix}, \begin{bmatrix} a_{11}^+ & a_{12}^+ \\ a_{21}^+ & a_{22}^+ \end{bmatrix} = \begin{bmatrix} 0.01 & 0.2 \\ 0.01 & 0.01 \end{bmatrix}, \\ \begin{bmatrix} b_{11}^+ & b_{12}^+ \\ b_{21}^+ & b_{22}^+ \end{bmatrix} &= \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, \begin{bmatrix} \alpha_{11}^+ & \alpha_{12}^+ \\ \alpha_{21}^+ & \alpha_{22}^+ \end{bmatrix} = \begin{bmatrix} 0.02 & 0.02 \\ 0.03 & 0.03 \end{bmatrix}, \\ \begin{bmatrix} \beta_{11}^+ & \beta_{12}^+ \\ \beta_{21}^+ & \beta_{22}^+ \end{bmatrix} &= \begin{bmatrix} 0.02 & 0.01 \\ 0.03 & 0.02 \end{bmatrix}. \end{aligned}$$

Let $d_i^*(t) = 1.2$, $\kappa_i = e^{\frac{2}{25}}$, then $e^{-\int_s^t d_i(\theta) d\theta} \leq e^{\frac{1}{25}} e^{-(t-s)}$, $i = 1, 2, t \geq s$. Choose $\varrho_1 = \varrho_2 = 1, \eta_1 = \eta_2 = 1$. Then

$$\begin{aligned} & \sup_{t \in \mathbb{R}} \left\{ -d_1^*(t) + d_1^+ \delta_1^+ \varrho_1 + \kappa_1 \left[\varrho_1^{-1} \sum_{j=1}^2 |a_{1j}(t)| L_j \varrho_j \right. \right. \\ & \left. \left. + \varrho_1^{-1} \sum_{j=1}^2 |\alpha_{1j}(t)| L_j \varrho_j + \varrho_1^{-1} \sum_{j=1}^2 |\beta_{1j}(t)| L_j \varrho_j \right] \right\} \\ & \leq \left[-1.2 + 2.8 \times 0.01 + e^{\frac{2}{25}} (0.01 + 0.01 + 0.01 \right. \\ & \quad \left. + 0.01 + 0.02 + 0.02 + 0.02 + 0.01) \right] = -1.0528 < -1, \\ & \sup_{t \in \mathbb{R}} \left\{ -d_2^*(t) + d_2^+ \delta_2^+ \varrho_2 + \kappa_2 \left[\varrho_2^{-1} \sum_{j=1}^2 |a_{2j}(t)| L_j \varrho_j \right. \right. \\ & \left. \left. + \varrho_2^{-1} \sum_{j=1}^2 |\alpha_{2j}(t)| L_j \varrho_j + \varrho_2^{-1} \sum_{j=1}^2 |\beta_{2j}(t)| L_j \varrho_j \right] \right\} \\ & \leq \left[-1.2 + 2.8 \times 0.02 + e^{\frac{2}{25}} (0.01 + 0.01 + 0.01 \right. \\ & \quad \left. + 0.01 + 0.03 + 0.03 + 0.03 + 0.02) \right] - 1.0140 < -1. \end{aligned}$$

Thus all conditions in Theorem 3.1 and Theorem 4.1 are satisfied, then system (27) has exactly one almost periodic solution $\tilde{x}(t)$, which is globally attractive (see Figure 1).

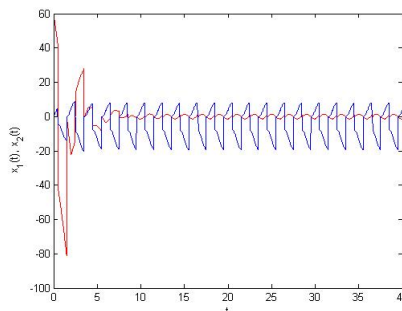


Figure 1: Numerical solutions of system (27), where the blue line denotes x_1 and the red line denotes x_2 .

6 Conclusions

We know that the existence of almost periodic solutions plays an important role in describing the behavior of nonlinear dynamical systems. Thus it has been extensively studied by many researchers during the past decades. In this article, we have discussed fuzzy cellular neural networks with leakage delays and proportional delays. By means of the differential inequality theory, fixed point theorem and almost periodic function theory, a set of sufficient conditions which guarantee the existence and global attractivity of a unique almost periodic solution of almost periodic solutions of fuzzy cellular neural networks with leakage delays and proportional delays are established. These sufficient conditions can be easily tested in practice. The obtained results complement some earlier publications (for example, [15-16,25]). In addition, the method in this article can be applied to some other neural networks with leakage delays and proportional delays. In addition, we point out that the establishment of neural networks with proportional delays further enrich and develop the neural network theory. Also it plays an important role in designing and optimizing networks. It is a pity that the research on synchronization, almost automorphic solution, pseudo almost periodic solutions, finite-time stability of neural networks are very rare. In the future, we will focus on this aspect.

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