Iranian Journal of Fuzzy Systems

Volume 17, Number 2, (2020), pp. 67-77

University of Sistan and Baluchestan

Fuzzy decision making in testing hypotheses: An introduction to the packages "FPV" and "Fuzzy.p.value" with practical examples

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Abstract

This paper reviews and compares two R packages "FPV" and "Fuzzy.p.value". These packages are designed for testing hypotheses in a fuzzy environment using a fuzzy p-value based approach. In fact, the packages "FPV" and "Fuzzy.p.value" propose some useful functions for testing hypotheses when the data / hypotheses are fuzzy rather than crisp. The proposed methods and functions have been presented together with illustrative practical examples.

Keywords: R package, testing fuzzy hypotheses, fuzzy data, fuzzy significance level, fuzzy p-value, extension principle.

1 Introduction

The field of testing statistical hypotheses plays an important role in decision making. Traditional testing hypotheses considers crisp decision, parameters, observation, random variables and significance level. However, we may confront vague definitions and fuzzy concepts in applied sciences, such as the threshold of patient tolerance, the degree of utility of life and the monthly income of a taxi driver. In this case, the classical methods can not be used and need to be generalized for use in fuzzy environments. The vagueness may be due to fuzzy data or fuzzy hypotheses. Therefore, this article discusses on testing hypotheses by fuzzy p-value approach for following three problems:
(i) testing crisp hypotheses with fuzzy observation (see e.g. [2]),
(ii) testing fuzzy hypotheses with crisp observation (see e.g. [10]),
(iii) testing fuzzy hypotheses with fuzzy observation (see e.g. [11]).

After the introduction of the notion of fuzzy sets by Zadeh [20], there have been efforts to extend statistical analyses / models to fuzzy environments using the fuzzy set theory, see e.g. [16] and [1]. The following approaches have been developed to solve the problem of testing hypotheses in fuzzy environment:
(a) Neyman-Pearson Lemma and its extensions (e.g. see [17]),
(b) Bayesian (e.g. see [15]),
(c) Likelihood ratio (e.g. see [18]),
(d) Minimax (e.g. see [12]), and
(e) p-value (e.g. see [2]).
Each of these approaches uses different assumptions and a well-defined optimality criterion.

For the first time, testing hypotheses with fuzzy data is discussed in [2] by a fuzzy p-value approach. Also, similar problem is considered in [4] and [5] based on non-parametric, UMP and UMPU tests, respectively. Testing fuzzy hypotheses for the two cases of crisp and fuzzy data are investigated in [10, 11] by fuzzy p-value approach. Another p-value-based approach is presented in [13] for this problem based on the Zadeh's probability measure.

Regarding the different methods of testing hypotheses, the classical p-value method is known as the easiest / most commonly used testing method among statistical users in various sciences. In applied sciences, the possibility of encountering non-precise / fuzzy concepts is high and testing hypotheses on the basis of the p-value approach in fuzzy

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Received: October 2018; Revised: May 2019; Accepted: August 2019.

environment causes / leads to the concept of the vague p-value. Computing the membership function of this vague p-value and making the final fuzzy decision is complicated and difficult, and this has led to the low growth in the use of fuzzy approach (in practical terms) among various users. The creation of new statistical softwares to work in fuzzy environment can facilitate the analysis of fuzzy data. "FPV" and "Fuzzy.p.value" are two packages in R (as an open source software) for this purpose. Although the fuzzy p-value approach is considered in these packages, users do not need to know / understand the special information and complicated formulas which are presented in [2, 19, 10, 11]. So, it is easy for the applied users to work with these packages for testing hypotheses in fuzzy environment.

This paper is organized as follows. In Section 2, we provide preliminary definitions / concepts about testing hypotheses in a fuzzy environment. We discuss the use of two R packages "Fuzzy.p.value" and "FPV" in Section 3 and Section 4, respectively. A comparison between two packages "Fuzzy.p.value" and "FPV" is provided in Section 5 and the final section is conclusions.

2 Preliminary concepts

To entrance the topic of testing hypotheses in fuzzy environment, some preliminaries and basic definitions are briefly reviewed and also the main discussed problem is presented in this section. Let R be the real line and $F(R) = \{A|A : R \to [0,1]\}$ is the set of all fuzzy sets on R. Also, let $F_C(R)$ is the set of all normal, convex and continuous fuzzy sets on R. The support and δ-cut of $A \in F(R)$ are respectively $Supp(A) = \{x|A(x) > 0\}$ and $A_\delta = \{x|A(x) \geq \delta\}$, for $\delta \in (0,1]$.

Any hypothesis of the form "$H : \theta \ is \ H(\theta)$" is called a *fuzzy hypothesis*, where "$H : \theta \ is \ H(\theta)$" implies that θ is in a fuzzy subset of parameter space Θ, with membership function $H(\theta)$ [17]. Moreover, see the definition of one/two-sided fuzzy hypothesis and also the definition of "the boundary of the fuzzy hypothesis" from [10].

The main problem: Let $\tilde{x}_1, ..., \tilde{x}_n \in F_C(R)$ be the fuzzy-valued observations of a random sample from $f_\theta(x)$. The membership function of test statistic $\mathbf{t} = h(\tilde{x}_1, ...\tilde{x}_n)$, which we denote by $\mathbf{t}(t)$, can be computed by *the extension principle*. The main problem discussed here is testing the fuzzy hypotheses

$$\tilde{H}_0 : \theta \ is \ H_0 \qquad vs. \qquad \tilde{H}_1 : \theta \ is \ H_1$$

with the fuzzy observations $\tilde{x}_1, ..., \tilde{x}_n$ by a p-value approach. Using R packages "FPV" and "Fuzzy.p.value", we are going to obtain the membership function of the fuzzy p-value, named $\mathbf{P}(p)$, for such a problem.

For more flexibility, the significance level is considered in this paper as the fuzzy set $\mathbf{S}$ on $(0,1)$ [6]. In the problem of testing hypotheses in fuzzy environment, $D(\mathbf{P} > \mathbf{S})$ is called *the degree of acceptance of* $\tilde{H}_0$, and $D(\mathbf{S} > \mathbf{P}) = 1 - D(\mathbf{P} > \mathbf{S})$ is called *the degree of rejection of* $\tilde{H}_0$ [10, 11], in which

$$D(M > N) = \frac{\Delta_{MN}}{\Delta_{MN} + \Delta_{NM}}, \quad for \ M, N \in F_C(R), \tag{1}$$

$$\Delta_{MN} = \int_{a^+_{M\delta} > a^-_{N\delta}} (a^+_{M\delta} - a^-_{N\delta}) \ d\delta + \int_{a^-_{M\delta} > a^+_{N\delta}} (a^-_{M\delta} - a^+_{N\delta}) \ d\delta, \tag{2}$$

$a^+_{M\delta} = \sup\{x|x \in M_\delta\}$ and $a^-_{M\delta} = \inf\{x|x \in M_\delta\}$, for $\delta \in (0,1]$.

Solving the problem of testing hypotheses in fuzzy environment by a p-value-based approach will be studied in two next sections by considering two R packages "Fuzzy.p.value" and "FPV".

3 "Fuzzy.p.value" package

Version 1.0 of package "Fuzzy.p.value" has been published on CRAN (Comprehensive R Archive Network) in 2016 [7)]. After review several methods of defining fuzzy sets by package "Fuzzy.p.value", some basic functions of this package have been presented by several numerical examples in this section.

3.1 Introducing fuzzy set

Fuzzy set $T \in F(R)$ is said to be *triangular fuzzy number*, and denoted by $T(a, b, c)$, if its membership function is

$$T(a,b,c)(x) = \begin{cases} \frac{x-a}{b-a} & \text{if} \quad a < x \leq b, \\ \frac{x-c}{b-c} & \text{if} \quad b < x \leq c, \\ 0 & \quad elsewhere \end{cases}$$

in which $a, b, c \in R$ and $a \leq b \leq c$. Triangular fuzzy number $T(a, b, c)$ can be easily defined in package "Fuzzy.p.value" by `T(a,b,c)`. Similarly, code `Tr(a,b,c,d)` is available to introduce trapezoidal fuzzy number $Tr(a, b, c, d)$ with membership function

$$Tr(a,b,c,d)(x) = \begin{cases} \frac{x-a}{b-a} & \text{if} \quad a < x \leq b, \\ 1 & \quad b < x \leq c, \\ \frac{x-d}{c-d} & \text{if} \quad c < x \leq d, \\ 0 & \quad elsewhere. \end{cases}$$

Moreover, it may be one decide to consider a linear membership function for fuzzy concepts "approximately smaller than" and "approximately bigger than" in fuzzy hypotheses. For such situations, functions `S` and `B` have been designed in the package "Fuzzy.p.value" to input the following one-sided fuzzy hypotheses

$$S(a,b)(x) = \begin{cases} 1 & \text{if} \quad x \leq a, \\ \frac{x-b}{a-b} & \text{if} \quad a < x \leq b, \\ 0 & \text{if} \quad x > b, \end{cases}$$

$$B(a,b)(x) = \begin{cases} 1 & \text{if} \quad b \leq x, \\ \frac{x-a}{b-a} & \text{if} \quad a \leq x < b, \\ 0 & \text{if} \quad x < a. \end{cases}$$

Remark 3.1. *Other non-linear and more complex fuzzy numbers such as LR fuzzy number, piecewise linear fuzzy number and power fuzzy number are available in the Package "FuzzyNumbers"; see Subsection 4.1 from [14] for more details.*

3.2 Computing the fuzzy p-value

Drawing the membership function of a fuzzy p-value can be done using the two functions `p_value.norm` and `p_value.pois`. The first function is used when the test statistic has a Normal distribution and the second when the test statistic has a Poisson distribution. Although the package "Fuzzy.p.value" is now only limited to the distributions Normal and Poisson (as two candidate for the class of all continue / discrete distributions), the idea of its program can be extended to test statistics with other continuous / discrete distributions. For instance, the usage of the `p_value.norm` function is as follows

```
p_value.norm(kind, H0, H1, t, s.d, n, sig)
```

in which `H0` and `H1` are the null and the alternative hypotheses, `t` is the observed value of the test statistic (mean), `s.d` is the standard deviation of the normal distribution, `n` is the sample size, and `sig` is the significance level of the test. Meanwhile, `kind` determine the kind of testing hypotheses with one of three allowable numbers 0, 1 and 2 according to the form of the alternative hypothesis. In other words, set `kind=0`, `kind=1` or `kind=2`, when the fuzzy hypothesis H_1 imply to "aproximately non-equality", "smaller than" and "bigger than", respectively.

3.3 Practical results

Several numerical examples are presented in this subsection to apply different types of fuzzy information in testing hypotheses by package "Fuzzy.p.value".

Example 3.2. *The lifetime X of lamps (in term of hour) produced by a factory is distributed normally with unknown mean μ and variance $\sigma^2 = 120^2$. Assume that the mean value of $n = 36$ sampled lamps is $\bar{x} = 1327$. We wish to test fuzzy hypotheses:*

$\tilde{H}_0$: *μ is approximately 1300,*

$\tilde{H}_1$: *μ is approximately bigger than 1300,*

at significance level $\boldsymbol{S} = T(0, 0.15, 0.3)$ where $\tilde{H}_0$ and $\tilde{H}_1$ are given by $T(1275, 1300, 1325)$ and $B(1275, 1325)$, respectively

*(Figure 1). Considering the type of $\tilde{H}_1$, the kind of rejection region **kind** $= 2$ and also the test statistic is $T(X_1, ..., X_n) = \bar{X} = \frac{\sum_{i=1}^{36} X_i}{36}$. To test fuzzy hypotheses at significance level $\boldsymbol{S} = T(0, 0.15, 0.3)$, one can use the following codes in Package "Fuzzy.p.value" which also constructs the fuzzy p-value (Figure 2)*

```
H0 = T(1275,1300,1325)
H1 = B(1275,1325)
t = T(1327,1327,1327)  # i.e., crisp t = 1327
sig = T(0.0,0.15,0.3)
p_value.norm(kind=2, H0, H1, t, s.d=120, n=36, sig)
```

After computing $\Delta_{\boldsymbol{PS}} = 0.17$ and $\Delta_{\boldsymbol{SP}} = 0.19$, Package "Fuzzy.p.value" finally accepts the alternative fuzzy hypothesis at level $\boldsymbol{S}$ with degree of acceptance $D(\mathbf{S} > \mathbf{P}) = 0.53$.

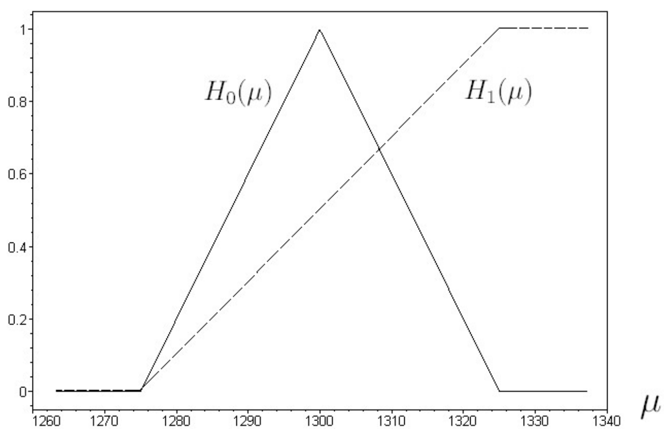

Figure 1: The membership function of fuzzy hypotheses in Example 3.2

Example 3.3. *In Example 3.2, suppose that the observed value of test statistic is $\boldsymbol{t} = T(1315, 1327, 1342)$ which is calculated by extension principle based fuzzy data. We wish to test:*

$\tilde{H}_0$: *μ is approximately 1300,*

$\tilde{H}_1$: *μ is approximately bigger than 1300,*

where $\tilde{H}_0$ and $\tilde{H}_1$ are given by $T(1275, 1300, 1325)$ and $B(1275, 1325)$, respectively (Figure 1).

To test above fuzzy hypotheses at level $\boldsymbol{S} = T(0, 0.05, 0.1)$, one can use the following codes which also draws the fuzzy p-value as depicted in Figure 2

```
H0 <- T(1275,1300,1325)
```

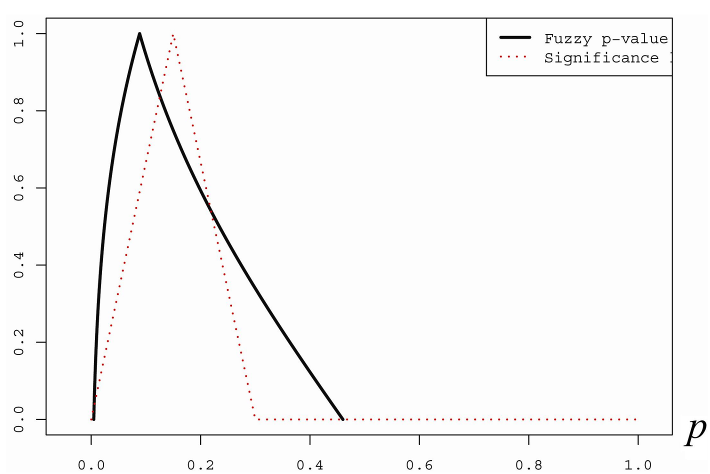

Figure 2: The membership functions of the fuzzy p-value and the fuzzy significance level in Example 3.2

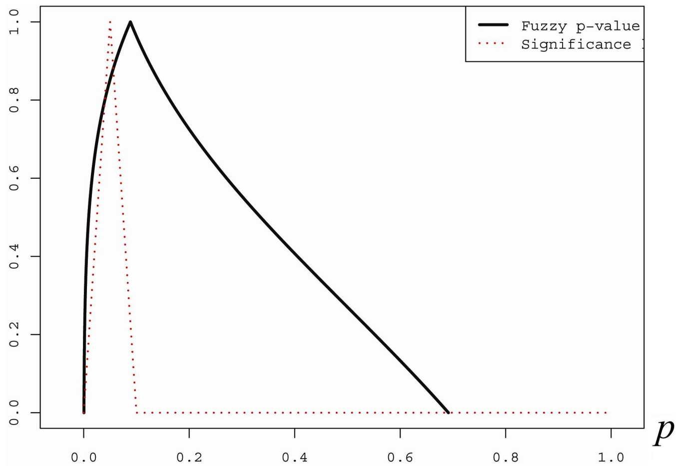

Figure 3: The membership functions of the fuzzy p-value and the fuzzy significance level in Example 3.3

```
H1 <- B(1275,1325)
t <- T(1315,1327,1342)
sig <- T(0,.05,.1)
p_value.norm(kind=2, H0, H1, t, s.d=120, n=36, sig)
```

After computing $\Delta_{PS} = 0.329$ and $\Delta_{SP} = 0.057$, Package "Fuzzy.p.value" finally accepts the null fuzzy hypothesis at level $\boldsymbol{S}$ with degree of acceptance $D(\mathbf{P} > \mathbf{S}) = 0.85$.

Example 3.4. *After reinstallation of a new system in the assembly line of a factory, the manager wants to test*
$\tilde{H}_0$: *λ is approximately bigger than 3,*
$\tilde{H}_1$: *λ is approximately smaller than 3,*
where λ is the mean of the monthly accidents, $\tilde{H}_0$ and $\tilde{H}_1$ are given by $B(2.75, 3.25)$ and $S(2.75, 3.25)$, respectively (Figure 4). Let random variables $X_1, ..., X_{12}$ are respectively the numbers of monthly accidents in past year. In this case, $T = \sum_{i=1}^{12} X_i$ statistic has Poisson distribution with unknown mean 12λ. Based on an especial manager definition for the accident concept, suppose that there occurred approximately 27 accidents after new system installation during the past year with triangular membership function $T(24, 27, 30)$. Considering the type of alternative fuzzy hypothesis, set **kind** $= 1$ *in function* **p_value.pois**. *Then, the fuzzy p-value and the fuzzy level $\boldsymbol{S} = T(0, 0.05, 0.1)$ are drawn in Figure 5 by the following code in Package "Fuzzy.p.value".*

```
H0 = B(12*2.75, 12*3.25)
H1 = S(12*2.75, 12*3.25)
t = T(24,27,30)
n = 12
sig = T(0.0,0.05,0.1)
p_value.pois(kind=1, H0, H1, t, n=12, sig)
```

After computing $\Delta_{PS} = 0.059$ and $\Delta_{SP} = 0.022$, Package "Fuzzy.p.value" finally accepts the null fuzzy hypothesis at level $\boldsymbol{S}$ with degree of acceptance $D(\mathbf{P} > \mathbf{S}) = 0.72$.

4 "FPV" package

The title of *FPV* is "Testing Hypotheses via Fuzzy p-value in Fuzzy Environment" and the version 0.5 published on CRAN in 2017 [8]. In this section we first review several methods for introducing fuzzy sets into the Package "FPV", and then discuss the most important of the function involved [8]. Finally, we solve all examples of Subsection 3.3 once again using Package "FPV", and present an extra numerical example at the end of this section.

4.1 Introducing fuzzy sets to "FPV"

Fuzzy sets in Package "FPV" are introduced using Package "FuzzyNumbers". The types of possible fuzzy numbers in Package "FuzzyNumbers" with their corresponding functions are listed in Table 1. For more details and numerical

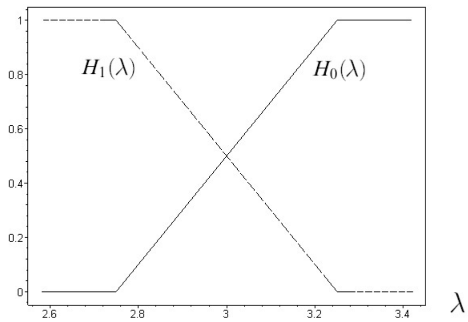

Figure 4: The membership functions of fuzzy hypotheses in Example 3.4

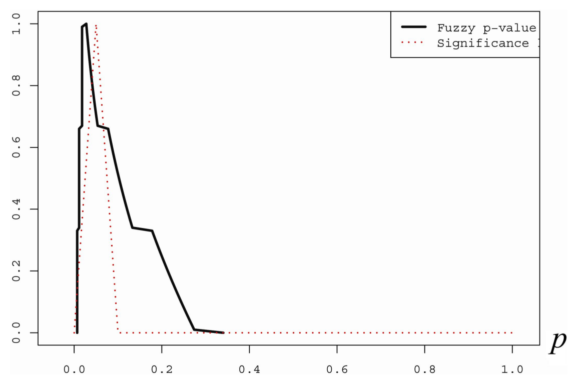

Figure 5: The membership functions of the fuzzy p-value and the fuzzy level in Example 3.4

examples, see [3] and Subsection 4.1 of [12].

Table 1: The possible fuzzy numbers in Package "FuzzyNumbers".

Fuzzy number	Function in Package "FuzzyNumbers"
triangular fuzzy number	`TriangularFuzzyNumber`
trapezoidal fuzzy number	`TrapezoidalFuzzyNumber`
LR fuzzy number	`FuzzyNumber`
piecewise linear fuzzy number	`PiecewiseLinearFuzzyNumber`
power fuzzy number	`PowerFuzzyNumber`

4.2 Computing the fuzzy p-value

The function `fuzzy.p.value` is the only important function in Package "FPV" which can be worked for any continuous / discrete distribution for testing hypotheses in fuzzy environment [8]. The usage of this function is as follows

```
fuzzy.p.value(t, H0b, sig, p.value, knot.n, fig, ...)
```

in which `t` is the observed value of test statistic, `H0b` is the boundary of null hypothesis, `sig` is the significance level of the test with default 0.05, and `knot.n` is the number of knots with default 10 (more details about `knot.n` has been published in [3] and [12]). Also, the argument `p.value` in function `fuzzy.p.value` is the p-value function for similar test in non-fuzzy environment which is a function of crisp `t` and crisp `H0b`. Meanwhile, `fig` is a numeric argument which can take only values 1, 2 or 3 with the following tasks:
If `fig` = 1, then just the membership function of fuzzy p-value will be shown in the exported figure from function `fuzzy.p.value`.
If `fig` = 2, then the membership functions of fuzzy p-value and fuzzy significance level will be shown in the figure.
If `fig` = 3, then three membership functions of `t`, `H0b` (inputted fuzzy numbers) and also the fuzzy p-value (outputted fuzzy number) are drawn in one figure.

4.3 Practical results

In this subsection, at first all examples of Subsection 4.3 have been resolved using Package "FPV". Then, some one more complex numerical example is given to show the flexible performance of the Package "FPV" with respect to Package "Fuzzy.p.value". It must be mentioned that a part of computations in Package "FPV" is using two packages "FuzzyNumbers" [3] and "FuzzyNumbers.Ext.2" [9]. Therefore, the user must install and load packages "FuzzyNumbers" and "FuzzyNumbers.Ext.2", before any computation in package "FPV".

Example 4.1. *Suppose that we are going to test the presented fuzzy hypotheses in Example 3.2 at the fuzzy level* $\boldsymbol{S} = T(0, 0.15, 0.3)$ *using Package "FPV" (see Figure 1). In a similar test, but in non-fuzzy environment, the p-value is equal to real number* $P_{\mu=H_{0b}}\left(\bar{X} \geq \bar{x}\right) = 1 - \Phi\left(\frac{\bar{x}-H_{0b}}{\sigma/\sqrt{n}}\right)$ *where* $\bar{x}$ *is the crisp observed value of test statistics and* H_{0b} *is the crisp boundary of the fuzzy hypothesis. Therefore, consider the following codes by "FPV" Package to solve the testing hypotheses in fuzzy environment.*

```
t = 1327
H0b = TriangularFuzzyNumber(1275,1300,1325)
sig = TriangularFuzzyNumber(0.0,0.15,0.3)
n = 36
sigma = 120
p.value = function(t,H0b) 1-pnorm((t-H0b)/(sigma/sqrt(n)))
fuzzy.p.value(t, H0b, sig, p.value, fig=2)
```

After computing $\Delta_{\boldsymbol{PS}} = 0.17$ *and* $\Delta_{\boldsymbol{SP}} = 0.19$*, Package "FPV" finally accepts* $\tilde{H}_1$ *at level* $\boldsymbol{S} = T(0, 0.15, 0.3)$ *with degree of acceptance* $D(\mathbf{S} > \mathbf{P}) = 0.53$*.*

Example 4.2. *Let we are going to test the fuzzy hypotheses of Example 3.3 by Package "FPV" where* $\boldsymbol{t} = T(1315, 1327, 1342)$*.*

```
fuzzy.p.value(t = TriangularFuzzyNumber(1315,1327,1342),
              H0b = TriangularFuzzyNumber(1275,1300,1325),
              sig = TriangularFuzzyNumber(0,.05,.1),
              p.value = function(t,H0b) 1-pnorm((t-H0b)/(120/sqrt(36))),
              fig=3 )
```

After computing $\Delta_{\boldsymbol{PS}} = 0.329$ *and* $\Delta_{\boldsymbol{SP}} = 0.057$*, Package "Fuzzy.p.value" finally accepts* $\tilde{H}_0$ *at level* $\boldsymbol{S}$ *with degree of acceptance* $D(\mathbf{P} > \mathbf{S}) = 0.85$*. Setting argument* $\boldsymbol{fig} = 3$*, causes to draw three membership functions of* $\boldsymbol{t}$*,* $\boldsymbol{H}_{0b}$ *and* $\boldsymbol{P}$ *in one figure (see Figure 6).*

Example 4.3. *Considering Example 3.4, we wish to test the fuzzy hypotheses depicted in Figure 4 using Package "FPV". Considering a similar test, but in non-fuzzy environment, the precise p-value is a function of* t *and* H_{0b} *as p-value* $= P_{\lambda=12*H_{0b}}(T \leq t) = F_T(t)$ *where* $t = \sum_{i=1}^{12} x_i$ *and* F_T *is the cdf of Poisson distribution with mean* $\lambda = 12 * H_{0b}$ *(i.e.,* $T = \sum_{i=1}^{12} X_i \sim Pois(\lambda = 12 * H_{0b})$*). Therefore, one can resolve the testing hypotheses in fuzzy environment by considering the following codes in "FPV" Package.*

```
fuzzy.p.value(t = TriangularFuzzyNumber(24,27,30),
              H0b = TriangularFuzzyNumber(2.75,3.25,3.25),
              sig = TriangularFuzzyNumber(0.0,0.05,0.1),
              p.value = function(t,H0b) ppois(t, lambda=12*H0b),
              fig = 2,
              knot = 100 )
```

Example 4.4. *Let* X *be distributed normally with unknown mean* μ *and unknown variance* σ^2*. Suppose that the following random sample of size* $n = 12$ *is observed and their membership functions are drawn in Figure 7 by gray:*
$T(3.16, 3.41, 3.57)$, $T(3.55, 3.79, 4.13)$, $T(2.10, 2.30, 2.45)$, $T(2.68, 3.06, 3.43)$, $T(2.67, 2.74, 2.97)$, $T(1.70, 1.71, 1.74)$, $T(2.31, 2.56, 2.73)$, $T(3.75, 3.86, 4.37)$, $T(0.99, 1.08, 1.11)$, $T(1.60, 1.63, 1.68)$, $T(2.61, 2.78, 2.87)$, *and* $T(2.00, 2.33, 2.36)$.

Moreover, the membership functions of mean, variance and standard deviation of fuzzy data are computed by Package "FuzzyNumbers" (see Figure 7). Based on the observed fuzzy data, we are going to answer the question that: Is the variance of population small or not?

First, we construct the following precise hypotheses to answer the above question:

$H_0 : \ \sigma^2 \leq \sigma_0^2$,

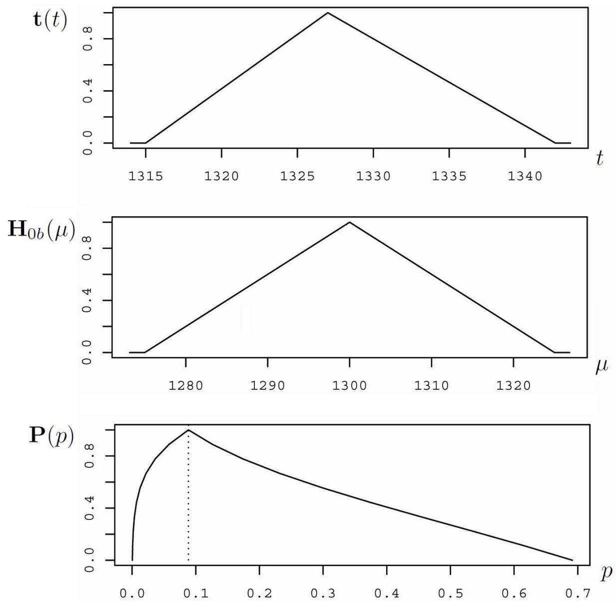

Figure 6: Three membership functions of the observed value of test statistics ($\mathbf{t}$), the boundary of the fuzzy null hypothesis ($\mathbf{H}_{0b}$) and the fuzzy p-value ($\mathbf{P}$) in Example 4.2

$H_1: \ \sigma^2 > \sigma_0^2,$

where $\sigma_0^2 = 0.5$. *Considering a similar test, in non-fuzzy environment, the precise p-value is equal to p-value =* $P(T > t) = 1 - F_T(t)$, *in which* $T = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$ *is the test statistic under the boundary of null hypothesis* $\sigma_0^2 = 0.5$ *and* t *is its observed value. So, one can test precise hypotheses at fuzzy significance level* $\boldsymbol{S} = T(0, 0.05, 0.10)$ *by the following codes in "FPV" Package. The result of test is shown in the second row of Table 2 which implied the null fuzzy hypothesis with 0.87 degree of acceptance.*

```
fuzzy.p.value( t = (n-1)*s2,
               H0b = 0.5,
               sig = TriangularFuzzyNumber(0.0,0.05,0.1),
               p.value = function(t,H0b)
                         pchisq(t/H0b, df=n-1, lower.tail=FALSE),
               fig=2,
               knot = 50,
               xlim = c(0,1),
               lwd=2 )
```

Table 2: The results of several tests, with the different vagueness for hypotheses, in Example 4.4.

$\tilde{H}_0$	$\tilde{H}_1$	$\Delta_{\mathbf{PS}}$	$\Delta_{\mathbf{SP}}$	Accepted hypothesis	Degree of acceptance
$S(0.5, 0.5)$	$B(0.5, 0.5)$	0.349	0.052	$\tilde{H}_0$	0.870
$S(0.45, 0.55)$	$B(0.45, 0.55)$	0.346	0.060	$\tilde{H}_0$	0.852
$S(0.4, 0.6)$	$B(0.4, 0.6)$	0.349	0.068	$\tilde{H}_0$	0.837
$S(0.35, 0.65)$	$B(0.35, 0.65)$	0.354	0.074	$\tilde{H}_0$	0.826

Now, we are going to construct the fuzzy concept "small" with a fuzzy sets, and therefore hypotheses are considered fuzzy rather than crisp as follows:

$\tilde{H}_0: \ \sigma^2$ *is approximately smaller than 0.5,*

$\tilde{H}_1: \ \sigma^2$ *is approximately bigger than 0.5,*

in which $\tilde{H}_0$ *and* $\tilde{H}_1$ *are given by* $S(0.45, 0.55)$ *and* $B(0.45, 0.55)$, *respectively. The result of this test at the fuzzy level* $\boldsymbol{S} = T(0, 0.05, 0.10)$ *is presented in the third row of Table 2. Moreover, Table 2 and Figure 8 contain the results of several tests with different vagueness for hypotheses.*

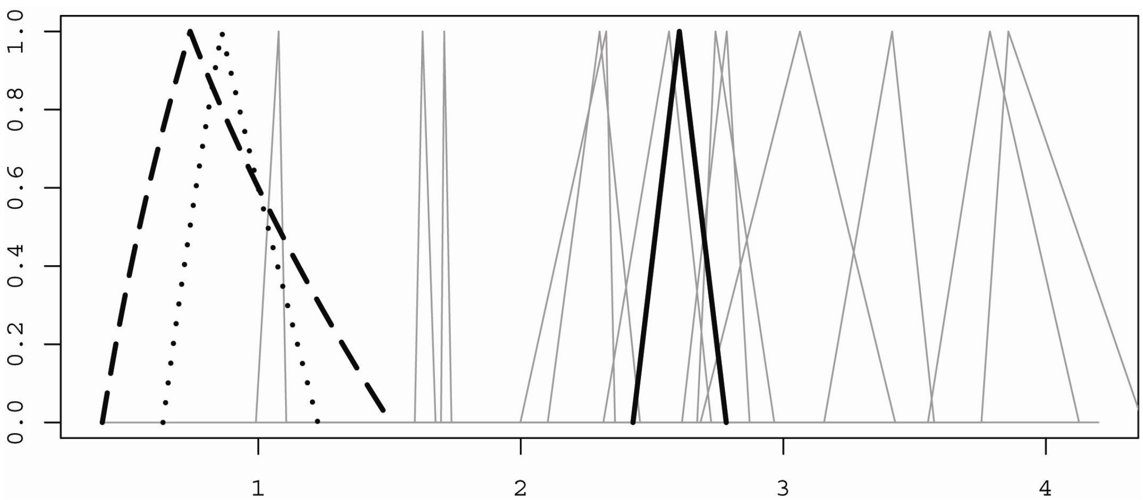

Figure 7: The membership functions of fuzzy mean (dark line), fuzzy variance (dash line) and fuzzy standard deviation (dot line) for the fuzzy data (gray line) in Example 4.4

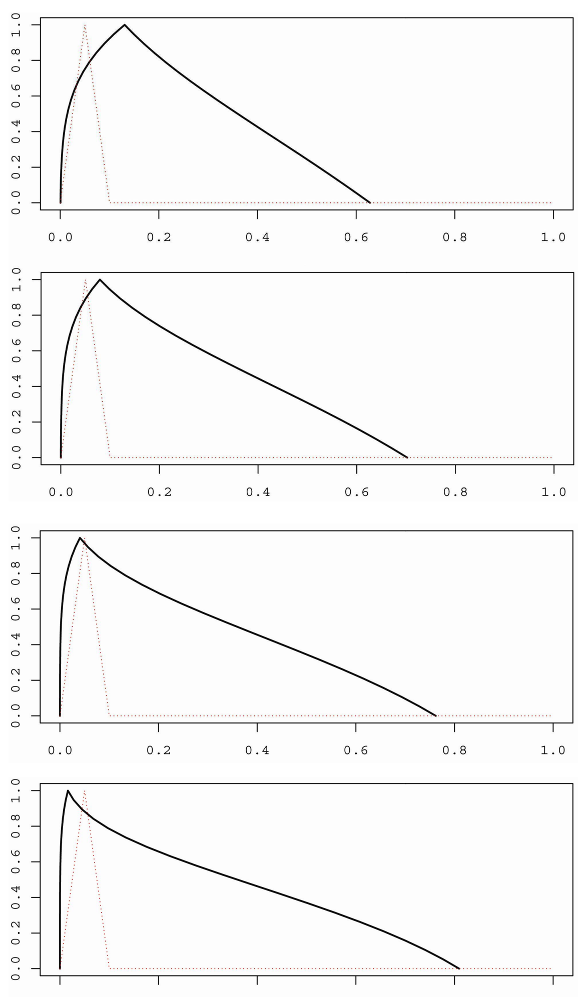

Figure 8: The membership functions of several fuzzy p-value and fuzzy significance level for the considered tests in Table 2 for Example 4.4

5 Comparison between two packages

The advantages and shortages of the two R packages "Fuzzy.p.value" and "FPV" are listed as follow:

1. Introducing a crisp / real value in package "FPV" is easier than in package "Fuzzy.p.value". For instance, to introduce the crisp test statistic $t = 7.4$, the user must consider code `T(7.4,7.4,7.4)` in the package "Fuzzy.p.value", while the code `t=7.4` is sufficient in the package "FPV".

2. Regarding the linear membership functions T, Tr, S and B which are introduced in Subsection 3.1 only for package "Fuzzy.p.value", analyses based on linear fuzzy sets is easier using package "Fuzzy.p.value" than package "FPV".

3. Using package "FPV" needs working with Package "FuzzyNumbers" to import fuzzy sets. So, package "Fuzzy.p.value" is more independently and more user-friendly rather than package "FPV" when the user apply linear fuzzy sets.

4. The exported figures from the package "FPV" have more variety. In other words, the user is able to draw three membership functions of `t`, `H0b` and the fuzzy p-value simultaneously in one figure using package "FPV" and this is not possible for package "Fuzzy.p.value".

5. Package "FPV" may be used to calculate fuzzy p-values from any statistical distribution, but package "Fuzzy.p.value" is applicable only when the test statistic is Normal or Poisson and this is a major weakness.

6. Additional arguments in function `fuzzy.p.value` are presented using the function `plot` in `R`. Therefore , the user can modify / control graphical arguments `lty`, `lwd`, `col`, `xlim` and etc. in package "FPV" and this is not possible in package "Fuzzy.p.value".

7. The accuracy level of calculations can be determined by argument `knot.n` in package "FPV", but package "Fuzzy.p.value" has not such ability to change the accuracy level of fuzzy p-value membership function.

6 Conclusions

Applying the fuzzy p-value approach for testing hypotheses based on fuzzy hypotheses or / and fuzzy data, needs a high dominance on at least one mathematical / statistical software which is usually impossible for the applied practitioners. Two R packages "FPV" and "Fuzzy.p.value" are introduced in this paper for easily solving such problems in R software. These packages able to compute and draw the membership function of fuzzy p-value. The comparison of fuzzy p-value with fuzzy significance level is another advantage of packages "FPV" and "Fuzzy.p.value" which is useful for decision making in a fuzzy environment. Several numerical examples are presented for comparison of the two packages.

Acknowledgement

The author thanks Professor M.B. Tata and anonymous reviewers for constructive suggestions.

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