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Extended and infinite ordered weighted averaging and sum operators with numerical examples

X. Pu[1], Z. Wang[2], R. Mesiar[3], R. R. Yager[4], J. Qin[5], X. Feng[6] and L. Jin[7]

[1] *College of Science, Nanjing Forestry University, Nanjing, China*
[2] *Development Committee, Nanjing Normal University, Nanjing, China*
[3] *Faculty of Civil Engineering, Slovak University of Technology, Radlinskho 11, Czech Republic*
[4] *Machine Intelligence Institute, Iona College, New Rochelle, NY 10801*
[5] *School of Management, Wuhan University of Technology, Wuhan, China*
[6,7] *Business School, Nanjing Normal University, Nanjing, China*

puxingting@163.com, 22028@njnu.edu.cn, radko.mesiar@stuba.sk,yager@panix.com, qinjindongseu@126.com, feng_xq@njnu.edu.cn, jls1980@163.com

Abstract

This study discusses some variants of Ordered Weighted Averaging (OWA) operators and related information aggregation methods. In detail, we define the Extended Ordered Weighted Sum (EOWS) operator and the Extended Ordered Weighted Averaging (EOWA) operator, which are applied in scientometrics evaluation where the preference is over finitely many representative works. As contrast, we also define the Infinite Ordered Weighted Sum (InOWS) operator and the Infinite Ordered Weighted Averaging (InOWA) operator, which are more suitable for the corresponding scientometrics evaluation where all of works of scholars are considered. We also define the family of Infinite Gaussian maxitive OWA weights function and the family of Infinite Gaussian OWA weights function, and discuss some of their mathematical properties. Some illustrative examples, comparisons and figures are provided to better expound their applicability in scientometrics evaluation.

Keywords: Aggregation function, decision making, evaluation, information fusion, ordered weighted averaging operators.

1 Introduction

Information Fusion Functions [4, 6, 7, 9, 8, 10, 11, 12, 19, 25, 27] are the most fundamental and powerful tools in a large range of evaluation problems and decision making practices. As a simpler paradigm of information fusion process, once the input information x to be aggregated is obtained, we need to select some suitable (family of) information fusion functions F to merge those input information, and then return back a final aggregated result $F(x)$.

Such fusion result can have different practical meanings in different decision making scenarios. For example, in venture investment, the fusion results can be expected values, meanwhile, the involved information fusion functions F are intimately related to some probability information; in preference involved decision making [6, 17], the aggregated results can be seen as the subjective anticipations of some decision makers; and in numerous evaluation problems, after merging collected single evaluations for each criterion, the resulting value is properly seen as the comprehensive evaluation result for certain object under evaluation. The applications of information fusion functions and techniques can be found in more areas such as pattern recognition [12], water quality evaluation [17], and scientometrics [1, 2, 3, 5, 20].

Without loss of generality, in this study, the involved input information under fusing will be conveniently and strictly defined by a non-negative function: $x^{<n>} : N^{<n>} \to [0, 1]$, called Input Function (of dimension n), where we take the denotations $N^{<n>} = \{1, ..., n\}$ $(n \geq 1)$. The set of all such functions $x^{<n>}$ is denoted by $\mathcal{X}^{<n>}$ $(n \geq 1)$; then, we also

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define $\mathcal{X} = \{x^{<n>} \in \mathcal{X}^{<n>} | n \in \mathrm{N}\}$, where N is the set of natural numbers, and we call $\mathcal{X}$ the extended set of all input functions of finite dimensions.

Ordered Weighted Averaging (OWA) operators [25] are widely used both in many areas of real applications and in theoretical studies [7, 8, 10, 12, 13, 14, 15, 16, 17, 21, 23, 22, 24, 28, 26, 27, 29]. Fixing n, an OWA operator to $x^{<n>} \in \mathcal{X}^{<n>}$ with OWA weights function $w^{<n>}$ is defined to be a mapping $OWA_{w^{<n>}} : \mathcal{X}^{<n>} \to [0, 1]$ such that

$$OWA_{w^{<n>}}(x^{<n>}) = \sum\nolimits_{r=1}^{n} w^{<n>}(r) \cdot x^{<n>}(\tau(r)) \tag{1}$$

where $w^{<n>} : N^{<n>} \to [0, 1]$ with $\sum_{r=1}^{n} w^{<n>}(r) = 1$, and $\tau : N^{<n>} \to N^{<n>}$ can be any suitable permutation such that $x^{<n>}(\tau(s)) \geq x^{<n>}(\tau(t))$ whenever $s < t$. In addition, the set of all such normalized OWA weights functions $w^{<n>}$ of dimension n is denoted by $\mathcal{W}^{(n)}$. Conventionally, a weights function $w^{<n>}$ with $\sum_{r=1}^{n} w^{<n>}(r) = 1$ is also known as an additive weights function, and, in comparison, a weights function $v^{<n>}$ with $\max(v^{<n>}(r))_{r=1}^{n} = 1$ is known as a maxitive weights function.

As an alternative information fusion method of OWA operator, we next review Ordered Weighted Sum (OWS) operators [22] with maxitive weights function. Fixing n, an OWS operator to $x^{<n>} \in \mathcal{X}^{<n>}$ with maxitive OWA weights function $v^{<n>}$ is defined to be a mapping $OWS_{v^{<n>}} : \mathcal{X}^{<n>} \to [0, n]$ such that

$$OWS_{v^{<n>}}(x^{<n>}) = \sum\nolimits_{r=1}^{n} v^{<n>}(r) \cdot x^{<n>}(\tau(r)) \tag{2}$$

where $v^{<n>} : N^{<n>} \to [0, 1]$ with $\max(v^{<n>}(r))_{r=1}^{n} = 1$. In addition, the set of all such maxitive weights OWA weights functions $v^{<n>}$ of dimension n is denoted by $\mathcal{V}^{<n>}$.

OWA and OWS operators can be appropriately used in different evaluation applications such as scientometrics and confidence evaluation [22].

Note that the above discussed OWA and OWS operators are all performed to input function $x^{<n>} \in \mathcal{X}^{<n>}$ with fixed n; thus, their usages in many evaluation problems is limited. For example, in scientometrics, the scientific outcome of a certain scholar (say, Scholar A) can be easily expressed by an input function, $x^{<n>} \in \mathcal{X}^{<n>}$, where n is fixed and represents the total number of scientific works of that scholar. Moreover, $x^{<n>}(r)$ is the single evaluation for her academic work r, which can be obtained or expressed using many different scientometric indices such as (i) the quality of journal that published it, (ii) the citation numbers of it, and (iii) the subjective evaluation from some expert. Similarly, for another scholar (say, Scholar B), she may have m ($m \neq n$) piece of scientific works; thus, $y^{<m>} \in \mathcal{X}^{<m>}$ represents the detailed performance of her works with both quantity and quality information contained in it.

When we need to compare Scholar A with Scholar B (considering both quantity and quality of their recorded works), a suitable choice is to use OWA operator (or OWS operator) to $x^{<n>}$ and $y^{<m>}$. However, when $m \neq n$, generally OWA and OWS operators of a fixed dimension can not be suitably performed to both of them. Therefore, this study will well consider this situation and propose some appropriate definitions to help perform corresponding fusions in a consistent way. Around this evaluation problem, in this study, we will propose the Extended Ordered Weighted Averaging and Sum operators, discuss their applicability in scientometrics evaluation, and analyze related mathematical properties that have practical explanations in some academic works evaluation problems.

The remainder of this work is organized as follows. Section 2 defines Extended Ordered Weighted Averaging and Sum operators with some illustrative examples. In Section 3, we define the family of Infinite Gaussian maxitive OWA weights functions and the family of Infinite Gaussian OWA weights function, and then discuss their usage in scientometric evaluation. Section 4 concludes this study by summarizing the main results.

2 Extended ordered weighted averaging and sum operators

The following definition for the extended OWS operator allows us to aggregate input functions $x^{<n>}$ where n is no longer restricted to be fixed.

Definition 2.1. *For any input function $x^{<n>} \in \mathcal{X}^{<n>}$ $(n \geq 1)$, the Extended Ordered Weighted Sum (EOWS) operator to $x^{<n>}$ with the maxitive OWA weights function $v^{<m>} \in \mathcal{V}^{(m)}$ $(m \geq 1)$, is defined to be a mapping $EOWS_{v^{<m>}} : \mathcal{X}^{<n>} \to [0, n]$ such that*

$$EOWS_{v^{<m>}}(x^{<n>}) = \sum\nolimits_{r=1}^{\min(m,n)} v^{<m>}(r) \cdot x^{<n>}(\tau(r)) \tag{3}$$

where $v^{<m>} : N^{<m>} \to [0, 1]$ with $\max\{v^{<m>}(r)\}_{r=1}^{m} = 1$; and $\tau : N^{<n>} \to N^{<n>}$ can be any suitable permutation such that $x^{<n>}(\tau(s)) \geq x^{<n>}(\tau(t))$ whenever $s < t$.

Example 2.2. *For any $v^{<m>} \in \mathcal{V}^{<m>}$ $(m \geq 1)$, if it is monotonic non-increasing, then, as an alternative evaluation method, we can suitably apply OWS operator to evaluate the academic works of different scholars. Suppose $v^{<5>} \in \mathcal{V}^{<5>}$ being a maxitive OWA weights function such that $(v^{<5>}(r))_{r=1}^{5} = (1, 0.8, 0.6, 0.4, 0.2)$, indicating that, in this evaluation, we actually only consider 5 representative works of any scholar under evaluation. Suppose $x_1^{<4>} \in \mathcal{X}^{<4>}$ with $(x_1^{<4>}(r))_{r=1}^{4} = (0.6, 0.3, 1, 0.6)$, representing Scholar 1 has only four published papers in total, each with quality $x_1^{<4>}(r)$, respectively; and suppose $x_2^{<6>} \in \mathcal{X}^{<6>}$ with $(x_2^{<6>}(r))_{r=1}^{6} = (0.4, 0.3, 0.4, 0.3, 0.6, 0.5)$, representing Scholar 2 has six published papers in total, each with quality $x_2^{<6>}(r)$, respectively.*

Then, using formula (3) we have the evaluation for up to 5 representative works of Scholar 1 by:

$$EOWS_{v^{<5>}}(x_1^{<4>}) = \sum_{r=1}^{\min\{5,4\}} v^{<5>}(r) \cdot x_1^{<4>}(\tau_1(r)) = (1)(1) + (0.8)(0.6) + (0.6)(0.6) + (0.4)(0.3) = 1.96$$

(where τ_1 is a suitable permutation related to $x_1^{<4>}$).
Similarly, we obtain the evaluation for up to 5 representative works of Scholar 2 by:

$$EOWS_{v^{<5>}}(x_2^{<6>}) = \sum_{r=1}^{\min\{5,6\}} v^{<5>}(r) \cdot x_2^{<6>}(\tau_2(r)) = (1)(0.6) + (0.8)(0.5) + (0.6)(0.4) + (0.4)(0.4) + (0.2)(0.3) = 1.46$$

(where τ_2 is a suitable permutation related to $x_2^{<6>}$).

Therefore, with the evaluations under such setting, we claim that the representative works (up to 5) of Scholar 1 is better than that of Scholar 2. Note that if $(v^{<5>}(r))_{r=1}^{5} = (1, 0, 0, 0, 0)$, then $EOWS_{v^{<5>}}(x_1^{<4>}) = \max(x_1^{<4>}(r))_{r=1}^{4} = 1$ and $EOWS_{v^{<5>}}(x_2^{<6>}) = \max(x_2^{<6>}(r))_{r=1}^{6} = 0.6$, indicating that the most representative one work of Scholar 1 is better than that of Scholar 2. Moreover, if $(v^{<5>}(r))_{r=1}^{5} = (1, 1, 1, 1, 1)$, then $EOWS_{v^{<5>}}(x_1^{<4>}) = \sum_{r=1}^{4} x_1^{<4>}(r) = 2.5$ and $EOWS_{v^{<5>}}(x_2^{<6>}) = \sum_{r=1}^{\min\{5,6\}} x_2^{<6>}(r) = 2.2$, showing that the summed performance of up to 5 representative works of Scholar 1 is still better than that of Scholar 2.

Remark 2.3. *Example 2.2 also shows that scholars (like Scholar 2) often may complain or question the fairness of certain evaluation method, since only few representative works are allowed to consider. In most of occasions, there is no absolutely perfect evaluation method; thus, devising more reasonable and well-designed evaluation methods to be flexibly and contingently chosen by decision makers, is meaningful and helpful.*

Definition 2.4. *For any input function $x^{<n>} \in \mathcal{X}^{<n>}$ $(n \geq 1)$, the Extended Ordered Weighted Averaging (EOWA) operator to $x^{<n>}$ with the OWA weights function $w^{<m>} \in \mathcal{W}^{<m>}$ $(m \geq 1)$, is defined to be a mapping $EOWA_{w^{<m>}} : \mathcal{X}^{<n>} \to [0, 1]$ such that*

$$EOWA_{w^{<m>}}(x^{<n>}) = \sum_{r=1}^{\min(m,n)} w^{<m>}(r) \cdot x^{<n>}(\tau(r)) \tag{4}$$

where $w^{<m>} : N^{<m>} \to [0, 1]$ with $\sum_{r=1}^{m} w^{<m>}(r) = 1$; and $\tau : N^{<n>} \to N^{<n>}$ can be any suitable permutation such that $x^{<n>}(\tau(s)) \geq x^{<n>}(\tau(t))$ whenever $s < t$.

Remark 2.5. *Given any maxitive OWA weights function $v^{<m>} \in \mathcal{V}^{(m)}$ $(m \geq 1)$, the direct normalization of it can generate corresponding OWA weights function $w^{<m>} \in \mathcal{W}^{<m>}$ with $w^{<m>}(r) = v^{<m>}(r) / \sum_{s=1}^{m} v^{<m>}(s)$. In addition, the $w^{<m>}$ obtained in such way, is then called the normalization of $v^{<m>}$. In Example 2.2, if we replace $v^{<m>}$ with its normalization, it is easy to observe that the comparisons for Scholar 1 and Scholar 2 remain the same.*

Remark 2.6. *Given any OWA weights function $w^{<m>}$, in EOWA environment we may, when necessary, still take Yager orness definition [25] or some of its variants; that is, $orness(w^{<m>}) = \sum_{r=1}^{m} w^{<m>}(r) \cdot \frac{n-r}{n-1}$. In scientometrics evaluation as discussed in Example 2.2, the orness of $w^{<m>}$ embodies some similar optimistic/pessimistic preference as usual.*

Remark 2.7. *For any $x_1^{<n_1>} \in \mathcal{X}^{<n_1>}$ and $x_2^{<n_2>} \in \mathcal{X}^{<n_2>}$, let $\tau_j : N^{<n>} \to N^{<n>}$ $(j = 1, 2)$ be any suitable permutation such that $x_j^{<n_j>}(\tau_j(s)) \geq x_j^{<n_j>}(\tau_j(t))$ whenever $s < t$. If (i) $n_1 \leq n_2$, and (ii) for any $r \in N^{<n_1>}$ we have $x_1^{<n_1>}(\tau_1(r)) \leq x_2^{<n_2>}(\tau_2(r))$, then $EOWS_{v^{<m>}}(x_1^{<n_1>}) \leq EOWS_{v^{<m>}}(x_2^{<n_2>})$ and $EOWS_{w^{<m>}}(x_1^{<n_1>}) \leq EOWS_{w^{<m>}}(x_2^{<n_2>})$ hold for any $v^{<m>} \in \mathcal{V}^{<m>}$ $(m \geq 1)$ and their normalizations $w^{<m>} \in \mathcal{W}^{<m>}$, respectively.*

As discussed in Example 2.2, $v^{<m>}$ $(w^{<m>})$ implies that we only consider up to m representative works of each scholar, though that consideration may contain different (relative) importance for evaluated works. However, that method in Example 2.2 clearly is not the only one to be accepted. In many research fields, both quality and productivity of works are important. Given any $m \geq 1$, ordinarily, there always exist some scholars who have n works (with $n > m$); thus, if one scholar has n good works, then there are $n - m$ works of her that can not be considered in EOWS or EOWA operators. Therefore, if we want to consider all of the works of any scholar who is under evaluation, and we are still willing to use OWA aggregation, then we may devise the corresponding versions of EOWS and EOWA operators in infinite environment.

Definition 2.8. *For any input function $x^{<n>} \in \mathcal{X}^{<n>}$ $(n \geq 1)$, the Infinite Ordered Weighted Sum (InOWS) operator to $x^{<n>}$ with the Infinite maxitive OWA weights function v, is defined to be a mapping $InOWS_v : \mathcal{X}^{<n>} \to [0, n]$ such that*

$$InOWS_v(x^{<n>}) = \sum_{r=1}^{n} v(r) \cdot x^{<n>}(\tau(r)) \tag{5}$$

where $v : \mathrm{N} \to [0, 1]$ satisfies $\sup (v(r))_{r\in\mathrm{N}} = 1$; and $\tau : N^{<n>} \to N^{<n>}$ can be any suitable permutation such that $x^{<n>}(\tau(s)) \geq x^{<n>}(\tau(t))$ whenever $s < t$. We also denote by $\mathcal{V}$ the space of all infinite maxitive OWA weights functions.

Definition 2.9. *For any input function $x^{<n>} \in \mathcal{X}^{<n>}$ $(n \geq 1)$, the Infinite Ordered Weighted Averaging (InOWA) operator to $x^{<n>}$ with the Infinite OWA weights function w, is defined to be a mapping $InOWA_w : \mathcal{X}^{<n>} \to [0, 1]$ such that*

$$InOWA_w(x^{<n>}) = \sum_{r=1}^{n} w(r) \cdot x^{<n>}(\tau(r)) \tag{6}$$

where $w : \mathrm{N} \to [0, 1]$ satisfies $\sum_{r\in\mathrm{N}} w(r) = 1$; and $\tau : N^{<n>} \to N^{<n>}$ can be any appropriate permutation such that $x^{<n>}(\tau(s)) \geq x^{<n>}(\tau(t))$ whenever $s < t$. We also denote by $\mathcal{W}$ the space of all infinite OWA weights functions.

Example 2.10. *For any $v \in \mathcal{V}$ (or $w \in \mathcal{W}$), if it is monotonic non-increasing and positive, then we can suitably apply InOWS (or InOWA) operator with it to evaluate the academic works of different scholars; hence, in this situation, actually we will consider "all" of the works of any certain scholar, irrespective of the number of the works she has published. Assuming $(v(r))_{r=1}^{\infty} = (r^{-1})_{r=1}^{\infty}$, $(w(r))_{r=1}^{\infty} = (2^{-r})_{r=1}^{\infty}$, and $(x^{<5>}(r))_{r=1}^{5} = (0.5, 0.6, 0.8, 0.6, 1)$, then we have*

$$InOWS_v(x^{<5>}) = \sum_{r=1}^{5} v(r) \cdot x^{<5>}(\tau(r)) = (1)(1) + (1/2)(0.8) + (1/3)(0.6) + (1/4)(0.6) + (1/5)(0.5) = 1.85;$$

$$InOWA_w(x^{<5>}) = \sum_{r=1}^{5} w(r) \cdot x^{<5>}(\tau(r)) = (1/2)(1) + (1/4)(0.8) + (1/8)(0.6) + (1/16)(0.6) + (1/32)(0.5) = 0.828125.$$

Remark 2.11. *In Example 2.10, once $v \in \mathcal{V}$ (or $w \in \mathcal{W}$) is positive, then all of the works of any scholar can be considered. In addition, different weights functions embody different preferences; for example, the preferences, from optimistic or representativeness considerations, to gradually moderate and comprehensive considerations over all works, will be discussed in Example 3.12.*

3 Families of gaussian weights functions

Families of OWA weights functions [26, 29] provided immense flexibility and controllability in decision aid practices, where OWA operators were used and played a pivotal role. On the other hand, some Gaussian distribution related OWA weights functions have been discussed and proved to be useful and reasonable [17, 24]. In this section, we will discuss the family of OWA weights functions based on Gaussian function; in the meanwhile, we will propose a practical usage of it in scientometrics evaluation.

Next, we briefly review some properties about Gaussian function. Gaussian function is defined by $f_a(x) = e^{-ax^2}$ $(x \in (-\infty, +\infty)$, $a > 0)$ and belongs to Schwartz space. More generally, one notes that Gaussian function is moderate decrease (i.e., for any $a > 0$, there exists a constant $A > 0$ such that $|f_a(x)| \leq A(1+x^2)^{-1}$ for all $x \in (-\infty, +\infty)$) [18]. In addition, by selecting $a = \pi$, we have the normalization of Gaussian function, i.e., $f_\pi(x) = e^{-\pi x^2}$ with $\int_{-\infty}^{+\infty} e^{-\pi x^2} dx = 1$. Furthermore, it is interesting that $\hat{f}_\pi(\xi) = f_\pi(\xi)$ where $\hat{f}_\pi$ is the Fourier transform of f_π, i.e., $\hat{f}_\pi(\xi) = \int_{-\infty}^{+\infty} f_\pi(x)e^{-2\pi i x\xi} dx$ $(\xi \in (-\infty, +\infty))$ with Fourier inversion $f_\pi(x) = \int_{-\infty}^{+\infty} \hat{f}_\pi(\xi)e^{2\pi i x\xi} d\xi$ [18]. Gaussian function is widely used in many fields of science and technology.

We next define the family of Gaussian maxitive OWA weights functions, and rephrase the family of Gaussian OWA weights function [17, 24].

Definition 3.1. *The family of Gaussian maxitive OWA weights function of dimension m with parameter $\sigma \in (0, +\infty)$, $\{v_\sigma^{<m>}\}_{\sigma\in(0,+\infty)}$ $(v_\sigma^{<m>} \in \mathcal{V}^{<m>})$, is defined such that $v_\sigma^{<m>}(r) = e^{-\pi((r-1)/\sigma)^2}$.*

Definition 3.2. *[17, 24] The family of Gaussian OWA weights function of dimension m with parameter $\sigma \in (0, +\infty)$, $\{w_\sigma^{<m>}\}_{\sigma\in(0,+\infty)}$ $(w_\sigma^{<m>} \in \mathcal{W}^{(m)})$, is defined such that $w_\sigma^{<m>}(r) = \frac{e^{-\pi((r-1)/\sigma)^2}}{\sum_{s=1}^{m} e^{-\pi((s-1)/\sigma)^2}}$.*

It is evident that Gaussian OWA weights function is the normalization of Gaussian maxitive OWA weights function. As the uniform limits, we have

(i) $\lim_{\sigma\to\infty} v_\sigma^{<m>} = (1, 1, ..., 1)$, i.e., $\lim_{\sigma\to\infty} v_\sigma^{<m>}(r) = 1$ $(r \in N^{<m>})$;

(ii) $\lim_{\sigma\to 0} v_\sigma^{<m>} = (1, 0, ..., 0)$, i.e., $\lim_{\sigma\to 0} v_\sigma^{<m>}(1) = 1$ and $\lim_{\sigma\to 0} v_\sigma^{<m>}(r) = 0$ $(r \neq 1)$;

(iii) $\lim_{\sigma\to\infty} w_\sigma^{<m>} = (1/m, 1/m, ..., 1/m)$, i.e., $\lim_{\sigma\to\infty} w_\sigma^{<m>}(r) = 1/m$ $(r \in N^{<m>})$; and
(iv) $\lim_{\sigma\to 0} w_\sigma^{<m>} = (1, 0, ..., 0)$, i.e., $\lim_{\sigma\to 0} w_\sigma^{<m>}(1) = 1$ and $\lim_{\sigma\to 0} w_\sigma^{<m>}(r) = 0$ $(r \neq 1)$.

Next, we discuss some monotonicities of both types of Gaussian OWA weights function with parameter $\sigma \in (0, +\infty)$.

Proposition 3.3. *For any $m \geq 1$, any $x^{<n>} \in \mathcal{X}^{<n>}$, and any two σ_1, σ_2 such that $0 < \sigma_1 < \sigma_2$, we have $v_{\sigma_2}^{<m>} > v_{\sigma_1}^{<m>}$ and $EOWS_{v_{\sigma_2}^{<m>}}(x^{<n>}) \geq EOWS_{v_{\sigma_1}^{<m>}}(x^{<n>})$.*

Lemma 3.4. *For any $m \geq 1$ and any σ_1, σ_2 such that $0 < \sigma_1 < \sigma_2$, we have $\sum_{k=1}^{r} w_{\sigma_2}^{<m>}(k) < \sum_{k=1}^{r} w_{\sigma_1}^{<m>}(k)$ $(r \in N^{<m>})$.*

Proof. We argue by contradiction. Suppose there exists one $r_0 \in N^{<m>}$ such that $\sum_{k=1}^{r_0} w_{\sigma_2}^{<m>}(k) \geq \sum_{k=1}^{r_0} w_{\sigma_1}^{<m>}(k)$. □

Since $\frac{w_{\sigma_1}^{<m>}(r)}{w_{\sigma_1}^{<m>}(r+1)} = \frac{e^{-\pi((r-1)/\sigma_1)^2}}{e^{-\pi(r/\sigma_1)^2}} = e^{(\pi/\sigma_1^2)(2r-1)} > e^{(\pi/\sigma_2^2)(2r-1)} = \frac{w_{\sigma_2}^{<m>}(r)}{w_{\sigma_2}^{<m>}(r+1)}$ for all $r \in \{1, 2, ..., m-1\}$, then we claim that $w_{\sigma_1}^{<m>}(r_0) < w_{\sigma_2}^{<m>}(r_0)$. If not so, then by easy deduction we have $w_{\sigma_1}^{<m>}(r_0) \geq w_{\sigma_2}^{<m>}(r_0)$ and $w_{\sigma_1}^{<m>}(k) > w_{\sigma_2}^{<m>}(k)$ for all $k \in \{1, 2, ..., r_0 - 1\}$, which leads to $\sum_{k=1}^{r_0} w_{\sigma_2}^{<m>}(k) \leq \sum_{k=1}^{r_0} w_{\sigma_1}^{<m>}(k)$.

Thus, $w_{\sigma_1}^{<m>}(r_0) < w_{\sigma_2}^{<m>}(r_0)$ holds. Then, it is easy to deduct that $w_{\sigma_1}^{<m>}(k) < w_{\sigma_2}^{<m>}(k)$ for all $k = \{r_0+1, r_0+2, ..., m\}$, which leads to $\sum_{k=r_0+1}^{m} w_{\sigma_2}^{<m>}(k) > \sum_{k=r_0+1}^{m} w_{\sigma_1}^{<m>}(k)$.

Consequently, we have

$$1 = \sum_{k=1}^{r_0} w_{\sigma_2}^{<m>}(k) + \sum_{k=r_0+1}^{m} w_{\sigma_2}^{<m>}(k) > \sum_{k=1}^{r_0} w_{\sigma_1}^{<m>}(k) + \sum_{k=r_0+1}^{m} w_{\sigma_1}^{<m>}(k) = 1,$$

which is absurd. We proved the lemma.

Proposition 3.5. *For any $x^{<n>} \in \mathcal{X}^{<n>}$ and any two σ_1, σ_2 such that $0 < \sigma_1 < \sigma_2$, $EOWA_{w_{\sigma_2}^{<m>}}(x^{<n>}) \leq EOWA_{w_{\sigma_1}^{<m>}}(x^{<n>})$.*

Proof. Firstly, we define $x^{<n>}(\tau(\min(m, n)+1)) = 0$, no matter $m \geq n$ or $m \leq n$, where $\tau : N^{<n>} \to N^{<n>}$ can be any suitable permutation such that $\tau(x^{<n>}(s)) \geq \tau(x^{<n>}(t))$ whenever $s < t$. □

Then, for both of the two situations ($m \geq n$ or $m \leq n$), we always have

$$\begin{aligned}
EOWA_{w_{\sigma_2}^{<m>}}(x^{<n>}) &= \sum_{r=1}^{\min(m,n)} w_{\sigma_2}^{<m>}(r) \cdot x^{<n>}(\tau(r)) \\
&= \sum_{r=1}^{\min(m,n)} w_{\sigma_2}^{<m>}(r) \cdot \sum_{k=r}^{n} \left(x^{<n>}(\tau(k)) - x^{<n>}(\tau(k+1))\right) \\
&= \sum_{k=1}^{\min(m,n)} \left(x^{<n>}(\tau(k)) - x^{<n>}(\tau(k+1))\right) \cdot \sum_{r=1}^{k} w_{\sigma_2}^{<m>}(r) \\
&\leq \sum_{k=1}^{\min(m,n)} \left(x^{<n>}(\tau(k)) - x^{<n>}(\tau(k+1))\right) \cdot \sum_{r=1}^{k} w_{\sigma_1}^{<m>}(r) \text{ (by Lemma 3.4)} \\
&= \sum_{r=1}^{\min(m,n)} w_{\sigma_1}^{<m>}(r) \cdot \sum_{k=r}^{n} \left(x^{<n>}(\tau(k)) - x^{<n>}(\tau(k+1))\right) \\
&= \sum_{r=1}^{\min(m,n)} w_{\sigma_1}^{<m>}(r) \cdot x^{<n>}(\tau(r)) \\
&= EOWA_{w_{\sigma_1}^{<m>}}(x^{<n>}).
\end{aligned}$$

Next, we define the family of Infinite Gaussian maxitive OWA weights functions and the family of Infinite Gaussian OWA weights function, respectively; and then discuss their usage in scientometric evaluation.

Definition 3.6. *The family of Infinite Gaussian maxitive OWA weights function with parameter $\sigma \in (0, +\infty)$, $\{v_\sigma\}_{\sigma\in(0,+\infty)}$ $(v_\sigma \in \mathcal{V})$, is defined such that $v_\sigma(r) = e^{-\pi((r-1)/\sigma)^2}$.*

Recall Poisson summation formula implies that $\sum_{r=-\infty}^{\infty} f_a(r) = \sum_{r=-\infty}^{\infty} \hat{f}_a(r)$ where $\hat{f}_a$ is the Fourier transform of f_a [18]. Thus, as a further implication, it is easy to conclude that for any $\sigma \in (0, +\infty)$, series $\sum_{s=1}^{\infty} e^{-\pi((s-1)/\sigma)^2}$ converges, i.e., sequence $\left(\sum_{s=1}^{r} e^{-\pi((s-1)/\sigma)^2}\right)_{r=1}^{\infty}$ converges to a fixed real number. Note that the convergence can be also derived by the property of moderate decrease, or by integral test for convergence. Therefore, it makes the following definition feasible.

Definition 3.7. *The family of Infinite Gaussian OWA weights function with parameter $\sigma \in (0, +\infty)$, $\{w_\sigma\}_{\sigma\in(0,+\infty)}$ $(w_\sigma \in \mathcal{W})$, is defined such that $w_\sigma(r) = \frac{e^{-\pi((r-1)/\sigma)^2}}{\sum_{s=1}^{\infty} e^{-\pi((s-1)/\sigma)^2}}$.*

Remark 3.8. *As the pointwise limit forms (note that (i) is not the uniform limit), we have*
(i) $\lim_{\sigma\to\infty} v_\sigma = (1, 1, ...)$, i.e., $\lim_{\sigma\to\infty} v_\sigma(r) = 1$ $(r \in \mathrm{N})$;
(ii) $\lim_{\sigma\to 0} v_\sigma = (1, 0, 0, ...)$, i.e., $\lim_{\sigma\to 0} v_\sigma(1) = 1$ and $\lim_{\sigma\to 0} v_\sigma(r) = 0$ $(r \neq 1)$;
(iii) $\lim_{\sigma\to\infty} w_\sigma = (0, 0, ...)$, i.e., $\lim_{\sigma\to\infty} w_\sigma(r) = 0$ $(r \in \mathrm{N})$, containing some further analysis about compactness and completeness which will not be discussed in this study; and
(iv) $\lim_{\sigma\to 0} w_\sigma = (1, 0, 0, ...)$, i.e., $\lim_{\sigma\to 0} w_\sigma(1) = 1$ and $\lim_{\sigma\to 0} w_\sigma(r) = 0$ $(r \neq 1)$.

Since in practice, $x^{<n>}$ is always supported on a finite set $N^{<n>}$, then the above discussions for limits are clearly enough for practical usage.

Some properties about related monotonicities are listed as follows.

Proposition 3.9. *For any $x^{<n>} \in \mathcal{X}^{<n>}$ and any two σ_1, σ_2 such that $0 < \sigma_1 < \sigma_2$, we have $v_{\sigma_2} > v_{\sigma_1}$ and $InOWS_{v_{\sigma_2}}(x^{<n>}) \geq InOWS_{v_{\sigma_1}}(x^{<n>})$.*

The lemma below has some similarity to Lemma 3.4, but it will deal with infiniteness rather than finiteness.

Lemma 3.10. *For any two σ_1, σ_2 such that $0 < \sigma_1 < \sigma_2$, we have $\sum_{k=1}^{r} w_{\sigma_2}(k) < \sum_{k=1}^{r} w_{\sigma_1}(k)$ $(r \in \mathrm{N})$.*

Proof. We still argue by contradiction which is similar to what has been done in Lemma 3.4. Suppose there exists one $r_0 \in \mathrm{N}$ such that $\sum_{k=1}^{r_0} w_{\sigma_2}(k) \geq \sum_{k=1}^{r_0} w_{\sigma_1}(k)$. □

Since $\frac{w_{\sigma_1}(r)}{w_{\sigma_1}(r+1)} = \frac{e^{-\pi((r-1)/\sigma_1)^2}}{e^{-\pi(r/\sigma_1)^2}} = e^{(\pi/\sigma_1^2)(2r-1)} > e^{(\pi/\sigma_2^2)(2r-1)} = \frac{w_{\sigma_2}^{(}r)}{w_{\sigma_2}(r+1)}$ for all $r \in \mathrm{N}$, then we claim that $w_{\sigma_1}(r_0) < w_{\sigma_2}(r_0)$. If not so, then by easy deduction we have $w_{\sigma_1}(r_0) \geq w_{\sigma_2}(r_0)$ and $w_{\sigma_1}(k) > w_{\sigma_2}(k)$ for all $k \in \{1, 2, ..., r_0 - 1\}$, which leads to $\sum_{k=1} r_0 w_{\sigma_2}(k) \leq \sum_{k=1} r_0 w_{\sigma_1}(k)$.

Thus, $w_{\sigma_1}(r_0) < w_{\sigma_2}(r_0)$ holds. Then, it is easy to deduct that $w_{\sigma_1}(k) < w_{\sigma_2}(k)$ for all $k = \{r_0 + 1, r_0 + 2, ...\}$, which leads to $\sum_{k=r_0+1}^{\infty} w_{\sigma_2}(k) > \sum_{k=r_0+1}^{\infty} w_{\sigma_1}(k)$.

Consequently, we have

$$1 = \sum_{k=1}^{r_0} w_{\sigma_2}(k) + \sum_{k=r_0+1}^{\infty} w_{\sigma_2}(k) > \sum_{k=1}^{r_0} w_{\sigma_1}(k) + \sum_{k=r_0+1}^{\infty} w_{\sigma_1}(k) = 1,$$

which is absurd. We proved the lemma.

Proposition 3.11. *For any $x^{<n>} \in \mathcal{X}^{<n>}$ and any two σ_1, σ_2 such that $0 < \sigma_1 < \sigma_2$, $InOWA_{w_{\sigma_2}}(x^{<n>}) \leq InOWA_{w_{\sigma_1}}(x^{<n>})$.*

Proof. Firstly, we define $x^{<n>}(\tau(n+1)) = 0$, where $\tau : N^{<n>} \to N^{<n>}$ can be any suitable permutation such that $\tau(x^{<n>}(s)) \geq \tau(x^{<n>}(t))$ whenever $s < t$. Then,

$$\begin{aligned}
InOWA_{w_{\sigma_2}(x^{<n>})} &= \sum{}_{r=1}^{n} w_{\sigma_2}(r) \cdot x^{<n>}(\tau(r)) \\
&= \sum_{r=1}^{n} w_{\sigma_2}(r) \cdot \sum_{k=r}^{n} \left(x^{<n>}(\tau(k)) - x^{<n>}(\tau(k+1))\right) \\
&= \sum_{k=1}^{n} \left(x^{<n>}(\tau(k)) - x^{<n>}(\tau(k+1))\right) \cdot \sum_{r=1}^{k} w_{\sigma_2}(r) \\
&\leq \sum_{k=1}^{n} \left(x^{<n>}(\tau(k)) - x^{<n>}(\tau(k+1))\right) \cdot \sum_{r=1}^{k} w_{\sigma_1}(r) \text{ (by Lemma 3.10} \\
&= \sum_{r=1}^{n} w_{\sigma_1}(r) \cdot \sum_{k=r}^{n} \left(x^{<n>}(\tau(k)) - x^{<n>}(\tau(k+1))\right) \\
&= \sum_{r=1}^{n} w_{\sigma_1}(r) \cdot x^{<n>}(\tau(r)) \\
&= EOWA_{w_{\sigma_1}(x^{<n>})}.
\end{aligned}$$

□

Example 3.12. *Suppose $n_1 = 10$ and $n_2 = 15$, the total numbers of published works of two scholars, Scholar 1 and Scholar 2, respectively. In addition, the qualities of them are represented by $x^{<n_1>} \in \mathcal{X}^{<n_1>}$ and $y^{<n_2>} \in \mathcal{X}^{<n_2>}$, where*

$$\begin{aligned}
(x^{<10>}(r))_{r=1}^{10} &= (\mathbf{1}, 1, 0.95, 0.9, 0.85, \mathbf{0.8}, 0.7, 0.6, 0.5, 0.5), \\
(y^{<15>}(r))_{r=1}^{15} &= (\mathbf{0.8}, 0.8, 0.8, 0.75, 0.75, \mathbf{0.7}, 0.7, 0.7, 0.7, 0.7, \mathbf{0.65}, 0.65, 0.6, 0.4, 0.3),
\end{aligned}$$

both already being ordered according to qualities from higher to lower.

If we use Infinite Ordered Weighted Averaging (InOWA) operator to $x^{<10>}$ and $y^{<15>}$ with Infinite Gaussian OWA weights function $\{w_\sigma\}_{\sigma\in(0,+\infty)}$ and specify parameters $\sigma_1 = 7, \sigma_2 = 12, \sigma_3 = 17, \sigma_4 = 22$ and $\sigma_5 = 27$, then we will obtain (also see Figure 1 and Figure 2 for the sketches of Gaussian OWA weights functions with different parameters):

(i) when $\sigma_1 = 7$,

$$InOWA_{w_7}(x^{<10>}) = \sum_{r=1}^{10} w_7(r) \cdot x^{<10>}(r) = 0.94, \quad \text{and} \quad InOWA_{w_7}(y^{<15>}) = \sum_{r=1}^{15} w_7(r) \cdot y^{<15>}(r) = 0.78;$$

(ii) when $\sigma_2 = 12$,

$$InOWA_{w_{12}}(x^{<10>}) = \sum_{r=1}^{10} w_{12}(r) \cdot x^{<10>}(r) = 0.83, \quad \text{and} \quad InOWA_{w_{12}}(y^{<15>}) = \sum_{r=1}^{15} w_{12}(r) \cdot y^{<15>}(r) = 0.75;$$

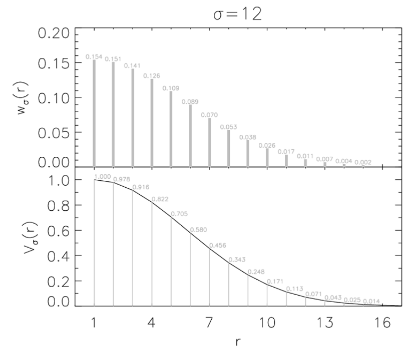

Figure 1: The Infinite Gaussian maxitive OWA weights function and Infinite Gaussian OWA weights function with parameter $\sigma_2 = 12$.

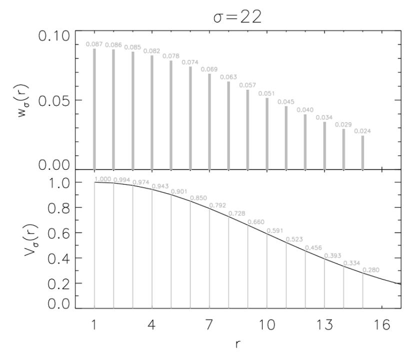

Figure 2: The Infinite Gaussian maxitive OWA weights function and Infinite Gaussian OWA weights function with parameter $\sigma_4 = 22$.

(iii) when $\sigma_3 = 17$,

$$InOWA_{w_{17}}(x^{<10>}) = \sum_{r=1}^{10} w_{17}(r) \cdot x^{<10>}(r) = 0.70, \quad \text{and} \quad InOWA_{w_{17}}(y^{<15>}) = \sum_{r=1}^{15} w_{17}(r) \cdot y^{<15>}(r) = 0.70;$$

(iv) when $\sigma_4 = 22$,

$$InOWA_{w_{22}}(x^{<10>}) = \sum_{r=1}^{10} w_{22}(r) \cdot x^{<10>}(r) = 0.59, \quad \text{and} \quad InOWA_{w_{22}}(y^{<15>}) = \sum_{r=1}^{15} w_{22}(r) \cdot y^{<15>}(r) = 0.64;$$

(v) when $\sigma_5 = 27$,

$$InOWA_{w_{27}}(x^{<10>}) = \sum_{r=1}^{10} w_{27}(r) \cdot x^{<10>}(r) = 0.51, \quad \text{and} \quad InOWA_{w_{27}}(y^{<15>}) = \sum_{r=1}^{15} w_{27}(r) \cdot y^{<15>}(r) = 0.58.$$

We observe that some different parameters selected for Gaussian OWA weights function will lead to different comparison results between the works of those two scholars. We find that when σ becomes smaller, we take more preference on fewer representative works of scholars; and conversely, when σ becomes larger, our considerations cover more works of scholars. This is why in our evaluation system the works of Scholar 1 is better than that of Scholar 2 when σ is relatively small, and Scholar 2's performance becomes better than that of Scholar 1 as σ is increasing. Particularly, when $\sigma_3 = 17$, the two scholars are considered to have the same performance. In further study and practice, we may further flexibly adjust or control the parameter σ in order to obtain more desirable comparison and evaluation results.

4 Conclusions

In this study, the Extended Ordered Weighted Sum (EOWS) operator and the Extended Ordered Weighted Averaging (EOWA) operator were defined. We showed that both operators can be well used in scientometrics evaluation if the numbers of representative works under evaluation is fixed by up to m.

In scientometrics evaluation, very often, there is no perfect and effective model to evaluate and compare the works of scholars. For example, when all of the works of scholars will be considered and under evaluation, EOWS and EOWA operators cannot suitably work. To provide more suitable models, we then proposed the Infinite Ordered Weighted Sum (InOWS) operator and the Infinite Ordered Weighted Averaging (InOWA) operator, and both of which consider all of works of scholars under evaluation.

We also defined the family of Infinite Gaussian maxitive OWA weights function and the family of Infinite Gaussian OWA weights function. An illustrative example in scientometrics evaluation compared the works of two scholars, and showed that the comparison results may be varied according to differently chosen parameter σ. That is, the larger σ is, the more works we consider in the evaluation; and conversely, the smaller σ is, the fewer representative works we pay attention to.

The extensions of OWA operators proposed in this study will draw more attentions in the theories of OWA operators and information fusion. The analyses for the evaluation methods also showed applicability and potentials of our proposals in a wider range of evaluation problems.

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