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A new approach to rank the decision making units in presence of infeasibility in intuitionistic fuzzy environment

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Abstract

Data envelopment analysis (DEA) is a linear programming based methodology to determine the relative performance efficiencies of homogeneous decision making units (DMUs). In real world applications, some input and output datas do not possess crisp/fuzzy essence but they possess intuitionistic fuzzy (IF) essence. So, in this study, we develop an IF BCC (IFBCC) and an IF super efficiency BCC (IFSEBCC) model with triangular IF numbers (TIFNs) input/output data. Firstly, the efficiencies of DMUs are determined using IFBCC model. If DMUs are efficient then the super efficiencies of DMUs are determined using IFSEBCC model. If DMUs do not have feasible (finite) efficiencies/super efficiencies in intuitionistic fuzzy environment, then we can not rank the DMUs. Therefore, we develop IF models to determine the efficiencies of DMUs and ranking the DMUs in the presence of infeasibility in intuitionistic fuzzy environment. Two illustrative examples and one real life application are provided to illustrate the applications of the proposed IF models. Finally, comparison has been performed with the existing crisp models.

Keywords: IF CCR efficiency, IF CCR super efficiency, IFBCC efficiency, IFBCC super efficiency, infeasibility, ranking.

1 Introduction

Data envelopment analysis (DEA) proposed by Charnes et.al. [7] is a non parametric linear programming (LP) based methodology to determine the relative performance efficiencies of homogeneous DMUs on the basis of multiple inputs-outputs. Without assuming prior weights associated with inputs and outputs.

CCR (Charnes, Cooper and Rhodes) [7] is the first model in DEA, which deals with proportional changes in inputs and outputs. Banker et.al. [6] extended CCR to returns to scale (RTS). DMUs can be any governmental agencies and non-profitable organizations like health centers, agriculture farms, educational institutions, banks, transportation etc. The crisp DEA model assigns an efficiency value less than one to an inefficient DMUs, from which a ranking can be derived. However, for an efficient DMU efficiency value is one [7]. Hence no ranking can be given to efficient DMUs. Andersen and Petersen [1] proposed a model for ranking efficient DMUs. They proposed the ranking method for efficient DMUs, so that the corresponding column of the DMU being ranked is removed from the coefficient matrix. Due to the occurrence of the infeasibility in Andersen and Petersen's [1] model, the difficulties do not end here. Andersen/Petersen's [1] model may be unstable because of extreme sensitivity to small changes in the data when some DMUs have relatively small values for some of its inputs. Jahanshahloo et.al. [16] proposed a model which successfully removes the mentioned difficulties arising from Andersen and Petersen's [1] model. Jahanshahloo et.al. [16] proposed a ranking extreme efficient model with constant returns to scale (CRS) and variable returns to scale (VRS) in DEA. Let CRS model denote by L1CRS and VRS denote by L1VRS.

In conventional DEA model, the data are taken as crisp (deterministic) quantities. However, in real world applications data is not available in crisp (exact) form, but it fluctuates. The fluctuation can take the form of fuzzy numbers and intuitionistic fuzzy numbers (IFNs) etc. So, to deal with such type of situations, the conventional DEA has been

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extended to fuzzy DEA (FDEA) by making use of fuzzy numbers. There are numerous studies on fuzzy approaches to efficiency in DEA literature.

Fuzzy set theory introduced by Zadeh [28] has been shown to be a useful tool to handle vagueness/uncertainties present in real world problems. In fuzzy set theory, the sum of membership (acceptance) and non-membership (rejection) degree of an element is one [30]. However, in real world problems, there is possibility that the sum of the acceptance and rejection values of an element may come out to be less than one. Thus, there remains some degree of hesitation. The fuzzy set theory [30] fails to deal with such type of problems; rather intuitionistic fuzzy set (IFS) theory more suitable for such type of problems. The IFS theory proposed by Atanassov [4]-[5], is an extension of fuzzy set theory and it is more useful to deal with vagueness/uncertainty.

Emrouznejad et. al. [10] presented the bibliography of DEA from 1978 to 2016 in which 10300 published research journal articles are included. Sengupta [24] proposed fuzzy measures and fuzzy mathematical programs in DEA models with vague and imprecise data. Emrouznejad et. al. [9] proposed a fuzzy DEA model using discrete approach. Dotoli et.al. [8] proposed a novel cross-efficiency FDEA model to determine the performance of the different elements under uncertainty. Kao and Liu [17] proposed the FDEA model using α- cut approach. Wang and Chin [27] proposed a fuzzy expected value approach to determine the optimistic and pessimistic efficiencies of DMUs in fuzzy environment. Wang et.al. [26] proposed two new FDEA models and fuzzy ranking approach to compare and rank the fuzzy efficiencies of the DMUs. Hatami et.al. [14] proposed a fuzzy ranking approach in FDEA environment. Sengupta [24] explained three types of fuzzy statics to illustrate the types of decisions and solutions that are achievable, when the data are vague/uncertain. Hatami et.al. [15] explained the taxonomy and gave a review of the FDEA methods. Guo and Tanaka [12] proposed an FDEA model with fuzzy data and extension of FDEA model to a more general form is also proposed by considering the relationship between DEA and regression analysis (RA). Mustafa et. al. [21] proposed a ranking approach using fuzzy concept in FDEA. Hajiagha et. al. [13] proposed IFDEA model with IF inputs and IF outputs using the aggregation operator. Arya and Yadav [2] proposed IF slack based measure (IFSBM) and IF super efficiency SBM (IFSESBM) models to determine the performances of DMUs. Arya and Yadav [3] proposed FDEA models to determine the lower and upper bound relative performance efficiencies of DMUs.

2 Preliminaries

2.1 Intuitionistic fuzzy set (IFS) [2]

Assume that X be a universe of discourse. Then an IFS is denoted by $\tilde{A}^I$ and is defined by $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x))\}$, where $\mu_{\tilde{A}^I} : X \to [0,1]$ is the membership and , $\nu_{\tilde{A}^I} : X \to [0,1]$ is non-membership function of an element x in $\tilde{A}^I$. The values of $\mu_{\tilde{A}^I}(x)$ and $\nu_{\tilde{A}^I}(x)$ of x being in $\tilde{A}^I$ with the condition $0 \le \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \le 1$, $\mu_{\tilde{A}^I}(x) \in [0,1]$ and $\nu_{\tilde{A}^I}(x) \in [0,1]$. The hesitation (indeterminacy) degree of an element x being in $\tilde{A}^I$ is defined as $\pi_{\tilde{A}^I}(x) = 1 - \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x)$ $\forall x \in X$. Obviously $0 \le \pi_{\tilde{A}^I}(x) \le 1$. If $\pi_{\tilde{A}^I}(x) = 0$, then $\tilde{A}^I$ is reduced to a fuzzy set.

2.2 Normal IFS [2]

Let $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)); x \in \mathbb{X}\}$ be an IFS. Then $\tilde{A}^I$ is called normal IFS if $\exists$ an $x \in \mathbb{X}$ such that $\mu_{\tilde{A}^I}(x_o) = 1$ and $\nu_{\tilde{A}^I}(x_o) = 0$. This x_o is called the mean or modal value of $\tilde{A}^I$.

2.3 Convex IFS [2]

Let $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)); x \in \mathbb{X}\}$ be an IFS. Then $\tilde{A}^I$ is called Convex IFS if

- $\mu_{\tilde{A}^I}$ is a quasi-concave function over $\mathbb{X}$, i.e., $min(\mu_{\tilde{A}^I}(x), \mu_{\tilde{A}^I}(y)) \le \mu_{\tilde{A}^I}(\lambda x + (1-\lambda)y)$, $\forall x, y \in \mathbb{X}$ and $\lambda \in [0,1]$.
- $\nu_{\tilde{A}^I}$ is a quasi-convex function over $\mathbb{X}$, i.e., $max(\nu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(y)) \ge \nu_{\tilde{A}^I}(\lambda x + (1-\lambda)y)$, $\forall x, y \in \mathbb{X}$ and $\lambda \in [0,1]$.

2.4 Intuitionistic fuzzy number (IFN) [2]

Let $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in \mathbb{R}\}$ be an IFS, where the real line $\mathbb{R}$ is the set of real numbers with its membership function $\mu_{\tilde{A}^I}$ and non-membership function $\nu_{\tilde{A}^I}$. Then $\tilde{A}^I$ is called an IFN if the following conditions hold:

- there exists a unique $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}^I}(x_0) = 1$ and $\nu_{\tilde{A}^I}(x_0) = 0$, i.e., $\tilde{A}^I$ is normal. x_0 is called the mean value of $\tilde{A}^I$.

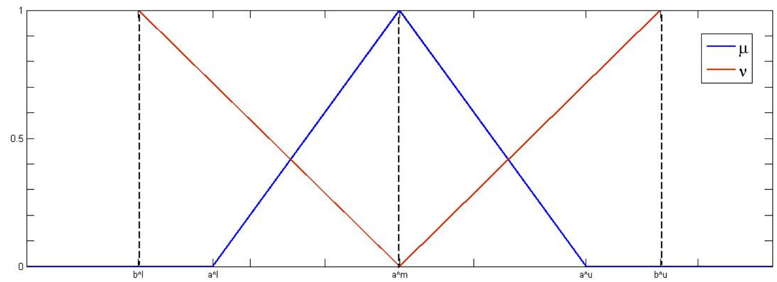

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Figure 1: TIFN $\tilde{A}^I = (a^l, a^m, a^u; a^{'l}, a^m, a^{'u})$

- $\tilde{A}^I$ is convex IFS over $\mathbb{R}$.

2.5 Triangular intuitionistic fuzzy number (TIFN) [2]

$\tilde{A}^I = (a^l, a^m, a^u; a'^l, a^m, a'^u)$ is an IFN with the membership function $\mu_{\tilde{A}^I}$ and non-membership function $\nu_{\tilde{A}^I}$ given by

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x-a^l}{a^m-a^l}, & a^l < x \le a^m, \\ \dfrac{a^u-x}{a^u-a^m}, & a^m \le x < a^u \\ 0 & , elsewhere. \end{cases} \qquad \nu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x-a^m}{a'^l-a^m}, & a'^l < x \le a^m, \\ \dfrac{a^m-x}{a^m-a'^u}, & a^m \le x < a'^u, \\ 1 & , elsewhere. \end{cases}$$

where $a^l, a^m, a^u, a'^l, a'^u \in \mathbb{R}$ such that $a'^l \le a^l \le a^m \le a^u \le a'^u$. Its graphical representation is given in Figure 1.

2.6 The expected values of TIFNs [22]

Let $\tilde{A}^I = (a^m; a^m - a^l, a^u - a^m; a^m - a'^l, a'^u - a^m)$ be an IFN with membership and non-membership functions $\mu_{\tilde{A}^I}(x)$ and $\nu_{\tilde{A}^I}(x)$ respectively. Then the expected interval [11] of $\tilde{A}^I$ is defined as the crisp interval $EI(\tilde{A}^I) = [E^L(\tilde{A}^I), E^U(\tilde{A}^I)]$, where

$$E^L(\tilde{A}^I) = \frac{a'^l + a^m}{2} + \frac{1}{2}\int_{a'^l}^{a^m} h_1(x)dx - \frac{1}{2}\int_{a^l}^{a^m} g_1(x)dx, \tag{1}$$

$$E^U(\tilde{A}^I) = \frac{a^m + a'^u}{2} + \frac{1}{2}\int_{a^m}^{a^u} g_2(x)dx - \frac{1}{2}\int_{a^m}^{a'^u} h_2(x)dx. \tag{2}$$

The expected value of an IFN $\tilde{A}^I$ is defined by $EV(\tilde{A}^I)$ and is defined by:

$$EV(\tilde{A}^I) = \frac{E^L(\tilde{A}^I) + E^U(\tilde{A}^I)}{2}. \tag{3}$$

Theorem 2.1. *Let $\tilde{A}^I = (a^l, a^m, a^u; a^{'l}, a^m, a^{'u})$ be a TIFN. Then* $EV(\tilde{A}^I) = \frac{a'^l + a^l + 4a^m + a^u + a^{'u}}{8}$.

Proof. Using (3.1), we have $E^L(\tilde{A}^I) = \frac{a^l + 2a^m + a^{'l}}{4}$.
Using (3.2), we have $E^U(\tilde{A}^I) = \frac{a^u + 2a^m + a^{'u}}{4}$.
Therefore, $EV(\tilde{A}^I) = \frac{a'^l + a^l + 4a^m + a^u + a^{'u}}{8}$. □

2.7 Ordering of TIFNs

Let $\tilde{A}^I = (a^l, a^m, a^u; a^{'l}, a^m, a^{'u})$ and $\tilde{B}^I = (b^l, b^m, b^u; b'^l, b^m, b'^u)$ be two TIFNs. Then
$\tilde{A}^I \ge \tilde{B}^I \iff EV(\tilde{A}^I) \ge EV(\tilde{B}^I)$,
$\tilde{A}^I \le \tilde{B}^I \iff EV(\tilde{A}^I) \le EV(\tilde{B}^I)$,
$\tilde{A}^I = \tilde{B}^I \iff EV(\tilde{A}^I) = EV(\tilde{B}^I)$,
$\min(\tilde{A}^I, \tilde{B}^I) = \tilde{A}^I \ if \tilde{A}^I \le \tilde{B}^I$,
$\max(\tilde{A}^I, \tilde{B}^I) = \tilde{A}^I \ if \ \tilde{A}^I \ge \tilde{B}^I$.

3 Background

DEA is a linear programming based methodology to determine the relative efficiencies of homogeneous DMUs when the production process consists of multiple inputs and multiple outputs. Assume that the performance of a set of n homogeneous DMUs $(DMU_j; j = 1, 2, 3, ..., n)$ is to be measured. Let $x_{ij}; i = 1, 2, 3, ..., m$ be the amount of the ith input utilized and $y_{rj}; r = 1, 2, 3, ..., s$ be the amount of the rth output produced. Then the efficiency of $(DMU_j; j = 1, 2, 3, ..., n)$ is given by

$$E_j = \frac{\sum_{r=1}^{s} v_r y_{rj}}{\sum_{i=1}^{m} u_i x_{ij}},$$

where u_i and v_r are the weights corresponding to x_{ij} and y_{rj} respectively.

In the CCR fractional program (FP) [7], the efficiency of the DMU_{jo} is to be maximized subject to the condition that the ratio of the virtual output to the virtual input of every DMU is less than or equal to unity.

The envelopment form of input-oriented BCC model [6] and super-efficiency BCC (SEBCC) model can be formulated as a linear program given in Model 1 and Model 2 respectively. If we remove the constraint $\sum_{j=1}^{n} \lambda_j = 1$ from both models, then we get CCR model and super-efficiency CCR (SECCR) model. The DMU_{j_o} is said to be BCC efficient, if $\theta_{j_o} = 1$; otherwise inefficient. If we get more than one efficient DMUs, then we apply super-efficiency method to rank the DMUs.

Model 1

$$\begin{aligned}
&min\ \theta_{j_o} \\
&s.t\ \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_{j_o} x_{ij_o}\ \forall i, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rj_o}\ \forall r \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \geq 0\ \forall j
\end{aligned}$$

Model 2

$$\begin{aligned}
&min\ \theta_{j_o} \\
&s.t\ \sum_{j=1, j\neq j_o}^{n} \lambda_j x_{ij} \leq \theta_{j_o} x_{ij_o}\ \forall i, \\
&\sum_{j=1, j\neq j_o}^{n} \lambda_j y_{rj} \geq y_{rj_o}\ \forall r \\
&\sum_{j=1, j\neq j_o}^{n} \lambda_j = 1, \\
&\lambda j \geq 0\ \forall j,\ j \neq j_o
\end{aligned},$$

For the BCC and SEBCC models in IFE, in which inputs and outputs are given as IFNs, we need IF BCC model. Assume that IF input is $\tilde{x}_{ij}^I$ and IF output is $\tilde{y}_{rj}^I$. Then IFBCC model and IFSEBCC model are given in Models 3 and 4 respectively. Mathematically, the BCC and SEBCC IF efficiencies of the DMU_{j_o} are denoted by $\tilde{\theta}_{j_o}^I$ and $\tilde{\theta}_{j_o}^{IS}$ respectively, and can be determined from the IFBCC (Model 3(a)) and IFSEBCC (Model 4(a)) models. In Models 5(a) and 6(a), $\tilde{y}_{rj_o}^I$ is the amount of the rth IF output produced by the DMU_{j_o}; $\tilde{x}_{ij_o}^I$ is the amount of the ith IF input used by the DMU_{j_o}. The input-oriented CCR models can be easily obtained by removing the condition $\sum_{j=1}^{n} \tilde{\lambda}_j^I = \tilde{1}^I$ from both the Models 5(a) and 6(a) respectively.

Model 3(a) (IFBCC)

$$\begin{aligned}
& min\ \tilde{\theta}^{I}_{j_o} \\
& s.t \sum_{j=1}^{n} \tilde{\lambda}^{I}_{j} \tilde{x}^{I}_{ij} \leq \tilde{\theta}^{I}_{j_o} \tilde{x}^{I}_{ij_o}\ \forall i, \\
& \sum_{j=1}^{n} \tilde{\lambda}^{I}_{j} \tilde{y}^{I}_{rj} \geq \tilde{y}^{I}_{rj_o}\ \forall r, \\
& \sum_{j=1}^{n} \tilde{\lambda}^{I}_{j} = \tilde{1}^{I}, \\
& \tilde{\lambda}^{I}_{j} \geq \tilde{0}^{I}\ \forall j.
\end{aligned}$$

Model 4(a) (IFSEBCC)

$$\begin{aligned}
& min\ \tilde{\theta}^{IS}_{j_o} \\
& s.t \sum_{j=1, j\neq j_o}^{n} \tilde{\lambda}^{I}_{j} \tilde{x}^{I}_{ij} \leq \tilde{\theta}^{I}_{j_o} \tilde{x}^{I}_{ij_o}\ \forall i, \\
& \sum_{j=1, j\neq j_o}^{n} \tilde{\lambda}^{I}_{j} \tilde{y}^{I}_{rj} \geq \tilde{y}^{I}_{rj_o}\ \forall r, \\
& \sum_{j=1, j\neq j_o}^{n} \tilde{\lambda}^{I}_{j} = \tilde{1}^{I}, \\
& \tilde{\lambda} j^{I} \geq \tilde{0}^{I}, \text{j=1,2,3,...,n}, j \neq j_o.
\end{aligned}$$

Let IF input-output data be TIFNs. Let $\tilde{x}^{I}_{ij} = (x^{l}_{ij}, x^{m}_{ij}, x^{u}_{ij}; x^{l'}_{ij}, x^{m}_{ij}, x^{u'}_{ij})$, $\tilde{y}^{I}_{rj} = (y^{l}_{rj}, y^{m}_{rj}, y^{u}_{rj}; y^{l'}_{rj}, y^{m}_{rj}, y^{u'}_{rj})$, $\tilde{x}^{I}_{ij_o} = (x^{l}_{ij_o}, x^{m}_{ij_o}, x^{u}_{ij_o}; x^{l'}_{ij_o}, x^{m}_{ij_o}, x^{u'}_{ij_o})$, $\tilde{y}^{I}_{rj_o} = (y^{l}_{rj_o}, y^{m}_{rj_o}, y^{u}_{rj_o}; y^{l'}_{rj_o}, y^{m}_{rj_o}, y^{u'}_{rj_o})$, $\tilde{\theta}^{I}_{j_o} = (\theta^{l}_{j_o}, \theta^{m}_{j_o}, \theta^{u}_{j_o}; \theta^{l'}_{j_o}, \theta^{m}_{j_o}, \theta^{u'}_{j_o})$ and $\tilde{\lambda}^{I}_{j} = (\lambda^{l}_{j}, \lambda^{m}_{j}, \lambda^{u}_{j}; \lambda^{l'}_{j}, \lambda^{m}_{j}, \lambda^{u'}_{j})$. Then, the IFBCC (Model 3(a)) and IFSEBCC (Model 4(a)) models are reduced to the following Models 5(b) and 6(b) respectively:

Model 3(b)

$$\begin{aligned}
& min\ (\theta^{l}_{j_o}, \theta^{m}_{j_o}, \theta^{u}_{j_o}, \theta^{l'}_{j_o}, \theta^{m}_{j_o}, \theta^{u'}_{j_o}) \\
& \text{subject to} \\
& \sum_{j=1}^{n} (\lambda^{l}_{j}, \lambda^{m}_{j}, \lambda^{u}_{j}; \lambda^{l'}_{j}, \lambda^{m}_{j}, \lambda^{u'}_{j}).(x^{l}_{ij}, x^{m}_{ij}, x^{u}_{ij}; x^{l'}_{ij}, x^{m}_{ij}, x^{u'}_{ij}) \leq (\theta^{l}_{j_o}, \theta^{m}_{j_o}, \theta^{u}_{j_o}, \theta^{l'}_{j_o}, \theta^{m}_{j_o}, \theta^{u'}_{j_o}).(x^{l}_{ij_o}, x^{m}_{ij_o}, \\
& x^{u}_{ij_o}, x^{l'}_{ij_o}, x^{m}_{ij_o}, x^{u'}_{ij_o}),\ \forall i, \\
& \sum_{j=1}^{n} (\lambda^{l}_{j}, \lambda^{m}_{j}, \lambda^{u}_{j}; \lambda^{l'}_{j}, \lambda^{m}_{j}, \lambda^{u'}_{j}).(y^{l}_{rj}, y^{m}_{rj}, y^{u}_{rj}; y^{l'}_{rj}, y^{m}_{rj}, y^{u'}_{rj}) \geq ({y_{rj_o}}^{l}, {y_{rj_o}}^{m}, {y_{rj_o}}^{u}; {y_{rj_o}}^{l'}, {y_{rj_o}}^{m}, {y_{rj_o}}^{u'})\ \forall r, \\
& \sum_{j=1}^{n} (\lambda^{l}_{j}, \lambda^{m}_{j}, \lambda^{u}_{j}; \lambda^{l'}_{j}, \lambda^{m}_{j}, \lambda^{u'}_{j}) = (1,1,1;1,1,1), \\
& \lambda j^{u'} \geq \lambda j^{u} \geq \lambda j^{m} \geq \lambda j^{l} \geq \lambda j^{l'}, \text{and } \lambda j^{l'} \geq 0\ \forall j.
\end{aligned}$$

Model 4(b)

$$min\ (\theta^{l}_{j_o}, \theta^{m}_{j_o}, \theta^{u}_{j_o}, \theta^{l'}_{j_o}, \theta^{m}_{j_o}, \theta^{u'}_{j_o})$$

subject to

$$\sum_{j=1, j\neq j_o}^{n} (\lambda^{l}_{j}, \lambda^{m}_{j}, \lambda^{u}_{j}; \lambda^{l'}_{j}, \lambda^{m}_{j}, \lambda^{u'}_{j}).(x^{l}_{ij}, x^{m}_{ij}, x^{u}_{ij}; x^{l'}_{ij}, x^{m}_{ij}, x^{u'}_{ij}) \leq (\theta^{l}_{j_o}, \theta^{m}_{j_o}, \theta^{u}_{j_o}, \theta^{l'}_{j_o}, \theta^{m}_{j_o}, \theta^{u'}_{j_o}).(x^{l}_{ij_o}, x^{m}_{ij_o},$$
$$x^{u}_{ij_o}, x^{l'}_{ij_o}, x^{m}_{ij_o}, x^{u'}_{ij_o}),\ \forall i,$$
$$\sum_{j=1, j\neq j_o}^{n} (\lambda^{l}_{j}, \lambda^{m}_{j}, \lambda^{u}_{j}; \lambda^{l'}_{j}, \lambda^{m}_{j}, \lambda^{u'}_{j}).(y^{l}_{rj}, y^{m}_{rj}, y^{u}_{rj}; y^{l'}_{rj}, y^{m}_{rj}, y^{u'}_{rj}) \geq ({y_{rj_o}}^{l}, {y_{rj_o}}^{m}, {y_{rj_o}}^{u}; {y_{rj_o}}^{l'}, {y_{rj_o}}^{m}, {y_{rj_o}}^{u'})\ \forall r,$$
$$\sum_{j=1, j\neq j_o}^{n} (\lambda^{l}_{j}, \lambda^{m}_{j}, \lambda^{u}_{j}; \lambda^{l'}_{j}, \lambda^{m}_{j}, \lambda^{u'}_{j}) = (1,1,1;1,1,1),$$
$$\lambda j^{u'} \geq \lambda j^{u} \geq \lambda j^{m} \geq \lambda j^{l} \geq \lambda j^{l'}, \text{and } \lambda j^{l'} \geq 0\ \forall j.$$

By using expected values of TIFNs (Section 3), we get the Models 7 and 8 from Model 3(b) and Model 4(b) respectively.

Model 5

$$min\ \frac{1}{8}(\theta^{l'}_{j_o} + \theta^{l}_{j_o} + 4\theta^{m}_{j_o} + \theta^{u}_{j_o} + \theta^{u'}_{j_o})$$

subject to

$$\sum_{j=1}^{n} (\lambda^{l'}_{j} x^{l'}_{ij} + \lambda^{l}_{j} x^{l}_{ij} + 4\lambda^{m}_{j} x^{m}_{ij} + \lambda^{u}_{j} x^{u}_{ij} + \lambda^{u'}_{j} x^{u'}_{ij}) \leq (\theta^{l'}_{j_o} x^{l'}_{ij_o} + \theta^{l}_{j_o} x^{l}_{ij_o} + 4\theta^{m}_{j_o} x^{m}_{ij_o} + \theta^{u}_{j_o} x^{u}_{ij_o} + \theta^{u'}_{j_o} x^{u'}_{ij_o}),\ \forall i,$$
$$\sum_{j=1}^{n} (\lambda^{l'}_{j} y^{l'}_{rj} + \lambda^{l}_{j} y^{l}_{rj} + 4\lambda^{m}_{j} y^{m}_{rj} + \lambda^{u}_{j} y^{u}_{rj} + \lambda^{u'}_{j} y^{u'}_{rj}) \geq ({y_{rj_o}}^{l'} + {y_{rj_o}}^{l} + 4{y_{rj_o}}^{m} + {y_{rj_o}}^{u} + {y_{rj_o}}^{u'})\ \forall r,$$
$$\sum_{j=1}^{n} ({\lambda_j}^{l'} + {\lambda_j}^{l} + 4{\lambda_j}^{m} + {\lambda_j}^{u} + {\lambda_j}^{u'}) = 8,$$
$$\lambda j^{u'} \geq \lambda j^{u} \geq \lambda j^{m} \geq \lambda j^{l} \geq \lambda j^{l'}, \text{and } \lambda j^{l'} \geq 0\ \forall j.$$

Model 6

$$min\ \frac{1}{8}(\theta^{l'}_{j_o} + \theta^{l}_{j_o} + 4\theta^{m}_{j_o} + \theta^{u}_{j_o} + \theta^{u'}_{j_o})$$

subject to

$$\sum_{j=1, j\neq j_o}^{n} (\lambda^{l'}_{j} x^{l'}_{ij} + \lambda^{l}_{j} x^{l}_{ij} + 4\lambda^{m}_{j} x^{m}_{ij} + \lambda^{u}_{j} x^{u}_{ij} + \lambda^{u'}_{j} x^{u'}_{ij}) \leq (\theta^{l'}_{j_o} x^{l'}_{ij_o} + \theta^{l}_{j_o} x^{l}_{ij_o} + 4\theta^{m}_{j_o} x^{m}_{ij_o} + \theta^{u}_{j_o} x^{u}_{ij_o} + \theta^{u'}_{j_o} x^{u'}_{ij_o}),\ \forall i,$$
$$\sum_{j=1, j\neq j_o}^{n} (\lambda^{l'}_{j} y^{l'}_{rj} + \lambda^{l}_{j} y^{l}_{rj} + 4\lambda^{m}_{j} y^{m}_{rj} + \lambda^{u}_{j} y^{u}_{rj} + \lambda^{u'}_{j} y^{u'}_{rj}) \geq ({y_{rj_o}}^{l'} + {y_{rj_o}}^{l} + 4{y_{rj_o}}^{m} + {y_{rj_o}}^{u} + {y_{rj_o}}^{u'})\forall r,$$
$$\sum_{j=1, j\neq j_o}^{n} ({\lambda_j}^{l'} + {\lambda_j}^{l} + 4{\lambda_j}^{m} + {\lambda_j}^{u} + {\lambda_j}^{u'}) = 8,$$
$$\lambda j^{u'} \geq \lambda j^{u} \geq \lambda j^{m} \geq \lambda j^{l} \geq \lambda j^{l'}, \text{and } \lambda j^{l'} \geq 0\ \forall j, j \neq j_o$$

The IFCCR and IFSECCR models can be easily obtained by removing the convexity condition from both models which are given in Models 7 and 8.

3.1 Advantage of the proposed approaches

1. Jahanshahloo et.al. [16] proposed a ranking extreme efficient model with CRS and VRS in DEA for crisp data, but did not develop the models for intuitionistic fuzzy data. Tavana et.al. [25] proposed a two-stage DEA model,

Khanjarpanah, H. and Jabbarzadeh, A. [18] proposed crisp and fuzzy cross-efficiency DEA models.
2. Jahanshahloo et.al. [16] proposed CRS and VRS in DEA for crisp data which is not applicable to fuzzy/ intuitionistic fuzzy environment. That is why we have developed models which is applicable to intuitionistic fuzzy environment as well as crisp and fuzzy.

4 Proposed IF models for efficient DMUs

Several authors (Andersen and Petersen (AP) [1], Mehrabian Alirezai and Jahanshahloo (MAJ) [20], Zhu [29] and Seiford and Zhu [23], etc) have proposed the ranking methods for ranking the best DMUs. In literature, the AP and MAJ models are infeasible [1],[19] for some cases. The AP model may be unstable [1] because of extreme sensitivity to small variations in the data. Jahanshahloo et.al. [16] proposed a ranking extreme efficient model with CRS and VRS in DEA. Let the Jahanshahloo et.al. [16] approach with CRS denote by L1CRS and with VRS denote by L1VRS. Assume that there are n DMUs to be determined, and each DMU is assumed to produce s multiple outputs from m multiple inputs. The L1CRS and L1VRS models are given in Models 9 and 10 respectively.

Model 7

$$MinE_{j_o}^{C} = \sum_{i=1}^{m} x_i - \sum_{r=1}^{s} y_r + \beta$$

$$\text{subject to} \quad \sum_{j=1, j\neq j_o}^{n} \lambda_j x_{ij} \leq x_i \ \forall i,$$

$$\sum_{j=1, j\neq j_o}^{n} \lambda_j y_{rj} \leq y_r \ \forall r,$$

$$x_i \geq x_{ij_o} \ \forall i, \ 0 \leq y_r \leq y_{rj_o} \ \forall r, \ \lambda_j \geq 0 \ \forall j, \, j \neq j_o$$

$$\text{where } \beta = \sum_{r=1}^{s} y_{rj_o} - \sum_{i=1}^{m} x_{ij_o} \text{ is a constant number.}$$

Model 8

$$MinE_{j_o}^{V} = \sum_{i=1}^{m} x_i - \sum_{r=1}^{s} y_r + \beta$$

$$\text{subject to} \quad \sum_{j=1, j\neq j_o}^{n} \lambda_j x_{ij} \leq x_i \ \forall i,$$

$$\sum_{j=1, j\neq j_o}^{n} \lambda_j y_{rj} \leq y_r \ \forall r,$$

$$\sum_{j=1, j\neq j_o}^{n} \lambda_j = 1,$$

$$x_i \geq x_{ij_o} \ \forall i, \ 0 \leq y_r \leq y_{rj_o} \ \forall r, \ \lambda_j \geq 0 \ \forall j, \, j \neq j_o$$

$$\text{where } \beta = \sum_{r=1}^{s} y_{rj_o} - \sum_{i=1}^{m} x_{ij_o} \text{ is a constant number.}$$

In Models 9 and 10, x_{ij} and y_{rj} are the amount of ith input and rth output respectively for DMU_j; $j = 1, 2, 3, ..., n$.

Remark 4.1. *L1CRS and L1VRS models are feasible and bounded, see [16].*

4.1 Fuzzy ranking approach with fuzzy weights (FWs)

In conventional DEA model, the weights of inputs and outputs are in crisp form. But, in real world problems, the weights may be fuzzy quantities. Therefore in this paper, weights of inputs and outputs are taken as TFNs. To describe

L1CRS and L1VRS efficiency evaluations in fuzzy environment, DMU_j utilizes m fuzzy inputs $\tilde{x}_{ij},\ i=1,2,3,...,m$ to produce s fuzzy outputs $\tilde{y}_{rj},\ r=1,2,3,...,s$. The fuzzy L1CRS (FL1CRS) efficiency of DMU_{j_o} is denoted by $\tilde{E}^C_{j_o}$ and fuzzy L1VRS (FL1VRS) efficiency of DMU_{j_o} is denoted by $\tilde{E}^V_{j_o}$ and can be measured from Model 9 and Model 10 respectively.

Model 9

$$Min\tilde{E}^C_{j_o}=\sum_{i=1}^{m}\tilde{x}_i-\sum_{r=1}^{s}\tilde{y}_r+\tilde{\beta}$$

$$\text{subject to } \sum_{j=1,j\neq j_o}^{n}\tilde{\lambda}_j\otimes\tilde{x}_{ij}\leq\tilde{x}_i\ \forall i,$$

$$\sum_{j=1,j\neq j_o}^{n}\tilde{\lambda}_j\otimes\tilde{y}_{rj}\leq\tilde{y}_r\ \forall r,$$

$$\tilde{x}_i\geq\tilde{x}_{ij_o}\ \forall i,\ \tilde{0}\leq\tilde{y}_r\leq\tilde{y}_{rj_o}\ \forall r,\ \tilde{\lambda}_j\geq\tilde{0}\ \forall j,\ j\neq j_o,$$

$$\text{where } \tilde{0}=(0,0,0),\ \text{and } \tilde{\beta}=\sum_{r=1}^{s}\tilde{y}_{rj_o}-\sum_{i=1}^{m}\tilde{x}_{ij_o}\ \text{is a constant number,}$$

Model 10

$$Min\tilde{E}^V_{j_o}=\sum_{i=1}^{m}\tilde{x}_i-\sum_{r=1}^{s}\tilde{y}_r+\tilde{\beta}$$

$$\text{subject to } \sum_{j=1,j\neq j_o}^{n}\tilde{\lambda}_j\otimes\tilde{x}_{ij}\leq\tilde{x}_i\ \forall i,$$

$$\sum_{j=1,j\neq j_o}^{n}\tilde{\lambda}_j\otimes\tilde{y}_{rj}\leq\tilde{y}_r\ \forall r,$$

$$\sum_{j=1,j\neq j_o}^{n}\tilde{\lambda}_j=\tilde{1},$$

$$\tilde{x}_i\geq\tilde{x}_{ij_o}\ \forall i,\ \tilde{0}\leq\tilde{y}_r\leq\tilde{y}_{rj_o}\ \forall r,\ \tilde{\lambda}_j\geq\tilde{0}\ \forall j,\ j\neq j_o,$$

$$\text{where } \tilde{0}=(0,0,0),\ \tilde{1}=(1,1,1),\ \text{and } \tilde{\beta}=\sum_{r=1}^{s}\tilde{y}_{rj_o}-\sum_{i=1}^{m}\tilde{x}_{ij_o}\ \text{is a constant number.}$$

4.2 Proposed IF ranking approach (PIFRA) with IF weights (IFWs):

To describe L1CRS and L1VRS efficiency evaluations in IFE, DMU_j uses m IF inputs $\tilde{x}^I_{ij}; i=1,2,...,m$ to produce s IF outputs $\tilde{y}^I_{rj}; r=1,2,...,s$. The IFL1CRS efficiency of DMU_{j_o} is denoted by $\tilde{E}^{IC}_{j_o}$ and IFL1VRS efficiency of DMU_{j_o} is denoted by $\tilde{E_{j_o}}^{IV}$ and can be measured from Models 13 and 14 respectively.

Model 11

$$Min\tilde{E}^{IC}_{j_o}=\sum_{i=1}^{m}\tilde{x}^I_i-\sum_{r=1}^{s}\tilde{y}^I_r+\tilde{\beta}^I$$

$$\text{subject to } \sum_{j=1,j\neq j_o}^{n}\tilde{\lambda}^I_j\otimes\tilde{x}^I_{ij}\leq\tilde{x}^I_i\ \forall i,$$

$$\sum_{j=1,j\neq j_o}^{n}\tilde{\lambda}^I_j\otimes\tilde{y}^I_{rj}\leq\tilde{y}^I_r\ \forall r,$$

$$\tilde{x}^I_i\geq\tilde{x}^I_{ij_o}\ \forall i,\ \tilde{0}^I\leq\tilde{y}^I_r\leq\tilde{y}^I_{rj_o}\ \forall r,\ \tilde{\lambda}^I_j\geq\tilde{0}^I\ \forall j,\ j\neq j_o,$$

$$\text{where } \tilde{0}^I=(0,0,0;0,0,0)\ \text{and } \tilde{\beta}^I=\sum_{r=1}^{s}\tilde{y}^I_{rj_o}-\sum_{i=1}^{m}\tilde{x}^I_{ij_o}\ \text{is a constant number.}$$

Model 12

$$Min\tilde{E}^{IV}_{j_o} = \sum_{i=1}^{m} \tilde{x}^I_i - \sum_{r=1}^{s} \tilde{y}^I_r + \tilde{\beta}^I$$

$$\text{subject to} \quad \sum_{j=1, j\neq j_o}^{n} \tilde{\lambda}^I_j \otimes \tilde{x}^I_{ij} \leq \tilde{x}^I_i \ \forall i,$$

$$\sum_{j=1, j\neq j_o}^{n} \tilde{\lambda}^I_j \otimes \tilde{y}^I_{rj} \leq \tilde{y}^I_r \ \forall r,$$

$$\sum_{j=1, j\neq j_o}^{n} \tilde{\lambda}^I_j = \tilde{1},$$

$$\tilde{x}^I_i \geq \tilde{x}^I_{ij_o} \ \forall i, \ \tilde{0}^I \leq \tilde{y}^I_r \leq \tilde{y}^I_{rj_o} \ \forall r, \ \tilde{\lambda}^I_j \geq \tilde{0}^I \ \forall j, \ j \neq j_o$$

where $\tilde{0}^I = (0,0,0;0,0,0)$, $\tilde{1}^I = (1,1,1;1,1,1)$ and $\tilde{\beta}^I = \sum_{r=1}^{s} \tilde{y}^I_{rj_o} - \sum_{i=1}^{m} \tilde{x}^I_{ij_o}$ is a constant number.

Theorem 4.2. *IFL1CRS model is feasible and bounded.*

Proof. For $k \neq j_0$, let
$\tilde{\lambda}^I_k = \tilde{1}^I, \ \tilde{\lambda}^I_j = \tilde{0}^I, \ \ \forall j, \ j \neq k, \ j_0,$
$\tilde{y}^I_r \leq \min\{\tilde{y}^I_{rk}, \tilde{y}^I_{rj_0}\}, \ \forall r,$
$\tilde{x}^I_i \geq \max\{\tilde{x}^I_{ik}, \tilde{x}^I_{ij_0}\}, \ \forall i.$
Then (X, Y, λ) with components $\tilde{x}^I_i \ (i = 1, 2, 3, ..., m)$, $\tilde{y}^I_r \ (\forall r)$ and $\tilde{\lambda}^I_k$ is a feasible solution of IFL1CRS model.
Since $\tilde{x}^I_i \geq \tilde{x}^I_{ij_0} \ (\forall i)$ and $\tilde{y}^I_r \leq \tilde{y}^I_{rj_0} \ (\forall r)$,
$\exists$ an TIFN $\tilde{\chi}^I$, s.t, $\sum_{i=1}^{m} \tilde{x}^I_i - \sum_{r=1}^{s} \tilde{y}^I_r \geq \tilde{\chi}^I$.
Hence, IFL1CRS model is bounded. □

Theorem 4.3. *IFL1VRS model is feasible and bounded.*

Proof. Similar to Theorem 4.2. □

4.3 Methodology for solving IFL1CRS and IFL1VRS models:

Assume that IF input, IF output data and all weights in Models 13 and 14 are TIFNs. Replacing $\tilde{x}^I_{ij}$ by $({x_{ij}}^l, {x_{ij}}^m, {x_{ij}}^u; {x_{ij}}^{l'}, {x_{ij}}^m, {x_{ij}}^{u'})$, $\tilde{y}^I_{rj}$ by $({y_{rj}}^l, {y_{rj}}^m, {y_{rj}}^u; {y_{rj}}^{l'}, {y_{rj}}^m, {y_{rj}}^{u'})$, $\tilde{x}^I_i$ by $({x_i}^l, {x_i}^m, {x_i}^u; {x_i}^{l'}, {x_i}^m, {x_i}^{u'})$, $\tilde{y}^I_r$ by $({y_r}^l, {y_r}^m, {y_r}^u; {y_r}^{l'}, {y_r}^m, {y_r}^{u'})$ and $\tilde{\lambda}^I_j$ by $({\lambda_j}^l, {\lambda_j}^m, {\lambda_j}^u; {\lambda_j}^{l'}, {\lambda_j}^m, {\lambda_j}^{u'})$, Model 13 is obtained from the IFL1CRS model.

Model 13

$$Min \ \tilde{E}^{IC}_{j_o} = \sum_{i=1}^{m} ({x_i}^l, {x_i}^m, {x_i}^u; {x_i}^{l'}, {x_i}^m, {x_i}^{u'}) - \sum_{r=1}^{s} ({y_r}^l, {y_r}^m, {y_r}^u; {y_r}^{l'}, {y_r}^m, {y_r}^{u'}) + (\beta^l, \beta^m, \beta^u; \beta^{l'}, \beta^m, \beta^{u'})$$

subject to

$$\sum_{j=1, j\neq j_o}^{n} (\lambda^l_j, \lambda^m_j, \lambda^u_j; \lambda^{l'}_j, \lambda^m_j, \lambda^{u'}_j) \otimes ({x_{ij}}^l, {x_{ij}}^m, {x_{ij}}^u; {x_{ij}}^{l'}, {x_{ij}}^m, {x_{ij}}^{u'}) \leq ({x_i}^l, {x_i}^m, {x_i}^u; {x_i}^{l'}, {x_i}^m, {x_i}^{u'}) \ \forall i,$$

$$\sum_{j=1, j\neq j_o}^{n} (\lambda^l_j, \lambda^m_j, \lambda^u_j; \lambda^{l'}_j, \lambda^m_j, \lambda^{u'}_j) \otimes ({y_{rj}}^l, {y_{rj}}^m, {y_{rj}}^u; {y_{rj}}^{l'}, {y_{rj}}^m, {y_{rj}}^{u'}) \leq ({y_r}^l, {y_r}^m, {y_r}^u; {y_r}^{l'}, {y_r}^m, {y_r}^{u'}) \ \forall r,$$

$$({x_i}^l, {x_i}^m, {x_i}^u; {x_i}^{l'}, {x_i}^m, {x_i}^{u'}) \geq ({x_{ij_o}}^l, {x_{ij_o}}^m, {x_{ij_o}}^u; {x_{ij_o}}^{l'}, {x_{ij_o}}^m, {x_{ij_o}}^{u'}) \ \forall i,$$

$$(0,0,0;0,0,0) \leq ({y_r}^l, {y_r}^m, {y_r}^u; {y_r}^{l'}, {y_r}^m, {y_r}^{u'}) \leq (y^l_{rj_o}, y^m_{rj_o}, y^u_{rj_o}; y^{l'}_{rj_o}, y^m_{rj_o}, y^{u'}_{rj_o}) \ \forall r,$$

$$(\lambda^l_j, \lambda^m_j, \lambda^u_j; \lambda^{l'}_j, \lambda^m_j, \lambda^{u'}_j) \geq (0,0,0;0,0,0); \ \forall j, \ j \neq j_o$$

where $(\beta^l, \beta^m, \beta^u; \beta^{l'}, \beta^m, \beta^{u'}) = \sum_{r=1}^{s} ({y_{rj_o}}^l, {y_{rj_o}}^m, {y_{rj_o}}^u; {y_{rj_o}}^{l'}, {y_{rj_o}}^m, {y_{rj_o}}^{u'}) - \sum_{i=1}^{m} ({x_{ij_o}}^l, {x_{ij_o}}^m, {x_{ij_o}}^u; {x_{ij_o}}^{l'}, {x_{ij_o}}^m, {x_{ij_o}}^{u'})$

is a constant.

Similarly, Model 14 is obtained from IFL1VRS model.

Model 14

$$Min\ \tilde{E}^{IV}_{j_o} = \sum_{i=1}^{m}({x_i}^l, {x_i}^m, {x_i}^u; {x_i}^{l'}, {x_i}^m, {x_i}^{u'}) - \sum_{r=1}^{s}({y_r}^l, {y_r}^m, {y_r}^u; {y_r}^{l'}, {y_r}^m, {y_r}^{u'}) + (\beta^l, \beta^m, \beta^u; \beta^{l'}, \beta^m, \beta^{u'})$$

subject to

$$\sum_{j=1, j\neq j_o}^{n}(\lambda_j^l, \lambda_j^m, \lambda_j^u; \lambda_j^{l'}, \lambda_j^m, \lambda_j^{u'}) \otimes ({x_{ij}}^l, {x_{ij}}^m, {x_{ij}}^u; {x_{ij}}^{l'}, {x_{ij}}^m, {x_{ij}}^{u'}) \leq ({x_i}^l, {x_i}^m, {x_i}^u; {x_i}^{l'}, {x_i}^m, {x_i}^{u'})\ \forall i,$$

$$\sum_{j=1, j\neq j_o}^{n}(\lambda_j^l, \lambda_j^m, \lambda_j^u; \lambda_j^{l'}, \lambda_j^m, \lambda_j^{u'}) \otimes ({y_{rj}}^l, {y_{rj}}^m, {y_{rj}}^u; {y_{rj}}^{l'}, {y_{rj}}^m, {y_{rj}}^{u'}) \leq ({y_r}^l, {y_r}^m, {y_r}^u; {y_r}^{l'}, {y_r}^m, {y_r}^{u'})\ \forall r,$$

$$\sum_{j=1, j\neq j_o}^{n}({\lambda_j}^l, {\lambda_j}^m, {\lambda_j}^u; \lambda_j^{l'}, \lambda_j^m, \lambda_j^{u'}) = (1,1,1;1,1,1),$$

$$({x_i}^l, {x_i}^m, {x_i}^u; {x_i}^{l'}, {x_i}^m, {x_i}^{u'}) \geq ({x_{ij_o}}^l, {x_{ij_o}}^m, {x_{ij_o}}^u; {x_{ij_o}}^{l'}, {x_{ij_o}}^m, {x_{ij_o}}^{u'})\ \forall i,$$

$$(0,0,0;0,0,0) \leq ({y_r}^l, {y_r}^m, {y_r}^u) \leq (y_{rj_o}^l, y_{rj_o}^m, y_{rj_o}^u; {y_{rj_o}}^{l'}, {y_{rj_o}}^m, {y_{rj_o}}^{u'})\ \forall r,$$

$$({\lambda_j}^l, {\lambda_j}^m, {\lambda_j}^u; \lambda_j^{l'}, \lambda_j^m, \lambda_j^{u'}) \geq (0,0,0;0,0,0)\ \forall j,\ j \neq j_o$$

where $(\beta^l, \beta^m, \beta^u; \beta^{l'}, \beta^m, \beta^{u'}) = \sum_{r=1}^{s}({y_{rj_o}}^l, {y_{rj_o}}^m, {y_{rj_o}}^u; {y_{rj_o}}^{l'}, {y_{rj_o}}^m, {y_{rj_o}}^{u'}) - \sum_{i=1}^{m}({x_{ij_o}}^l, {x_{ij_o}}^m, {x_{ij_o}}^u; {x_{ij_o}}^{l'}, {x_{ij_o}}^m, {x_{ij_o}}^{u'})$
is a constant number.

By using the arithmetic operations Sub-section 2.8, Model 13 can be transformed into the following model:

Model 15

$$MinE^{IC}_{j_o} = \sum_{i=1}^{m}({x_i}^l, {x_i}^m, {x_i}^u; {x_i}^{l'}, {x_i}^m, {x_i}^{u'}) - \sum_{r=1}^{s}({y_{rj}}^l, {y_{rj}}^m, {y_{rj}}^u; {y_{rj}}^{l'}, {y_{rj}}^m, {y_{rj}}^{u'}) + (\beta^l, \beta^m, \beta^u; \beta^{l'}, \beta^m, \beta^{u'})$$

subject to

$$\sum_{j=1, j\neq j_o}^{n}(\lambda_j^l {x_{ij}}^l, \lambda_j^m {x_{ij}}^m, \lambda_j^u {x_{ij}}^u; \lambda_j^{l'} {x_{ij}}^{l'}, \lambda_j^m {x_{ij}}^m, \lambda_j^{u'} {x_{ij}}^{u'}) \leq ({x_i}^l, {x_i}^m, {x_i}^u; {x_i}^{l'}, {x_i}^m, {x_i}^{u'})\ \forall i,$$

$$\sum_{j=1, j\neq j_o}^{n}(\lambda_j^l {y_{rj}}^l, \lambda_j^m {y_{rj}}^m, \lambda_j^u {y_{rj}}^u; \lambda_j^{l'} {y_{rj}}^{l'}, \lambda_j^m {y_{rj}}^m, \lambda_j^{u'} {y_{rj}}^{u'}) \leq ({y_r}^l, {y_r}^m, {y_r}^u; {y_r}^{l'}, {y_r}^m, {y_r}^{u'})\ \forall r,$$

$$({x_i}^l, {x_i}^m, {x_i}^u; {x_i}^{l'}, {x_i}^m, {x_i}^{u'}) \geq ({x_{ij_o}}^l, {x_{ij_o}}^m, {x_{ij_o}}^u; {x_{ij_o}}^{l'}, {x_{ij_o}}^m, {x_{ij_o}}^{u'}),\ i = 1,2,3,...,m,$$

$$(0,0,0;0,0,0) \leq ({y_r}^l, {y_r}^m, {y_r}^u; {y_r}^{l'}, {y_r}^m, {y_r}^{u'}) \leq (y_{rj_o}^l, y_{rj_o}^m, y_{rj_o}^u; y_{rj_o}^{l'}, y_{rj_o}^m, y_{rj_o}^{u'})\ \forall r,$$

$$({\lambda_j}^l, {\lambda_j}^m, {\lambda_j}^u; {\lambda_j}^{l'}, {\lambda_j}^m, {\lambda_j}^{u'}) \geq (0,0,0;0,0,0)\ \forall j,\ j \neq j_o$$

where $(\beta^l, \beta^m, \beta^u; \beta^{l'}, \beta^m, \beta^{u'}) = \sum_{r=1}^{s}({y_{rj_o}}^l, {y_{rj_o}}^m, {y_{rj_o}}^u; {y_{rj_o}}^{l'}, {y_{rj_o}}^m, {y_{rj_o}}^{u'}) - \sum_{i=1}^{m}({x_{ij_o}}^l, {x_{ij_o}}^m, {x_{ij_o}}^u; {x_{ij_o}}^{l'}, {x_{ij_o}}^m, {x_{ij_o}}^{u'})$.

Model 15 is an IF linear programming problem (LPP) (IFLPP), we obtain crisp LPP model (Model 16) from Model 15 by using expected value method.

Model 16

$$MinEV(E_{j_o}^C) = EV(\sum_{i=1}^{m}(x_i{}^l, x_i{}^m, x_i{}^u; x_i{}^{l'}, x_i{}^m, x_i{}^{u'}) - \sum_{r=1}^{s}(y_{rj}{}^l, y_{rj}{}^m, y_{rj}{}^u; y_{rj}{}^{l'}, y_{rj}{}^m, y_{rj}{}^{u'}) + (\beta^l, \beta^m, \beta^u; \beta^{l'}, \beta^m, \beta^{u'}))$$

subject to

$$EV(\sum_{j=1, j\neq j_o}^{n}(\lambda_j^l x_{ij}{}^l, \lambda_j^m x_{ij}{}^m, \lambda_j^u x_{ij}{}^u; \lambda_j^{l'} x_{ij}{}^{l'}, \lambda_j^m x_{ij}{}^m, \lambda_j^{u'} x_{ij}{}^{u'})) \leq EV((x_i{}^l, x_i{}^m, x_i{}^u; x_i{}^{l'}, x_i{}^m, x_i{}^{u'}))\ \forall i,$$

$$EV(\sum_{j=1, j\neq j_o}^{n}(\lambda_j^l y_{rj}{}^l, \lambda_j^m y_{rj}{}^m, \lambda_j^u y_{rj}{}^u; \lambda_j^{l'} y_{rj}{}^{l'}, \lambda_j^m y_{rj}{}^m, \lambda_j^{u'} y_{rj}{}^{u'})) \leq EV((y_r{}^l, y_r{}^m, y_r{}^u; y_r{}^{l'}, y_r{}^m, y_r{}^{u'}))\ \forall r,$$

$$EV((x_i{}^l, x_i{}^m, x_i{}^u; x_i{}^{l'}, x_i{}^m, x_i{}^{u'})) \geq EV((x_{ij_o}{}^l, x_{ij_o}{}^m, x_{ij_o}{}^u; x_{ij_o}{}^{l'}, x_{ij_o}{}^m, x_{ij_o}{}^{u'}))\ \forall i,$$

$$EV((0,0,;0,0,0)) \leq EV((y_r{}^l, y_r{}^m, y_r{}^u; y_r{}^{l'}, y_r{}^m, y_r{}^{u'})) \leq EV((y_{rj_o}^l, y_{rj_o}^m, y_{rj_o}^u; y_{rj_o}^{l'}, y_{rj_o}^m, y_{rj_o}^{u'}))\ \forall r,$$

$$EV((\lambda_j{}^l, \lambda_j{}^m, \lambda_j{}^u; \lambda_j{}^{l'}, \lambda_j{}^m, \lambda_j{}^{u'})) \geq EV((0,0,0;0,0,0))\ \forall j,\ j \neq j_o$$

where $EV((\beta^l, \beta^m, \beta^u; \beta^{l'}, \beta^m, \beta^{u'})) = EV(\sum_{r=1}^{s}(y_{rj_o}{}^l, y_{rj_o}{}^m, y_{rj_o}{}^u; y_{rj_o}{}^{l'}, y_{rj_o}{}^m, y_{rj_o}{}^{u'})) - EV(\sum_{i=1}^{m}(x_{ij_o}{}^l, x_{ij_o}{}^m, x_{ij_o}{}^u; x_{ij_o}{}^{l'}, x_{ij_o}{}^m, x_{ij_o}{}^{u'}))$.

By using the expected values of TIFNs, Model 16 reduces to the following crisp LPP:

Model 17

$$MinE_{j_o}^{PC} = \frac{1}{8}(\sum_{i=1}^{m}(x_i{}^{l'} + x_i{}^l + 4x_i{}^m + x_i{}^u + x_i{}^{u'}) - \sum_{r=1}^{s}(y_{rj}{}^{l'} + y_{rj}{}^l + 4y_{rj}{}^m + y_{rj}{}^u + y_{rj}{}^{u'}) + (\beta^{l'} + \beta^l + 2\beta^m + \beta^u + \beta^{u'}))$$

subject to

$$\sum_{j=1, j\neq j_o}^{n}(\lambda_j^{l'} x_{ij}{}^{l'} + \lambda_j^l x_{ij}{}^l + 4\lambda_j^m x_{ij}{}^m + \lambda_j^u x_{ij}{}^u + \lambda_j^{u'} x_{ij}{}^{u'}) \leq (x_i{}^{l'} + x_i{}^l + 4x_i{}^m + x_i{}^u + x_i{}^{u'})\ \forall i,$$

$$\sum_{j=1, j\neq j_o}^{n}(\lambda_j^{l'} y_{rj}{}^{l'} + \lambda_j^l y_{rj}{}^l + 4\lambda_j^m y_{rj}{}^m + \lambda_j^u y_{rj}{}^u + \lambda_j^{u'} y_{rj}{}^{u'}) \leq (y_r{}^{l'} + y_r{}^l + 4y_r{}^m + y_r{}^u + y_r{}^{u'})\ \forall r,$$

$$(x_i{}^{l'} + x_i{}^l + 4x_i{}^m + x_i{}^u + x_i{}^{u'}) \geq (x_{ij_o}{}^{l'} + x_{ij_o}{}^l + 4x_{ij_o}{}^m + x_{ij_o}{}^u + x_{ij_o}{}^{u'})\ \forall i,$$

$$0 \leq (y_r{}^{l'} + y_r{}^l + 4y_r{}^m + y_r{}^u + y_r{}^{u'}) \leq (y_{rj_o}^{l'} + y_{rj_o}^l + 4y_{rj_o}^m + y_{rj_o}^u + y_{rj_o}^{u'})\ \forall r,$$

$$(\lambda_j{}^{l'} + \lambda_j{}^l + 4\lambda_j{}^m + \lambda_j{}^u + \lambda_j{}^{u'}) \geq 0\ \forall j,\ j \neq j_o$$

where $(\beta^{l'} + \beta^l + 4\beta^m + \beta^u + \beta^{u'}) = \sum_{r=1}^{s}(y_{rj_o}{}^{l'} + y_{rj_o}{}^l + 4y_{rj_o}{}^m + y_{rj_o}{}^u + y_{rj_o}{}^{u'}) - \sum_{i=1}^{m}(x_{ij_o}{}^{l'} + 4x_{ij_o}{}^m + x_{ij_o}{}^u + x_{ij_o}{}^{u'})$.

Model 17 is the proposed IFL1CRS (PIFL1CRS) model and $E_{j_o}^{PC}$ is known as PIFL1CRS efficiency. Similarly, IFL1VRS model (Model 12) is transformed to the crisp LPP (Model 18) on the basis of expected values:

Model 18

$$MinE_{j_o}^{PV} = \frac{1}{8}(\sum_{i=1}^{m}(x_i{}^{l'} + x_i{}^l + 4x_i{}^m + x_i{}^u + x_i{}^{u'}) - \sum_{r=1}^{s}(y_{rj}{}^{l'} + y_{rj}{}^l + 4y_{rj}{}^m + y_{rj}{}^u + y_{rj}{}^{u'}) + (\beta^{l'} + \beta^l + 4\beta^m + \beta^u + \beta^{u'}))$$

subject to

$$\sum_{j=1, j\neq j_o}^{n}(\lambda_j^{l'} x_{ij}{}^{l'} + \lambda_j^l x_{ij}{}^l + 4\lambda_j^m x_{ij}{}^m + \lambda_j^u x_{ij}{}^u + \lambda_j^{u'} x_{ij}{}^{u'}) \leq (x_i{}^{l'} + x_i{}^l + 4x_i{}^m + x_i{}^u + x_i{}^{u'})\ \forall i,$$

$$\sum_{j=1, j\neq j_o}^{n}(\lambda_j^{l'} y_{rj}{}^{l'} + \lambda_j^l y_{rj}{}^l + 4\lambda_j^m y_{rj}{}^m + \lambda_j^u y_{rj}{}^u + \lambda_j^{u'} y_{rj}{}^{u'}) \leq (y_r{}^{l'} + y_r{}^l + 4y_r{}^m + y_r{}^u + y_r{}^{u'})\ \forall r,$$

$$\sum_{j=1, j\neq j_o}^{n}(\lambda_j{}^{l'} + \lambda_j{}^l + 4\lambda_j{}^m + \lambda_j{}^u + \lambda_j{}^{u'}) = 8,$$

$$(x_i{}^{l'} + x_i{}^l + 4x_i{}^m + x_i{}^u + x_i{}^{u'}) \geq (x_{ij_o}{}^{l'} + x_{ij_o}{}^l + 4x_{ij_o}{}^m + x_{ij_o}{}^u + x_{ij_o}{}^{u'})\ \forall i,$$

$$0 \leq (y_r{}^{l'} + y_r{}^l + 4y_r{}^m + y_r{}^u + y_r{}^{u'}) \leq (y_{rj_o}^{l'} + y_{rj_o}^l + 4y_{rj_o}^m + y_{rj_o}^u + y_{rj_o}^{u'})\ \forall r,$$

$$({\lambda_j}^{l'} + {\lambda_j}^{l} + 4{\lambda_j}^{m} + {\lambda_j}^{u} + {\lambda_j}^{u'}) \geq 0 \ \forall j,\ j \neq j_o$$

where $(\beta^{l'} + \beta^{l} + 4\beta^{m} + \beta^{u} + \beta^{u'}) = \sum_{r=1}^{s} ({y_{rj_o}}^{l'} + {y_{rj_o}}^{l} + 4{y_{rj_o}}^{m} + {y_{rj_o}}^{u} + {y_{rj_o}}^{u'}) - \sum_{i=1}^{m} ({x_{ij_o}}^{l} + {x_{ij_o}}^{l'} + 4{x_{ij_o}}^{m} + {x_{ij_o}}^{u} + {x_{ij_o}}^{u'})$.

Model 18 is the proposed IFL1VRS (PIFL1VRS) model and $E_{j_o}^{PV}$ is known as PIFL1VRS efficiency.

4.4 Ranking approach

- **Step 1:** Find the IFCCR and IFBCC efficiencies (Model 5). If efficiencies lie in [0,1], then DMUs are ranked according to the decreasing values of efficiencies. If not, then go to Step 2.
- **Step 2:** Find the IFSECCR and IFSEBCC efficiencies (Model 6). If feasible (finite) efficiencies are obtained, then DMUs are ranked according to their decreasing values of efficiencies. If infeasible efficiency/efficiencies are obtained, then go to Step 3.
- **Step 3:** Find the PIFL1CRS and PIFL1VRS efficiencies Models 17 and 18. DMUs are ranked according to the decreasing values of PIFL1CRS and PIFL1VRS efficiencies.

5 Numerical examples

In this section, to ensure the validity of the proposed models, we consider two different examples and one real life application. The efficiencies obtained by the proposed models will be termed as proposed efficiencies (PEs). In the first example, input-output data are taken as crisp. In the second example, input-output data are taken as fuzzy and crisp. In the third example (Real life application), input-output data are taken as IF. Using these examples, we can conclude that the proposed models are valid for crisp data and fuzzy data as well as for IF data.

Example 5.1. *Let there be 12 DMUs having two inputs and two outputs is listed in Table 1. Since any real number (b) can be written in the form of TIFN as $\tilde{b}^I = (b, b, b; b, b, b)$, the proposed models are also applicable for crisp data.*

Table 1: Input-output data for 12 DMUs

DMUs		1	2	3	4	5	6	7	8	9	10	11	12
inputs	x_1	4	14	24	20	48	35	3	17	23	20	45	48
	x_2	3	6	3	2	4	7.5	3	7	3	2.5	4.5	8
outputs	y_1	1	2	3	2	4	5	1	1.8	2.9	1.9	3.9	4.8
	y_2	2	6	12	6	16	30	1	7	13	5	17	31

In Table 2, according to super efficiency method in CCR model, DMUs 4, 5, 6 and 7 has infeasible solution and BCC model, DMU6 has infeasible solution. So, we can not rank these DMUs. Thus, for such type of efficiencies we can find feasible solution by proposed method (Models 17 and 18). The PIFL1VRS efficiency, i.e., $E_{j_o}^{PV}$ of each DMU is evaluated using Model 18 and listed in Table 2. The ranks of the DMUs according to the decreasing values of efficiencies are presented in Table 2. In CCR, DMUs are ranked in the order of $DMU4 > DMU12 > DMU7 > DMU5 > DMU6 > DMU3 > DMU9 > DMU10 > DMU2 > DMU8$ whereas in BCC, they are ranked in the order of $DMU12 > DMU6 > DMU7 > DMU4 > DMU5 > DMU9 > DMU1 > DMU3 > DMU11 > DMU10 > DMU2 > DMU8$.

Comparison of efficiencies: In Table 2, the PEs of DMUs are found to be finite (feasible) by PIFL1CRS and PIFL1VRS. In Table 2, DMU 4 has infeasible super efficiency in the crisp CCR DEA but has feasible PE with score 31.1 using PIFL1CRS. In Table 2, DMU 6 has infeasible super efficiency in the crisp BCC DEA but has feasible PE with score 7.5 using PIFL1VRS. Therefore, PIFL1CRS and PIFL1VRS is more realistic rather than the crisp DEA. The crisp DEA does not deal with the infeasibility while PIFL1VRS deals with the infeasibility. Therefore, PIFL1VRS is more efficient rather than the crisp DEA.

Example 5.2. *Consider the performance assessment problem of 19 DMUs in terms of one fuzzy and one crisp inputs, and three fuzzy and one crisp outputs. The fuzzy input-output data is presented in Table 3. Since any triangular fuzzy number $\tilde{b}^I = (b1, b2, b3)$ can also be written in the form of a TIFN $\tilde{b}^I = (b1, b2, b3; b1, b2, b3)$, the proposed methodology is also applicable for fuzzy data. The only difference is that by using the proposed methodology, we get IF weights instead of fuzzy weights.*

Table 2: Efficiency, Super-Efficiency & Proposed Efficiency results of 12 DMUs

	CCR form				BCC form			
DMUs	Es	SEs	PEs	Ranks	Es	SEs	PEs	Ranks
1	.99	-	-	9	1	1.045	-	7
2	.728	-	-	11	.7857	-	-	11
3	1	1.056	-	6	1	1.0386	-	8
4	1	Infeasible	31.1	1	1	1.2813	-	4
5	1	Infeasible	7.488	4	1	1.19	-	5
6	1	Infeasible	3.99	5	1	Infeasible	7.5	2
7	1	Infeasible	15.17	3	1	1.33	-	3
8	.5675	-	-	12	.5675	-	-	12
9	1	1.1	-	7	1	1.08	-	6
10	.991	-	-	8	.9914	-	-	9
11	.928	-	-	10	.9337	-	-	10
12	1	Infeasible	20.1	2	1	Infeasible	20.2	1

Es: Efficiencies, SEs: Super efficiencies and PEs: Proposed efficiencies

Table 3: Input-output data for 19 DMUs

DMUs	Inputs		Outputs			
	x_1	x_2	y_1	y_2	y_3	y_4
1	(13.17, 13.9, 14.2)	5	42	(43, 45.3, 47.6)	(13.5, 14.2, 14.9)	(28.6, 30.1, 31.6)
2	(15.64, 16.46, 17.28)	4.5	39	(38.1, 40.1, 42.1)	(12.4, 13, 13.7)	(28.3, 29.8, 31.3)
3	(9.17, 9.76, 10.35)	6	26	(38.6, 39.6, 40.6)	(13.1, 13.8, 14.5)	(23.3, 24.5, 25.7)
4	(9.99, 10.52, 11.05)	4	22	(34.2, 36, 37.8)	(10.7, 11.3, 11.9)	(16.8, 25, 26.3)
5	(9.03, 9.5, 9.98)	3.8	21	(32.5, 34.2, 35.9)	(11.4, 12, 12.6)	(19.4, 20.4, 21.4)
6	(8.55, 8.79, 9.03)	5.4	10	(18.1, 18.9, 19.3)	(4.8, 5, 5.3)	(15.7, 16.5, 17.3)
7	(5.9, 6.21, 6.52)	6.2	14	(25.2, 26.5, 27.8)	(6.7, 7, 7.4)	(18.7, 19.7, 20.7)
8	(11.8, 12.42, 12.68)	6	25	(34.1, 35.9, 37.7)	(8.6, 9, 9.5)	(23.5, 24.7, 25.9)
9	(3.49, 3.67, 3.85)	8	4	(16.5, 17.4, 18.3)	(0.1, 0.1, 0.1)	(17.2, 18.1, 19)
10	(8.48, 8.93, 9.38)	7	16	(32.6, 34.3, 36)	(6.2, 6.5, 6.8)	(19.6, 20.6, 21.6)
11	(16.85, 17.74, 18.63)	7.1	43	(43.3, 45.6, 47.9)	(13.3, 14, 14.7)	(29.5, 31.1, 32.7)
12	(14.11, 14.85, 15.59)	6.2	27	(36.8, 38.7, 40.6)	(13.1, 13.8, 14.5)	(24.1, 25.4, 26.7)
13	(8, 8.4, 8.7)	5.2	40	(42, 42.5 42.9)	(14,14.5, 14.9)	(28,28.5,28.9)
14	(12, 12.5, 12.9)	6	37	(36.8, 38.7, 40.6)	(13,13.2,13.8)	(29,29.3,29.9)
15	(15,15.8,16.2)	3.9	22	(12.9,13.4,14.2)	(13.1, 13.8, 14.5)	(23,23.7,24.3)
16	(9,9.3,9.8)	4.3	25	(36.8, 38.7, 40.6)	(10,10.4,10.9)	(24,24.6, 25.1)
17	(13.3,13.9,14.5)	5.1	19	(36.8, 38.7, 40.6)	(10,10.3,10.9)	(17.8,18.5,19)
18	(16,16.6,17.1)	3.5	15	(36.8, 38.7, 40.6)	(4.9,5.4,5.8)	(16,16.8,17.2)
19	(12.05,12.3,12.9)	6	13	(36.8, 38.7, 40.6)	(6.6,6.9,7.4)	(19, 19.2 19.9)

In Table 4, according to super-efficiency method in BCC model DMUs 1, 2, 5, 6, 10, 11, 13, 14, 15 and 17 have infeasible solution and in CCR model DMU11 has infeasible solution. So, we can not rank these DMUs. Thus, this type of efficiencies we can find feasible solution by the proposed Model 18. The efficiency $E^{PV}_{j_o}$ *of each DMUs are evaluated using Model 6, and are listed in Table 4.*

Comparison of efficiencies:

In Table 4, the PEs of DMUs are found to be finite (feasible) by PIFL1CRS and PIFL1VRS. In Table 4, DMUs 1, 2, 5, 6, 10, 11, 13, 14, 15 and 17 have infeasible super efficiencies in the crisp/fuzzy BCC and DMU 11 has infeasible super efficiencies in the crisp/fuzzy CCR DEA but have feasible PEs using PIFL1VRS and PIFL1CRS respectively. Therefore, PIFL1CRS and PIFL1VRS are more realistic rather than the crisp/fuzzy DEA. The crisp/fuzzy DEA does not deal with the infeasibility while PIFL1CRS and PIFL1VRS deals with the infeasibility. Therefore, PIFL1CRS and

Table 4: Efficiency, Super-Efficiency & Proposed Efficiency results of 19 DMUs

DMUs	BCC form				CCR form			
	Es	SEs	PE	Ranks	Es	SEs	PEs	Ranks
1	1	Infeasible	1.12e+2	1	1	1.275	-	2
2	1	Infeasible	1.01e+2	5	1	1.068		5
3	0.9659	-	-	18	0.875	-	-	13
4	1	1.16	-	12	0.9794	-	-	9
5	1	Infeasible	74.33	6	0.9987	-	-	7
6	1	Infeasible	36.21	10	.8385	-	-	14
7	0.9762	-	-	15	0.9424	-	-	10
8	0.9753	-	-	16	0.7229	-	-	16
9	1	1.73	-	11	.9909	-	-	6
10	1	Infeasible	60.58	9	.7476	-	-	15
11	1	Infeasible	1.08e+2	4	1	Infeasible	1.08e+2	1
12	0.8561	-	-	19	0.6746	-	-	19
13	1	Infeasible	1.109e+2	2	0.9903	-	-	8
14	1	Infeasible	1.02e+2	3	.9393	-	-	11
15	1	Infeasible	73.91	7	.7115	-	-	17
16	1	1.084	-	13	1	1.0719	-	4
17	1	Infeasible	64.625	8	0.8765	-	-	12
18	0.972	-	-	17	0.6993	-	-	18
19	1	1.0545	-	14	1	1.0857	-	3

Es: Efficincies, SEs: Super efficiencies and PEs: Proposed efficiencies

PIFL1VRS are more efficient rather than the crisp/fuzzy DEA.

Example 5.3. ***(Real life application)***

In health sector, the managers/other authorities reorganize, time to time, medical staff, non-medical staff and beds. Therefore, the uncertainty comes in medical staff, non-medical staff and beds, i.e., input data is uncertain. Some patients leave the hospital without treatment due to insufficient beds for hospitalization and impossibility to provide appropriate care etc. Therefore, output data is uncertain. Thus, the input- output data are taken as IF numbers (IFNs) to deal with uncertainty in health sector. Let us consider a real life application to illustrate the proposed model. Let there be two IF inputs: (i) number of doctors = $\tilde{x}^I_{1j}$*, (ii) number of pharmacists =* $\tilde{x}^I_{2j}$*, and two fuzzy outputs: (i) number of inpatients =* $\tilde{y}^I_{1j}$*, (ii) number of outpatients =* $\tilde{y}^I_{2j}$ *of community health centers (CHCs) in Meerut district of Uttar Pradesh, India and IF input-output of CHCs are shown in Table 5. In Table 6, according to super-efficiency method*

Table 5: IF input and IF output data

DMU	IF input		IF output	
	$\tilde{x}^I_{1j}$	$\tilde{x}^I_{2j}$	$\tilde{y}^I_{1j}$	$\tilde{y}^I_{2j}$
H1	(12,13,16;9,12,17)	(3,5,8;2,5,10)	(3640,3650,3665;3635,3650,3695)	(134130,134137,134140;134125,134137,134145)
H2	(12,12,12;12,12,12)	(3,5,7;2,5,9)	(4150,4160,4175;4148,4160,4195)	(116060,116062,116070;116055,116062,116075)
H3	(11,12,15;9,12,18)	(2,4,5;1,4,6)	(4360,4370,4380;4357,4370,4398)	(94060,94066,94070;94055,94066,94075)
H4	(5,8,11;2,8,14)	(1,1,3;1,1,5)	(485,492,500;483,492,515)	(24325,24329,24334;24322,24329,24338)
H5	(7,10,13;4,10,16)	(3,4,6;2,4,10)	(2460,2464,2470;2458,2464,2475)	(99745,99748,99750;99742,99748,99755)
H6	(8,11,14;5,11,17)	(2,3,4;1,3,5)	(1360,1368,1378;1358,1368,1398)	(49398,49401,49405;49395,49401,49409)
H7	(7,10,11;5,10,15)	(1,2,6;1,2,18)	(1055,1062,1080;1050,1062,1083)	(37769,37772,37776;37765,37772,37779)
H8	(7,11,14;5,11,17)	(2,4,7;1,4,19)	(1295,1302,1310;1290,1302,1325)	(82838,82841,82845;82835,82841,82849)
H9	(9,12,15;6,12,18)	(2,5,7;1,5,15)	(1660,1671,1690;1657,1671,16105)	(100590,100596,100600;100586,100596,100605)
H10	(13,16,19;10,16,22)	(2,4,6;1,4,18)	(1010,1018,1035;1008,1018,1045)	(64349,64351,64358;64345,64351,64360)
H11	(8,11,14;5,11,17)	(3,5,8;2,5,20)	(1500,1504,1515;1495,1504,1535)	(80050,80056,80060;80045,80056,80065)
H12	(5,5,11;2,8,15)	(3,4,6;1,4,7)	(1960,1965,1972;1958,1965,1985)	(58160,58167,58170;58157,58167,58174)

in CCR model hospital 2 has infeasible solution. So, we can not rank this hospital. Thus, this type of efficiencies we can find feasible solution by the proposed Model 17. The efficiency $E^{PC}_{j_o}$ *of each hospital is evaluated using Model 5, and are listed in Table 6. In BCC, hospitals are ranked in the order of Hospital* $2 >$ *Hospital* $1 >$ *Hospital* $9 >$ *Hospital* $8 >$ *Hospital* $3 >$ *Hospital* $11 >$ *Hospital* $5 >$ *Hospital* $12 >$ *Hospital* $7 >$ *Hospital* $4 >$

Hospital 6 > *Hospital* 10*, whereas in CCR, they are ranked in the order of* *Hospital* 2 > *Hospital* 1 > *Hospital* 3 > *Hospital* 5 > *Hospital* 7 > *Hospital* 4 > *Hospital* 9 > *Hospital* 8 > *Hospital* 12 > *Hospital* 11 > *Hospital* 6 > *Hospital* 10.

Table 6: Efficiency, Super-Efficiency & Proposed Efficiency results of 12 hospitals

	BCC form			CCR form			
Hospitals	Es	SEs	Ranks	Es	SEs	PEs	Ranks
1	0.8833	-	2	0.7794	-	-	2
2	1	1.09	1	1	Infeasible	1.1	1
3	0.7583	-	5	0.6806	-	-	3
4	0.5741	-	10	0.5714	-	-	6
5	0.649	-	7	0.6489	-	-	4
6	0.5322	-	11	0.5236	-	-	11
7	0.59	-	9	0.575	-	-	5
8	0.7713	-	4	0.5434	-	-	8
9	0.7992	-	3	0.56	-	-	7
10	0.4213	-	12	0.382	-	-	12
11	0.7503	-	6	0.536	-	-	10
12	0.6205	-	8	0.5417	-	-	9

Es: Efficiencies, SEs: Super efficiencies and PEs: Proposed efficiencies

Comparison of efficiencies: In Table 6, the PEs of DMUs are found to be finite (feasible) by PIFL1CRS. In Table 6, Hospital 2 has infeasible super efficiency in the crisp/fuzzy CCR DEA but have feasible PEs with score 1.1 using PIFL1CRS. Therefore, PIFL1CRS is more realistic rather than the crisp/fuzzy DEA. The crisp/fuzzy DEA does not deal with the infeasibility while PIFL1CRS deals with the infeasibility. Therefore, PIFL1VRS is more efficient rather than the crisp/fuzzy DEA.

6 Conclusion

In fact, crisp input and output data are not always available in real world applications, and some inputs and outputs may possess IF essence instead of precise and fuzziness. In this paper, crisp L1CRS and crisp L1VRS models are extended to IF L1CRS and IF L1VRS models respectively with TIFN input-output data in IF environment. CRS and VRS models are extended to proposed IF CRS (PIFL1CRS) and proposed IF VRS (PIFL1VRS) models respectively to determine the efficiencies of the DMUs in IFEs. In addition, the crisp LPP forms PIFL1CRS (Model 17) and PIFL1VRS (Model 18) of the IFL1CRS and IFL1VRS models are presented using the expected value method for ease of solution implementation and ranking all the DMUs according to their efficiencies. The PIFL1CRS and PIFL1VRS models have been tested with two illustrative examples and one real life application. The proposed approaches are effective and easy to implement in real life problems.
The uncertainty in this paper is limited to TIFNs. The suggestion for future research is to apply the propose models for real life applications which can take care subjective and linguistic input-output data.

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