Iranian Journal of Fuzzy Systems

Volume 17, Number 1, (2020), pp. 155-169

University of Sistan and Baluchestan

BL-general fuzzy automata and minimal realization: Based on the associated categories

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Abstract

The present paper is an attempt to study the minimal BL-general fuzzy automata which realizes the given fuzzy behavior. Of two methods applied for construction of such automaton presented here, one has been based on Myhill-Nerode's theory while the other has been based on derivatives of the given fuzzy behavior. Meanwhile, the categories of BL-general fuzzy automata and fuzzy behavior, along with a functorial relationship between them, are introduced.

Keywords: Component, BL-general fuzzy automata, fuzzy behavior, category, functor, realization, reachability, observability.

1 Introduction

The study of fuzzy automata has been initiated by Wee and Santos [16] in 1960 after the introduction of fuzzy set theory by Zadeh [22]. As automata constitute a mathematical model of computation, fuzzy finite automata [14] may be considered as an extended model which includes the notions frequently encountered in the study of natural languages, namely vagueness and imprecision. In view of the fact that the study of fuzzy finite automata reduces the gap between formal languages and natural languages, somewhat different notions were introduced subsequently [4,8-12]. In these studies, the membership values in the closed interval [0, 1] were considered. In general, general fuzzy automata provide an attractive systematic way for generalizing discrete applications [5]. Moreover, general fuzzy automata are able to create capabilities which are hardly achievable by other tools. On the other hand, the contribution of GFA to neural networks has been considerable, and dynamical fuzzy systems are becoming more and more popular and useful. Basic logic, or BL for short, has been introduced by Hajek in order to provide a general framework for formalizing statements of fuzzy nature. It is an adequate calculus when we have to do with statements about which we may say that they are true principally only to a certain degree, that is, to which it is in general unreasonable or impossible to assign a sharp yes or no. Formulas of propositional BL may be interpreted by means of BL-algebras [19, 20]. In 2012, by using the definitions of general fuzzy automata and BL-algebra, Abolpour and Zahedi [1, 2] suggested a new definition, that is a generalization of GFA; accordingly, named it as BL-general fuzzy automata. A fairly general definition of a machine in a category has been introduced in the literature by Arbib and Manes [3], which in particular cases encompasses the linear systems (used in control theory), the standard automata with outputs, tree automata and stochastic machines. An interesting problem in this context is the realization problem, which says that given a behavior, we can design a machine which realizes it. This problem has been studied in fairly general category-theoretic setup by Goguen [6] and Arbib and Manes [3]. In particular, Arbib and Manes have provided a minimal realization in the cases of linear systems, automata with outputs and tree automata. After the introduction of fuzzy languages [13, 17, 21], recently, the similar concept has been studied in [7, 15] for fuzzy languages in algebraic setup. In the current study, chiefly inspired from [3, 6], the concept of minimal realization (i. e., minimal BL-general fuzzy automaton) which realizes the given fuzzy behavior in category-theoretic setup has been investigated. For such construction, two concepts have been

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Received: November 2018; Revised: April 2019; Accepted: June 2019.

used, specifically, one is based on Myhill-Nerode's theory and the other is on the basis of derivatives of the given fuzzy behavior. In between, the study also introduces the concepts of reachability, observability, category of BL-general fuzzy automata and category of fuzzy behaviors, as well as the functors between these two categories.

2 Preliminaries

In this section, we recall the concept of BL-general fuzzy automata, which is to be used in the next sections. Throughout, a nonempty set Σ, Σ^* denotes the free monoid generated by Σ. We shall denote by Λ, the identity elemet of Σ^*.

Definition 2.1. [5] *A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted by $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ where (i) Q is a finite set of states, $Q = \{q_1, q_2, ..., q_n\}$, (ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, ..., a_m\}$, (iii) $\tilde{R}$ is the set of fuzzy start states, (iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, ..., b_k\}$, (v) $\omega : Q \to Z$ is the output function, (vi) $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \to [0, 1]$ is the augmented transition function, (vii) $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ is called membership assignment function. Function $F_1(\mu, \delta)$, as seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k, is represented as:*

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

It indicates that membership value (mv) of the state q_j at time $t + 1$, is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition. There are many options which can be used for the function $F_1(\mu, \delta)$. It can be, for example, $max\{\mu, \delta\}$, $min\{\mu, \delta\}$, $\frac{\mu+\delta}{2}$ or any other applicable mathematical function. (viii) $F_2 : [0, 1]^* \to [0, 1]$ is called multi-membership resolution function. The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Example 2.2. [2] *Let Q be a nonempty set. Then $(P(Q), *, \to, \bigcap, \bigcup, \phi, Q)$ is a BL-algebra where $P(Q)$ is a power set of Q.*

Definition 2.3. [2] *Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton and $\overline{Q} = (P(Q), *, \to, \bigcap, \bigcup, \phi, Q)$ be a BL-algebra in Example 2.2. We define the BL-GFA as ten-tuple machine denoted with $\tilde{F}_l = (\overline{Q}, \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z},$
$\omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$, where*
(i) $\overline{Q} = P(Q)$ is a finite set of states,
(ii) Σ is a finite set of input symbols,
(iii) $\tilde{R}$ is the set of fuzzy start states,
(iv) $\overline{Z}$ is a finite set of output symbols where $\overline{Z}$ is a power set of Z,
(v) $\omega_l : \overline{Q} \to \overline{Z}$ is the output function defined by, $\omega_l(Q_i) = \bigcup_{q \in Q_i} \omega(q)$, (vi) $\delta_l : \overline{Q} \times \Sigma \times \overline{Q} \to [0, 1]$ is the transition function defined by: $\delta_l(\{p\}, a, \{q\}) = \delta(p, a, q)$ and $\delta_l(Q_i, a, Q_j) = \bigvee_{q_i \in Q_i, q_j \in Q_j} \delta(q_i, a, q_j)$ for all $Q_i, Q_j \in P(Q)$ and $a \in \Sigma$,
(vii) $f_l : \overline{Q} \times \Sigma \to \overline{Q}$ is the next state map defined by, $f_l(Q_i, a) = \bigcup_{q_i \in Q_i} \{q_j : \delta(q_i, a, q_j) \in \triangle\}$,
(viii) $\tilde{\delta}_l : (\overline{Q} \times [0, 1]) \times \Sigma \times \overline{Q} \to [0, 1]$ is the augmented transition function defined by:

$$\mu^{t+1}(Q_j) = \tilde{\delta}_l((Q_i, \mu^t(Q_i)), a, Q_j) = F_1(\mu^t(Q_i), \delta_l(Q_i, a, Q_j)),$$

(ix) $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ is called membership assignment function,
(x) $F_2 : [0, 1]^* \to [0, 1]$ is called multi-membership resolution function.

The reason of nomination BL-GFA, is that $\overline{Q}$, the set of states of BL-GFA, is a BL-algebra.

Remark 2.4. [2] *We let $\phi \xrightarrow{\Lambda, 0} \{q_0\}$ and $Q_i = Q_{act}(t_i)$ if and only if Q_i contains the active states of BL-GFA at time t_i.*

Definition 2.5. [2] *A derivation of an input string $X (X \in \Sigma^*)$ denoted as $der_i(X)$, is an ordered set of states which are passed successively upon entrance of each symbol of the string, starting from an initial state. i is an arbitrary index usually starting from 1. Let $X = a_1a_2...a_ka_{k+1}...a_m$. Then we have:*

$$der_i(X) = \left\{ Q_{i_0}Q_{i_1}...Q_{i_m} \middle| Q_{i_0} \in \tilde{R}, Q_{i_0} \xrightarrow{\delta_l(Q_{i_0}, a_1, Q_{i_1})} Q_{i_1} \to Q_{i_2} ... \to Q_{i_{m-1}} \xrightarrow{\delta_l(Q_{i_{m-1}}, a_m, Q_{i_m})} Q_{i_m} \right\},$$

where $Q_{i_0}, Q_{i_1}, ..., Q_{i_m} \in P(Q)$.

A string may have several derivations. The set of all derivations of string X is denoted as $D_{der}(X)$. Then, a threshold derivation of X is a member of $D_{der}(X)$ subject to a threshold, as follows:

$$der_i(X, \frac{\tau_1}{\tau_2}) = \left\{ Q_{i_0} Q_{i_1} ... Q_{i_m} \in D_{der}(X) \left| \tau_1 \leq \delta_l(Q_{i_k}, a_k, Q_{i_{k+1}}) \leq \tau_2,\ 0 \leq k < m \right. \right\}.$$

Similarly, the set of all threshold derivation of X is denoted as $D_{der}(X, \frac{\tau_1}{\tau_2})$. Obviously, $D_{der}(X, \frac{\tau_1}{\tau_2}) \subseteq D_{der}(X)$. Actually, we are interested in the active states of a BL-GFA upon entry of a string X. Without any loss of generality, we can use the same notation used for the active state set at a specific time ($Q_{act}(t)$), and denote the active state set of string X as $Q_{act}(X)$, since $Q_{act}(X) \equiv Q_{act}(t_0 + |X|)$, where t_0 is the starting time of operation of the BL-GFA and $|X|$ is the length of X. Then $Q_{act}(X)$ can be defined as:

Definition 2.6. [2] *(Active state set of an input string) The active state set of an input X, is the fuzzy set of all active states, after string X has entered the BL-general fuzzy automata,*

$$Q_{act}(X) = \left\{ (Q_{i_m}, \mu^{t_0+|X|}(Q_{i_m})) \,|Q_{i_0} Q_{i_1} ... Q_{i_m} \in D_{der}(X) \right\}.$$

Similarly, the threshold active state set of an input string is defined as:

$$Q_{act}(X, \frac{\tau_1}{\tau_2}) = \left\{ (Q_{i_m}, \mu^{t_0+|X|}(Q_{i_m})) \left| Q_{i_0} Q_{i_1} ... Q_{i_m} \in D_{der}(X, \frac{\tau_1}{\tau_2}) \right. \right\}.$$

It is obvious that $Q_{act}(X, \frac{\tau_1}{\tau_2}) \subseteq Q_{act}(X)$.

Here also, the $\frac{0}{\tau}, \frac{\tau}{1}, \frac{\tau}{\tau}$ *thresholds will give the threshold derivations and active sates, where all transitions are less than or equal to* τ, *greater than or equal to* τ, *and exactly equal to* τ, *respectively.*

Definition 2.7. [2] *(State set of an output label) In the BL-GFA, the state set of the output label $Z_l \in P(Z)$ denoted as Q_{z_l}, is the set of all states whose associated output is Z_l, $Q_{Z_l} = \{Q_{i_m} | \omega_l(Q_{i_m}) = Z_l\}$. In BL-general fuzzy automata, however, it makes more sense to talk about the belonging/disbelonging to an output label.*

Definition 2.8. [2] *In the BL-GFA $\tilde{F}_l = (\overline{Q}, \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ a string $X \in \Sigma^*$ is said to belong to the output label $Z_l \in P(Z)$, if starting from an initial state, at least one of the states in the active state set of X generates output label Z_l. Otherwise X is said to disbelong to Z_l.*

$$X \text{ belongs to } Z_l \Leftrightarrow\ \exists Q_{i_m} \,|\, Q_{i_m} \in Dom\,(Q_{act}(X)) \bigcap Q_{Z_l}$$
$$X \text{ disbelogs to } Z_l \Leftrightarrow Dom\,(Q_{act}(X)) \bigcap Q_{Z_l} = \phi$$

The reaction of BL-general fuzzy automata to sequences of input symbols, i. e., to elements of Σ^*, can be described formally as follows:

Definition 2.9. [2] *(Run map and behavior) The run map (with threshold $\frac{\tau_1}{\tau_2}$) of BL-general fuzzy automata $\tilde{F}_l = (\overline{Q}, \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ is the map $\rho : \Sigma^* \to \overline{Q}$ defined by the following induction:*
$\rho(\Lambda) = \{q_0\}$ and $\rho(\sigma_1 \sigma_2 ... \sigma_n) = Q_{i_n}$, where

$$(Q_{i_n}, \mu^{t_0+n}(Q_{i_n})) \in Q_{act}(\sigma_1 \sigma_2 ... \sigma_n) \left[(Q_{i_n}, \mu^{t_0+n}(Q_{i_n})) \in Q_{act}(\sigma_1 \sigma_2 ... \sigma_n, \frac{\tau_1}{\tau_2}) \right]$$

implies that $\rho(\sigma_1 \sigma_2 ... \sigma_n \sigma_{n+1}) = f_l(Q_{i_n}, \sigma_{n+1})$ for all $\sigma_1, \sigma_2, ..., \sigma_n, \sigma_{n+1} \in \Sigma$. $\tilde{F}_l$ is called reachable if ρ is onto.

The behavior (with threshold $\frac{\tau_1}{\tau_2}$) of $\tilde{F}_l$ is the map $\beta = \omega_l o \rho : \Sigma^* \to \overline{Z}$. Thus, when receiving inputs $\sigma_1 \sigma_2 ... \sigma_n$, the BL-GFA terminates in the state $\rho(\sigma_1 \sigma_2 ... \sigma_n) = Q_{i_n}$, and final output $\beta(\sigma_1 \sigma_2 ... \sigma_n) = \omega_l(Q_{i_n}) = Z_l$ where $\sigma_1 \sigma_2 ... \sigma_n$ belongs to Z_l.

For some applications, it may be useful to use the following definition to assign membership value (mv) to a string. It is adapted from the mv definition of strings in a Regular Fuzzy Grammar (RFG). This definition is sometimes called max-min rule.

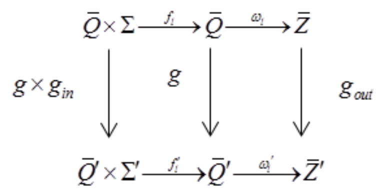
Definition 2.10. [2] *(Membership value of a string) The mv of a string X denoted as $\mu(X)$, is the maximum membership value among all its derivations, where the mv of a derivation is the minimum transition weight encountered in that derivation.*

Given that X has n derivation, and $X = a_1a_2...a_ka_{k+1}...a_m$ and that the ith derivation is $der_i(X) = Q_{i_0}...Q_{i_k}Q_{i_{k+1}} \cdots Q_{i_m}$, where $Q_{i_0} \in \tilde{R}$, the mv of $der_i(X)$ is computed as:

$$\mu(der_i(X)) = \delta_l(Q_{i_0}, a_1, Q_{i_1}) \wedge ... \wedge \delta_l(Q_{i_k}, a_{k+1}, Q_{i_{k+1}}) \wedge ... \wedge \delta_l(Q_{i_{m-1}}, a_m, Q_{i_m}).$$

Definition 2.11. [2] *Given $(\overline{Q}, \Sigma, f_l, \delta_l)$ and $(\overline{Q}', \Sigma', f_l', \delta_l')$, a homomorphism $h : (\overline{Q}, \Sigma, f_l, \delta_l) \to (\overline{Q}', \Sigma', f_l', \delta_l')$ with threshold $\frac{\tau_1}{\tau_2}$ are maps $g : \overline{Q} \to \overline{Q}'$ and $g_{in} : \Sigma \to \Sigma'$ which commutes with the operations and hold the following property:*
(i) $gof_l = f_l'o(g \times g_{in})$,
(ii) $\tau_1 \leq \delta_l(f_l(Q_i, \sigma_1), \sigma_2, Q_j) \leq \tau_2$ if and only if $\tau_1 \leq {\delta_l}'(g(f_l(Q_i, \sigma_1)), g_{in}(\sigma_2), g(Q_j)) \leq \tau_2$.

Definition 2.12. [2] *Given two BL-general fuzzy automata $\tilde{F}_l = (\overline{Q}, \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ and $\tilde{F}_l' = (\overline{Q}', \Sigma', \tilde{R}' = (\{q_0'\}, \mu^{t_0}(\{q_0'\})), \overline{Z}', \omega_l', \delta_l', f_l', \tilde{\delta}_l', F_1, F_2)$ a morphism of $\tilde{F}_l$ to $\tilde{F}_l'$ with threshold $\frac{\tau_1}{\tau_2}$ is a triple (g, g_{in}, g_{out}) where $g : \overline{Q} \to \overline{Q}'$, $g_{in} : \Sigma \to \Sigma'$ and $g_{out} : \overline{Z} \to \overline{Z}'$ are maps such that:*
(i) $h : (\overline{Q}, \Sigma, f_l, \delta_l) \to (\overline{Q}', \Sigma', f_l', \delta_l')$ is a homomorphism with threshold $\frac{\tau_1}{\tau_2}$,
(ii) g and g_{out} commute with the outputs, i. e., $g_{out}o\omega_l = \omega_l'og$,
(iii) g preserves the initial state, i. e., $g(\{q_0\}) = \{q_0'\}$.



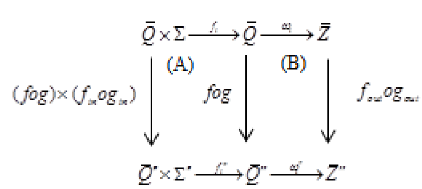
Theorem 2.13. [2] *For each morphism $(g, g_{in}, g_{out}) : \tilde{F}_l \to \tilde{F}_l'$ with threshold $\frac{\tau_1}{\tau_2}$ of BL-general fuzzy automata, the run map ρ of $\tilde{F}_l$ is related to the run map ρ' of $\tilde{F}_l'$ by $\rho' = go\rho$, hence the behavior β of $\tilde{F}_l$ is related to the behavior β' of $\tilde{F}_l'$ by $\beta' = g_{out}o\beta$.*

3 The functor between the BL-GFA and fuzzy behavior

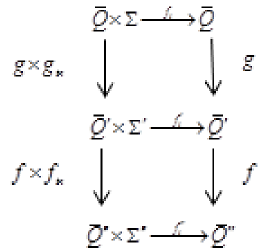
In this section, the categories of BL-general fuzzy automata and fuzzy behavior are investigated. In between, the concepts as reachability map and observability map of BL-GFA are also studied. Lastly, a functor from the category of BL-GFAs to the category of fuzzy behaviors is introduced. We begin with the concept of BL-general fuzzy automata, which is similar to the concept of deterministic fuzzy automata in the sense of [18, 19], with the difference that here augmented transition function $(\tilde{\delta}_l)$ is used instead of transition map.

Proposition 3.1. *The class of all BL-general fuzzy automata and their morphisms forms a category under component-wise composition of maps.*

Proof. Let $\tilde{F}_l$, $\tilde{F}_l{}'$ and $\tilde{F}_l{}''$ be BL-general fuzzy automata and $(g, g_{in}, g_{out}) : \tilde{F}_l \to \tilde{F}_l'$ and $(f, f_{in}, f_{out}) : \tilde{F}_l{}' \to \tilde{F}_l{}''$ be their morphisms. In order to demonstrate that composition $(fog, f_{in}og_{in}, f_{out}og_{out}) : \tilde{F}_l \to \tilde{F}_l{}''$ is a morphism, it is enough to show that the following diagrams commute.



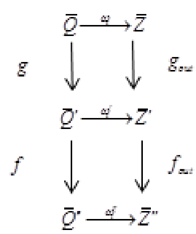
The commutativity of square (A) follows from the commutativity of both the upper and lower parts of the following diagram.



As, for all $(Q_i, \sigma) \in \overline{Q} \times \Sigma$

$[(fog)of_l](Q_i, \sigma) = f(g(f_l(Q_i, \sigma))) = f(f_l'((g \times g_{in})(Q_i, \sigma))) = f_l^{''}((f \times f_{in})(g(Q_i), g_{in}(\sigma))) = (f_l^{''}o[(f \times f_{in})o(g \times g_{in})])(Q_i, \sigma).$

The commutativity of square (B) follows from the commutativity of both the upper and lower parts of the following diagram.



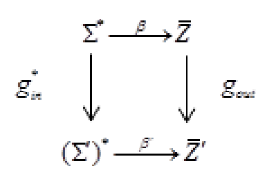
As, for all $Q_i \in \overline{Q}$, $[(f_{out}og_{out})o\omega_l](Q_i) = f_{out}(g_{out}(\omega_l(Q_i))) = f_{out}(\omega_l'(g(Q_i))) = \omega_l''(f(g(Q_i))) = [\omega_l''o(fog)](Q_i)$. Also, we have

$$\begin{aligned}
&\tau_1 \leq \delta_l(f_l(Q_i, \sigma_1), \sigma_2, Q_j) \leq \tau_2 \Leftrightarrow \\
&\tau_1 \leq \delta_l'(g(f_l(Q_i, \sigma_1)), g_{in}(\sigma_2), g(Q_j)) \leq \tau_2 \Leftrightarrow \\
&\tau_1 \leq \delta_l''(f(g(f_l(Q_i, \sigma_1))), f_{in}(g_{in}(\sigma_2)), f(g(Q_j))) \leq \tau_2 \Leftrightarrow \\
&\tau_1 \leq \delta_l''(((fog)of_l)(Q_i, \sigma_1), (f_{in}og_{in})(\sigma_2), (fog)(Q_j)) \leq \tau_2,
\end{aligned}$$

and $fog(\{q_0\}) = f(g(\{q_0\})) = f(\{q_0'\}) = \{q_0''\}$.
We shall denote by BGFA, the category of BL-general fuzzy automata. □

Remark 3.2. *Let B be the class of BGFA-objects and BGFA-morphisms with the restriction that the second component of each BGFA-morphism (i. e., g_{in}) is onto. Then B is also a category. Obviously, it is a subcategory of BGFA.*

Definition 3.3. *Let $\tilde{F}_l$ and $\tilde{F}_l^{'}$ be BL-general fuzzy automata. A morphism between two behaviors $\beta : \Sigma^* \to \overline{Z}$ and $\beta' : (\Sigma')^* \to \overline{Z}^{'}$ of $\tilde{F}_l$ and $\tilde{F}_l^{'}$ respectively is a pair (g_{in}, g_{out}) where $g_{in} : \Sigma^* \to (\Sigma')^*$ and $g_{out} : \overline{Z} \to \overline{Z}^{'}$ are maps such that the following diagram commutes.*



Where g_{in}^* is the free extension of the map g_{in} defined inductively by $g_{in}^*(\Lambda) = \Lambda$ and $g_{in}^*(wx) = g_{in}^*(w)g_{in}(x)$ for all $w \in \Sigma^*$ and $x \in \Sigma$.

Definition 3.4. *Let $\tilde{F}_l = (\overline{Q}, \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-general fuzzy automaton. The free extension f_l^* of f_l is a map $f_l^* : \overline{Q} \times \Sigma^* \to \overline{Q}$ such that*
(i) $f_l^*(Q_i, \Lambda) = Q_i$,
(ii) $f_l^*(Q_i, wx) = f_l(f_l^*(Q_i, w), x)$,
for all $Q_i \in \overline{Q}$, $w \in \Sigma^*$ and $x \in \Sigma$.

Definition 3.5. *Let* $\tilde{F}_l = (\overline{Q}, \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ *be a BL-general fuzzy automaton. We define the fuzzy output map* $\gamma : \overline{Q} \times \overline{Z} \to [0,1]$ *as follows:*

$$\gamma(Q_i, Z_l) = \begin{cases} \mu(w), & if \quad Q_i \in Dom(Q_{act}(w)) \bigcap Q_{Z_l} \\ 0, & otherwis \end{cases}$$

where $w \in \Sigma^*$, $Q_i \in \overline{Q}$ *and* $Z_l \in \overline{Z}$.

Definition 3.6. *Given a BL-general fuzzy automaton*

$$\tilde{F}_l = (\overline{Q}, \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$$

the fuzzy behavior β_{Q_i} *of* $\tilde{F}_l$ *in state* Q_i *is a map* $\beta_{Q_i} : \Sigma^* \times \overline{Z} \to [0,1]$ *such that* $\beta_{Q_i}(w, Z_l) = \gamma(f_l^*(Q_i, w), Z_l)$.

Remark 3.7. *It can be easily seen that* $\beta_{\{q_0\}} = \gamma o(\rho \times id_{\overline{Z}})$.

Definition 3.8. *The fuzzy behavior of* $\tilde{F}_l$ *in initial state* $\{q_0\}$ *is simply called the fuzzy behavior of* $\tilde{F}_l$.

Definition 3.9. *(a) Given a BL-general fuzzy automaton* $\tilde{F}_l$, *the observability map* σ *is a map* $\sigma : \overline{Q} \to L^{\Sigma^* \times \overline{Z}}$ *such that* $\sigma(Q_i) = \beta_{Q_i}$, *where* $L^{\Sigma^* \times \overline{Z}} = \{ f \mid f : \Sigma^* \times \overline{Z} \to [0,1] \}$.
(b) $\tilde{F}_l$ *is called observable if* σ *is one-to-one.*

Remark 3.10. *Note that* $\sigma(\{q_0\}) = \beta_{\{q_0\}}$, *where*

$$\beta_{\{q_0\}}(w, Z_l) = [\gamma o(\rho \times id_{\overline{Z}})](w, Z_l) = \gamma(\rho(w), Z_l) = \gamma(f_l^*(\{q_0\}, w), Z_l)$$

for all $w \in \Sigma^*$ *and* $Z_l \in \overline{Z}$.

Example 3.11. *Consider the GFA in Fig. 1. It is specified as:*
$\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, *where*
$Q = \{q_0, q_1, q_2\}$ *is the set of states,*
$\Sigma = \{a, b\}$ *is the set of input symbols,*
$\tilde{R} = \{(q_0, 1)\}$,
$Z = \{z_1, z_2, z_3\}$ *s.t.* $\omega(q_0) = z_1, \omega(q_1) = z_2, \omega(q_2) = z_3$,
$F_1(\mu, \delta) = \delta$, $F_2() = \mu^{t+1}(q_m) = \wedge_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)))$.

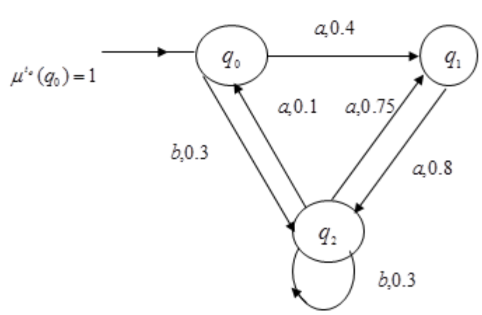


Figure 1: The GFA of Example 3.11

According to the definition of BL-general fuzzy automata, the following figure is concluded:

For example if we choos $F_1(\mu, \delta_l) = \delta_l$ and $F_2() = \mu^{t+1}(Q_j) = \wedge_{i=1}^{n} F_1(\mu^t(Q_i), \delta_l(Q_i, a, Q_j))$ then by the definition of BL-general fuzzy automata we have the active states and $mv's$ at different times upon the input string "ba^2ba" as follows:
$\{q_0\} \xrightarrow{b} \{q_2\} \xrightarrow{a} \{q_0, q_1\} \xrightarrow{a} \{q_1, q_2\} \xrightarrow{b} \{q_2\} \xrightarrow{a} \{q_0, q_1\}$
$\mu^{t_0}(\{q_0\}) = 1$,
$\mu^{t_1}(\{q_2\}) = \tilde{\delta}_l((\{q_0\}, \mu^{t_0}(\{q_0\})), b, \{q_2\}) = F_1(\mu^{t_0}(\{q_0\}), \delta_l(\{q_0\}, b, \{q_2\}))$
$= \delta_l(\{q_0\}, b, \{q_2\}) = \delta(q_0, b, q_2) = 0.3$,

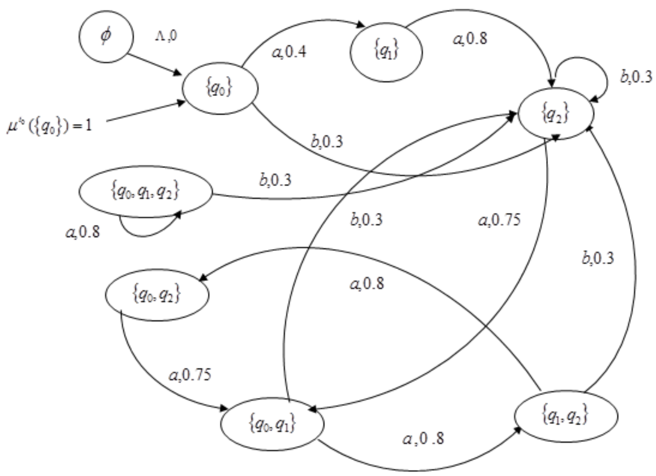


Figure 2: The BL- GFA of Example 3.11

$$\begin{aligned}
\mu^{t_2}(\{q_0,q_1\}) &= \tilde{\delta}_l((\{q_2\},\mu^{t_1}(\{q_2\})),a,\{q_0,q_1\}) = F_1(\mu^{t_1}(\{q_2\}),\delta_l(\{q_2\},a,\{q_0,q_1\}))\\
&= \delta_l(\{q_2\},a,\{q_0,q_1\}) = \delta(q_2,a,q_0)\vee\delta(q_2,a,q_1) = 0.1\vee 0.75 = 0.75,\\
\mu^{t_3}(\{q_1,q_2\}) &= \tilde{\delta}_l((\{q_0,q_1\},\mu^{t_2}(\{q_0,q_1\})),a,\{q_1,q_2\}) = F_1(\mu^{t_2}(\{q_0,q_1\}),\delta_l(\{q_0,q_1\},a,\{q_1,q_2\}))\\
&= \delta_l(\{q_0,q_1\},a,\{q_1,q_2\}) = \delta(q_0,a,q_1)\vee\delta(q_0,a,q_2)\vee\delta(q_1,a,q_1)\vee\delta(q_1,a,q_2)\\
&= 0.4\vee 0\vee 0\vee 0.8 = 0.8,\\
\mu^{t_4}(\{q_2\}) &= \tilde{\delta}_l((\{q_1,q_2\},\mu^{t_3}(\{q_1,q_2\})),b,\{q_2\}) = F_1(\mu^{t_3}(\{q_1,q_2\}),\delta_l(\{q_1,q_2\},b,\{q_2\}))\\
&= \delta_l(\{q_1,q_2\},b,\{q_2\}) = \delta(q_1,b,q_2)\vee\delta(q_2,b,q_2) = 0\vee 0.3 = 0.3,\\
\mu^{t_5}(\{q_0,q_1\}) &= \tilde{\delta}_l((\{q_2\},\mu^{t_4}(\{q_2\})),a,\{q_0,q_1\}) = F_1(\mu^{t_4}(\{q_2\}),\delta_l(\{q_2\},a,\{q_0,q_1\}))\\
&= \delta_l(\{q_2\},a,\{q_0,q_1\}) = \delta(q_2,a,q_0)\vee\delta(q_2,a,q_1) = 0.1\vee 0.75 = 0.75.
\end{aligned}$$

Table 1: The active states and $mv's$ at different times of BL-GFA in Example 3.11

time	t_0	t_1	t_2	t_3	t_4	t_5
input	Λ	b	a	a	b	a
$Q_{act}(t)$	$\{q_0\}$	$\{q_2\}$	$\{q_0,q_1\}$	$\{q_1,q_2\}$	$\{q_2\}$	$\{q_0,q_1\}$
mv	1	0.3	0.75	0.8	0.3	0.75

The following can be easily verified:

$$\phi \xrightarrow{\Lambda} \{q_0\} \xrightarrow{a} \{q_1\} \xrightarrow{a} \{q_2\} \xrightarrow{b} \{q_2\} \xrightarrow{a} \{q_0,q_1\}$$

Thus, $der(a^2ba) = D_{der}(a^2ba) = \{q_0\}\{q_1\}\{q_2\}\{q_2\}\{q_0,q_1\}$.
Then $Q_{act}(a^2ba) = (\{q_0,q_1\},\mu^{t_4}(\{q_0,q_1\}))$.
By the definitions of run map ρ and behavior β of BL-GFA we have:

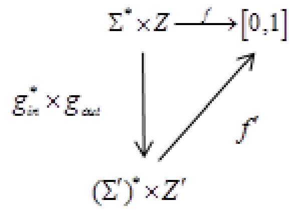
$$\begin{aligned}
&\rho(\Lambda) = \{q_0\},\\
&\rho(\Lambda a) = f_l(\rho(\Lambda),a) = f_l(\{q_0\},a) = \{q_1\},\\
&\rho(a^2) = f_l(\rho(a),a) = f_l(\{q_1\},a) = \{q_2\},\\
&\rho(a^2b) = f_l(\rho(a^2),b) = f_l(\{q_2\},b) = \{q_2\},\\
&\rho(a^2ba) = f_l(\rho(a^2b),a) = f_l(\{q_2\},a) = \{q_0,q_1\}.
\end{aligned}$$

Then, $\beta(a^2ba) = \omega_l o\rho(a^2ba) = \omega_l(\rho(a^2ba)) = \omega_l(\{q_0,q_1\}) = \{z_1,z_2\}$.
Hence, a^2ba belongs to $\{z_1,z_2\}$ since, $\{q_0,q_1\}\in Dom(Q_{act}(a^2ba))\bigcap Q_{\{z_1,z_2\}}$.
Therefore,

$$\begin{aligned}
\gamma(\{q_0,q_1\},\{z_1,z_2\}) = \mu(a^2ba) &= \delta_l(\{q_0\},a,\{q_1\})\wedge\delta_l(\{q_1\},a,\{q_2\})\wedge\delta_l(\{q_2\},b,\{q_2\})\wedge\delta_l(\{q_2\},a,\{q_0,q_1\})\\
&= 0.4\wedge 0.8\wedge 0.3\wedge 0.75 = 0.3,
\end{aligned}$$

But $\gamma(\{q_0, q_1\}, \{z_3\}) = 0$.

Definition 3.12. *(a) For two sets Σ and Z, fuzzy behavior is a map $f : \Sigma^* \times Z \to [0, 1]$.*
(b) A morphism between two fuzzy behaviors $f : \Sigma^* \times Z \to [0, 1]$ and $f' : (\Sigma')^* \times Z' \to [0, 1]$ is a pair (g_{in}, g_{out}), where $g_{in} : \Sigma \to \Sigma'$ and $g_{out} : Z \to Z'$ are maps such that the following diagram commutes:



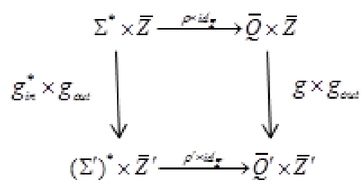
Proposition 3.13. *For given two sets Σ and Z, the class of fuzzy behaviors and their morphisms forms a category.*

Proof. The proof is similar to that proposition 3.1. □

We shall denote by FB the category of fuzzy behaviors.

Remark 3.14. *Let D be the class of FB-objects and FB-morphisms with the restriction that the first component of each FB-morphism (i. e., g_{in}) is onto. Then D is also category. Obviously, it is a subcategory of FB.*

Proposition 3.15. *Given two BL-general fuzzy automata $\tilde{F}_l$ and $\tilde{F}'_l$ with run maps $\rho : \Sigma^* \to \overline{Q}$ and $\rho' : (\Sigma')^* \to \overline{Q}'$ respectively, and BGFA-morphism $(g, g_{in}, g_{out}) : \tilde{F}_l \to \tilde{F}'_l$, then the following diagram commutes.*



Proof. We prove this by induction on the length of strings in Σ^*. Let for $|w| = 0$

$$[(g \times g_{out})o(\rho \times id_{\overline{Z}})](\Lambda, Z_l) = (g \times g_{out})(\rho(\Lambda), Z_l) = (g \times g_{out})((\{q_0\}), Z_l) = (g(\{q_0\}), g_{out}(Z_l)) = (\{q'_0\}, g_{out}(Z_l))$$

and

$$[(\rho' \times id_{\overline{Z}})o(g^*_{in} \times g_{out})](\Lambda, Z_l) = (\rho' \times id_{\overline{Z}})(g^*_{in}(\Lambda), g_{out}(Z_l)) = (\rho' \times id_{\overline{Z}})(\Lambda, g_{out}(Z_l)) = (\rho'(\Lambda), g_{out}(Z_l)) = (\{q'_0\}, g_{out}(Z_l)).$$

Thus, the diagram commutes for $|w| = 0$.
We now assume that the result is true for all strings of length less than or equal to n, i. e., for all $w \in \Sigma^*$ such that $|w| \leq n$,

$$[(g \times g_{out})o(\rho \times id_{\overline{Z}})](w, Z_l) = [(\rho' \times id_{\overline{Z}})o(g^*_{in} \times g_{out})](w, Z_l) \text{ i. e., } (g(\rho(w)), g_{out}(Z_l)) = (\rho'(g^*_{in}(w)), g_{out}(Z_l)).$$

Then,

$$\begin{aligned}[(g \times g_{out})o(\rho \times id_{\overline{Z}})](wx, Z_l) &= (g \times g_{out})(\rho(wx), Z_l) = (g(\rho(wx)), g_{out}(Z_l)) = (g(f_l(\rho(w), x)), g_{out}(Z_l)) \\ &= (f'_l(g(\rho(w), g_{in}(x))), g_{out}(Z_l))\end{aligned}$$

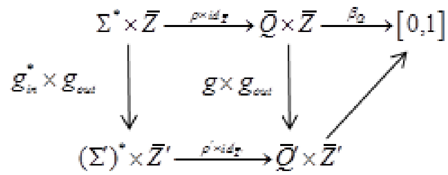
Furthermore,

$$\begin{aligned}&[(\rho' \times id_{\overline{Z}})o(g^*_{in} \times g_{out})](wx, Z_l) = (\rho' \times id_{\overline{Z}})(g^*_{in}(wx), g_{out}(Z_l)) \\ &= (\rho' \times id_{\overline{Z}})(g^*_{in}(w)g_{in}(x), g_{out}(Z_l)) = (\rho'(g^*_{in}(w)g_{in}(x)), g_{out}(Z_l)) \\ &= (f'_l(\rho'(g^*_{in}(w)), g_{in}(x)), g_{out}(Z_l))) = (f'_l(g(\rho(w), g_{in}(x)), g_{out}(Z_l)).\end{aligned}$$

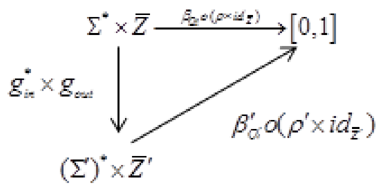
The above equality follows from the fact that the result is true for all strings of length less than or equal to n. Hence the diagram commutes for $|w| = n + 1$. □

Proposition 3.16. *Let $E : BGFA \to FB$ such that $E(\tilde{F}_l) = \beta_{\{q_0\}}$, for all $\tilde{F}_l \in BGFA$ and for all $BGFA$-morphism $(g, g_{in}, g_{out}) : \tilde{F}_l \to \tilde{F}'_l$, $E(g, g_{in}, g_{out}) = (g_{in}, g_{out})$. Then E is a functor.*

Proof. We will only show that (g_{in}, g_{out}) is FB-morphism. Since (g, g_{in}, g_{out}) is a BGFA-morphism, each of the following diagrams commute:



The above diagram leads us to the following



i. e., $[\beta'_{Q'_i} o(\rho' \times id_{\overline{Z}'})]o(g^*_{in} o g_{out}) = \beta_{Q_i} o(\rho \times id_{\overline{Z}})$. Also, as $\beta_{\{q_0\}} = \gamma o(\rho \times id_{\overline{Z}})$ and $\beta'_{\{q'_0\}} = \gamma' o(\rho' \times id_{\overline{Z}'})$, the diagram states that (g_{in}, g_{out}) is indeed for a FB-morphism. ☐

Definition 3.17. *$\tilde{F}_l \in BGFA$ is a realization of fuzzy behavior f if $E(\tilde{F}_l) = f$. Further the realization $\tilde{F}_l$ is canonical if $\tilde{F}_l$ is both reachable and observable.*

4 Minimal realization of fuzzy behavior

In order to present two approaches for construction of minimal BL-general fuzzy automaton for the given fuzzy behavior of $\tilde{F}_l$, we divide this section into two subsections.

4.1 Minimal realization based on Myhill-Nerode's theory

This subsection is towards the construction of minimal realization of the given fuzzy behavior of $\tilde{F}_l$ based on Myhill-Nerode's theory. We begin with the following.

Proposition 4.1. *Given a fuzzy behavior $\beta_{\{q_0\}}$ of $\tilde{F}_l$, there exists a BL-general fuzzy automaton $(\tilde{F}_l)_\beta$ which realizes it.*

Proof. Define a relation $\approx$ on Σ^* by $w_1 \approx w_2$ if and only if $\beta_{\{q_0\}}(w_1 w, Z_l) = \beta_{\{q_0\}}(w_2 w, Z_l)$, $\forall\, w \in \Sigma^*$ and $\forall Z_l \in \overline{Z}$. Then, $\approx$ is an equivalence relation on Σ^*. Now, let $Q_\beta = \frac{\Sigma^*}{\approx} = \{[w]\,|w \in \Sigma^*\}$, where $[w] = \{v \in \Sigma^* \,|\, w \approx v\}$, and define the maps f_β, ω_β, δ_β, $\tilde{\delta}_\beta$ and γ_β as:
$f_\beta : Q_\beta \times \Sigma \to Q_\beta$ such that $f_\beta([w]\,, x) = [wx]$, $\forall\, [w] \in Q_\beta$ and $\forall x \in \Sigma$,
$\gamma_\beta : Q_\beta \times \overline{Z} \to [0,1]$ such that $\gamma_\beta([w]\,, Z_l) = \beta_{\{q_0\}}(w, Z_l)$,$\forall\, [w] \in Q_\beta$ and $\forall Z_l \in \overline{Z}$,
$\omega_\beta : Q_\beta \to \overline{Z}$ such that $\omega_\beta([w]) = \beta(w)$, $\forall\, [w] \in Q_\beta$,
$\delta_\beta : Q_\beta \times \Sigma \times Q_\beta \to [0,1]$ such that $\delta_\beta([w]\,, x, [w']) = \mu(w\,x\,w')$,$\forall\, [w]\,, [w'] \in Q_\beta$ and $\forall x \in \Sigma$,
and $\tilde{\delta}_\beta(([w]\,, \mu^t([w])), x, [w']) = F_1(\mu^t([w]), \delta_\beta([w]\,, x, [w']))$ where $\mu^t([w]) = \mu(w)$.
We will only show that f_β is well-defined. The proofs for others are the same.
Let $w_1,\, w_2 \in \Sigma^*$ such that $[w_1] = [w_2]$. Then, $w_1 \approx w_2$. Now,

$$\begin{aligned} w_1 \approx w_2 &\Rightarrow \beta_{\{q_0\}}(w_1 v, Z_l) = \beta_{\{q_0\}}(w_2 v, Z_l)\ \forall v \in \Sigma^*,\ \ Z_l \in \overline{Z} \\ &\Rightarrow \beta_{\{q_0\}}(w_1 x t, Z_l) = \beta_{\{q_0\}}(w_2 x t, Z_l)\ \forall x \in \Sigma,\ t \in \Sigma^*,\ \ Z_l \in \overline{Z} \\ &\Rightarrow w_1 x \approx w_2 x\,,\ \forall x \in \Sigma \\ &\Rightarrow [w_1 x] = [w_2 x]\,,\ \forall x \in \Sigma \\ &\Rightarrow f_\beta([w_1]\,, x) = f_\beta([w_2]\,, x) \end{aligned}$$

And so f_β is well-defined.
Thus $(\tilde{F}_l)_\beta = (Q_\beta, \Sigma, \tilde{R}_\beta = ([\Lambda]\,, \mu^{t_0}([\Lambda])), \overline{Z}, f_\beta, \omega_\beta, \delta_\beta, \tilde{\delta}_\beta, F_1, F_2)$ is a BL-general fuzzy automaton. Also, by induction it is easy to verify that f_β can be extended to $(f^*)_\beta : Q_\beta \times \Sigma^* \to Q_\beta$ such that $(f^*)_\beta([w]\,, w') = [ww']\,,\ \forall w' \in \Sigma^*$.
Finally, $E((\tilde{F}_l)_\beta)(w, Z_l) = \beta_{[\Lambda]}(w, Z_l) = \gamma_\beta((f^*)_\beta([\Lambda]\,, w), Z_l) = \gamma_\beta([w]\,, Z_l) = \beta_{\{q_0\}}(w, Z_l)$.

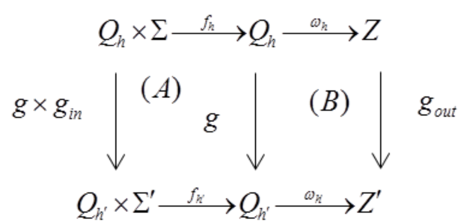
Therefore, $E((\tilde{F}_l)_\beta) = \beta_{\{q_0\}}$. Hence, $(\tilde{F}_l)_\beta$ realizes $\beta_{\{q_0\}}$. □

Proposition 4.2. *The realization $(\tilde{F}_l)_\beta$ of fuzzy behavior $\tilde{F}_l$ is canonical.*

Proof. Let $[w] \in Q_\beta$. Then, $\rho_\beta(w) = (f^*)_\beta([\Lambda], w) = [\Lambda w] = [w]$. Thus, ρ_β is onto, and so $(\tilde{F}_l)_\beta$ is reachable. Also, let $w_1, w_2 \in \Sigma^*$ such that $\sigma([w_1]) = \sigma([w_2])$. Thus, $\beta_{[w_1]} = \beta_{[w_2]}$. Now, $\beta_{[w_1]} = \beta_{[w_2]}$, implying that for all $(w, Z_l) \in \Sigma^* \times \overline{Z}$, $\beta_{[w_1]}(w, Z_l) = \beta_{[w_2]}(w, Z_l)$ such that $\gamma_\beta((f^*)_\beta([w_1], w), Z_l) = \gamma_\beta((f^*)_\beta([w_2], w), Z_l)$, i. e., $\gamma_\beta([w_1 w], Z_l) = \gamma_\beta([w_2 w], Z_l)$, i. e., $\beta_{\{q_0\}}(w_1 w, Z_l) = \beta_{\{q_0\}}(w_2 w, Z_l)$, i. e., $w_1 \approx w_2$ or $[w_1] = [w_2]$, whereby $\sigma : Q_\beta \to L^{\Sigma^* \times \overline{Z}}$ is one-to-one. Hence, $(\tilde{F}_l)_\beta$ is canonical realization of fuzzy behavior of $\tilde{F}_l$. □

Proposition 4.3. *Let $N : D \to B$ be a map which sends each $h \in D$ to $(\tilde{F}_l)_h$, and to each D-morphism $(g_{in}, g_{out}) : h \to h'$, to $(g, g_{in}, g_{out}) : (\tilde{F}_l)_h \to (\tilde{F}_l)_{h'}$, where g is the map $g : Q_h \to Q_{h'}$ such that $g([w]) = [g^*_{in}(w)]$. Then, N is a functor.*

Proof. We only show that for given D-morphism $(g_{in}, g_{out}) : h \to h'$, $(g, g_{in}, g_{out}) : (\tilde{F}_l)_h \to (\tilde{F}'_l)_{h'}$ where g is the map $g : Q_h \to Q_{h'}$ such that $g([w]) = [g^*_{in}(w)]$, is a B-morphism.
First, we show that the map $g : Q_h \to Q_{h'}$ is well-defined, i. e., for given $[w_1] = [w_2]$, $[g^*_{in}(w_1)] = [g^*_{in}(w_2)]$, and it holds if $h'(g^*_{in}(w_1)w', y) = h'(g^*_{in}(w_2)w', y), \forall w' \in (\Sigma')^*$. Now, as g_{in} is onto, g^*_{in} is onto and thus for given $w' \in (\Sigma')^*$ there exist $w \in \Sigma$ such that $g^*_{in}(w) = w'$. Thus, we have $h'(g^*_{in}(w_1)g^*_{in}(w), y) = h'(g^*_{in}(w_2)g^*_{in}(w), y)$, i. e., $h'(g^*_{in}(w_1 w), y) = h'(g^*_{in}(w_2 w), y)$, or that $h(w_1 w, y) = h(w_2 w, y)$ (it follows from the fact that (g_{in}, g_{out}) is a D-morphism, cf., Definition 3.12) which is true since $[w_1] = [w_2]$. Thus, the map g is well-defined.
Again, in order to prove that $(g, g_{in}, g_{out}) : (\tilde{F}_l)_h \to (\tilde{F}_l)_{h'}$ is a B-morphism, we have to show that the following diagrams commute.



To show the commutativity of square (A), let $([w], x) \in Q_h \times \Sigma$. Thus,

$$[f_{h'} o(g \times g_{in})]([w], x) = f_{h'}(g([w]), g_{in}(x)) = f_{h'}([g^*_{in}(w)], g_{in}(x)) = [g^*_{in}(w) g_{in}(x)] = [g^*_{in}(wx)] .$$

Also, $(gof_h)([w], x) = g(f_h([w], x)) = g([wx]) = [g^*_{in}(wx)]$. Thus, square (A) commutes. To show the commutativity of square (B), let $[w] \in Q_h$. Then,

$$\begin{aligned}(g_{out} o\omega_h)([w]) &= g_{out}(\omega_h([w])) = g_{out}(\beta(w)) = (g_{out} o\beta)(w) = (\beta' o g^*_{in})(w) = \beta'(g^*_{in}(w)) \\ &= \omega_{h'}([g^*_{in}(w)]) = \omega_{h'}(g(w)) = (\omega_{h'} o g)(w)\end{aligned}$$

(cf., Definition 3.3). Thus, square (B) commutes. Also, let $u, v \in \Sigma^*$ then we have

$$\begin{aligned}&\tau_1 \le \delta_{h'}(g((f_h([u], \sigma_1)), g_{in}(\sigma_2), g([v])) \le \tau_2 \Leftrightarrow \\ &\tau_1 \le \delta_{h'}(g([u\sigma_1]), g_{in}(\sigma_2), g([v])) \le \tau_2 \Leftrightarrow \\ &\tau_1 \le \delta_{h'}(g(\rho(u\sigma_1)), g_{in}(\sigma_2), g(\rho(v))) \le \tau_2 \Leftrightarrow \\ &\tau_1 \le \delta_{h'}(\rho'(u\sigma_1), g_{in}(\sigma_2), \rho'(v)) \le \tau_2 \Leftrightarrow \\ &\tau_1 \le \delta_{h'}([u\sigma_1], \sigma_2, [v]) \le \tau_2 \Leftrightarrow \\ &\tau_1 \le \mu(u\sigma_1\sigma_2 v) \le \tau_2 \Leftrightarrow \\ &\tau_1 \le \delta_h(\rho(u\sigma_1), \sigma_2, \rho(v)) \le \tau_2 \Leftrightarrow \\ &\tau_1 \le \delta_h([u\sigma_1], \sigma_2, [v]) \le \tau_2 \Leftrightarrow \\ &\tau_1 \le \delta_h(f_h([u], \sigma_1), \sigma_2, [v]) \le \tau_2,\end{aligned}$$

and $g([\Lambda]) = [g^*_{in}(\Lambda)] = [\Lambda]$.
The following is required to show that the canonical realization discussed previously is minimal.

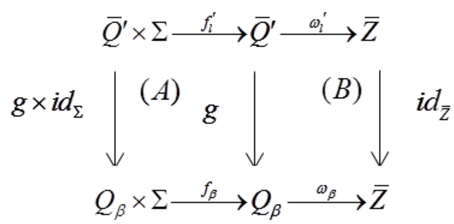
□

Proposition 4.4. *For any fuzzy behavior of $\tilde{F}_l$, the BL-general fuzzy automaton* $(\tilde{F}_l)_\beta = (Q_\beta, \Sigma, ([\Lambda], \mu^{t_0}([\Lambda])), \overline{Z}, f_\beta, \omega_\beta, \delta_\beta, \tilde{\delta}_\beta, F_1, F_2)$ *is a homomorphic image of any reachable BL-general fuzzy automaton which realizes it.*

Proof. Let $\tilde{F}_l' = (\overline{Q}', \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z}, \omega_l', \delta_l', f_l', \tilde{\delta}_l', F_1, F_2)$ be a reachable BL-general fuzzy automaton which realizes $\beta_{\{q_0\}}$ by its fuzzy output γ', i. e., $\gamma'(Q_i', Z_l) = \beta_{\{q_0\}}(w, Z_l)$, $\forall (Q_i', Z_l) \in \overline{Q}' \times \overline{Z}$ where $(f_l')^*(\{q_0\}, w) = Q_i'$ for some $w \in \Sigma^*$. Then, for all $Q_i' \in \overline{Q}'$ there exists $w \in \Sigma^*$ such that $(f_l')^*(\{q_0\}, w) = Q_i'$. Define a map $(g, id_\Sigma, id_{\overline{Z}}) : \tilde{F}_l' \to (\tilde{F}_l)_\beta$ such that $g(Q_i') = [w]$ iff $(f_l')^*(\{q_0\}, w) = Q_i'$ $\forall Q_i{}' \in \overline{Q}'$ and $w \in \Sigma^*$. The map $(g, id_\Sigma, id_{\overline{Z}}) : \tilde{F}_l' \to (\tilde{F}_l)_\beta$ is well-defined as shown below: Let $u, v \in \Sigma^*$ such that $(f_l')^*(\{q_0\}, u) = Q_i' = (f_l')^*(\{q_0\}, v)$. Then, for all $w \in \Sigma^*$

$$\begin{aligned}\beta_{\{q_0\}}(uw, Z_l) &= \gamma'((f_l')^*(\{q_0\}, uw), Z_l) = \gamma'((f_l')^*((f_l')^*(\{q_0\}, u), w), Z_l)\\ &= \gamma'((f_l')^*(Q_i', w), Z_l) = \gamma'((f_l')^*((f_l')^*(\{q_0\}, v), w), Z_l)\\ &= \gamma'((f_l')^*(\{q_0\}, vw), Z_l) = \beta_{\{q_0\}}(vw, Z_l).\end{aligned}$$

Thus, $u \approx v$, i. e., $[u] = [v]$, whereby $(g, id_\Sigma, id_{\overline{Z}}) : \tilde{F}_l' \to (\tilde{F}_l)_\beta$ is well-defined. Also, for given $[u] \in Q_\beta$, $\exists Q_i' \in \overline{Q}'$ such that $g(Q_i') = [u]$, because $g(Q_i') = g((f_l')^*(\{q_0\}, u))$. Thus, $(g, id_\Sigma, id_{\overline{Z}}) : \tilde{F}_l' \to (\tilde{F}_l)_\beta$ is onto. Finally, to show that $(g, id_\Sigma, id_{\overline{Z}}) : \tilde{F}_l' \to (\tilde{F}_l)_\beta$ is a homomorphism, we have to show that the following diagrams commute.



To show the commutativity of square (A), let $(Q_i', x) \in \overline{Q}' \times \Sigma$, where $f_l'(\{q_0\}, u) = Q_i'$, for some $u \in \Sigma$.
Then, $[f_\beta o(g \times id_\Sigma)](Q_i', x) = f_\beta((g \times id_\Sigma)(Q_i', x)) = f_\beta(g(Q_i'), x) = f_\beta([u], x) = [ux]$.
Also, $(gof_l')(Q_i', x) = g(f_l'(Q_i', x)) = g(f_l'((f_l')^*(\{q_0\}, u)), x) = g((f_l')^*(\{q_0\}, ux)) = [ux]$. To show the commutativity of square (B), let $Q_i' \in \overline{Q}'$, where $f_l{}'(\{q_0\}, u) = Q_i{}'$, for some $u \in \Sigma$. Then,
$(\omega_\beta og)(Q_i') = \omega_\beta(g(Q_i')) = \omega_\beta([u]) = \beta(u) = (id_{\overline{Z}} o\beta')(u) = \beta'(u) = (\omega_l' o\rho')(u) = \omega_l'(\rho'(u)) = \omega_l'(f_l{}'(\{q_0\}, u)) = \omega_l'(Q_i')$.
Also, let Q_i', $Q_j' \in \overline{Q}'$, where $f_l'(\{q_0\}, u) = Q_i'$ for some $u \in \Sigma$ and $f_l'(\{q_0\}, v) = Q_j'$ for some $v \in \Sigma$. Therefore, we have

$$\begin{aligned}&\tau_1 \le \delta_\beta(g(f_l'(Q_i', \sigma_1)), \sigma_2, g(Q_j')) \le \tau_2\\ &\Leftrightarrow \tau_1 \le \delta_\beta(g(f_l'(f_l'(\{q_0\}, u), \sigma_1)), \sigma_2, g(f_l'(\{q_0\}, v)) \le \tau_2\\ &\Leftrightarrow \tau_1 \le \delta_\beta(g(\rho'(u\sigma_1)), \sigma_2, g(\rho'(v))) \le \tau_2\\ &\Leftrightarrow \tau_1 \le \delta_\beta(\rho(u\sigma_1), \sigma_2, \rho(v)) \le \tau_2\\ &\Leftrightarrow \tau_1 \le \delta_\beta([u\sigma_1], \sigma_2, [v]) \le \tau_2\\ &\Leftrightarrow \tau_1 \le \mu(u\sigma_1\sigma_2 v) \le \tau_2\\ &\Leftrightarrow \tau_1 \le \delta_l'(\rho'(u\sigma_1), \sigma_2, v) \le \tau_2\\ &\Leftrightarrow \tau_1 \le \delta_l'(f_l'(f_l'(\{q_0\}, u), \sigma_1), \sigma_2, f_l'(\{q_0\}, v)) \le \tau_2\\ &\Leftrightarrow \tau_1 \le \delta_l'(f_l'(Q_i', \sigma_1), \sigma_2, Q_j') \le \tau_2\end{aligned}$$

Thus, $\tau_1 \le \delta_l'(f_l'(Q_i', \sigma_1), \sigma_2, Q_j') \le \tau_2 \Leftrightarrow \tau_1 \le \delta_\beta(g(f_l'(Q_i', \sigma_1)), \sigma_2, g(Q_j')) \le \tau_2$ and $g(\{q_0\}) = g((f_l')(\phi, \Lambda)) = [\Lambda]$. Hence, $(\tilde{F}_l)_\beta$ is a homomorphic image of $\tilde{F}_l{}'$. □

Definition 4.5. *A BL-general fuzzy automaton*

$$\tilde{F}_l = (\overline{Q}, \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$$

realizes fuzzy behavior $f : \Sigma^* \times Z \to [0,1]$ *is called minimal if for all BL-general fuzzy automata* $\tilde{F}_l' = (\overline{Q}', \Sigma, \tilde{R}', \overline{Z}, \omega_l', \delta_l', f_l', \tilde{\delta}_l', F_1, F_2)$ *realizes* f, $|\overline{Q}| \le |\overline{Q}'|$, *where* $|\overline{Q}|$ *denotes number of states in BL-general fuzzy automata* $\tilde{F}_l$.
The following shows that the canonical realization discussed in proposition 4.1 and 4.2 is minimal.

Proposition 4.6. *For given fuzzy behavior of $\tilde{F}_l$, the BL-general fuzzy automata $(\tilde{F}_l)_\beta$ is minimal BL-general fuzzy automata.*

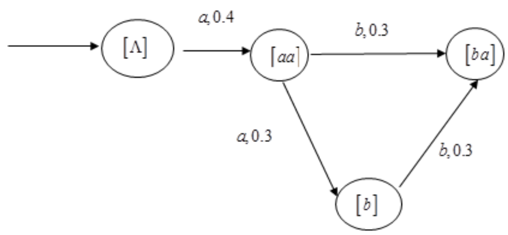


Figure 3: The BL- GFA of Example 4.7

Proof. Let $\tilde{F}_l' = (\overline{Q}', \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z}, \omega_l', \delta_l', f_l', \tilde{\delta}_l', F_1, F_2)$ be a BL-general fuzzy automaton, which realizes $\beta_{\{q_0\}}$ by its fuzzy output. Also, let

$$\tilde{F}_l'' = (\overline{Q}'', \Sigma, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \overline{Z}, \omega_l'', \delta_l'', f_l'', \tilde{\delta}_l'', F_1, F_2)$$

be another BL-general fuzzy automaton, where $\overline{Q}'' \subseteq \overline{Q}'$ such that for all $Q_i'' \in \overline{Q}''$, $\exists u \in \Sigma^*$ such that $f_l'(\{q_0\}, u) = Q_i''$, $f_l'' = f_l' \Big| \overline{Q}'' \times \Sigma$. Then, $\tilde{F}_l''$ is obviously a reachable BL-general fuzzy automaton which realizes fuzzy behavior $\beta_{\{q_0\}}$ with fuzzy output γ''. Now, from proposition 4.4, $(\tilde{F}_l)_\beta$ is homomorphic image of $\tilde{F}_l''$. Thus, $|Q_{\beta_i}| \leq \left|\overline{Q}''\right| \leq \left|\overline{Q}'\right|$. Hence, $(\tilde{F}_l)_\beta$ is minimal BL-general fuzzy automata, which realizes fuzzy behavior $\beta_{\{q_0\}}$. □

Example 4.7. *Consider a fuzzy behavior $\beta_{\{q_0\}} : \Sigma^* \times \overline{Z} \to [0,1]$ of $\tilde{F}_l$ in Example 3.11 such that:*

$$\beta_{\{q_0\}}(\Lambda, \{z_1\}) = 1,\ \beta_{\{q_0\}}(a^2, \{z_3\}) = 0.4,\ \beta_{\{q_0\}}(ba, \{z_1, z_2\}) = 0.3,\ \beta_{\{q_0\}}(b, \{z_3\}) = 0.3,$$

overmonoid generated by $\Sigma = \{a, b\}$. Then, the minimal BL-general fuzzy automata which realizes $\beta_{\{q_0\}}$ is given in Figure 3. Also, it can easily verified that the reachability map $\rho : \Sigma^* \to Q_\beta$ is onto, i. e., $(\tilde{F}_l)_\beta$ is reachable and the observability map σ is one-to-one. Thus, realization $(\tilde{F}_l)_\beta$ of $\beta_{\{q_0\}}$ is canonical.

4.2 Minimal realization based on derivatives of fuzzy behavior

This subsection is towards the construction of minimal realization of a given fuzzy behavior of $\tilde{F}_l$, based on derivatives of the fuzzy behavior. We begin with the following concept of derivatives.

Definition 4.8. *Let $\beta_{\{q_0\}} : \Sigma^* \times \overline{Z} \to [0,1]$ be a fuzzy behavior of $\tilde{F}_l$. The derivative of $\beta_{\{q_0\}}$, with respect to $u \in \Sigma^*$, is a map $\beta^u_{\{q_0\}} : \Sigma^* \times \overline{Z} \to [0,1]$ such that $\beta^u_{\{q_0\}}(w, Z_l) = \beta_{\{q_0\}}(uw, Z_l)$ $\forall w \in \Sigma^*$, $Z_l \in \overline{Z}$. For given a fuzzy behavior $\beta_{\{q_0\}}$ of $\tilde{F}_l$ we put $Q^\beta = \left\{\beta^u_{\{q_0\}} \mid u \in \Sigma^*\right\}$.*

Proposition 4.9. *Given a fuzzy behavior of $\tilde{F}_l$, there exists $(\tilde{F}_l)^\beta \in BGFA$ which realizes it.*

Proof. Let $\beta_{\{q_0\}} : \Sigma^* \times \overline{Z} \to [0,1]$ be a fuzzy behavior of $\tilde{F}_l$. Define the maps $f^\beta,\ \omega^\beta,\ \delta^\beta,\ \tilde{\delta}^\beta$ as:
$f^\beta : Q^\beta \times \Sigma \to Q^\beta$ such that $f^\beta(\beta^u_{\{q_0\}}, v) = \beta^{uv}_{\{q_0\}}$, $\forall v \in \Sigma$ and $\forall \beta^u_{\{q_0\}} \in Q^\beta$,
$\omega^\beta : Q^\beta \to \overline{Z}$ such that $\omega^\beta(\beta^u_{\{q_0\}}) = \beta(u)$,$\forall \beta^u_{\{q_0\}} \in Q^\beta$,
$\delta^\beta : Q^\beta \times \Sigma \times Q^\beta \to [0,1]$ such that $\delta^\beta(\beta^u_{\{q_0\}}, w, \beta^v_{\{q_0\}}) = \beta^{uwv}_{\{q_0\}}$, $\forall w \in \Sigma$ and $\forall \beta^u_{\{q_0\}}, \beta^v_{\{q_0\}} \in Q^\beta$,
$\tilde{\delta}^\beta : (Q^\beta \times [0,1]) \times \Sigma \times Q^\beta \to [0,1]$ such that

$$\tilde{\delta}^\beta((\beta^u_{\{q_0\}}, \mu^t(\beta^u_{\{q_0\}})), w, \beta^v_{\{q_0\}}) = F_1(\mu^t(\beta^u_{\{q_0\}}), \delta^\beta(\beta^u_{\{q_0\}}, w, \beta^v_{\{q_0\}}))$$

where $\mu^t(\beta^u_{\{q_0\}}) = \mu(u)$. The maps $f^\beta,\ \omega^\beta,\ \delta^\beta,\ \tilde{\delta}^\beta$ are well-defined. We will only show that f^β is well-defined.
Let $g,\ h \in Q^\beta$ such that $g = h$. Then, there exists $u,\ v \in \Sigma^*$ such that $g = \beta^u_{\{q_0\}},\ h = \beta^v_{\{q_0\}}$ and $\beta^u_{\{q_0\}} = \beta^v_{\{q_0\}}$. Now,

$$\begin{aligned} g = h \Rightarrow \beta^u_{\{q_0\}} = \beta^v_{\{q_0\}} &\Rightarrow \beta^u_{\{q_0\}}(w, Z_l) = \beta^v_{\{q_0\}}(w, Z_l)\ \forall w \in \Sigma^*,\ Z_l \in \overline{Z} \\ &\Rightarrow \beta^u_{\{q_0\}}(\sigma t, Z_l) = \beta^v_{\{q_0\}}(\sigma t, Z_l)\ \forall t \in \Sigma^*,\ \sigma \in \Sigma,\ Z_l \in \overline{Z} \\ &\Rightarrow \beta^{u\sigma}_{\{q_0\}}(t, Z_l) = \beta^{v\sigma}_{\{q_0\}}(t, Z_l)\ \forall t \in \Sigma^*,\ Z_l \in \overline{Z} \\ &\Rightarrow \beta^{u\sigma}_{\{q_0\}} = \beta^{v\sigma}_{\{q_0\}} \Rightarrow f^\beta(\beta^u_{\{q_0\}}, \sigma) = f^\beta(\beta^v_{\{q_0\}}, \sigma) \Rightarrow f^\beta(g, \sigma) = f^\beta(h, \sigma),\ \forall \sigma \in \Sigma. \end{aligned}$$

Thus, $(\tilde{F}_l)^{\beta} = (Q^{\beta}, \Sigma, \tilde{R}^{\beta} = (\beta^{\Lambda}_{\{q_0\}}, \mu^{t_0}(\beta^{\Lambda}_{\{q_0\}})), \overline{Z}, \omega^{\beta}, \delta^{\beta}, f^{\beta}, \tilde{\delta}^{\beta}, F_1, F_2)$ is a BL-general fuzzy automaton.
Also, by induction, it is easy to verify that f^{β} can be extended to $(f^*)^{\beta} : Q^{\beta} \times \Sigma^* \to Q^{\beta}$ such that $(f^*)^{\beta}(\beta^{u}_{\{q_0\}}, w) = \beta^{uw}_{\{q_0\}}$, $\forall w \in \Sigma^*$. Finally, it remains to show that $E((\tilde{F}_l)^{\beta}) = \beta_{\{q_0\}}$. For this, let $w \in \Sigma^*$ and $Z_l \in \overline{Z}$, then

$$E((\tilde{F}_l)^{\beta})(w, Z_l) = \beta^{\Lambda}_{\{q_0\}}(w, Z_l) = \beta_{\{q_0\}}(\Lambda w, Z_l) = \beta_{\{q_0\}}(w, Z_l).$$

Hence, $E((\tilde{F}_l)^{\beta}) = \beta_{\{q_0\}}$, whereby $(\tilde{F}_l)^{\beta}$ realizes $\beta_{\{q_0\}}$. □

Proposition 4.10. *The realization* $(\tilde{F}_l)^{\beta} = (Q^{\beta}, \Sigma, \tilde{R}^{\beta} = (\beta^{\Lambda}_{\{q_0\}}, \mu^{t_0}(\beta^{\Lambda}_{\{q_0\}})), \overline{Z}, \omega^{\beta}, \delta^{\beta}, f^{\beta}, \tilde{\delta}^{\beta}, F_1, F_2)$ *of a fuzzy behavior* $\tilde{F}_l$ *is canonical.*

Proof. According to the definition of observability map σ ($\sigma(\beta^{u}_{\{q_0\}}) = \beta^{u}_{\{q_0\}}$), it is clear that σ is one-to-one. To show that reachability map ρ is onto, let $g \in Q^{\beta}$. Then, there exists $w \in \Sigma^*$ such that $g = \beta^{w}_{\{q_0\}}$.
Now, $\rho(w) = (f^*)^{\beta}(\beta^{\Lambda}_{\{q_0\}}, w) = \beta^{\Lambda w}_{\{q_0\}} = \beta^{w}_{\{q_0\}} = g$. Thus, for each $g \in Q^{\beta}$ there exists $w \in \Sigma^*$ such that $\rho(w) = g$, whereby ρ is onto. Hence, $(\tilde{F}_l)^{\beta}$ of fuzzy behavior $\tilde{F}_l$ is canonical. □

Now, we provide another functorial relationship between the categories D and B.

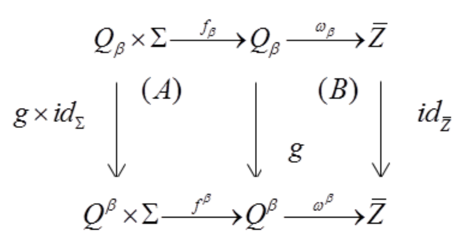
Proposition 4.11. *Let* $H : D \to B$ *be a map which sends each* $f \in D$ *to* $(\tilde{F}_l)^{f}$*, and to each D-morphism*$(g_{in}, g_{out}) : f \to f'$*, to* $(g, g_{in}, g_{out}) : (\tilde{F}_l)^{f} \to (\tilde{F}_l)^{f'}$ *, where* $g : Q^{f} \to Q^{f'}$ *is the map such that* $g(f^{w}) = f^{g^*_{in}(w)}$*. Then,* H *is a functor.*

Proof. It is similar to that of Proposition 4.3. □

We close this subsection by providing an isomorphism between $(\tilde{F}_l)_{\beta}$ and $(\tilde{F}_l)^{\beta}$.

Proposition 4.12. *Given a fuzzy behavior* $\beta_{\{q_0\}}$ *of* $\tilde{F}_l$*, the BGFA-object* $(\tilde{F}_l)^{\beta}$ *is isomorphic to BGFA-object* $(\tilde{F}_l)_{\beta}$*.*

Proof. Define a map $\varphi = (g, id_{\Sigma}, id_{\overline{Z}}) : (\tilde{F}_l)_{\beta} \to (\tilde{F}_l)^{\beta}$ such that $g([w]) = \beta^{w}_{\{q_0\}}$, $\forall w \in \Sigma^*$. Then for given $[w] \in Q_{\beta}$, there exists $\beta^{w}_{\{q_0\}} \in Q^{\beta}$ such that $g([w]) = \beta^{w}_{\{q_0\}}$. Thus, φ is onto. Also, it is clear that φ is one-to-one. Finally, in order to demonstrate that $\varphi = (g, id_{\Sigma}, id_{\overline{Z}}) : (\tilde{F}_l)_{\beta} \to (\tilde{F}_l)^{\beta}$ is a homomorphism, we have to show that the following diagrams commute.



To show the commutativity of square (A), let $([w], \sigma) \in Q_{\beta} \times \Sigma$.
Then, $[f^{\beta} o (g \times id_{\Sigma})]([w], \sigma) = f^{\beta}(g([w]), \sigma) = f^{\beta}(\beta^{w}_{\{q_0\}}, \sigma) = \beta^{w\sigma}_{\{q_0\}}$.
Also, $(g o f_{\beta})([w], \sigma) = g(f_{\beta}([w], \sigma)) = g([w\sigma]) = \beta^{w\sigma}_{\{q_0\}}$. Thus, square (A) commutes. Commutativity of square (B) follows from the fact that

$$(\omega^{\beta} o g)([w]) = \omega^{\beta}(g([w])) = \omega^{\beta}(\beta^{w}_{\{q_0\}}) = \beta(w) = \omega_{\beta}([w]) = (id_{\overline{Z}} o \omega_{\beta})([w]).$$

Finally, $g([\Lambda]) = \beta^{\Lambda}_{\{q_0\}}$ and

$$\begin{aligned}
&\tau_1 \leq \delta^{\beta}(g(f_{\beta}([u], \sigma_1)), \sigma_2, g([v])) \leq \tau_2 \\
&\Leftrightarrow \tau_1 \leq \delta^{\beta}(g([u\sigma_1]), \sigma_2, g([v])) \leq \tau_2 \\
&\Leftrightarrow \tau_1 \leq \delta^{\beta}(\beta^{u\sigma_1}_{\{q_0\}}, \sigma_2, \beta^{v}_{\{q_0\}}) \leq \tau_2 \\
&\Leftrightarrow \tau_1 \leq \beta^{u\sigma_1\sigma_2 v}_{\{q_0\}} \leq \tau_2 \\
&\Leftrightarrow \tau_1 \leq \mu(u\sigma_1\sigma_2 v) \leq \tau_2 \\
&\Leftrightarrow \tau_1 \leq \delta_{\beta}([u\sigma_1], \sigma_2, [v]) \leq \tau_2 \\
&\Leftrightarrow \tau_1 \leq \delta_{\beta}(f_{\beta}([u], \sigma_1), \sigma_2, [v]) \leq \tau_2.
\end{aligned}$$

Hence, the BGFA-object $(\tilde{F}_l)^{\beta}$ is isomorphic to BGFA-object $(\tilde{F}_l)_{\beta}$. □

5 Conclusion

The current study was an attempt to answer the question that ”given a fuzzy behavior, whether we could design a minimal BL-general fuzzy automaton, which realized it” in category theoretic setup. Specifically, two minimal BL-general fuzzy automata which realized the given fuzzy behavior were introduced and studied. The construction of one such automaton was based on Myhill-Nerode’s theory, while construction of the other was based on derivative of the given fuzzy behavior. In future, we will try to introduce the concept of realization of fuzzy behavior by monoids and soft BCK-algebra.

References

[1] K. Abolpour, M. M. Zahedi, *BL-general fuzzy automata and accept behavior*, Journal of Applied Mathematics and Computing, **38** (2012), 103–118.

[2] K. Abolpour, M. M. Zahedi, *Isomorphism between two BL-general fuzzy automata*, Soft Computing, **16** (2012), 729–736.

[3] M. A. Arbib, E. G. Manes, *Machines in a category: An expository introduction*, SIAM Review, **16** (1974), 285–302.

[4] Y. Cao, Y. Ezava, *Nondeterministic fuzzy automata*, Information Sciences, **191** (2012), 86–97.

[5] M. Doostfatemeh, S. C. Kremer, *New directions in fuzzy automata*, International Journal of Approximate Reasoning, **38** (2005), 175–214.

[6] J. A. Goguen, *Minimal realization of machines in closed categories*, Bulletin of American Mathematical Society, **78** (1972), 777–783.

[7] J. Ignjatovic, M. Ciric, S. Bogdanovic, T. Petkovic, *Myhill-Nerode type theory of fuzzy languages and automata*, Fuzzy Sets and Systems, **161** (2010), 1288–1324.

[8] Y. B. Jun, *Intuitionistic fuzzy finite state machins*, Journal of Applied Mathematics and Computing, **17** (2005), 109–120.

[9] Y. B. Jun, *Intuitionistic fuzzy finite swichboard state machines,* Journal of Applied Mathematics and Computing, **20** (2006), 315–325.

[10] Y. B. Jun, *Quotient structures of intuitionistic fuzzy finite state machins,* Information Sciences, **177** (2007), 4977–4986.

[11] Y. H. Kim, J. G. Kim, S. J. Cho, *Products of T-generalized state machines and T-generalized transformation semigroups*, Fuzzy Sets and Systems, **93** (1998), 87–97.

[12] H. V. Kumbhojkar, S. R. Chaudhri, *On proper fuzzification of fuzzy finite state machines*, International Journal of Fuzzy Mathematics, **4** (2008), 1019–1027.

[13] E. T. Lee, L. A. Zadeh, *Note on fuzzy languages*, Information Sciences, **1** (1969), 421–434.

[14] D. S. Malik, J. N. Mordeseon, M. K. Sen, *Submachines of fuzzy finite state machine*, Journal of Fuzzy Mathematics, **2** (1994), 781–792.

[15] D. W. Qiu, *Automata theory based on complete residuated latticed-valued logic (I)*, Science China (Ser. F), **44** (2001), 419–429.

[16] E. S. Santos, *Maximin automata*, Information and Control, **12** (1968), 367–377.

[17] M. G. Thomason, P. N. Marinos, *Deterministic acceptors of regular fuzzy languages*, IEEE Transactions Systems Man Cybernetics, **4** (1974), 228–230.

[18] S. P. Tiwari, A. K. Singh, *On minimal realization of fuzzy behavior and associated categories*, Journal of Applied Mathematics and Computing, **45** (2014), 223–234.

[19] S. P. Tiwari, V. K. Yadav, A. K. Singh, *Construction of a minimal realization and monoid for a fuzzy language; a categorical approch*, Journal of Applied Mathematics and Computing, **47** (2015), 401–416.

[20] E. Turunen, *Boolean deductive systems of BL-algebras*, Archive for Mathematical Logic, **40** (2001), 467–473

[21] V. K. Yadav, V. Gautam, S. P. Tiwari, *On minimal realization of IF-languages: A categorical approach*, Iranian Journal of Fuzzy Systems, **13**(3) (2016), 19–34.

[22] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–35.