Iranian Journal of Fuzzy Systems

Volume 17, Number 3, (2020), pp. 1-18

University of Sistan and Baluchestan

A new quadratic deviation of fuzzy random variable and its application to portfolio optimization

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Abstract

The aim of this paper is to propose a convex risk measure in the framework of fuzzy random theory and verify its advantage over the conventional variance approach. For this purpose, this paper defines the quadratic deviation (QD) of fuzzy random variable as the mathematical expectation of QDs of fuzzy variables. As a result, the new risk criterion essentially describes the variation of a fuzzy random variable around its expected value. For triangular and trapezoidal fuzzy random variables as well as their linear combinations, we establish the analytical expressions of their QDs, and obtain the desirable convexity about the analytical expressions with respect to critical parameters. To explore the practical value of the proposed QD, we apply it to a portfolio selection problem to quantify the investment risk, and develop three mean-QD models to find the optimal allocation of the fund in different risky securities. Due to the convexity of our QD, the original three mean-QD models can be turned into their equivalent convex parametric quadratic programming problems, which can be solved by conventional optimization methods. The computational results clearly demonstrate that our new QD significantly reduces the computational complexity that cannot be avoided when variance is used as a risk criterion. Finally, the numerical comparison between the proposed mean-QD model and mean-variance model is conducted to show the consistency between the optimal results in both techniques. Meanwhile, the comparison between the proposed QD, variance, spread, and second moment is made to summarize the similarities and differences between them, distinguish these four risk criteria and determine their respective application scopes in decision systems.

Keywords: Risk criterion, hybrid uncertainty, mean-QD model, convexity, computational complexity, portfolio optimization.

1 Introduction

There is no doubt that uncertainty is always present and involved in various kinds of information that spread all over the decision making process. The potential risk caused by the uncertainty is inevitable in each decision making system, and necessarily influences the possibility of achieving optimality. Meanwhile, by the principle of risk-return tradeoff, potential return rises with an increase in risk. That is, the low levels of the investment risk are associated with the low potential returns and vice versa. Thus, the potential risk should not be ignored when decisions are made to achieve optimality in practical optimization problems, such as service system design problems with interruption risk.

Under such situations with risk, identifying a valid risk criterion becomes more important for each optimization problem. How to identify the suitable risk criterion depends on the uncertainty that gives rise to potential risk. As we know, uncertainty mainly involves randomness caused by objective factors such as inflation and economic trend, or fuzziness caused by subjective factors such as the estimations based on the knowledge of experts and human opinions. In stochastic decision systems, a series of typical risk criteria have been studied to deal with potential risk deriving from randomness, such as variance [11, 32], VaR and CVaR [4, 8], and absolute deviation [25, 33]. Based on fuzzy

Corresponding Author: X. Wu
Received: December 2018; Revised: July 2019; Accepted: September 2019.

theory [5, 16, 34], risk criteria effectively measuring potential risk stemming from fuzziness have also been documented in the literature, including variance [9, 36], VaR and CVaR [14, 22], absolute deviation [3, 24, 35], spread and moment [1, 29, 30, 31] and so on. However, it is usually difficult to distinguish between randomness and fuzziness in practical optimization problems, as these two types of uncertainty might arise at the same time and gradually influence each other. In the literature, this phenomena is usually called hybrid or mixed uncertainty [10, 17, 23].

Hybrid uncertainty in practice is common, because it can reflect the impacts of both objective and subjective factors on the development process and lead to practical optimization problems. For example, in the spacecraft structural system [27], the heat flux density is time-varying and associated with expert opinions, so it can be characterized by possibility distribution, while the heat conductivity and volumetric heat capacity are system inherent factors that can be described by probability distributions. Therefore, fuzziness and randomness influence the optimality of decisions simultaneously, and it is significative to make decisions under hybrid uncertainty. It is imperative to find appropriate risk criteria under fuzzy random environment. Just as stated in the existing literature, variance [6, 12] and VaR [26, 28] are also valid to quantify the potential risk under hybrid uncertainty.

However, taking variance as a risk criterion, the classical mean-variance technique is challenged in computational complexity under hybrid uncertainty. For instance, Hao and Liu [7] solved the developed mean-variance models by genetic algorithm (GA); Li et al. [13] designed a hybrid intelligent algorithm by integrating simulated annealing algorithm, neural network and fuzzy simulation techniques to solve the nonlinear portfolio optimization models, such as mean-variance model and chance-constrained model. However, intelligent algorithms usually provide approximate optimal solutions, and it is difficult to identify the errors between approximate solutions and optimal solutions. In addition, approximate optimal solutions to some extent prevent the realization of optimality in practical optimization problems. Therefore, it is important and attractive to find an appropriate risk criterion to measure hybrid uncertainty. Such a risk criterion is expected to essentially measure the potential risk and significantly reduce the computational complexity that cannot be avoided when variance is used as a risk criterion.

Motivated by this idea, this paper proposes the QD of fuzzy random variable as a novel risk criterion under hybrid uncertainty. The QD of a fuzzy random variable is defined as the mathematical expectation of QDs of fuzzy variables, so it essentially describes the variation of a fuzzy random variable around its expected value. For trapezoidal and triangular fuzzy random variables as well as their linear combinations, the analytical expressions of the QDs are established, and the convexity of the QD functions with respect to parameters is obtained. The obtained results show that the proposed QD has some advantages over variance in the quadratic forms and convexity, which motivate us to verify the feasibility and validity of the QD as a risk criterion in fuzzy random environment from the perspective of practical application.

There are two reasons why it is portfolio optimization problems in which the QD of fuzzy random variable is employed as a new risk criterion. The first reason is that portfolio selection problem is a traditional optimization problem in the fund industry with hybrid uncertainty, in which fuzziness reflects the impacts of specific risk such as the operational conditions of the companies that ensure these securities on potential investment return, while randomness indicates the influences of systematic risk such as market fluctuation and inflation on potential investment return. The second reason is to demonstrate the advantage of the proposed mean-QD technique by comparing it with the mean-variance technique [7]. According to the fundamental work in modern portfolio theory [20], mean-variance technique has been the most common mathematical framework for analyzing how to optimally allocate limited fund for different risky securities. To illustrate the validity and computational advantages, a comparison between the proposed mean-QD technique and mean-variance technique [7] is carried out in Section 5.

Furthermore, this paper takes the QD as a risk criterion to develop three mean-QD optimization models for portfolio selection problems, in which the investment return is quantified by the expected value and the investment risk is characterized by the QD of fuzzy random portfolio return. Based on the analytical expressions, the mean-QD optimization models are converted to their equivalent convex parametric quadratic programming problems, which can be solved by conventional optimization methods [21]. From the perspective of computational complexity, the solution results demonstrate the efficiency of the QD as a risk criterion, and the solution procedure shows the advantage of the QD. To demonstrate the significance made by this work, two comparative studies are conducted. The first one is the numerical comparison between the proposed mean-QD technique and mean-variance technique [7], and the second one is the comparison between the proposed QD and the existing risk criteria, including variance [7], spread [19, 29, 30], and second moment [31], to show their similarities and differences. These comparison studies provide a valuable guidance for decision makers on how to identify the most suitable risk criterion for practical optimization problems.

The major contributions of this study are: 1) A feasible risk criterion, called the QD, is proposed to measure hybrid uncertainty. As a new risk criterion, the analytical expressions of QDs have favorable convexity properties for triangular and trapezoidal fuzzy random variables as well as their linear combinations. 2) Three mean-QD optimization models are developed for portfolio selection problems. Under mild assumptions on uncertain returns, the proposed mean-QD models can be turned into their convex parametric quadratic programming problems, which can be solved by conventional solution methods to find the global optimal portfolio. 3) We conduct the comparison between mean-QD technique and mean-variance technique, and the comparison between the proposed QD, variance, spread, and second moment. The comparative results contribute to identifying the best risk criterion and technique for portfolio optimization problems in the presence of uncertainty, especially hybrid uncertainty.

The rest of this paper is organized as follows. Section 2 introduces the new concept QD of fuzzy random variable and discusses its convexity for common fuzzy random variables. Section 3 applies the proposed QD as a risk measure in portfolio selection problems and develops three new mean-QD optimization models. In Section 4, two numerical examples are presented to illustrate the practicability and the efficiency of the proposed optimization models and the solution method. Section 5 focuses on the comparative studies between the proposed mean-QD optimization method with the existing optimization methods. Finally, Section 6 gives the conclusions.

2 The QD of fuzzy random variable

This section introduces the concept of quadratic deviation (QD) for a fuzzy random variable, and then derives the analytical expressions and matrix forms of the QD for trapezoidal and triangular fuzzy random variables as well as their linear combinations, respectively. Following that, it is important and useful to prove the convexity of the QD functions with respect to the critical parameters.

Based on the concept of the QD and the theorems proposed in Section 2, this paper verifies the feasibility and validity of the QD as a new risk criterion for three reasons: i) The QD is really able to measure potential risk in the presence of hybrid uncertainty. ii) The matrix forms of the QD can be derived for common fuzzy random variables as well as their linear combinations, and the covariance matrix included in the QD can reflect the interdependency between securities in portfolio optimization problems. iii) The QD is convex quadratic function of the critical parameter. Especially, the convex property endows the mean-QD optimization models to be developed with low computational complexity, global optimal solution, and noteworthy managerial insights. These important results to be analyzed in detail in Subsection 3.1 are exactly the motivation and rationale behind the concept of the QD.

2.1 Definition of the QD

This subsection formally defines the QD of a fuzzy random variable, and presents the relevant results for trapezoidal and triangular fuzzy random variables so as to deduce the analytical expressions of the QD.

Definition 2.1. Let ξ be a fuzzy random variable defined on a probability space $(\Omega, \Sigma, \mathrm{Pr})$ with finite expected value $\mathrm{E}[\xi]$. Then the QD of ξ is defined as the expected value of $\mathrm{D}[\xi(\omega)]$, that is,

$$\mathrm{D}[\xi] = \int_{\Omega} \mathrm{D}[\xi(\omega)]\mathrm{Pr}(\mathrm{d}\omega), \tag{1}$$

where the integrand is the following L–S integral [2]

$$\mathrm{D}[\xi(\omega)] = \int_{(-\infty,+\infty)} (r - \mathrm{E}[\xi])^2 \mathrm{d}\Phi_{\xi(\omega)}(r),$$

and $\Phi_{\xi(\omega)}(r) = \mathrm{Cr}\{\xi(\omega) \leq r\}$ is the monotone increasing function of r for fuzzy variable $\xi(\omega)$.

By the definition of QD, the analytical expression of $\mathrm{D}[\xi(\omega)]$ should be derived first so as to obtain the analytical expression of the QD for each given fuzzy random variable. Firstly, if $\xi(\omega)$ is a trapezoidal fuzzy variable, then the analytical expression of $\mathrm{D}[\xi(\omega)]$ is established in the following lemma.

Lemma 2.2. *Let $\xi(\omega)$ be a trapezoidal fuzzy variable $(X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega))$. Then the analytical expression of $\mathrm{D}[\xi(\omega)]$ is*

$$\mathrm{D}[\xi(\omega)] = \tfrac{1}{6}[X_1^2(\omega) + X_2^2(\omega) + X_3^2(\omega) + X_4^2(\omega) + X_1(\omega)X_2(\omega) + X_3(\omega)X_4(\omega) - 3m(X_1(\omega) + X_2(\omega) + X_3(\omega) + X_4(\omega)) + 6m^2],$$

where m is the expected value of ξ.

Proof. The credibility of event $\{\xi(\omega) \leq r\}$ is computed by

$$\Phi_{\xi(\omega)}(r) = \mathrm{Cr}\{\xi(\omega) \leq r\} = \begin{cases} 0, & r < X_1(\omega) \\ \frac{r - X_1(\omega)}{2(X_2(\omega) - X_1(\omega))}, & X_1(\omega) \leq r \leq X_2(\omega) \\ \frac{1}{2}, & X_2(\omega) \leq r \leq X_3(\omega) \\ 1 - \frac{X_4(\omega) - r}{2(X_4(\omega) - X_3(\omega))}, & X_3(\omega) \leq r \leq X_4(\omega) \\ 1, & r > X_4(\omega). \end{cases}$$

In order to simplify the notation, denote by $m = \mathrm{E}[\xi]$. Then, by Definition 2.1 and the expression of monotone increasing function $\Phi_{\xi(\omega)}(r)$, the QD of the fuzzy variable $\xi(\omega)$ is calculated by

$$\begin{aligned}\mathrm{D}[\xi(\omega)] &= \int_{(-\infty,+\infty)}(r-m)^2\mathrm{d}\Phi_{\xi(\omega)}(r)\\ &= \int_{[X_1(\omega),X_2(\omega)]}(r-m)^2\mathrm{d}\left(\frac{r-X_1(\omega)}{2(X_2(\omega)-X_1(\omega))}\right)+\int_{[X_3(\omega),X_4(\omega)]}(r-m)^2\mathrm{d}\left(1-\frac{X_4(\omega)-r}{2(X_4(\omega)-X_3(\omega))}\right)\\ &= \frac{1}{6}[X_1^2(\omega)+X_2^2(\omega)+X_3^2(\omega)+X_4^2(\omega)+X_1(\omega)X_2(\omega)+X_3(\omega)X_4(\omega)-3m(X_1(\omega)+X_2(\omega)+X_3(\omega)+X_4(\omega))+6m^2].\end{aligned}$$

The proof of the lemma is complete. □

Secondly, when the trapezoidal fuzzy variable $(X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega))$ reduces to a triangular fuzzy variable $(X_1(\omega), X_2(\omega), X_3(\omega))$ with $X_2(\omega) = X_3(\omega)$, by Lemma 2.2, we obtain the following useful corollary about the analytical expression of QD for a triangular fuzzy variable $\xi(\omega)$.

Corollary 2.3. *Let $\xi(\omega)$ be a triangular fuzzy variable $(X_1(\omega), X_2(\omega), X_3(\omega))$. Then the analytical expression of $\mathrm{D}[\xi(\omega)]$ is*

$$\mathrm{D}[\xi(\omega)] = \frac{1}{6}[X_1^2(\omega)+2X_2^2(\omega)+X_3^2(\omega)+X_1(\omega)X_2(\omega)+X_2(\omega)X_3(\omega)-3m(X_1(\omega)+2X_2(\omega)+X_3(\omega))+6m^2],$$

where m is the expected value of ξ.

To demonstrate that the QD is a feasible risk measure, we provide an interesting property in the following theorem.

Theorem 2.4. *Suppose that ξ is a fuzzy random variable with finite expected value $\mathrm{E}[\xi]$. If $\mathrm{D}[\xi] = 0$, then $\xi = \mathrm{E}[\xi]$ holds almost sure with respect to mean chance [18].*

Proof. First, by the nonnegativity of random variable $D[\xi(\omega)]$, we rewrite the QD of ξ as follows

$$\mathrm{D}[\xi] = \int_0^\infty \Pr\{\omega\in\Omega \mid D[\xi(\omega)] \geq r\}\mathrm{d}r.$$

If $\mathrm{D}[\xi] = 0$, for every $r > 0$, one has $\Pr\{\omega \in \Omega \mid D[\xi(\omega)] \geq r\} = 0$. Since $\{\omega \in \Omega \mid D[\xi(\omega)] > 0\} = \bigcup_{n=1}^{\infty}\{\omega \in \Omega \mid D[\xi(\omega)] \geq 1/n\}$, and Pr is countable subadditivity, one has $\Pr\{\omega \in \Omega \mid D[\xi(\omega)] > 0\} = 0$. As a consequence, $\Pr\{\omega \in \Omega \mid D[\xi(\omega)] = 0\} = 1$, i.e., $D[\xi(\omega)] = 0$ holds almost sure with respect to probability.

In this case, ξ reduces to a fuzzy variable, and $D[\xi(\omega)]$ can be regarded as the spread of fuzzy variable [30], i.e., $D[\xi(\omega)] = \mathrm{Sp}[\xi(\omega)]$ holds almost sure with respect to probability. According to the property of L–S integral [2], we have $\mathrm{Cr}\{\xi(\omega) = \mathrm{E}[\xi]\} = 1$ provided that $\mathrm{Sp}[\xi(\omega)] = 0$, i.e., for the given ω, $\xi(\omega) = \mathrm{E}[\xi]$ holds almost sure with respect to credibility measure.

Based on the above analysis, $\mathrm{Ch}\{\xi = \mathrm{E}[\xi]\} = \int_\Omega \mathrm{Cr}\{\xi(\omega) = \mathrm{E}[\xi]\}\Pr(\mathrm{d}\omega) = 1$, where Ch is the mean chace of a fuzzy random event [18], i.e., $\xi = \mathrm{E}[\xi]$ holds almost sure with respect to mean chance. The proof of theorem is complete. □

2.2 Matrix representations of the QD

For trapezoidal and triangular fuzzy random variables, the analytical expressions of the QD can be represented as the matrix forms.

For the sake of presentation, the following notations are adopted, $X_1(\omega) = X(\omega), X_2(\omega) = X(\omega) + r_1, X_3(\omega) = X(\omega) + r_2$, and $X_4(\omega) = X(\omega) + r_3$. Thus, a trapezoidal fuzzy random variable ξ is denoted by $(X(\omega), X(\omega) + r_1, X(\omega) + r_2, X(\omega) + r_3)$, where $r_i, i = 1, 2, 3$, are real numbers with $r_3 \geq r_2 \geq r_1 \geq 0$. If $r_1 = r_2$, then the trapezoidal fuzzy random variable ξ reduces to the triangular fuzzy random variable $(X(\omega), X(\omega) + r_1, X(\omega) + r_2)$.

Firstly, the following theorem deals with the matrix representation of the QD for a trapezoidal fuzzy random variable.

Theorem 2.5. *Let ξ be a trapezoidal fuzzy random variable such that for each $\omega \in \Omega$, $\xi(\omega) = (X(\omega), X(\omega)+r_1, X(\omega)+r_2, X(\omega)+r_3)$ with $r_3 \geq r_2 \geq r_1 \geq 0$, and $X \sim \mathcal{N}(\mu, \sigma^2)$. Then the QD of ξ can be represented as the following matrix form $\mathrm{D}[\xi] = r^T Pr + \sigma^2$, where $r = (r_1, r_2, r_3)^T$, and*

$$P = \begin{bmatrix} \frac{5}{48} & -\frac{1}{16} & -\frac{1}{16}\\ -\frac{1}{16} & \frac{5}{48} & \frac{1}{48}\\ -\frac{1}{16} & \frac{1}{48} & \frac{5}{48}\end{bmatrix}.$$

Moreover, the quadratic function $\mathrm{D}[\xi]$ is convex with respect to parameter vector $r = (r_1, r_2, r_3)^T$.

Proof. On the one hand, since ξ is a trapezoidal fuzzy random variable, its expected value is

$$\mathrm{E}[\xi] = \mathrm{E}[X] + \tfrac{1}{4}(r_1 + r_2 + r_3) = \mu + \tfrac{1}{4}(r_1 + r_2 + r_3).$$

Denoting $m = \mathrm{E}[\xi]$, by Lemma 2.2, the analytical expression of $\mathrm{D}[\xi(\omega)]$ is as follows

$$\begin{aligned}\mathrm{D}[\xi(\omega)] &= \frac{1}{6}[6X^2(\omega) + 3(r_1 + r_2 + r_3 - 4m)X(\omega) + r_1^2 + r_2^2 + r_3^2 + r_2r_3 - 3m(r_1 + r_2 + r_3) + 6m^2] \\ &= (X(\omega) - \mu)^2 + \frac{5}{48}(r_1^2 + r_2^2 + r_3^2) - \frac{1}{8}r_1r_2 - \frac{1}{8}r_1r_3 + \frac{1}{24}r_2r_3.\end{aligned}$$

As a consequence, by Definition 2.1, the QD of the trapezoidal fuzzy random variable ξ is calculated by

$$\begin{aligned}\mathrm{D}[\xi] &= \int_{-\infty}^{+\infty} [(x - \mu)^2 + \frac{5}{48}(r_1^2 + r_2^2 + r_3^2) - \frac{1}{8}r_1r_2 - \frac{1}{8}r_1r_3 + \frac{1}{24}r_2r_3] \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x \\ &= \sigma^2 + \frac{5}{48}(r_1^2 + r_2^2 + r_3^2) - \frac{1}{8}r_1r_2 - \frac{1}{8}r_1r_3 + \frac{1}{24}r_2r_3 = r^TPr + \sigma^2.\end{aligned}$$

On the other hand, the Hessian matrix of the quadratic form $\mathrm{D}[\xi] = r^TPr + \sigma^2$ is P. Furthermore, each leading principal minor of the matrix P is positive, so the Hessian matrix P is positive-definite. Therefore, the quadratic function $\mathrm{D}[\xi]$ is convex with respect to parameter vector $r = (r_1, r_2, r_3)^T$. The proof of theorem is complete. □

Secondly, as a particular case of trapezoidal fuzzy random variable, the matrix representation of the QD for a triangular fuzzy random variable is summarized in the following corollary.

Corollary 2.6. *Let ξ be a triangular fuzzy random variable such that for each $\omega \in \Omega$, $\xi(\omega) = (X(\omega), X(\omega)+r_1, X(\omega)+r_2)$ with $r_2 \geq r_1 \geq 0$, and $X \sim \mathcal{N}(\mu, \sigma^2)$. Then the QD of ξ can be represented as the following matrix form $\mathrm{D}[\xi] = r^TQr + \sigma^2$, where $r = (r_1, r_2)^T$ and $Q = \begin{bmatrix} \frac{1}{12} & -\frac{1}{24} \\ -\frac{1}{24} & \frac{5}{48} \end{bmatrix}$. Moreover, the quadratic function $\mathrm{D}[\xi]$ is convex with respect to parameter vector $r = (r_1, r_2)^T$.*

2.3 Convex property of the QD functions

This subsection is aimed at deducing the matrix representations of the QD for the linear combination of trapezoidal and triangular fuzzy random variables, and verifying the convexity of the QD function with respect to critical parameter vector. It is worth noting that such a desirable convex property will enable the mean-QD optimization models (to be developed in the next section) to be converted into their equivalent convex parametric quadratic programming problems and further facilitate the decision makers to find appropriate solution methods to find global optimal solutions.

The following theorem gives the matrix form of the QD for the linear combination of trapezoidal fuzzy random variables, and obtains the convexity with respect to parameter vector.

Theorem 2.7. *Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ be an n-dimensional fuzzy random vector such that for each $\omega \in \Omega$, $\xi_i(\omega) = (X_i(\omega), X_i(\omega)+r_{i1}, X_i(\omega)+r_{i2}, X_i(\omega)+r_{i3})$ with $r_{i3} \geq r_{i2} \geq r_{i1} \geq 0, i = 1, 2, \ldots, n$, are mutually independent trapezoidal fuzzy variables. If $X = (X_1, X_2, \ldots, X_n)^T \sim \mathcal{N}(\mu_0, \Sigma)$ where $\mu_0 = (\mu_1, \mu_2, \ldots, \mu_n)^T$ and Σ is the covariance matrix of X, then the QD of the linear combination $\sum_{i=1}^n x_i\xi_i$ can be represented by the following matrix form*

$$\mathrm{D}\left[\sum_{i=1}^n x_i\xi_i\right] = x^T(H + \Sigma)x,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $H = S^TPS$,

$$S = \begin{bmatrix} r_{11} & r_{21} & \cdots & r_{n1} \\ r_{12} & r_{22} & \cdots & r_{n2} \\ r_{13} & r_{23} & \cdots & r_{n3} \end{bmatrix}_{3\times n}, \tag{2}$$

and

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}_{n\times n}. \tag{3}$$

Moreover, the quadratic function $\mathrm{D}\left[\sum_{i=1}^n x_i\xi_i\right]$ is convex with respect to the parameter vector x.

Proof. Let $Y=\sum_{i=1}^n x_iX_i$, and $\eta=\sum_{i=1}^n x_i\xi_i$. Then, for each $\omega\in\Omega$, we have

$$Y=(x_1,x_2,\ldots,x_n)\begin{bmatrix}X_1\\X_2\\\vdots\\X_n\end{bmatrix}\sim\mathcal{N}(\mu,\sigma^2),$$

where the parameters μ and σ^2 are represented as

$$\mu=(x_1,x_2,\ldots,x_n)\begin{bmatrix}\mu_1\\\mu_2\\\vdots\\\mu_n\end{bmatrix},\quad \sigma^2=(x_1,x_2,\ldots,x_n)\begin{bmatrix}\sigma_{11}&\sigma_{12}&\cdots&\sigma_{1n}\\\sigma_{21}&\sigma_{22}&\cdots&\sigma_{2n}\\\vdots&\vdots&&\vdots\\\sigma_{n1}&\sigma_{n2}&\cdots&\sigma_{nn}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=x^T\Sigma x.$$

If we denote $r_1=\sum_{i=1}^n r_{i1}x_i, r_2=\sum_{i=1}^n r_{i2}x_i$, and $r_3=\sum_{i=1}^n r_{i3}x_i$, then the following equation holds

$$\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix}=\begin{bmatrix}r_{11}&r_{21}&\cdots&r_{n1}\\r_{12}&r_{22}&\cdots&r_{n2}\\r_{13}&r_{23}&\cdots&r_{n3}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}.$$

Since $\xi_i(\omega), i=1,2,\ldots,n$, are mutually independent trapezoidal fuzzy variables, by the property of independence [15], $\eta(\omega)$ is the following trapezoidal fuzzy variable,

$$\eta(\omega)=(Y(\omega),Y(\omega)+r_1,Y(\omega)+r_2,Y(\omega)+r_3),$$

where $r_3\geq r_2\geq r_1\geq 0$ and $Y\sim\mathcal{N}(\mu,\sigma^2)$.

According to Theorem 2.5, the QD of the trapezoidal fuzzy variable $\eta(\omega)$ can be represented by

$$\mathrm{D}[\eta]=\mathrm{D}\left[\sum_{i=1}^n x_i\xi_i\right]=(r_1,r_2,r_3)P(r_1,r_2,r_3)^T+\sigma^2. \tag{4}$$

Let $x=(x_1,x_2,\ldots,x_n)^T$, then $(r_1,r_2,r_3)^T=Sx$. Furthermore, by introducing the matrix H, that is,

$$H=S^TPS, \tag{5}$$

it yields that $\mathrm{D}\left[\sum\limits_{i=1}^n x_i\xi_i\right]=(r_1,r_2,r_3)P(r_1,r_2,r_3)^T+\sigma^2=x^TS^TPSx+x^T\Sigma x=x^THx+x^T\Sigma x=x^T(H+\Sigma)x$.

Next we will prove the convexity of the QD with respect to decision vector x. Firstly, Eq. (4) indicates that the quadratic function $\mathrm{D}\left[\sum_{i=1}^n x_i\xi_i\right]$ is convex with respect to parameter vector (r_1,r_2,r_3) because the matrix P is positive-definite. In other words, the quadratic function is a positive definite quadratic form, that is, $\mathrm{D}\left[\sum_{i=1}^n x_i\xi_i\right]\geq 0$ holds for any parameter vector (r_1,r_2,r_3). Furthermore, there exists a linear relation between the parameter vector (r_1,r_2,r_3) and decision vector x, that is, $(r_1,r_2,r_3)^T=Sx$. Therefore, for any given decision vector $x\geq 0$ in practical optimization problems, it yields that $\mathrm{D}\left[\sum_{i=1}^n x_i\xi_i\right]=x^T(H+\Sigma)x\geq 0$, which means that $(H+\Sigma)$ is a positive-definite matrix. Meanwhile, $(H+\Sigma)$ is also the Hessian matrix of the quadratic function that $\mathrm{D}\left[\sum_{i=1}^n x_i\xi_i\right]=x^T(H+\Sigma)x$. Thus, the quadratic function $\mathrm{D}\left[\sum_{i=1}^n x_i\xi_i\right]$ is convex with respect to decision vector x. The proof of theorem is complete. □

The following corollary deals with the matrix representation of the QD for the linear combination of triangular fuzzy random variables and its convex property with respect to decision vector.

Corollary 2.8. *Let $\xi=(\xi_1,\xi_2,\ldots,\xi_n)^T$ be an n-dimensional fuzzy random vector such that for each $\omega\in\Omega$, $\xi_i(\omega)=(X_i(\omega),X_i(\omega)+r_{i1},X_i(\omega)+r_{i2})$ with $r_{i2}\geq r_{i1}\geq 0, i=1,2,\ldots,n$, are mutually independent triangular fuzzy variables. If $X=(X_1,X_2,\ldots,X_n)^T\sim\mathcal{N}(\mu_0,\Sigma)$ with $\mu_0=(\mu_1,\mu_2,\ldots,\mu_n)^T$, and Σ the covariance matrix of X, then the QD of linear combination $\sum_{i=1}^n x_i\xi_i$ can be represented by the following matrix form*

$$\mathrm{D}\left[\sum_{i=1}^n x_i\xi_i\right]=x^T(H+\Sigma)x,$$

where $x = (x_1, x_2, \ldots, x_n)^T, H = S^T Q S$,

$$S = \begin{bmatrix} r_{11} & r_{21} & \cdots & r_{n1} \\ r_{12} & r_{22} & \cdots & r_{n2} \end{bmatrix}_{2\times n}, \ \textit{and} \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}_{n\times n}.$$

Moreover, the quadratic function $\mathrm{D}\left[\sum_{i=1}^{n} x_i\xi_i\right]$ *is convex with respect to the decision vector* x*.*

3 Application of the QD to portfolio selection problem

In this section, the quadratic deviation (QD) is applied to portfolio selection problem as a new risk criterion, and three novel mean-QD optimization models are developed to find the optimal allocation of the investment fund in the invested securities. To solve the proposed models, the mean-QD optimization models are converted to their equivalent parametric programming problems. Due to the desirable convex property and the structural characteristics, the equivalent programming problems can be solved by conventional solution methods to obtain the global optimal solutions.

3.1 Feasibility of the QD as a risk criterion

The new risk criterion should be feasible and valid to quantify the degree of the potential investment risk and have some advantages over variance from the computationally tractable perspective. With this in mind, this paper formally defines the QD of a fuzzy random variable to measure the potential investment risk in fuzzy random portfolio selection problem. Before describing the problem, the feasibility and validity of the QD as a new risk criterion are verified from the following four aspects to help readers get a more comprehensive understanding.

(1) According to Definition (1), the QD describes the variation of the values that a fuzzy random variable takes around its expected value. In virtue of Theorem 2.4, if $D[\xi]$ is zero, then fuzzy random variable ξ degenerates a constant. If $D[\xi] > 0$, then the values of ξ are scattered around the expected value. In that case, the fuzzy random variable takes the values with the lower stability, which means greater uncertainty. In the portfolio selection problem, greater uncertainty indicates the higher level of investment risk along with the portfolio return. Therefore, the QD is able to measure the investment risk in portfolio optimization problems in the presence of hybrid uncertainty. The greater the QD of the portfolio return is, the higher the accompanying investment risk is; Otherwise, the lower investment risk always comes with the lower portfolio return.

(2) The interdependency between securities is reflected by the covariance matrix included in the QD formulas. In practical security markets, the securities usually influence each other because of the uncertain factors, such as the limited investment fund and the subjective assessments of security returns. Actually, the increase in the return of one security might lead to the decrease of other securities, and the interaction effect between securities has an immediate impact on the total portfolio return. Thus, the interdependency between securities cannot be ignored when investors make their optimal investment plans. However, mean-variance model usually provides the optimal allocation of investment fund without addressing the interaction effect between securities.

(3) The matrix representations of the QD can be obtained for trapezoidal and triangular fuzzy random variables as well as their linear combinations, given that the random term is the realization of a normally distributed random variable with known mean and variance. It is important to note that, in practical decision problems, trapezoidal and triangular fuzzy random variables are frequently used to describe the subjective assessments of items with uncertainty on the basis of human opinions while the normally distributed random variable is also commonly used to measure the objective potential variation that might be positive or negative with a certain probability. Therefore, the matrix representations of the QD play an important role in modeling the potential risk, because such clear quadratic forms are favorable from the computationally tractable perspective, and provide some noteworthy managerial insights for decision makers.

(4) For trapezoidal and triangular fuzzy random variables as well as their linear combinations, the QD functions are convex with respect to the critical parameters. From optimization viewpoint, the convex property is obviously important in identifying the risk criterion and formulating the decision-making models. Actually, as a new risk criterion in portfolio selection problem, the convex property ensures that the QD of fuzzy random portfolio return is a convex quadratic function with respect to decision vector. Thus, the original mean-QD optimization models can be turned into its equivalent convex parametric quadratic programming problems, and the equivalent programming problems can be solved by conventional optimization methods to obtain the global optimal portfolio. Such global optimal solution provides the valuable guidance for decision makers to make their limited fund to get the maximum rise in value, which is the major motivation for investors to engage in the investments.

3.2 Mean-QD models for portfolio selection problems

In the framework of classical portfolio optimization problems, an investor with limited fund naturally expects to achieve the maximum expected return by seeking for the optimal allocation of the fund in different risky securities in the financial market. Given a collection of potential securities indexed from 1 to n, let the vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ be the fuzzy random return of a portfolio in the next time period, where ξ_i represents the return of security i. Denote the real number $x_i \geq 0$ as the investment proportion in security i, $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^n x_i = 1$. If $x_i = 0$, then the security i should not be included in the portfolio; If $0 < x_i \leq 1$, then the security i should be chosen as one of the portfolio and the investor should allocate his fund for it by the value of the investment proportion x_i. As a consequence, the investment return that the investor would obtain by using this portfolio is represented by $R(x, \xi) = \sum_{i=1}^n x_i\xi_i$, where $x = (x_1, x_2, \ldots, x_n)^T$. As stated in the existing literature, the potential reward associated with this portfolio is quantified by the expected return,

$$\mathrm{E}[R(x,\xi)] = \mathrm{E}\left[\sum_{i=1}^n x_i\xi_i\right].$$

It is well-known that the security with high return usually comes with a high level of risk. So, it is necessary to identify an appropriate risk measure for the portfolio return $R(x, \xi)$. In this section, the investment risk associated with the return $R(x, \xi)$ is gauged by the QD of the potential investment return, that is,

$$\mathrm{D}[R(x,\xi)] = \mathrm{D}\left[\sum_{i=1}^n x_i\xi_i\right].$$

Using $E[R(x, \xi)]$ and $D[R(x, \xi)]$ as the optimization indexes, three formulations about portfolio selection problem will be proposed according to the investor's attitude towards risk. Firstly, the investor has to suffer the greater potential risk during the investment process if he would like to achieve the higher expected return. In that case, the moderate investor might tend to seek for a tradeoff between the expected return and the potential associated risk. Thus, the following mean-QD optimization model can be built by weighing the return and the risk, that is,

$$\left\{\begin{array}{ll} \min & \mathrm{D}[R(x,\xi)] - \phi\mathrm{E}[R(x,\xi)] \\ \text{subject to:} & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0, i = 1, 2, \ldots, n, \end{array}\right. \tag{6}$$

which is named the DE model for convenience, because the objective function is formed by combining the quadratic deviation and the expected return.

In the DE model (6), the parameter ϕ is the weight coefficient satisfying $\phi \geq 0$, and the value of ϕ reflects the degree of the investor's attention to the expected return. Generally speaking, the greater the value of parameter ϕ is, the more important the expected return is in the decision making process. The setting where $\phi = 0$ means that the investor is completely reluctant to bear the investment risk regardless of the potential return. In reality, this is an extreme and even inexistent case in which the investor avoids the investment risk by completely sacrificing the investment return. If $\phi = 1$, then the investor pays equal attention to the potential investment return and risk, i.e., the return and the risk play equally important roles in the determination of the optimal investment portfolio.

Secondly, if the investor is a risk taker, then he is keen on pursuing the maximum rise in value as long as the potential risk is lower than the upper limit, which is the maximum level of the risk that the investor can bear. The adventurous investor usually prefers to accept the bargain with an uncertain payoff rather than attempt to reduce the uncertainty by accepting a bargain with a more certain but possibly lower expected payoff. For the adventurous investor with the given maximum acceptable risk level ψ, the following mean-QD optimization model can be developed to seek for the optimal investment decision, that is,

$$\left\{\begin{array}{ll} \max & \mathrm{E}[R(x,\xi)] \\ \text{subject to:} & \mathrm{D}[R(x,\xi)] \leq \psi \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0, i = 1, 2, \ldots, n, \end{array}\right. \tag{7}$$

which is named the E model for convenience, because the optimization index in objective function is expected return.

In the E model (7), the parameter ψ reflects the maximum level of the risk that the investor is able to bear. Moreover, the objective function is a monotone increasing function with respect to the parameter ψ. In other words, the higher the bearable risk is, the greater the expected return is. Especially, when $\psi = 0$, the setting with the constraint $\mathrm{D}[R(x,\xi)] = 0$ is also an extreme and even inexistent case, in which the investor pursues overly the maximum expected return by completely turning a blind eye to the associated investment risk. In that case, the investor either obtains the maximum investment return or completely goes bankrupt. The investor might do so in some desperate conditions.

Thirdly, if the investor is averse to the potential risk resulting from the uncertainty in securities market, then he prefers to obtain the expected return with a lower uncertainty provided that the potential expected return is larger than the lower limit, which is the minimum return that the investor can accept. What such an investor really cares about is the certainty rather than the maximum expected return. Therefore, for the investor who is seeking for the portfolio with minimum investment risk under the given minimum acceptable level φ of expected return, the following mean-QD optimization model can be established to determine the optimal investment decision, that is,

$$\begin{cases} \min & \mathrm{D}[R(x,\xi)] \\ \text{subject to:} & \mathrm{E}[R(x,\xi)] \geq \varphi \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, i = 1, 2, \ldots, n, \end{cases} \tag{8}$$

which is called the D model for convenience, because the optimization index in objective function is quadratic deviation.

In the D model (8), the parameter φ stands for the minimum acceptable expected return. The investor might give up the investment opportunity if the expected return is too low to make a profit. However, if such a psychological expectation is set too high as a threshold value, then the higher potential risk inevitably comes together with it. That is, the investor needs to make a balance between the expectation and the associated risk. Theoretically, the objective function monotonously increases with respect to the parameter φ. In finance, the optimal objective value as a function of φ plays an important role in financial decision making. Its graph is called the efficient frontier with horizontal axis corresponding to risk and the vertical axis corresponding to return.

3.3 Convex parametric quadratic programming

Based on three proposed mean-QD optimization models, i.e., DE model (6), E model (7) and D model (8), this subsection will derive their equivalent parametric quadratic programming models. It is worth noting that the equivalent parametric quadratic programming problems are convex with respect to decision vector.

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ be the fuzzy random return of a portfolio. In general, for each $\omega \in \Omega$, $\xi_i(\omega) = (X_i(\omega), X_i(\omega) + r_{i1}, X_i(\omega) + r_{i2}, X_i(\omega) + r_{i3})$ with $r_{i3} \geq r_{i2} \geq r_{i1} \geq 0, i = 1, 2, \ldots, n$, are mutually independent trapezoidal fuzzy variables. If $X = (X_1, X_2, \ldots, X_n)^T \sim \mathcal{N}(\mu_0, \Sigma)$ with $\mu_0 = (\mu_1, \mu_2, \ldots, \mu_n)^T$, and Σ the covariance matrix of X, then the portfolio return is $R(x,\xi) = \sum_{i=1}^{n} x_i \xi_i$.

According to Theorem 2.7, the QD of the portfolio return is expressed as follows

$$\mathrm{D}[R(x,\xi)] = x^T (H + \Sigma) x = \frac{1}{2} x^T M x,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $M = 2(H + \Sigma)$, and the parametric matrices Σ and H are defined by Eqs. (3) and (5), respectively. Furthermore, the expected portfolio return is calculated by

$$\mathrm{E}[R(x,\xi)] = \sum_{i=1}^{n} \left(\mu_i + \frac{1}{4} r_{i1} + \frac{1}{4} r_{i2} + \frac{1}{4} r_{i3} \right) x_i = u^T x,$$

where the parametric vector

$$u = \left(\mu_1 + \frac{1}{4} r_{11} + \frac{1}{4} r_{12} + \frac{1}{4} r_{13}, \mu_2 + \frac{1}{4} r_{21} + \frac{1}{4} r_{22} + \frac{1}{4} r_{23}, \ldots, \mu_n + \frac{1}{4} r_{n1} + \frac{1}{4} r_{n2} + \frac{1}{4} r_{n3} \right)^T.$$

Consequently, given parametric matrices H, Σ and u, $\mathrm{D}[R(x,\xi)]$ and $\mathrm{E}[R(x,\xi)]$ can be equivalently represented as the functions of decision vector x,

$$\mathrm{D}[R(x,\xi)] = \frac{1}{2} x^T M x, \tag{9}$$

and

$$\mathrm{E}[R(x,\xi)] = u^T x. \tag{10}$$

Obviously, the expected return $\mathrm{E}[R(x,\xi)]$ is a linear function of decision vector x, while the quadratic function $\mathrm{D}[R(x,\xi)]$ as the risk criterion is convex with respect to decision vector x by Theorem 2.7.

Based on the optimization indexes (9) and (10), DE model (6) can be equivalently rewritten as the following convex parametric quadratic programming problem

$$\begin{cases} \min & \frac{1}{2}x^T Mx + c^T x \\ \text{subject to:} & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, i = 1, 2, \ldots, n, \end{cases} \tag{11}$$

where M is a symmetric $n \times n$ matrix, and $c = -\phi u$.

Similarly, E model (7) can be equivalently converted to the following convex parametric quadratic programming problem

$$\begin{cases} \max & u^T x \\ \text{subject to:} & \frac{1}{2}x^T Mx \leq \psi \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, i = 1, 2, \cdots, n. \end{cases} \tag{12}$$

Finally, D model (8) can be equivalently represented as the following convex parametric quadratic programming problem

$$\begin{cases} \min & \frac{1}{2}x^T Mx \\ \text{subject to:} & u^T x \geq \varphi \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, i = 1, 2, \cdots, n. \end{cases} \tag{13}$$

In optimization problems (11), (12) and (13), $M = 2(S^T PS + \Sigma)$. Specifically, the matrix S represents the return information about each security, and the covariance matrix Σ reflects the interdependency between the risky securities that will be invested in. Besides, the constant matrix P is positive definite (see Theorem 2.5). An important point to note is that the parametric programming strengthens the capacity to respond to the change of the influencing factors. The optimal decision can be updated rapidly and expediently by modifying the parametric data in the proposed optimization models when the influencing factors have changed to generate the new information about the security return and the degree of the interdependency between the selected securities. Such effective response to the change cannot be achieved in nonparametric programming problems.

Furthermore, optimization models (11), (12) and (13) are convex quadratic programming problems due to the convexity of quadratic function (9). Thus, these programming problems can be solved by the conventional optimization methods to obtain the global and accurate optimal solutions. The solution procedure can be achieved by the general purpose softwares such as Lingo when the dimensionality of the decision vector is large.

4 Numerical examples

This section will show two numerical examples to illustrate the validity of the proposed work. Specifically, the first portfolio optimization problem is formulated by the proposed DE model (11) and solved by the active-set method [21]. Meanwhile, the second problem is modeled by the proposed E model (12)and the D model (13) respectively, and the solution procedures are achieved by Lingo software owing to the large dimensionality of decision vector.

Example 4.1. Suppose an investor intends to invest his fund in three securities. Let x_i be the investment proportion in security i, and $\xi = (\xi_1, \xi_2, \xi_3)^T$ the uncertain return vector of the portfolio. For each $\omega \in \Omega$, $\xi_i(\omega), i = 1, 2, 3$, are the following mutually independent trapezoidal fuzzy variables

$$\xi_1(\omega) = (X_1(\omega), X_1(\omega) + 0.5, X_1(\omega) + 1.0, X_1(\omega) + 1.5),$$

$$\xi_2(\omega) = (X_2(\omega), X_2(\omega) + 0.8, X_2(\omega) + 2.0, X_2(\omega) + 2.4),$$

$$\xi_3(\omega) = (X_3(\omega), X_3(\omega) + 0.8, X_3(\omega) + 1.0, X_3(\omega) + 2.5),$$

where $X = (X_1, X_2, X_3)^T$ is a 3-dimensional normal random vector with the expected value $\mu_0 = (1.00, 1.20, 1.25)^T$, and the covariance matrix

$$\Sigma = \begin{bmatrix} \frac{1}{16} & -\frac{1}{48} & -\frac{1}{24} \\ -\frac{1}{48} & \frac{1}{12} & \frac{1}{48} \\ -\frac{1}{24} & \frac{1}{48} & \frac{1}{8} \end{bmatrix}. \tag{14}$$

The above portfolio optimization problem is solved by the proposed DE model (11). According to Theorem 2.7, the optimization problem is equivalent to the following convex quadratic programming

$$\begin{cases} \min & \frac{1}{2}x^T M x + c^T x \\ \text{subject to:} & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0, i = 1, 2, 3, \end{cases} \tag{15}$$

where the vector $c = \left(-\frac{7}{2}, -5, -\frac{93}{20}\right)^T$, and the symmetric matrix is calculated as follows

$$M = \begin{bmatrix} \frac{2}{3} & \frac{109}{120} & \frac{11}{16} \\ \frac{109}{120} & \frac{139}{75} & \frac{34}{25} \\ \frac{11}{16} & \frac{34}{25} & \frac{673}{480} \end{bmatrix}.$$

The active-set method is applied to solve the optimization problem (15), and the iteration process is summarized as follows in the case where $\phi = 2$.

Iteration 0: At the feasible starting point $x^{(0)} = (1, 0, 0)^T$, the constraints 1, 3 and 4 are active, so set $W_0 = \{1, 3, 4\}$. Solve the following quadratic programming problem

$$\begin{cases} \min & \frac{1}{2}y^T M y + g_0^T y \\ \text{subject to:} & y_1 + y_2 + y_3 = 0 \\ & y_2 = 0 \\ & y_3 = 0, \end{cases} \tag{16}$$

where $g_0 = \left(-\frac{23}{6}, -\frac{491}{120}, -\frac{235}{60}\right)^T$. The solution of problem (16) is $y^{(0)} = (0, 0, 0)^T$, so the Lagrange multipliers are computed by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \lambda_3 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \lambda_4 = \begin{bmatrix} -\frac{23}{6} \\ -\frac{491}{120} \\ -\frac{235}{60} \end{bmatrix},$$

and $(\lambda_1, \lambda_3, \lambda_4) = \left(-\frac{23}{6}, -\frac{31}{120}, -\frac{1}{12}\right)$. Thus, $j = 3, x^{(1)} = x^{(0)} = (1, 0, 0)^T$, and $W_1 = \{1, 4\}$. Set $k = 1$, and go to the next iteration.

Iteration 1: Solve the following quadratic programming problem

$$\begin{cases} \min & \frac{1}{2}y^T M y + g_1^T y \\ \text{subject to:} & y_1 + y_2 + y_3 = 0 \\ & y_3 = 0, \end{cases} \tag{17}$$

where $g_1 = \left(-\frac{23}{6}, -\frac{491}{120}, -\frac{235}{60}\right)^T$. The solution of problem (17) is $y^{(1)} = \left(-\frac{155}{422}, \frac{155}{422}, 0\right)^T \neq 0$, so compute

$$\alpha_1 = \min\left\{1, \min_{i \notin W_1, a_i^T y^{(1)} < 0} \frac{b_i - a_i^T x^{(1)}}{a_i^T y^{(1)}}\right\} = \min\left\{1, \frac{422}{155}\right\} = 1.$$

Thus, $x^{(2)} = x^{(1)} + \alpha_1 y^{(1)} = \left(\frac{267}{422}, \frac{155}{422}, 0\right)^T$, and $W_2 = \{1, 4\}$. Set $k = 2$, and go to the next iteration.

Iteration 2: Solve the following quadratic programming problem

$$\begin{cases} \min & \frac{1}{2}y^T M y + g_2^T y \\ \text{subject to:} & y_1 + y_2 + y_3 = 0 \\ & y_3 = 0, \end{cases} \tag{18}$$

where $g_2 = \left(-\frac{27797}{10128}, -\frac{37925}{10128}, -\frac{25087}{6752}\right)^T$. The solution of problem (18) is $y^{(2)} = \left(-\frac{300}{211}, \frac{300}{211}, 0\right)^T \neq 0$, so compute

$$\alpha_2 = \min\left\{1, \min_{i \notin W_2, a_i^T y^{(2)} < 0} \frac{b_i - a_i^T x^{(2)}}{a_i^T y^{(2)}}\right\} = \min\left\{1, \frac{267}{600}\right\} = \frac{267}{600}.$$

Thus, $x^{(3)} = x^{(2)} + \alpha_2 y^{(2)} = (0, 1, 0)^T$, and $W_3 = \{1, 2, 4\}$. Set $k = 3$, and go to the next iteration.

Iteration 3: Solve the following quadratic programming problem

$$\begin{cases} \min & \frac{1}{2}y^T M y + g_3^T y \\ \text{subject to:} & y_1 + y_2 + y_3 = 0 \\ & y_1 = 0 \\ & y_3 = 0, \end{cases} \tag{19}$$

where $g_3 = \left(-\frac{311}{120}, -\frac{236}{75}, -\frac{329}{100}\right)^T$. The solution of problem (19) is $y^{(3)} = (0,0,0)^T = 0$, so the Lagrange multipliers are computed by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \lambda_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \lambda_4 = \begin{bmatrix} -\frac{311}{120} \\ -\frac{236}{75} \\ -\frac{329}{100} \end{bmatrix},$$

and $(\lambda_1, \lambda_2, \lambda_4) = \left(-\frac{236}{75}, \frac{111}{200}, -\frac{43}{300}\right)$. Thus, $j = 4, x^{(4)} = x^{(3)} = (0,1,0)^T$, and $W_4 = \{1,2\}$. Set $k = 4$, and go to the next iteration.

Iteration 4: Solve the following quadratic programming problem

$$\begin{cases} \min & \frac{1}{2}y^T My + g_4^T y \\ \text{subject to:} & y_1 + y_2 + y_3 = 0 \\ & y_1 = 0, \end{cases} \tag{20}$$

where $g_4 = \left(-\frac{311}{120}, -\frac{236}{75}, -\frac{329}{100}\right)^T$. The solution of problem (20) is $y^{(4)} = \left(0, -\frac{344}{1285}, \frac{344}{1285}\right)^T \neq 0$, so compute

$$\alpha_4 = \min\left\{1, \min_{i \notin W_4, a_i^T y^{(4)} < 0} \frac{b_i - a_i^T x^{(4)}}{a_i^T y^{(4)}}\right\} = \min\left\{1, \frac{1285}{344}\right\} = 1.$$

Thus, $x^{(5)} = x^{(4)} + \alpha_4 y^{(4)} = \left(0, \frac{941}{1285}, \frac{344}{1285}\right)^T$, and $W_5 = \{1,2\}$. Set $k = 5$, and go to the next iteration.

Iteration 5: Solve the following quadratic programming problem

$$\begin{cases} \min & \frac{1}{2}y^T My + g_5^T y \\ \text{subject to:} & y_1 + y_2 + y_3 = 0 \\ & y_1 = 0, \end{cases} \tag{21}$$

where $g_5 = \left(-\frac{408751}{154200}, -\frac{315988}{96375}, -\frac{315988}{96375}\right)^T$. The solution of problem (21) is $y^{(5)} = (0,0,0)^T = 0$, so the Lagrange multipliers are computed by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \lambda_2 = \begin{bmatrix} -\frac{408751}{154200} \\ -\frac{315988}{96375} \\ -\frac{315988}{96375} \end{bmatrix},$$

and $(\lambda_1, \lambda_2) = \left(-\frac{315988}{96375}, \frac{161383}{257000}\right)$. $W_5 \bigcap \{2,3,4\} = \{2\}$ and $\lambda_2 > 0$, so stop and the global optimal solution to the portfolio optimization problem (15) is as follows

$$x^{(5)} = \left(0, \frac{941}{1285}, \frac{344}{1285}\right)^T. \tag{22}$$

By Eq. (22), security 1 will not be included in the portfolio while security 2 and security 3 should be invested in at the rate of (941/1285) and (344/1285), respectively. This optimal decision is effective and credible owing to the following three reasons: 1) Security 1 exerts a negative influence on both security 2 and security 3 according to the negative covariance shown in the matrix (14). 2) There is a positive correlation between security 2 and security 3 due to the positive covariance (1/48). 3) The coefficient of variation of security 2 is less than that of security 3, so security 2 will generate a more stable potential return. As a result, it is advisable for the investor to collect security 2 with a higher proportion and security 3 with a lower proportion in the investment portfolio. Besides, it is easy to verify that the optimal objective value in optimization problem (15) monotonically decreases with respect to the parameter ϕ. Therefore, the solution procedure will not be repeated for different values of the parameter ϕ.

Example 4.2. Suppose an investor intends to invest his fund in ten securities. Let x_i be the investment proportion to security i, and $\xi = (\xi_1, \xi_2, \ldots, \xi_{10})^T$ the uncertain return vector of the portfolio. For each $\omega \in \Omega$, $\xi_i(\omega), i = 1,2,\ldots,10$, are the following mutually independent trapezoidal fuzzy variables

$$\xi_1(\omega) = (X_1(\omega), X_1(\omega)+0.3, X_1(\omega)+0.6, X_1(\omega)+0.9), \xi_2(\omega) = (X_2(\omega), X_2(\omega)+0.4, X_2(\omega)+0.7, X_2(\omega)+1.0),$$

$$\xi_3(\omega) = (X_3(\omega), X_3(\omega)+0.5, X_3(\omega)+0.6, X_3(\omega)+1.5), \xi_4(\omega) = (X_4(\omega), X_4(\omega)+0.6, X_4(\omega)+0.9, X_4(\omega)+1.7),$$

$$\xi_5(\omega) = (X_5(\omega), X_5(\omega)+0.6, X_5(\omega)+1.0, X_5(\omega)+1.9), \xi_6(\omega) = (X_6(\omega), X_6(\omega)+0.8, X_6(\omega)+1.5, X_6(\omega)+2.3),$$

$$\xi_7(\omega) = (X_7(\omega), X_7(\omega)+0.9, X_7(\omega)+1.7, X_7(\omega)+2.5), \xi_8(\omega) = (X_8(\omega), X_8(\omega)+1.0, X_8(\omega)+2.0, X_8(\omega)+2.9),$$

$\xi_9(\omega)=(X_9(\omega), X_9(\omega)+1.1, X_9(\omega)+2.3, X_9(\omega)+3.3), \xi_{10}(\omega)=(X_{10}(\omega), X_{10}(\omega)+1.2, X_{10}(\omega)+2.7, X_{10}(\omega)+3.8)$, where $X=(X_1, X_2, \ldots, X_{10})^T$ is a normal random vector with the expected value

$$\mu_0=(1.0109, 1.0137, 1.0143, 1.0144, 1.0149, 1.0154, 1.0172, 1.0178, 1.0182, 1.0197)^T,$$

and the covariance matrix

$$\Sigma=10^{-4}\times\begin{bmatrix} 1.8308 & -0.3636 & -1.4714 & 0.1487 & 1.2816 & -1.4605 & -0.5323 & 0.5965 & 2.9169 & -1.2269\\ -0.3636 & 3.9106 & -0.1342 & 0.0631 & -0.2694 & -0.2036 & -2.3608 & 4.1608 & -4.6358 & 3.0292\\ -1.4714 & -0.1342 & 5.2536 & -2.0527 & -3.0990 & 3.7798 & 6.7902 & -2.5452 & -1.4131 & 4.2781\\ 0.1487 & 0.0631 & -2.0527 & 7.0641 & 7.6329 & -1.7617 & -2.6202 & 0.6444 & 0.5765 & -6.3165\\ 1.2816 & -0.2694 & -3.0990 & 7.6329 & 9.1891 & -2.4171 & -3.1581 & 0.5798 & 3.4927 & -7.0019\\ -1.4605 & -0.2036 & 3.7798 & -1.7617 & -2.4171 & 10.1658 & 1.8840 & -2.3015 & 2.8006 & 9.4219\\ -0.5323 & -2.3608 & 6.7902 & -2.6202 & -3.1581 & 1.8840 & 14.2569 & -9.1860 & 2.6510 & 2.1856\\ 0.5965 & 4.1608 & -2.5452 & 0.6444 & 0.5798 & -2.3015 & -9.1860 & 15.6602 & -8.2035 & -1.1481\\ 2.9169 & -4.6358 & -1.4131 & 0.5765 & 3.4927 & 2.8006 & 2.6510 & -8.2035 & 16.2252 & -2.1352\\ -1.2269 & 3.0292 & 4.2781 & -6.3165 & -7.0019 & 9.4219 & 2.1856 & -1.1481 & -2.1352 & 25.9052 \end{bmatrix}.$$

The above portfolio selection problem is formulated by E model (12) and D model (13), respectively. By the fuzzy random security returns, the vector u and the matrix M in optimization models (12) and (13) are computed as follows

$$u=(1.4609, 1.5387, 1.6643, 1.8144, 1.8899, 2.1654, 2.2922, 2.4928, 2.6682, 2.9447)^T$$

and

$$M=\begin{bmatrix} 0.1954 & 0.2124 & 0.2748 & 0.3350 & 0.3826 & 0.4898 & 0.5372 & 0.6326 & 0.7280 & 0.8522\\ 0.2126 & 0.2328 & 0.2992 & 0.3650 & 0.4162 & 0.5342 & 0.5868 & 0.6904 & 0.7920 & 0.9294\\ 0.2748 & 0.2992 & 0.4094 & 0.4846 & 0.5520 & 0.6942 & 0.7590 & 0.8886 & 1.0206 & 1.1932\\ 0.3350 & 0.3650 & 0.4846 & 0.5848 & 0.6666 & 0.8430 & 0.9228 & 1.0852 & 1.2468 & 1.4572\\ 0.3828 & 0.4162 & 0.5520 & 0.6666 & 0.7606 & 0.9620 & 1.0532 & 1.2390 & 1.4244 & 1.6648\\ 0.4898 & 0.5342 & 0.6942 & 0.8430 & 0.9622 & 1.2336 & 1.3512 & 1.5888 & 1.8280 & 2.1426\\ 0.5372 & 0.5858 & 0.7588 & 0.9228 & 1.0532 & 1.3512 & 1.4848 & 1.7420 & 2.0060 & 2.3500\\ 0.6326 & 0.6904 & 0.8886 & 1.0852 & 1.2390 & 1.5888 & 1.7420 & 2.0552 & 2.3588 & 2.7660\\ 0.7280 & 0.7920 & 1.0206 & 1.2468 & 1.4244 & 1.8280 & 2.0060 & 2.3588 & 2.7186 & 3.1826\\ 0.8524 & 0.9294 & 1.1934 & 1.4572 & 1.6648 & 2.1426 & 2.3500 & 2.7660 & 3.1826 & 3.7372 \end{bmatrix}.$$

As a consequence, the proposed E model (12) is solved by Lingo software and the optimal solutions are reported in Table 1. Moreover, for each given value of the parameter ψ, the optimal solutions shown in Table 1 are global due to the convexity of optimization problem (12).

Table 1: Allocation proportions to 10 securities for model (12) (%)

ψ	x_1	x_2	x_7	x_8	x_{10}
0.0977	100	0	0	0	0
0.099	92.41078	7.58922	0	0	0
0.108	42.69589	57.30411	0	0	0
0.197	0	80.19929	19.80071	0	0
0.475	0	33.06397	66.93603	0	0
0.690	0	5.93569	94.06431	0	0
0.789	0	0	82.22562	17.77438	0
0.987	0	0	13.11673	86.88327	0
1.182	0	0	0	79.07968	20.92032
1.588	0	0	0	30.10729	69.89271
1.701	0	0	0	17.67586	82.32414
1.869	0	0	0	0	100

From Table 1, it is always wise for the investor with medium acceptable risk level to invest his fund in different securities to avoid the potential risk. However, the extreme cases inevitably exist for the risk-averse investor and the adventurous investor. Based on the safety-critical thought, the former would like to invest all the fund in the security with the lowest risk in spite of the lowest expected return. Meanwhile, the adventurous investor prefers to invest all the fund in the security with the highest expected return even if he has to bear the worst and most imprudent risk. Furthermore, the optimal investment portfolio shifts from the securities with the lower expected return to those with the higher expected return as the maximum acceptable risk level increases. Such a computational result is consistent with the theoretical analysis that the optimal value of the objective function in the optimization problem (12) monotonically increases with respect to the acceptable risk level ψ.

On the other hand , the D model (13) is also solved by Lingo software and, owing to the convexity of the optimization problem (13), the global optimal solutions are collected in Table 2 for each given value of parameters φ.

Table 2: Allocation proportions to 10 securities for model (13) (%)

φ	x_1	x_2	x_7	x_8	x_{10}
1.300	100	0	0	0	0
1.475	81.87661	18.12339	0	0	0
1.536	3.47044	96.52956	0	0	0
1.678	0	81.51294	18.48706	0	0
1.864	0	56.82814	43.17186	0	0
2.291	0	0.15926	99.84074	0	0
2.312	0	0	90.12961	9.87039	0
2.396	0	0	48.25523	51.74477	0
2.488	0	0	2.39282	97.60718	0
2.588	0	0	0	85.57203	14.42797
2.944	0	0	0	0.15490	99.84510
2.9447	0	0	0	0	100

Table 2 shows that investment diversification can effectively reduce the potential risk to portfolio. If all the fund is invested in only one security, then the portfolio will sustain the full brunt of the decline when the security suffers a serious downturn. Therefore, splitting the investment fund in different securities can enhance the capacity of the investor to take the potential risk. The extreme cases still exist for conservative or greedy investors, and the optimal portfolio still shifts from the securities with the lower expected return to those with the higher expected return due to the increasing objective function in the problem (13) with respect to the minimum acceptable level φ. This result is in accordance with the principle of risk-return tradeoff that potential return rises with an increase in risk. The low levels of the investment risk are associated with the low potential returns and vice versa. Therefore, the invested fund can render a higher profit only if investor is willing to accept the possibility of the losses probably caused by potential risk.

5 Comparative results

This section will carry out the comparative study empirically in two aspects. One is the comparison between the proposed QD and variance to show the similarities and the essential differences, and then to distinguish the proposed mean-QD model from the mean-variance model in [7]. The other one is the full comparison between the proposed QD and the existing risk criteria to show how to identify the most suitable risk criterion for each specific agency problem.

Firstly, both the QD and variance [7] are valid risk criteria and are minimized in optimization problems in the presence of hybrid uncertainty. Both techniques have the same modeling idea that is aimed at balancing potential risk and return, that is, seeking for the risk-return tradeoff. Such a modeling idea is in accordance with the psychology of decision makers in practical decision-making activities. However, there are some differences between them in analytical expressions and mathematical properties, just as stated in Table 3. The differences between the proposed QD and variance is exactly the reason why mean-QD technique and mean-variance technique perform differently. As above, both techniques are distinguished from each other in certain aspects, such as mathematical structure and properties, computational complexity, and solution methods, just as reported in Table 4.

Secondly, the proposed QD is empirically compared with three typical existing risk criteria, that is, variance [7] in the framework of fuzzy random theory, spread [30] and second moment [31] in the framework of fuzzy theory. On the one hand, variance and the QD are based on the same theoretical basis, but differ in mathematical properties and computational complexity. The comparison between variance and the QD is discussed in Table 3, so it will not be

Table 3: Comparison between QD and variance in [7]

	Variance	Quadratic deviation (QD)
Analytical expression	Nonlinear function	Quadratic function
Fuzzy term	Mutually independent	Mutually independent
Random term	Mutually independent	Interdependent
Convex or concave	Neither	convex

Table 4: Comparison between mean-QD model and mean-variance model in [7]

	Mean-variance model	Mean-QD model
Risk criterion	Variance	Quadratic deviation (QD)
Equivalent programming	Nonlinear programming	Convex quadratic programming
Solution method	Genetic algorithm	Conventional solution methods
Optimal solution	Approximate solution	Global and accurate solution

repeated here. On the other hand, spread and second moment are different from the QD in theoretical bases, but they have the similarities in mathematical properties and advantages in computing. Actually, the QD is an extension of spread and second moment in the framework of fuzzy random theory. Therefore, these three concepts are essentially the same, but defined in different theoretical environments.

To quantify potential risk in fuzzy environment, spread of a fuzzy variable ξ was proposed by using Lebesgue–Stieltjes integral in [30], that is,

$$\mathrm{Sp}[\xi] = \int_{(-\infty,+\infty)} (r - \mathrm{E}[\xi])^2 \mathrm{d}\Phi(r),$$

where $\Phi(r) = \mathrm{Cr}\{\xi \leq r\}$ is a monotone increasing function of r associated with the fuzzy variable ξ. The research findings in [30] show that, spread is quadratic convex function with respect to fuzzy parameters for trapezoidal and triangular fuzzy variables, respectively. Based on the definition of spread and the interconnection between variance and it [29], spread is valid as a new risk criteria in fuzzy decision systems, and the expectation-spread models established for portfolio optimization problems with fuzzy returns can be equivalently turned to convex quadratic programming problems, given that the return parametric matrix is full row rank.

Motivated by such great mathematical properties, the concept of spread was extended to describe the variation of a reduced fuzzy variable in the framework of type-2 fuzzy theory by defining second moment [31], that is,

$$\mathrm{M}_2[\xi] = \int_{(-\infty,+\infty)} (t - \mathrm{E}[\xi])^2 \mathrm{d}\left(\mathrm{Cr}\{\xi \leq t\}\right),$$

where ξ is a reduced fuzzy variable of type-2 fuzzy variable, and $\mathrm{Cr}\{\xi \leq t\}$ is a monotone increasing function of t associated with ξ. According to the research results in [31], as an extension of spread, second moment has the same mathematical properties as spread, and the reward-risk and risk-reward models based on second moment are consistent with expectation-spread models in essence including similar structure and computational advantages.

Following the existing work, this paper further extends the essential core of spread and second moment from fuzzy environment to fuzzy random environment, and defines the more general risk criterion, that is, the QD of a fuzzy random variable. As expected, the convexity and main results about spread and second moment still hold for the QD, in spite of the different theoretical basis. Meanwhile, the convex properties and computational advantages of mean-QD optimization models are also verified in both theoretical and practical aspects. Therefore, the proposed QD is feasible and valid as a risk criterion to deal with hybrid uncertainty. The extension of risk criteria made in this paper not only verifies the robustness of the essential core, but also contributes to quantifying potential risk in optimization problems.

Based on the brief introductions, the main similarities and differences between variance, spread, second moment, and the proposed QD are basically summarized in Table 5. In a word, all these risk criteria associated with different theoretical bases are applicable to measure potential risk level, but in different settings. Therefore, identifying the suitable risk criterion for each specific optimization problem is very crucial. The comparative results in Table 5 provide decision makers with a valuable guidance on how to select the suitable risk criterion for a certain optimization problem.

Table 5: Comparison between this work and the existing work in [7],[30] and [31]

	[7]	[30]	[31]	This work
Year	2009	2011	2012	2019
Risk index	Variance	Spread	Second moment	Quadratic deviation
Object	Fuzzy random variable	Fuzzy variable	Reduced fuzzy variable of a type-2 fuzzy variable	Fuzzy random variable
Theoretical basis	Fuzzy random theory	Fuzzy theory	Fuzzy theory	Fuzzy random theory
Analytical expression	Nonlinear function	Quadratic function	Quadratic function	Quadratic function
Convex or concave	Neither	Convex	Convex	Convex
Type of security returns	Fuzzy random return	Fuzzy return	Type-2 fuzzy return	Fuzzy random return
Solution method	Genetic algorithm	Conventional solution methods, e.g. Lemke's complementary pivoting algorithm	Conventional solution methods, e.g. cutting plane method	Conventional solution methods, e.g. active-set method
Optimal solution	Approximate	Global and accurate	Global and accurate	Global and accurate

6 Conclusions

This paper defined a new risk criterion, called the QD, under hybrid uncertainty. It is important that the analytical formula of the new risk criterion is convex with respect to critical parameters. Just because of the desirable convex property, the proposed three mean-QD optimization models for portfolio selection problems can be converted equivalently to their convex parametric quadratic programming problems, which can be solved by conventional optimization methods such as the active-set method to get the global and accurate optimal solutions. Consequently, the computational results illustrated the efficiency of the QD as a risk criterion and that it really reduces the computational complexity that cannot be avoided when variance is used as a risk criterion. Furthermore, the comparative results between the proposed mean-QD technique and mean-variance technique demonstrated the consistency of the optimal results in both the techniques from three aspects, that is, investment diversification, the degree of investment, and the trend of optimal allocation. The comparative results between these four types of risk criteria including the proposed QD, variance, spread, and second moment reported the similarities and differences between them, which can guide decision makers on how to distinguish their respective application scopes and how to identify the suitable risk criterion for each optimization problem. In short, the proposed QD as a risk criterion has some advantages over variance in quadratic analytical expression and convexity, which are the essential factors in achieving the low computational complexity and the global optimal solutions. Especially, the realization of the optimality is the major motivation for decision makers to engage in economic activities.

Acknowledgement

The work of Dr. Wu was supported by the National Natural Science Foundation of China under Grant No.71601141, the State Scholarship Fund from China Scholarship Council, and the Program for Innovative Research Team in University of Tianjin (No. TD13-5038). The work of Dr. Liu was supported by the National Natural Science Foundation of China under Grant No.61773150.

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