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Switching fuzzy modelling and control scheme using T-S fuzzy systems with nonlinear consequent parts

M. J. Koopaee[1] and V. J. Majd[2]

[1,2] *School of Electrical and Computer Engineering, Tarbiat Modares University, Tehran, Iran*

mohammadali.javaherikoopaee@pg.canterbury.ac.nz, majd@modares.ac.ir

Abstract

This paper extends the idea of switching T-S fuzzy systems with linear consequent parts to nonlinear ones. Each nonlinear subsystem is exactly represented by a T-S fuzzy system with Lure type consequent parts, which allows to model and control wider classes of switching systems and also reduce the computation burden of control synthesis. With the use of a switching fuzzy Lyapunov function, the LMI conditions for asymptotic stability of the system with maximum decay-rate and disturbance attenuation properties under arbitrary switching law are derived. Moreover, several numerical examples are provided to demonstrate the effectiveness of the proposed approaches to reduce the computational burden of control synthesis and improving the closed-loop performance of the system.

Keywords: Switching nonlinear systems, fuzzy systems, sector-bounded nonlinearity, linear matrix inequality, Lure type systems.

1 Introduction

Switching systems are a particular class of hybrid dynamical systems, which consist of several subsystems and a switching law that selects the active subsystem among them [15]. Recently, this type of systems have been under investigation because of their extensive capability for modelling mechanical systems, chemical processes and power systems [14, 35]. Furthermore, there exist even some non-switching nonlinear systems that cannot be globally stabilized by any continuous feedback controller [15, 4]; thus, designing a switching controller for stabilizing such systems is inevitable.

Generally speaking, controller design for switching nonlinear systems is a difficult task, mainly because in contrary to linear systems, there is no straightforward method for dealing with nonlinear subsystems [7]. To deal with this problem, the fuzzy modelling framework is extended to model hybrid systems in [21] for the first time. Shortly after, in [25], the idea of switching fuzzy systems has been studied and a continuous-time switching Takagi Sugeno (T-S) fuzzy controller based on parallel distribution compensation (PDC) method is proposed for a system subject to second order non-holonomic constraint. From that time, many works have been reported both in continuous- and discrete-time domain. (See [20] and the references therein).

Available literature in discrete-time T-S fuzzy switching systems, which is the main topic of this paper, can be categorized into two different classes. In [6] and more recently in [36], a switching PDC controller is proposed to obtain less conservative stability conditions for non-switching T-S fuzzy systems. In this first group of papers, initially, the state space is partitioned for the construction of a piecewise quadratic Lyapunov function and then, LMI based conditions for controller design are derived. In the second class, such as [37, 5, 3, 12], the authors have employed T-S fuzzy models as a modeling framework for switching nonlinear systems. In these works, T-S fuzzy systems are used to capture the nonlinearity of each subsystem with a set of rules with linear consequent parts. The resulting system remains nonlinear, however each subsystem is represented with linear local model, thus the conventional linear control methods can be employed and the stability can be guaranteed by some LMI conditions. More precisely, in [37, 5], switched fuzzy systems are considered, where the switching signal is employed as an input. Then, a common quadratic

Corresponding Author: V. J. Majd

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Lyapunov function (CQLF) is used to design both a state feedback controller and a state-dependent switching law. In [3], to reduce the conservatism of the stability conditions, a switching fuzzy Lyapunov function is employed for the switching T-S fuzzy system and stability conditions under arbitrary switching law are obtained. More recently, in [12] using a fuzzy Lyapunov function candidate, less conservative LMI stabilization conditions are derived, in [33] control design is investigated, in [32] the problem of observer based controller design is studied and in [34] the problem of filter design for switching T-S fuzzy systems with imprecise modes is addressed.

The major drawback of the previously mentioned works (i.e., [37, 5, 3, 12, 33, 32, 34]) is that, when the complexity of the system increases, the number of necessary rules for exact representation of the system by T-S models and as a result the number of LMI conditions grows exponentially. This problem is even more concerning when one deals with switching systems, where each subsystem is modeled by a T-S fuzzy system. Solving the mentioned problem was our first motivation for considering a nonlinear term in the consequent parts of each rule.

Moreover, there is a growing attention toward the study of switching Lure type systems, where each subsystem consists of a sector-bounded measurable nonlinear term in addition to the linear terms [10, 11, 13]. Particularly, in [10], in addition to considering the switching discrete-time Lure type system, actuator saturation is considered and in [11] the problem of control design for such systems without saturation is investigated and in [13] an observer based stabilization controller with intermittent measurements is designed. Although these methods can be employed to control wide class of switching systems, they are totally incapable, when the linear terms of the switching Lure systems are also extended to nonlinear terms to cover wider classes of the systems. In this case even conventional T-S fuzzy models with linear consequent part can no longer be used to capture the nonlinearities due to the fact that, the analytical model of the measurable sector-bounded nonlinearities is unknown. In fact, this is our second motivation for considering nonlinear term in the consequent part of each rules.

To overcome the two mentioned limitations associated with T-S fuzzy systems with linear consequent part, one can use the idea of sector-bounded nonlinear then-parts proposed in [9]. In [9], to reduce the number of rules, a part of nonlinearity of the system is lumped to the then-parts of the rules in addition to the linear terms. The common nonlinear term that appears in the consequent of each rule must satisfy a sector-bounded condition. This work has been extended in [19] to consider constraint on system output for non-switching fuzzy systems. More recently, a model based predictive control for complex nonlinear system is proposed in [27], which result in more relaxed and robust solution with lower online computation. Similar controllers are also designed in [23] and [17] for systems with varying local nonlinear models and wind turbines with variable speed and pitch, respectively, showing the effectiveness of such fuzzy systems with nonlinear local models for dealing with complex nonlinear systems.

In this paper, the discrete-time switching fuzzy systems with nonlinear local models are introduced for the first time, which allow to represent a class of switching systems with unknown-structure but measurable sector-bounded nonlinearities. This new modeling framework allows to exactly represent nonlinear systems with unknown structure sector bounded nonlinear terms. It is not possible to represent this type of systems with conventional T-S fuzzy models, hence the results in this paper can be viewed as an extension of the available works in the field of Lure type switching system such as [11, 13]. The stability of the proposed model is investigated and the control synthesis conditions based on a fuzzy switching Lyapunov function are derived in the form of LMIs. Another novel aspect of this paper is the method of obtaining LMI stability conditions. To show the superiority of this method over similar ones, feasibility region of the obtained LMI conditions and that of [9], is compared for the special case, when the system does not switch, due to the limitations of design in [9]. Moreover, the method in [3] is compared to our proposed method in terms of the number of rules, and as a result, the number of sufficient conditions to guarantee the asymptotical stability of the system. To strengthen the claim that with the use of this method, a better closed loop performance can be achieved, decay-rate optimization and controller design is also incorporated into the problem and tested by numerical simulations.

The organization of this paper is as follows. In section 2, some necessary mathematical backgrounds are presented. In section 3 switching T-S fuzzy systems with nonlinear local models are introduced, and in section 4, LMI based control synthesis conditions are obtained. Numerical examples are presented in section 5 to show the validity of the results and the concluding remarks are provided in section 6.

2 The problem statement and preliminaries

Consider the following discrete-time switching nonlinear system:

$$\begin{aligned} x(k+1) &= f_{1\sigma(k)}(x(k)) + g_{1a\sigma(k)}(x(k))\, w(k) + g_{1b\sigma(k)}(x(k))\, u(k), \\ y(k) &= f_{2\sigma(k)}(x(k)) + g_{2a\sigma(k)}(x(k))\, w(k) + g_{2b\sigma(k)}(x(k))\, u(k), \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$ is the state vector, $u \in \mathcal{R}^{m_u}$ is the input vector, $w(k) \in \mathcal{R}^{m_w}$ is the disturbance, $y(k) \in \mathcal{R}^{m_y}$ is the output, $\sigma(k) \in \{1, 2, .., L\}$ is the switching law and the simplified notation $f_{1l}(.)$, $g_{1al}(.)$, $g_{1bl}(.)$, $f_{2l}(.)$, $g_{2al}(.)$, $g_{2bl}(.)$ denote nonlinear function matrices with $l = 1, \ldots, L$ is the index of the active subsystem according to the switching law. Moreover, $f_{1l}(.), f_{2l}(.)$ is structured as a combination of two terms as follows:

$$f_{1l}(x(k)) = f_{1al}(x(k)) + f_{1bl}(x(k))\,\overline{\phi}_l(x(k)),$$

$$f_{2l}(x(k)) = f_{2al}(x(k)) + f_{2bl}(x(k))\,\overline{\phi}_l(x(k)), \tag{2}$$

where $f_{1al} \in R^n, f_{2al} \in R^{m_y}$, $f_{1bl} \in R^{n\times p}, f_{2bl} \in R^{m_y \times p}$ are known nonlinear matrix functions and

$$\overline{\phi}_l(x(k))_{p\times 1} = [\phi_{l1}(x(k)), \phi_{l2}(x(k)), \ldots, \phi_{lp}(x(k))]^T, \tag{3}$$

is a sector bounded nonlinear vector function that fulfills the following assumption:

Assumption 2.1. *Function $\overline{\phi}_{lq}(.)$ satisfies the following sector-bounded condition:*

$$\overline{\phi}_{lq}(x(k)) \in Co\left\{E_{L_{lq}}x(k), E_{U_{lq}}x(k)\right\} \qquad \forall q=1, .., p, \tag{4}$$

where $E^T_{L_{lq}}$, $E^T_{U_{lq}} \in \mathcal{R}^n$ and $Co\{.\}$ denotes the set of all convex combinations of two points $E_{L_{lq}}x(k)$ and $E_{U_{lq}}x(k)$ defined as follows [19]*:*

$$\left\{\theta_1 E_{U_{lq}}x(k) + \theta_2 E_{L_{lq}}x(k) \;\middle|\; \theta_1 + \theta_2 = 1, \theta_i > 0\right\}. \tag{5}$$

As a result of (4) each $\phi_{lq}(x(k))$ satisfies the following inequality:

$$\left(E_{L_{lq}}x(k) - \overline{\phi}_{lq}(x(k))\right)\left(E_{U_{lq}}x(k) - \overline{\phi}_{lq}(x(k))\right) \le 0. \tag{6}$$

Remark 2.2. *If functions $f_{1l}(.)$, $g_{1al}(.)$, $g_{1bl}(.)$ in (1) be linear, the system reduces to the well-known discrete-time switching Lure' System. This, shows that this model is more general than the one considered in* [10, 11]*.*

Remark 2.3. *It should be noted that, $\overline{\phi}_l(x(k))$ could be used to represent measurable state dependent nonlinearities with unknown structure in addition to the known nonlinear terms* [30]*. However, when the model of $\overline{\phi}_l(x(k))$ is also available, the advantages of considering the form given in (2) for $f_l(.)$, is to move the nonlinear function matrix $\overline{\phi}_l(x(k))$ to the consequent parts of each rule of the T-S fuzzy model of the system, hence reduce the number of the rules. This will be discussed in more detail in the following section.*

Without loss of generality one may assume that:

$$\overline{\phi}_{lq}(x(k)) \in Co\left\{0, \overline{E}_{U_{lq}}x(k)\right\} \quad \forall q = 1, .., p. \tag{7}$$

Hence, it is not difficult to show that inequality (6) can be written as follows [9]:

$$\overline{\phi}_l^T(x(k))\,\Lambda_l E_l x(k) - \overline{\phi}_l^T(x(k))\,\Lambda_l \overline{\phi}_l(x(k)) \ge 0, \tag{8}$$

where $E_l^T = \left[\overline{E}^T{}_{U_{l1}}, \overline{E}^T{}_{U_{l2}}, \cdots, \overline{E}^T{}_{U_{lp}}\right] \in \mathcal{R}^{n\times p}$ and is assumed to be known, $\Lambda_l \in \mathcal{R}^{p\times p}$ is a positive definite diagonal matrix.

3 Discrete time switching nonlinear systems modeling

In this section, we extend discrete-time T-S fuzzy models with nonlinear consequent parts given in [9], to model the switching system. Moreover, one can consider it as the extension of switching T-S fuzzy systems with linear consequent parts to that with nonlinear consequent parts. It should be noted that whether the form of measurable nonlinear vector function $\overline{\phi}_l(x(k))$ is assumed to be known or unknown, the modelling and controller design methods remain the same. However, in both cases, the value of $\overline{\phi}_l(x(k))$ is assumed to be available for measurement, which is a very common assumption and is considered in many works such as [11, 9].

The nonlinear switching system (1) can be represented by a set of fuzzy subsystems where the l-th subsystem is exactly represented with r_l rules as follows:

$$\begin{aligned}&\text{Rule i:}\\ &\quad \text{If} \quad z_l^1(k)\, is\ M_l^{i1} and \ldots and\ z_l^{v_l}(k)\, is\ M_l^{iv_l}\\ &\quad \text{Then} \quad x(k+1) = A_l^i x(k) + B_{1l}^i u(k) + B_{2l}^i w(k) + G_{1l}^i \overline{\phi}_l(x(k))\\ &\quad y(k) = C_l^i x(k) + D_{1l}^i u(k) + D_{2l}^i w(k) + G_{2l}^i \overline{\phi}_l(x(k)),\end{aligned} \tag{9}$$

where $i = 1, \ldots, r_l$, $l = 1, \ldots, L$, $z_l^1(k), \ldots, z_l^{v_l}(k)$ are premise variables, $M_l^{i1}, \ldots, M_l^{iv_l}$ are fuzzy sets, $A_l^i, B_{1l}^i, B_{2l}^i, G_{1l}^i$, $C_l^i, D_{1l}^i, D_{2l}^i, G_{2l}^i$, are constant matrices and $\overline{\phi}_l(x(k))$ is the unknown structure sector-bounded nonlinear vector function which satisfies (8).

Remark 3.1. *Unlike conventional T-S fuzzy systems with linear consequent parts, in which the mathematical structure of each nonlinearity should be known to exactly represent the system, using fuzzy rules (9) it is no longer necessary to know the structure of sector bounded nonlinear term $\overline{\phi}_l(x(k))$.*

With the use of the PDC method, the following switching controller is proposed for the system (9) where i-th rule of l-th sub controller is as follows:

$$\begin{aligned}&\text{Rule i:}\\ &\quad \text{If } z_l^1(k)\, is\ M_l^{i1} and \ldots and\ z_l^{v_l}(k)\, is\ M_l^{iv_l}\\ &\quad \text{Then } u(k) = K_l^i x(k) + \Upsilon_l^i \overline{\phi}_l(x(k)).\end{aligned} \tag{10}$$

Using fuzzy inference method with singleton fuzzifier, product inference engine and center average defuzzifier, the analytical representation of (9) and (10) is obtained as follows respectively:

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r_l} h_l^i(z(k)) \left\{ A_l^i x(k) + B_{1l}^i u(k) + B_{2l}^i w(k) + G_{1l}^i \overline{\phi}_l(x(k)) \right\},\\ y(k) &= \sum_{i=1}^{r_l} h_l^i(z(k)) \left\{ C_l^i x(k) + D_{1l}^i u(k) + D_{2l}^i w(k) + G_{2l}^i \overline{\phi}_l(x(k)) \right\},\end{aligned} \tag{11}$$

$$u(k) = \sum_{i=1}^{r_l} h_l^i(z(k)) \left\{ K_l^i x(k) + \Upsilon_l^i \overline{\phi}_l(x(k)) \right\}, \tag{12}$$

where $h_l^i(z(k)) = \frac{\kappa_l^i(z(k))}{\sum_{j=1}^{r_l} \kappa_l^j(z(k))}$ and $\kappa_l^i(z(k)) = \prod_{j=1}^{v_l} M_l^{ij}\left(z_l^j(k)\right)$. Substituting (12) into (11), the l-th subsystem can be written in closed-loop form as follows:

$$\begin{aligned} x(k+1) &= (A_l(h) + B_{1l}(h) K_l(h)) x(k) + (G_{1l}(h) + B_{1l}(h) \Upsilon_l(h)) \overline{\phi}_l(x(k)) + B_{2l}(h)\ w(k),\\ y(k) &= (C_l(h) + D_{1l}(h) K_l(h)) x(k) + (G_{2l}(h) + D_{1l}(h) \Upsilon_l(h)) \overline{\phi}_l(x(k)) + D_{2l}(h)\ w(k),\end{aligned} \tag{13}$$

where:

$$\begin{aligned} &A_l(h) = \sum_{i=1}^{r_l} h_l^i(z(k)) A_l^i, \quad B_{1l}(h) = \sum_{i=1}^{r_l} h_l^i(z(k)) B_{1l}^i, \quad G_{1l}(h) = \sum_{i=1}^{r_l} h_l^i(z(k)) G_{1l}^i,\\ &B_{2l}(h) = \sum_{i=1}^{r_l} h_l^i(z(k)) B_{2l}^i, \quad C_l(h) = \sum_{i=1}^{r_l} h_l^i(z(k)) C_{1l}^i, \quad D_{1l}(h) = \sum_{i=1}^{r_l} h_l^i(z(k)) D_{1l}^i,\\ &D_{2l}(h) = \sum_{i=1}^{r_l} h_l^i(z(k)) D_{2l}^i, \quad G_{2l}(h) = \sum_{i=1}^{r_l} h_l^i(z(k)) G_{2l}^i, \quad K_l(h) = \sum_{i=1}^{r_l} h_l^i(z(k)) K_l^i,\end{aligned} \tag{14}$$

As it can be seen in (12), control feedback signal is a combination of conventional state feedback and the sector-bounded nonlinear term, i.e.,$\overline{\phi}_l(x(k))$, which allows to achieve better close-loop performance as will be discussed later.

Remark 3.2. *With the use of fuzzy model (9) to model the switching nonlinear system (1), then-part of each rule can be considered as a Lure' type system, hence the results in related works such as* [10, 11] *can easily be extended to the more general case, where each subsystem is the combination of known nonlinear terms and sector-bounded nonlinearities. Also one can see that this model is more general than the considered model in* [1].

4 Switching fuzzy controller design

In this section we derive the conditions to design switching fuzzy controller in form of (10) for the system (9) to guarantee asymptotic stability under arbitrary switching. In other words, the switching law is not known a priori, and thus, a switching controller should be designed to stabilize the system under any switching law. The following lemma is employed frequently in the rest of the paper:

Lemma 4.1. *If for all* $l, m = 1, \ldots, L,\ i = 1,\ \ldots, r_l,\ q = 1,\ \ldots, r_m$ *and* $j = 1,\ \ldots, r_l$*, such that* $j \neq i$ *there exists matrices* W_{lm}^{ijq} *such that the following conditions hold:*

$$\begin{gathered} W_{lm}^{iiq} > 0, \\ \frac{1}{r_l - 1} W_{lm}^{iiq} + \frac{1}{2}\left(W_{lm}^{ijq} + W_{lm}^{jiq}\right) > 0, \end{gathered} \tag{15}$$

then, the following inequality will be satisfied:

$$\sum_{q=1}^{r_m}\sum_{i=1}^{r_l}\sum_{j=1}^{r_l} h_l^i h_l^j h_m^q W_{lm}^{ijq} > 0, \tag{16}$$

where $0 < h_l^i, h_l^j,\ h_m^q < 1$ *and* $\sum_{i=1}^{r_l} h_l^i = 1,\ \sum_{j=1}^{r_l} h_l^j = 1,\ \sum_{q=1}^{r_m} h_m^q = 1.$

Proof. The proof is similar to the one given in [29] when extended to switching fuzzy systems hence, it is omitted for the sake of brevity. □

Remark 4.2. *Double sum inequality (16) is frequently used in the fuzzy control literature, because many stability problems can be written in this form. Note that, this lemma, only gives sufficient conditions, however compared to* [22]*, which gives necessary and sufficient conditions, the computation burden of Lemma 4.1 is considerably lower.*

4.1 Stabilizing controller design for the systems without disturbance

4.1.1 Stability analysis

The following theorem gives sufficient conditions for closed-loop stability of the switching fuzzy system under arbitrary switching law:

Theorem 4.3. *Consider the switching fuzzy system (9) without disturbance and via the fuzzy controller (10). If for all* $l, m = 1, \ldots, L$*,* $i = 1,\ \ldots, r_l$*,* $q = 1,\ \ldots, r_m$ *and* $j=1, \ldots, r_l$*, there exist positive definite matrices* $P_1^1, \ldots,\ P_1^{r_1}, \ldots,$ $P_l^1, \ldots,\ P_l^{r_l}, \ldots,\ P_L^1, \ldots,\ P_L^{r_L} \in R^{n\times n}$*, diagonal positive definite matrices* $\Gamma_l \in R^{p\times p}$*, and matrices* $K_l^i \in R^{m\times n}$*,* $\Upsilon_l^i \in R^{m\times p}$ *such that matrices:*

$$W_{lm}^{ijq} = \begin{bmatrix} P_l^i & -E_l^T\Gamma_l & \left(A_l^j + B_{1l}^j K_l^i\right)^T P_m^q \\ * & 2\Gamma_l & \left(G_{1l}^j + B_{1l}^j \Upsilon_l^i\right)^T P_m^q \\ * & * & P_m^q \end{bmatrix} \tag{17}$$

satisfies the conditions in (15) then the switching fuzzy system (9) via the fuzzy controller (10) is asymptotically stable under arbitrary switching law.

Proof. Considering Lemma 4.1, and substituting (17) into (16) follows that:

$$\sum_{q=1}^{r_m}\sum_{j=1}^{r_l}\sum_{i=1}^{r_l} h_l^i\left(x(k)\right) h_l^j\left(x(k)\right) h_m^q\left(x(k+1)\right) \times \begin{bmatrix} P_l^i & -E_l^T\Gamma_l & \left(A_l^j + B_{1l}^j K_l^i\right)^T P_m^q \\ * & 2\Gamma_l & \left(G_{1l}^j + B_{1l}^j \Upsilon_l^i\right)^T P_m^q \\ * & * & P_m^q \end{bmatrix} > 0. \tag{18}$$

Using notation (14), inequality (18) can be rewritten as:

$$\begin{bmatrix} P_l(h) & -E_l^T\Gamma_l & \left(A_l(h) + B_{1l}(h) K_l(h)\right)^T P_m^+(h) \\ * & 2\Gamma_l & \left(G_{1l}(h) + B_{1l}(h)\Upsilon_l(h)\right)^T P_m^+(h) \\ * & * & P_m^+(h) \end{bmatrix} > 0, \tag{19}$$

where $P_m^+(h) = \sum_{q=1}^{r_m} h_m^q(x(k+1)) P_m^q$. Now, applying Schur complement to (19) it follows that:

$$-\begin{bmatrix} P_l(h) & -E_l^T\Gamma_l \\ * & 2\Gamma_l \end{bmatrix} + \begin{bmatrix} H_l^T(h) \\ F_l^T(h) \end{bmatrix} P_m^+(h) \begin{bmatrix} H_l(h) & F_l(h) \end{bmatrix} < 0, \tag{20}$$

where $H_l(h) = A_l(h) + B_{1l}(h) K_l(h)$and $F_l(h) = G_{1l}(h) + B_{1l}(h)\Upsilon_l(h)$. Pre- and post-multiplying (20) by $\begin{bmatrix} x(k) & \overline{\phi}_l(x(k)) \end{bmatrix}^T$ and its' transpose respectively, yields:

$$\begin{gathered}\left(H_l(h)x(k) + F_l(h)\overline{\phi}_l(x(k))\right)^T P_m^+(h)\left(H_l(h)x(k) + F_l(h)\overline{\phi}_l(x(k))\right) \\ -x(k)^T P_l(h)x(k) + 2\overline{\phi}_l^T(x(k))\Gamma_l E_l x(k) - 2\overline{\phi}_l^T\Gamma_l\overline{\phi}_l(x(k)) < 0.\end{gathered} \tag{21}$$

Defining:

$$V(x(k)) = x(k)^T P_{\sigma(k)}(h)x(k) \tag{22}$$

as the switching fuzzy Lyapunov function where:

$$P_{\sigma(k)}(h) = \sum_{i=1}^{r_l} h_{\sigma(k)}^i(z(k)) P_{\sigma(k)}^i, \tag{23}$$

then inequality (21) can be equivalently written as:

$$\Delta V(x(k)) + 2\overline{\phi}_l^T(x(k))\Gamma_l E_l x(k) - 2\overline{\phi}_l^T\Gamma_l\overline{\phi}_l(x(k)) < 0. \tag{24}$$

Finally, because Γ_l is chosen to be a positive diagonal matrix, inequality (8) and (24) follows that $\Delta V(x(k)) < 0$, hence switching fuzzy system (9) with controller (10) is asymptotically stable under arbitrary switching law. □

Remark 4.4. *In the proof of Theorem 4.3, no restriction is imposed on the switching law. Specifically, it can be seen from inequality (21), when the l-th subsystem is active, the active subsystem in the next time-step is arbitrary, which clearly shows that the variation of Lyapunov function is negative even when the switching occurs, thus the theorem guarantees the stability of the closed-loop system under arbitrary switching law*

4.1.2 Controller synthesis

Obviously, Conditions in (17) are not in LMI form. To obtain LMI based conditions, we introduce the following theorem.

Theorem 4.5. *If for all $l, m = 1, \ldots, L$, $i = 1, \ldots, r_l$, $q = 1, \ldots, r_m$ and $j = 1, \ldots, r_l$, there exist positive definite matrices $Q_1^1, \ldots, Q_L^{r_L} \in R^{n\times n}$, positive definite diagonal matrices $\psi_l \in R^{p\times p}$ and matrices $\Theta_l^i \in R^{m\times p}$, $T_l^i \in R^{m\times n}$ such that the matrices:*

$$W_{lm}^{ijq} = \begin{bmatrix} Q_l^i & -Q_l^i E_l^T & \left(A_l^j Q_l^i + B_{1l}^j\Theta_l^i\right)^T \\ * & 2\psi_l & \left(G_{1l}^j\psi_l + B_{1l}^j \mathrm{T}_l^i\right)^T \\ * & * & Q_m^q \end{bmatrix} \tag{25}$$

satisfies the conditions in (15), then the switching fuzzy system (9) via the fuzzy controller (10) with:

$$\Upsilon_l^i = \mathrm{T}_l^i\psi_l^{-1},\ K_l^i = \Theta_l^i\left(Q_l^i\right)^{-1} \tag{26}$$

is asymptotically stable under arbitrary switching law.

Proof. considering Lemma 4.1 and substituting (25) into (16) follows that:

$$\sum_{q=1}^{r_m}\sum_{j=1}^{r_l}\sum_{i=1}^{r_l} h_l^i(x(k))\, h_l^j(x(k))\, h_m^q(x(k+1)) \times \begin{bmatrix} Q_l^i & -Q_l^i E_l^T & \left(A_l^j Q_l^i + B_{1l}^j\Theta_l^i\right)^T \\ * & 2\psi_l & \left(G_{1l}^j\psi_l + B_{1l}^j T_l^i\right)^T \\ * & * & Q_m^q \end{bmatrix} > 0. \tag{27}$$

Using notation (14) solving (26) for T_l^i and Θ_l^i, and substituting them into inequality (27) result in:

$$\begin{bmatrix} Q_l(h) & -Q_l(h)E_l^T & Q_l(h){H_l}^T(h) \\ * & 2\psi_l & \psi_l{F_l}^T(h) \\ * & * & Q_m^+(h) \end{bmatrix} > 0, \tag{28}$$

where $Q_m^+(h) = \sum_{q=1}^{r_m} h_m^q(x(k+1))Q_m^q$, $H_l(h)$ and $F_l(h)$ are defined as in (20). Applying Schur complement on (28) and using $Q_l(h) = P_l^{-1}(h)$ and $Q_m^+(h) = (P_m^+(h))^{-1}$yields:

$$\begin{bmatrix} -E_lP_l^{-1}(h) \\ H_l(h)P_l^{-1}(h) \end{bmatrix} P_l(h) \begin{bmatrix} -P_l^{-1}(h)E_l^T & P_l^{-1}(h){H_l}^T(h) \end{bmatrix} - \begin{bmatrix} 2\psi_l & \psi_l{F_l}^T(h) \\ * & (P_m^+(h))^{-1} \end{bmatrix} > 0. \tag{29}$$

Pre- and post-multiply (29) by $diag\left(\psi_l^{-1}, P_m^+(h)\right)$ and its' transpose respectively and using Schur complement again, gives:

$$\begin{bmatrix} P_l(h) & -E_l^T\psi_l^{-1} & {H_l}^T(h)P_m^+(h) \\ * & 2\psi_l^{-1} & {F_l}^T(h)P_m^+(h) \\ * & * & P_m^+(h) \end{bmatrix} > 0. \tag{30}$$

Since ψ_l is a positive definite diagonal matrix; let $\Gamma_l = \psi_l^{-1}$. Using this notation, inequality (30) reduces to:

$$\begin{bmatrix} P_l(h) & -E_l^T\Gamma_l & {H_l}^T(h)P_m(h^+) \\ * & 2\Gamma_l & {F_l}^T(h)P_m(h^+) \\ * & * & P_m^+(h) \end{bmatrix} > 0. \tag{31}$$

and thus condition (19) of Theorem 4.3 is satisfied, which completes the proof. □

Remark 4.6. *It can be inferred from inequality (25) that Theorem 4.5 is an extension of the result presented in* [3] *for linear consequent parts case. Compared to that of* [3]*, the size of LMIs has been increased; however, because of the reduction of the number of rules as a result of using T-S fuzzy systems with nonlinear consequent parts, solving the LMI's with the use of Theorem 4.3 has less computation burden. This is also verified in an example given in section 5.*

Remark 4.7. *Considering Theorem 4.5 with $l = m = 1$, the stability conditions for the non-switched T-S fuzzy systems with nonlinear consequent part can be obtained In Example 5.1, it will be shown that this theorem will reduce the conservativeness of the LMI conditions compared to those presented in* [9]*. The reason is that the approach used in* [9]*, which is also employed by more recent works such as* [18] *requires additional variables to be used in order to obtain the stability conditions in terms of LMIs. This increases the conservativeness of the solution and in some cases can cause the solver to fail.*

4.2 Incorporating closed-loop performance

In addition to the stabilization problem discussed in the previous section, controller design with desirable closed-loop performance should also be considered. Maximum decay-rate and H_∞ controller design conditions will be discussed in this section.

4.2.1 Decay-rate performance

To design a stabilizing controller with maximum decay-rate, the minimum value of $0 < \alpha \leq 1$ should be found that satisfies the following inequality [26]:

$$\Delta V(x(k)) < \left(\alpha^2 - 1\right)V(x(k)), \tag{32}$$

where $V(x(k))$ is the switching fuzzy Lyapunov function(22) .

It can be seen from condition (32) that, a common α for all subsystems should be found. However, there may exist a subsystem which can be stabilized with a decay-rate smaller than α. Hence, one can find the minimum value of $0 < \alpha_{\sigma(k)} \leq 1$ at each switching phase satisfying the following conditions:

$$\Delta V(x(k)) < \left(\alpha_{\sigma(k)}^2 - 1\right)V(x(k)), \tag{33}$$

Remark 4.8. *It is important to note that although the maximum decay-rate of the entire switching system is considered as* $\alpha = max_l\,\alpha_l$ *, however, the controller for the* l*-th subsystem is designed by minimizing* α_l*, not* α*. With this approach, a better closed-loop performance will be achieved for the entire system.*

To obtain LMI based condition to design switching fuzzy controller with maximum decay-rate for the system the following corollary is presented:

Corollary 4.9. *If there exist positive definite matrices* $Q_1^1, \ldots, Q_L^{r_L} \in R^{n\times n}$*, positive diagonal matrix* $\Gamma_l \in R^{p\times p}$ *and matrices* $\Theta_l^i \in R^{m\times p}$*,* $T_l^i \in R^{m\times n}$ *and* $0 < \beta_l < 1$*, the solution to the following optimization problem with* $\boldsymbol{\beta} = [\beta_1,\ \beta_2, \ldots,\ \beta_L]$

$$\min_{\boldsymbol{Q_l^i},\boldsymbol{\Gamma_l},\boldsymbol{\Theta_l^i},\mathbf{T_l^i}} \boldsymbol{\beta}\ ,$$

subject to the conditions in (15) with

$$W_{lm}^{ijq} = \begin{bmatrix} \beta_l Q_l^i & -Q_l^i E_l^T & \left(A_l^j Q_l^i + B_{1l}^j \Theta_l^i\right)^T \\ * & 2\Gamma_l & \left(G_{1l}^j \Gamma_l + B_{1l}^j \mathrm{T}_l^i\right)^T \\ * & * & Q_m^q \end{bmatrix}. \tag{34}$$

Then, the switching fuzzy system (9) when $w(k) = 0$ *via the fuzzy controller (10) with* $\Upsilon_l^i = \mathrm{T}_l^i \Gamma_l^{-1}$*,* $K_l^i = \Theta_l^i Q_l^{i^{-1}}$ *is asymptotically stable under arbitrary switching law, and the switching system (13) is guaranteed to have a maximum decay-rate equal to* $\alpha = \sqrt{\max_l \beta_l}$ *.*

Proof. Considering decay-rate condition (33) for l-th subsystem with $\alpha_l^2 = \beta_l$ and switching Lyapunov function candidate (22), the proof is straightforward and omitted. □

4.2.2 H_∞ Performance

In addition to maximum decay-rate condition, disturbance rejection is another control specification that should be considered in real world systems.

Theorem 4.10. *If for all* $l, m = 1, \ldots, L$*,* $i = 1, \ \ldots, r_l$ *,* $q = 1, \ \ldots, r_m$ *and* $j = 1, \ldots, r_l$*, there exist positive definite matrices* $Q_1^1, \ldots, Q_L^{r_L} \in R^{n\times n}$*, positive definite diagonal matrices* $\psi_l \in R^{p\times p}$ *and matrices* $\Theta_l^i \in R^{m\times p}$ *,* $T_l^i \in R^{m\times n}$ *such that matrices:*

$$W_{lm}^{ijq} = \begin{bmatrix} Q_l^i & -Q_l^i E_l^T & 0 & \left(A_l^j Q_l^i + B_{1l}^j \Theta_l^i\right)^T & \left(C_l^j Q_l^i + D_{1l}^j \Theta_l^i\right)^T \\ * & 2\psi_l & 0 & \left(G_{1l}^j \psi_l + B_{1l}^j T_l^i\right)^T & \left(G_{2l}^j \psi_l + D_{1l}^j T_l^i\right)^T \\ * & * & \gamma^2 I & B_{2l}^{T^j} & D_{2l}^{T^j} \\ * & * & * & Q_m^q & 0 \\ * & * & * & * & I \end{bmatrix}, \tag{35}$$

satisfy the conditions in (15), then the switching fuzzy system (9) via the fuzzy controller (10) with $\Upsilon_l^i = \mathrm{T}_l^i \Gamma_l^{-1}$*,* $K_l^i = \Theta_l^i Q_l^{i^{-1}}$ *is asymptotically stable under arbitrary switching law and the following* H_∞ *performance will be satisfied:*

$$\sum_{K=0}^{\infty} y^T(k)\, y(k) - \sum_{K=0}^{\infty} \gamma_\infty^2 w^T(k)\, w(k) < 0. \tag{36}$$

Proof. Considering Lemma 4.1, and substituting (15) into (16) follows that

$$\sum_{q=1}^{r_m}\sum_{j=1}^{r_l}\sum_{i=1}^{r_l} h_l^i(x(k))\, h_l^j(x(k))\, h_m^q(x(k+1)) \times \begin{bmatrix} Q_l^i & -Q_l^i E_l^T & 0 & \left(A_l^j Q_l^i + B_{1l}^j \Theta_l^i\right)^T & \left(C_l^j Q_l^i + D_{1l}^j \Theta_l^i\right)^T \\ * & 2\psi_l & 0 & \left(G_{1l}^j \psi_l + B_{1l}^j T_l^i\right)^T & \left(G_{2l}^j \psi_l + D_{1l}^j T_l^i\right)^T \\ * & * & \gamma^2 I & B_{2l}^{T^j} & D_{2l}^{T^j} \\ * & * & * & Q_m^q & 0 \\ & * & * & * & I \end{bmatrix}. \tag{37}$$

Using the same notation as previous and substituting T_l^i and Θ_l^i into inequality (37), result in:

$$\begin{bmatrix} Q_l(h) & -Q_l(h)E_l^T & 0 & Q_l(h){H_{1l}}^T(h) & Q_l(h){H_{2l}}^T(h) \\ * & 2\psi_l & 0 & \psi_l F_{1l}^T(h) & \psi_l F_{2l}^T(h) \\ * & * & \gamma^2 I & B_{2l}^T(h) & D_{2l}^T(h) \\ * & * & * & Q_m^+(h) & 0 \\ * & * & * & * & I \end{bmatrix} > 0, \tag{38}$$

where $Q_m^+(h) = \sum_{q=1}^{r_m} h_m^q(x(k+1))\,Q_m^q$ and:

$$H_{1l}(h) = A_l(h) + B_{1l}(h)K_l(h), \qquad F_{1l}(h) = G_{1l}(h) + B_{1l}(h)\Upsilon_l(h),$$
$$H_{2l}(h) = C_l(h) + D_{1l}(h)K_l(h), \qquad F_{1l}(h) = G_{2l}(h) + D_{1l}(h)\Upsilon_l(h)$$

Applying Schur complement to (37), and using $Q_l(h) = P_l^{-1}(h)$ and $Q_m^+(h) = (P_m^+(h))^{-1}$ yields:

$$\begin{bmatrix} -E_l P_l^{-1}(h) \\ 0 \\ H_{1l}(h)P_l^{-1}(h) \\ H_{2l}(h)P_l^{-1}(h) \end{bmatrix} P_l(h) \begin{bmatrix} -P_l^{-1}(h)E_l^T & 0 & P_l^{-1}(h)H_{1l}^T(h) & P_l^{-1}(h)H_{2l}^T(h) \end{bmatrix} - \begin{bmatrix} 2\psi_l & 0 & \psi_l F_{1l}^T(h) & \psi_l F_{2l}^T(h) \\ * & \gamma^2 I & B_{2l}^T(h) & D_{2l}^T(h) \\ * & * & (P_m^+(h))^{-1} & 0 \\ * & * & * & I \end{bmatrix} < 0. \tag{39}$$

Pre- and post-multiply (39) by $diag\left(\psi_l^{-1}, I, P_m^+(h),\ I\right)$ and its' transpose, respectively, and using Schur complement again, gives:

$$\begin{bmatrix} P_l(h) & -E_l^T\psi_l^{-1} & 0 & H_{1l}^T(h)P_m^+(h) & H_{2l}^T(h) \\ * & 2\psi_l^{-1} & 0 & F_{1l}^T(h)P_m^+(h) & F_{2l}^T(h) \\ * & * & \gamma^2 I & B_{2l}^T(h)P_m^+(h) & D_{2l}^T(h) \\ * & * & * & P_m^+(h) & 0 \\ * & * & * & * & I \end{bmatrix} > 0, \tag{40}$$

where $P_m^+(h) = \sum_{q=1}^{r_m} h_m^q(x(k+1))\,P_m^q$. Now, applying Schur complement to (40) , it follows that:

$$\begin{bmatrix} -P_l(h) & E_l^T\Gamma_l & 0 \\ * & -2\Gamma_l & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} H_{1l}^T(h) \\ F_{1l}^T(h) \\ B_{2l}^T(h) \end{bmatrix} P_m^+(h) \begin{bmatrix} H_{1l}(h) & F_{1l}(h) & B_{2l}(h) \end{bmatrix} + \begin{bmatrix} H_{2l}^T(h) \\ F_{2l}^T(h) \\ D_{2l}^T(h) \end{bmatrix} \begin{bmatrix} H_{2l}(h) & F_{2l}(h) & D_{2l}(h) \end{bmatrix} < 0 \tag{41}$$

Pre- and post-multiplying (40) by $\begin{bmatrix} x(k) & \overline{\phi}_l(x(k)) & w(k) \end{bmatrix}^T$ and its' transpose, respectively, yields:

$$\begin{aligned} &\left(H_{1l}(h)x(k) + F_{1l}(h)\overline{\phi}_l(x(k))\right)^T P_m^+(h)\left(H_{1l}(h)x(k) + F_{1l}(h)\overline{\phi}_l(x(k))\right) \\ &\quad + (B_{2l}(h)w(k))^{\mathrm{T}}\ P_m^+(h)(B_{2l}(h)w(k)) - x(k)^T P_l(h)x(k) + 2\overline{\phi}_l^T(x(k))\Gamma_l E_l x(k) \\ &\quad - 2\overline{\phi}_l^T\Gamma_l\overline{\phi}_l(x(k)) - \gamma^2 w^T(k)w(k) + y^T(k)y(k) < 0. \end{aligned} \tag{42}$$

Calculating $\Delta V(x(k))$, inequality (42) can be equivalently written as:

$$\Delta V(x(k)) + 2\overline{\phi}_l^T(x(k))\Gamma_l E_l x(k) - 2\overline{\phi}_l^T\Gamma_l\overline{\phi}_l(x(k)) + y^T(k)y(k) - \gamma^2 w^T(k)w(k) < 0. \tag{43}$$

Considering (8), (43) immediately results in:

$$\Delta V(x(k)) + y^T(k)y(k) - \gamma^2 w^T(k)w(k) < 0. \tag{44}$$

Calculating the sum of (44) from k= 0 to ∞ and under zero initial condition yields:

$$y^T(k)\,y(k) - \gamma^2 w^T(k)w(k) < 0. \tag{45}$$

which completes the proof.

□

In the next section, it will be shown that with the use of Theorem 3 and Corollary 1 we can design a controller with better closed loop performance.

5 Simulation results

In this section, we provide numerical examples to show the validity of the previously presented results. In Example 5.1, we discuss the feasibility region of the conditions of the proposed method in Theorem 4.5, and the claims in Remark 4.7.

Example 5.1. *Consider the switching T-S fuzzy system (9) with* $L = 1$, $w(k) = 0$ *and the following matrices:*

$$A_1^1 = \begin{bmatrix} -1 & 0 \\ -1 & -0.5 \end{bmatrix},\ A_1^2 = \begin{bmatrix} 1 & -1.6 \\ 2 & b \end{bmatrix}, B_1^1 = \begin{bmatrix} 1 \\ a \end{bmatrix},\ B_1^2 = \begin{bmatrix} 1.3 \\ 1 \end{bmatrix},$$

$$G_1^1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},\ G_1^2 = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix},\ E = [1 \quad 4]$$

where $a, b \in \mathbb{R}^n$ *are the system parameters. As mentioned in Remark 4.7, Theorem 4.5 can easily be used for non-switching systems as well. To support our claim about conservatism of the proposed approach, the feasibility region of Theorem 4.5 and Theorem 8 in [9] for* $1.3 < a < 3.4$ *and* $-2 < b < 20$ *is found using Yalmip as a parser and Mosek as a solver [16, 23]. The comparative results are depicted in Figure 1, which clearly supports our claims in Remark 4.7.*

In both theorems, the form of Lyapunov function is the same; however, in our proof for Theorem 4.5, we have used a different method to obtain LMI conditions, which is based on fewer LMI variables. Hence, as shown in Figure 1, the feasibility region of Theorem 4.5 covers a considerable region compared to the conditions in [9], giving more relaxed stability conditions.

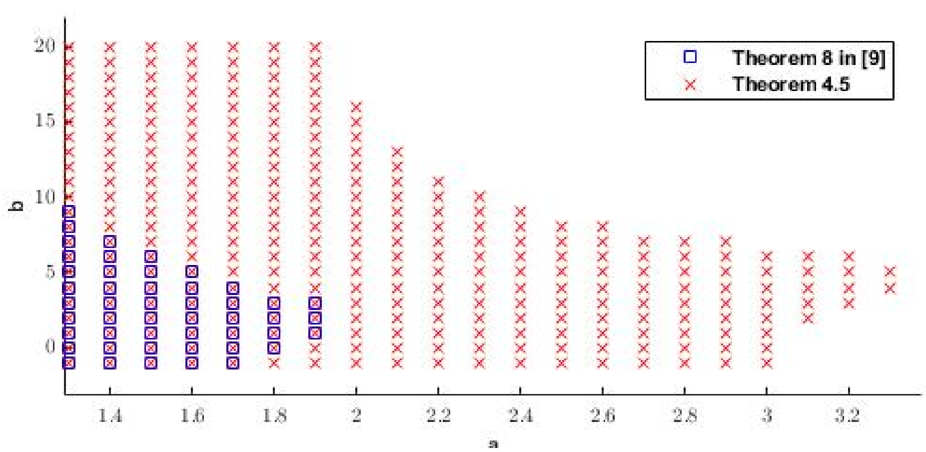

Figure 1: Feasibility Region of conditions of Theorem 4.5 and Theorem 8 in [9].

In the next two examples, the computation burden of Theorem 4.5 and Theorem 4.3 in [3] and the closed loop performance of the system using the extended versions of Theorem 4.3 in [3] with decay-rate and H_∞ performance will be investigated.

Example 5.2. *Consider the following switching nonlinear system:*

$$x(k+1) = f_{la}(x(k)) + f_{lb}(x(k))\,\overline{\phi}_l(x(k)) + g_l(x(k))\,u(k), \tag{46}$$

where:

$$f_{1a}\left(x\left(k\right)\right)=\left[\begin{array}{c} 0.1x_1-0.3x_2 \\ -\left(5+\left(x_2cos\left(x_1\right)\right)^2\right)\frac{1+x_2+x_2(cos(x_1))^2}{1+\left(cos\left(x_1\right)\right)^2} \end{array}\right],$$

$$f_{1b}\left(x\left(k\right)\right)=\left[\begin{array}{c} 1 \\ \frac{-0.5}{1+(cos(x_1))^2} \end{array}\right],\ g_1\left(x\left(k\right)\right)=\left[\begin{array}{c} 0 \\ 1.5+0.3(x_2cos\left(x_1\right))^2 \end{array}\right],$$

$$f_{2a}\left(x\left(k\right)\right)=\left[\begin{array}{c} 0.2x_1-3x_2 \\ \frac{-x_1x_2^2-5x_1}{1+(sin(x_1))^2} \end{array}\right], f_{2b}\left(x\left(k\right)\right)=\left[\begin{array}{c} 0 \\ \frac{1}{1+(sin(x_1))^2} \end{array}\right],\ g_2\left(x\left(k\right)\right)=\left[\begin{array}{c} -1 \\ x_2^2+5 \end{array}\right]$$

choose:

$$\overline{\phi}_1=x_2^2\sin\left(2x_1\right)+2x_2\ ,\overline{\phi}_2=\sin\left(x_1\right)\ -\frac{2}{\pi}x_1, \tag{47}$$

which satisfies the following bounds:

$$0\leq\overline{\phi}_1\leq 4x_2\ ,0\leq\overline{\phi}_2\leq\frac{\pi-2}{\pi}x_1. \tag{48}$$

To verify that $\overline{\phi}_1$ satisfies (7), the plane $z(x_1,x_2)=4x_2$ and $\overline{\phi}_1$ is plotted in Figure 2.

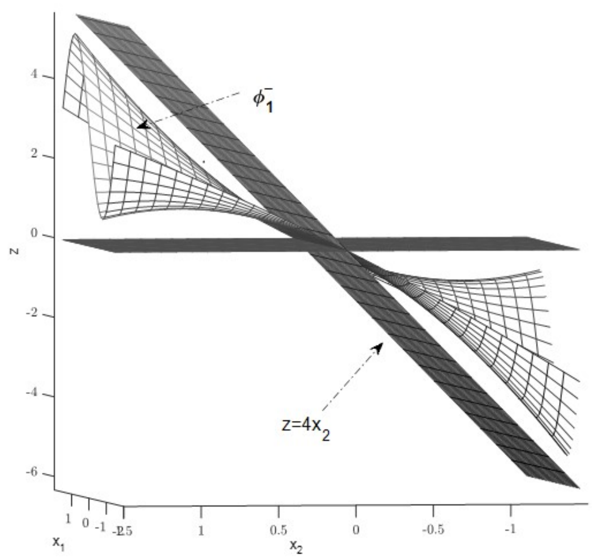

Figure 2: Sector bound of $\overline{\boldsymbol{\phi}}_{\mathbf{1}}$.

It should be noted that the zero lower bound of $\overline{\phi}_1$ and $\overline{\phi}_2$ in (48) is not a restrictive coincidence as one can always add and remove the non-zero lower bounds to the system equations and obtain the same inequality with zero lower bounds. By considering the disturbance free form of (9) , the exact representation of the switching nonlinear system (46) is obtained with four rules for each subsystem, with G_{1l}^i, B_{1l}^i, A_{1l}^i as follows:

$$A_1^1=\left[\begin{array}{cc} 0.1 & -0.3 \\ -\mathrm{a}_1b_1 & -b_1 \end{array}\right],A_1^2=\left[\begin{array}{cc} 0.1 & -0.3 \\ -\mathrm{a}_1b_2 & -b_2 \end{array}\right],A_1^3=\left[\begin{array}{cc} 0.1 & -0.3 \\ -\mathrm{a}_2b_1 & -b_1 \end{array}\right],A_1^4=\left[\begin{array}{cc} 0.1 & -0.3 \\ -\mathrm{a}_2b_2 & -b_2 \end{array}\right],$$

$$B_{11}^1=\left[\begin{array}{c} 0 \\ 0.3b_1 \end{array}\right],\ B_{11}^2=\left[\begin{array}{c} 0 \\ 0.3b_2 \end{array}\right],\ B_{11}^3=\left[\begin{array}{c} 0 \\ 0.3b_1 \end{array}\right],B_{11}^4=\left[\begin{array}{c} 0 \\ 0.3b_2 \end{array}\right],$$

$$G_{11}^1=\left[\begin{array}{c} 1 \\ -0.5\mathrm{a}_1 \end{array}\right],\ G_{11}^2=\left[\begin{array}{c} 1 \\ -0.5\mathrm{a}_1 \end{array}\right],\ G_{11}^3=\left[\begin{array}{c} 1 \\ -0.5\mathrm{a}_2 \end{array}\right],\ G_{11}^4=\left[\begin{array}{c} 1 \\ -0.5\mathrm{a}_2 \end{array}\right],$$

$$A_2^1=\left[\begin{array}{cc} 0.2 & -3 \\ -\mathrm{c}_1d_1 & -2 \end{array}\right],\ A_2^2=\left[\begin{array}{cc} 0.2 & -3 \\ -\mathrm{c}_1d_2 & -2 \end{array}\right],A_2^3=\left[\begin{array}{cc} 0.2 & -3 \\ -\mathrm{c}_2d_1 & -2 \end{array}\right],A_2^4=\left[\begin{array}{cc} 0.2 & -3 \\ -\mathrm{c}_2d_2 & -2 \end{array}\right],$$

$$B_{12}^1 = B_{12}^2 = \begin{bmatrix} 0 \\ c_1 \end{bmatrix}, B_{12}^3 = B_{12}^4 = \begin{bmatrix} 0 \\ c_2 \end{bmatrix}, G_{12}^1 = \begin{bmatrix} -1 \\ d_1 \end{bmatrix}, G_{12}^2 = \begin{bmatrix} -1 \\ d_2 \end{bmatrix}, G_{12}^3 = \begin{bmatrix} -1 \\ d_1 \end{bmatrix}, G_{12}^4 = \begin{bmatrix} -1 \\ d_2 \end{bmatrix}.$$

with the following fuzzy sets:

$$M_1^{11} = \frac{z_1^1 - a_2}{a_1 - a_2}, \quad M_1^{12} = \frac{z_1^2 - b_2}{b_1 - b_2}, \quad M_1^{21} = \frac{a_1 - z_1^1}{a_1 - a_2}, \quad M_1^{21} = \frac{b_1 - z_1^2}{b_1 - b_2},$$

$$M_2^{11} = \frac{z_2^1 - c_2}{c_1 - c_2}, \quad M_2^{12} = \frac{z_2^2 - d_2}{d_1 - d_2}, \quad M_2^{21} = \frac{c_1 - z_2^1}{c_1 - c_2}, \quad M_2^{22} = \frac{d_1 - z_2^2}{d_1 - d_2},$$

where $a_1 = 1$, $a_2 = 0.5$, $b_1 = 49$, $b_2 = 5$, $c_1 = 1$, $c_2 = 0.5$, $d_1 = 49$, $d_2 = 5$ and $z_1^1 = \frac{1}{1+(cos(x_1))^2}$, $z_1^2 = (x_2 cos\,(x_1))^2 + 5$, $z_2^2 = {x_2}^2 + 5$, $z_2^1 = \frac{1}{1+(sin(x_1))^2}$.

Notice that by employing conventional T-S model for modelling system (46) with known $\overline{\phi}_l\,(x\,(k))$, eight rules will be derived for modelling each subsystem, however using T-S fuzzy systems with nonlinear consequent parts in the form of (9) with unknown-structure $\overline{\phi}_l\,(x\,(k))$, only four rules are needed. To show the superiority of our work in terms of computation burden compared to [3] which considers switching T-S fuzzy systems with linear consequent parts, Theorems 4.5 and Theorem 4.3 in [3] are applied to the system in (46) and the results are given in the Table 1, where Yalmip [16] is used as parser.

Table 1: Comparison between computation time of Theorem 4.5 and Theorem 4.3 in [3].

			Total Solver Time (Sec)		
	Number of Conditions	**Number of Variables**	**Sdpt3 [28]**	**Sedumi [24]**	**Mosek [2]**
Theorems 2	90	50	4.65	4.84	0.725
Theorem 4.3 in [3]	304	80	7.48	14.2	0.97

The results in Table 1 show that, if one use conventional T-S fuzzy systems with linear consequent parts for modeling the system (46) and the method in [3] for designing a stabilizing controller, the number of control synthesis conditions and LMI computation time are considerably higher compared to our proposed method. It should be noted that in [3] instead of using Lemma 4.1, the method in [31] is used, hence for more realistic comparison, a version of Theorem 4.5 with the same method is employed in this paper to clearly show that the reduction in computation burden is due to employing nonlinear consequent parts. It may seem that the reduction of computation burden in such an offline procedure is not important, however in order to solve an LMI problem, when the number of conditions increases, the solvers usually cannot find a feasible solution not because the problem is infeasible but because of numerical issues.

To compare the closed loop performance of the system resulting from the proposed controller, Theorem 4.5 and Corollary 4.9 are applied to the system (46). Employing Theorem 4.5, the following controller coefficients have been obtained:

$$K_1^1 = [3.3292,\ 3.3340]\,, K_1^2 = [3.3282,\ 3.3400]\,,\ K_1^3 = [1.6666,\ 3.3342]\,,\ K_1^4 = [\ 1.6666,\ 3.3207]\,,$$

$$\Upsilon_1^1 = 0.0329,\ \Upsilon_1^2 = \ 0.3227, \Upsilon_1^3 = 0.0157,\ \Upsilon_1^4 = 0.0412,$$

$$K_2^1 = [48.996,\ 2.0408]\,, K_2^2 = [4.9959,\ 2.0408]\,,\ K_2^3 = [48.9933,\ 4.0895]\,, K_2^4 = [\ 4.9933,\ 4.0895]\,,$$

$$\Upsilon_2^1 = -\ 48.9823,\ \Upsilon_2^2 = \ -4.9823, \Upsilon_2^3 = -97.9186 \quad , \Upsilon_2^4 = -9.9723.$$

Moreover, using Corollary 4.9 the following results have been derived:

$$K_1^1 = [3.329,\ 3.3331]\,, K_1^2 = [3.3295,\ 3.3313]\,,\ K_1^3 = [1.6667,\ 3.3334]\,, K_1^4 = [\ 1.6667,\ 3.3333]\,,$$

$$\Upsilon_1^1 = 0.0343,\ \Upsilon_1^2 = 0.3361, \Upsilon_1^3 = 0.0057,\ \Upsilon_1^4 = 0.0560,$$

$$K_2^1 = [49.001,\ 1.9808\ ]\,, K_2^2 = [5.0012,\ 1.9808]\,,\ K_2^3 = [49.002,\ 3.9608]\,, K_2^4 = [\ 5.002,\ 3.9608]\,,$$

$$\Upsilon_2^1 = -49.0060,\ \Upsilon_2^2 = \ -5.0060 \ \ , \Upsilon_2^3 -\ 5.0060,\ \Upsilon_2^4 = -10.0059,$$

when the minimum value of $\beta_1,\ \beta_2$ are obtained to be 0.441 and 0.034 respectively.

For simulation purposes, consider the arbitrary switching law as depicted in Figure 3, where its value at each time-step is the index of the active subsystem. With the initial condition $x_1 = 1$ and $x_2 = 2$, closed-loop system based on Theorem 4.5 and Corollary 4.9 is simulated, and the state trajectories of the system are depicted in Figure 4 and Figure 5. To illustrate the closed-loop performance using our proposed method, the closed-loop system based on extended version of Theorem 4.3 in [3] with decay-rate is simulated and the results are included in these figures.

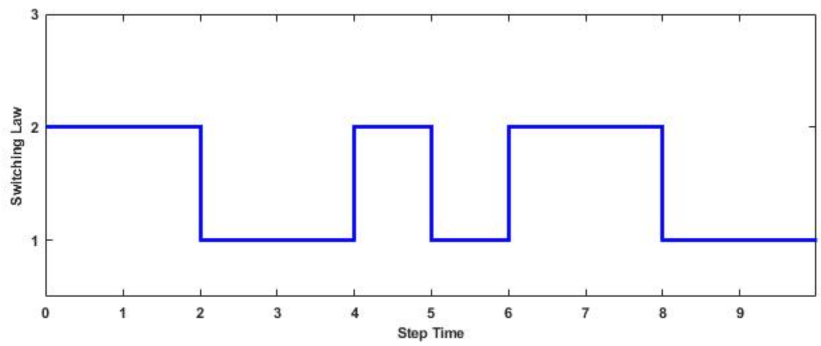

Figure 3: The given switching law.

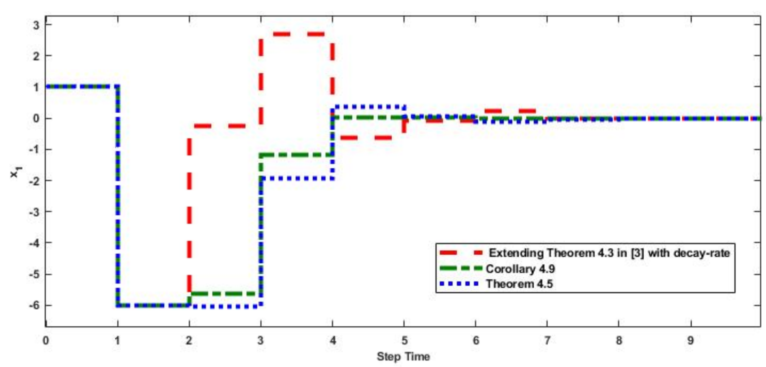

Figure 4: State trajectory $\mathbf{x_1}$.

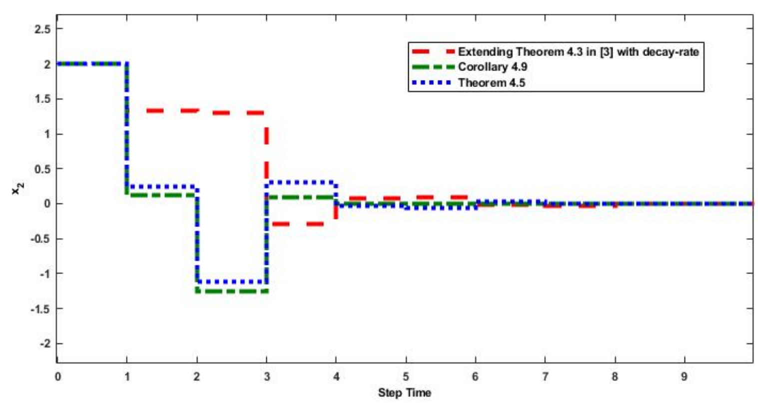

Figure 5: State trajectory $\mathbf{x_2}$.

It can be seen from the state trajectories in Figure 4 and Figure 5 that, by using Theorem 4.5 the system has been stabilized. Furthermore, it can be verified that with the use of Corollary 4.9, the performance of the closed-loop system is improved, compared to Theorem 4.5. Moreover, the result shows the superiority of our approach compared to the extended version of Theorem 4.3 in [3] with decay-rate as well. This is because in our approach, the measured value of sector bounded nonlinearity is also used in the feedback signal.

Example 5.3. *Consider the same system as in the previous example, but with disturbance.*

$$w(k) = sin(k)exp(-0.1k),\ B_{2l}^{i} = [0\ 1]^{T},\ C_{l}^{i} = [0, 0.1],\ D_{1l}^{i} = D_{2l}^{i} = G_{2l}^{i} = 0,$$

Both Theorem 4.10 and the first part of Theorem 1 of [33], which propose H_∞ control design for switching T-S fuzzy systems with linear consequent part, were applied to the system and the output of the system subject to disturbance and the value of $\frac{\sum_{k=0}^{K} y^T(k)y(k)}{\sum_{k=0}^{K} d^T(k)d(k)}$ with zero initial conditions are depicted in Figure 6, Figure 7 respectively.

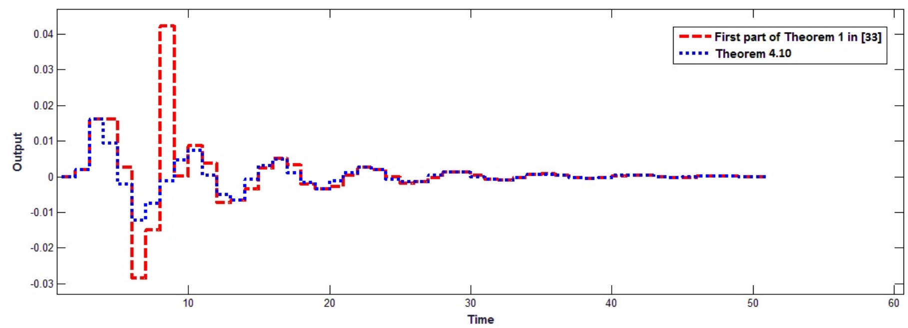

Figure 6: The System's output with zero initial conditions.

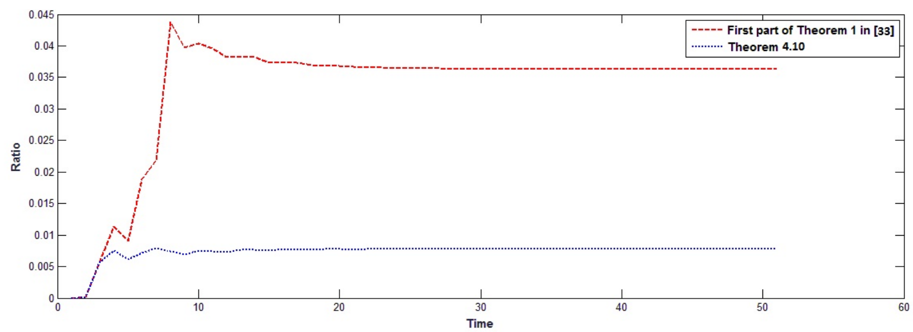

Figure 7: The Value of $\frac{\sum_{k=0}^{K} y^T(k)y(k)}{\sum_{k=0}^{K} d^T(k)d(k)}$ with zero initial conditions.

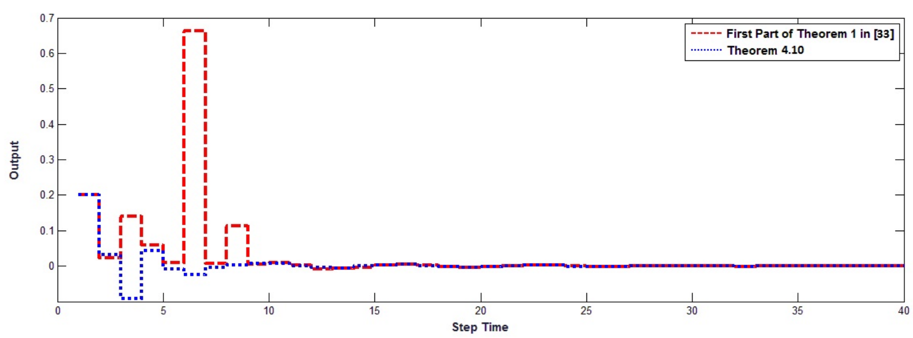

Figure 8: The output of the system with non-zero initial condition.

As it can be seen from Figure 6, the proposed controller results in a better disturbance attenuation, as a direct consequence of using the sector bounded term in the feedback signal. Moreover to better compare the disturbance rejection properties of the closed-loop system, the ratio between $\|y\|_2$ and $\|w\|_2$ is also depicted in Figure 7. Moreover with the initial conditions $x_1 = 1$ and $x_2 = 2$, Theorem 4.10 and the first part of theorem 1 in [33] applied to the system and the output of the system is depicted in Figure 8, which also shows that using switching T-S fuzzy systems with sector bounded consequent parts results a better closed-loop performance.

It should be noted that, the comparison is made for the case that the structure of the sector-bounded nonlinearities is known, otherwise when the structure of nonlinearity (for instance sensor nonlinearity) is unknown and it is only measurable, conventional T-S fuzzy models with linear consequent parts could not be used at all. Hence, the current work can be considered as an extension of switching fuzzy systems with linear consequent parts, such as [12] to nonlinear ones.

6 Conclusions

In this work, the switching T-S fuzzy systems with nonlinear consequent parts were presented for the first time. It was shown that, with the use of this novel modeling framework, a wider class of systems can be modeled, and because of the use of nonlinear consequent parts, the available design for discrete-time switching Lure' systems can be easily extended to a wider class of systems. Furthermore, sufficient LMI conditions for asymptotic stability under arbitrary switching law based on a switching fuzzy Lyapunov function were presented. The results compared to several works in the literature, which showed that the proposed method not only can be used to model wider class of systems, but also results in a more relaxed design with less computational burden and better closed-loop performances compared to similar work in [33].

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