

## Using necessarily weak efficient solutions for solving a biobjective transportation problem with fuzzy objective functions coefficients

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### Abstract

This paper considers a biobjective transportation problem with various fuzzy objective functions coefficients. Fuzzy coefficients can be of different types such as triangular, trapezoidal, (semi)  $L - R$ , or flat (semi)  $L - R$  fuzzy numbers. First, we convert the problem to a parametric interval biobjective transportation problem using  $\gamma$ -cuts of fuzzy coefficients. Then, we consider a fix  $\gamma$ -cut and obtain a necessarily weak efficient solution to the yielded interval biobjective program by a new algorithm. It uses basic feasible solutions and the parametric simplex algorithm. Furthermore, we suggest another algorithm for finding a reasonable solution, called  $\gamma^*$ -necessarily weak efficient, to the main biobjective transportation problem. To illustrate the validity and performance of the proposed algorithms, we present some numerical examples.

*Keywords:* Biobjective transportation problem, interval biobjective linear programming problem, necessarily weak efficient solution, fuzzy membership function,  $\gamma$ -cut.

## 1 Introduction

The classical transportation problem has a single objective function. It is a special type of network optimization problems which has wide practical applications in the transportation industry, supply chain management, production planning, etc. [2, 6]. In the transportation problem, we seek an optimal distribution of available supplies in certain sources to some destinations for satisfying their demands. Further, we need to optimize an objective function. The optimization usually includes minimizing shipping cost or time, or maximizing shipping profit [2].

In the real world and practical applications, optimization problems often involve more than one objective function. Multicriteria optimization is a basic tool for formulating these kinds of problems [9]. In this field, multiobjective transportation problem is investigated comprehensively. It is due to the fact that the decision maker can consider delivery cost, delivery time, transportation profit, safety, demandant satisfaction, etc. as several distinct objective functions simultaneously.

The biobjective transportation problem is a special case of multiobjective transportation problems with the least number of objective functions. Indeed, it has just two objective functions, and thus, the vectors of objective function values can be realized by the decision maker easily. Aneja and Nair [1] studied a biobjective transportation problem and presented a method which finds extreme points in the criterion space. Tayi [33] proposed an approach to biobjective transportation problems, which gains a compromise solution from the set of efficient solutions.

There are many situations in reality where the coefficients of single or multiobjective optimization problems are not exactly known [29, 30]. It is due to uncertain environments of real life situations. In fact, uncertainty is a substantial feature of decision making problems that arise from multiple sources [3]. Interval programming and fuzzy

programming are two approaches which can deal with uncertain coefficients in mathematical programming problems [4, 8, 10, 15, 29, 34].

The concept of fuzzy sets was first expressed by Zadeh [36] to tackle uncertainty. Bellman and Zadeh [3] applied fuzzy set theory to decision making problems. Zimmerman [37, 38] also used fuzzy set theory to single and multiobjective optimization models. In fuzzy programming, uncertain coefficients are modeled by fuzzy sets or fuzzy numbers. Luhandjula and Rangoage [21] presented an approach to a multiobjective program with fuzzy valued objective functions based on interval approximation. Some researchers have considered biobjective transportation problem with fuzzy objective functions coefficients. Das and Bera [7] developed a method for a biobjective transportation problem with fuzzy costs and fuzzy times and some other fuzzy parameters. Keshavarz and Khorram [18] solved a biobjective transportation problem with fuzzy delivery time and fuzzy transportation profit by two bi-level programming problems. Keshavarz [19] suggested a revised algorithm to the Keshavarz and Khorram's method. Kumar et al. [20] solved a fuzzy biobjective fixed charge transportation problem. Singh and Singh [32] considered a biobjective transportation problem with fuzzy costs and fuzzy times.

Interval objective function coefficients are highly used in optimization problems [4, 13, 23, 24, 28, 26, 35]. Bitran [4] introduced two kinds of solutions to multiobjective linear programming problems with interval objective functions coefficients. Hladik [13] investigated necessarily efficient solutions and their complexity [14] in multiobjective linear programming. Moreover, he also introduced [12] some sufficient conditions which speed up the problem of checking necessarily efficient solutions. Inuiguchi and Sakawa [17] extended the concept of possibly and necessarily efficiency from interval multiobjective linear programming to the fuzzy coefficients case. They have also proposed a nonlinear program to check the degree of possibly efficiency in the fuzzy case. Mishmast Nehi and Alinezhad [22] solved a multiobjective linear program with interval coefficients with respect to necessarily efficient points. Rivaz and Yaghoobi [28, 26, 27, 25] focused on interval multiobjective linear programs. They have introduced some models for recognizing different solutions and some approaches for solving such problems. A new type of solutions called necessarily weak efficient is also introduced by them [26, 27]. Although different types of solutions are defined to interval multiobjective linear programs, necessarily (weak) efficient solutions are the most important ones since these solutions are (weak) efficient for all values of interval data [13, 17, 27]. Necessarily weak efficient solutions set is a larger set than necessarily efficient solutions set, since the second set is contained in the first one. We focus on necessarily weak efficient solutions in this paper.

The goal of this paper is proposing a new method for solving biobjective transportation problem with fuzzy objective functions coefficients. Our method is based on a bisection method. It tries to find a necessarily weak efficient solution to a fixed  $\gamma$ -cut version of the problem. An important feature of the new method is that it does not change the constraints of the transportation problem. Thus, we can use the transportation tableaus and network simplex method [2]. Moreover, for obtaining a necessarily weak efficient solution, we use the basic feasible solutions of the problem and introduce a new optimization model which can check necessarily weak efficiency. Note that finding a necessarily weak efficient solution is not an easy task. Indeed, it is an NP-hard problem [14]. The given numerical examples show the strength of the final solutions of our proposed algorithms in comparison with the other methods.

Rest of the paper is as follows. In Section 2, some preliminaries and a fuzzy biobjective transportation problem are introduced. In Section 3, a new method for finding a necessarily weak efficient solution of an interval biobjective linear program is suggested. We propose an algorithm for solving a fuzzy biobjective transportation problem in Section 4. Moreover, some numerical examples are given to show the validity and performance of the new algorithms. Finally, Section 5 is devoted to the concluding remarks.

## 2 Preliminaries

A biobjective linear programming problem can be formulated as follows:

$$\begin{aligned} \min \quad & Cx = (c_1x, c_2x)^t \\ \text{s.t.} \quad & x \in X = \{x \in \mathbb{R}^n \mid Ax=b, x \geq 0\}, \end{aligned} \quad (1)$$

where  $c_1x = \sum_{j=1}^n c_{1j}x_j$  and  $c_2x = \sum_{j=1}^n c_{2j}x_j$  are linear real-valued objective functions. Thus,  $C$  is a  $2 \times n$  matrix with two rows  $c_1 = (c_{11}, \dots, c_{1n})$  and  $c_2 = (c_{21}, \dots, c_{2n})$ .  $A$  is an  $m \times n$  matrix, the right hand side vector and the vector of variables are  $b \in \mathbb{R}^m$  and  $x \in \mathbb{R}^n$ , respectively. The superscript  $t$  over a vector or matrix denotes the transpose.

Let  $y^1 = (y_1^1, y_2^1)$  and  $y^2 = (y_1^2, y_2^2)$  be two dimensional vectors. Then:

- $y^1 \preceq y^2$  if  $y_i^1 \leq y_i^2$  for  $i = 1, 2$  and  $y^1 \neq y^2$ .
- $y^1 \prec y^2$  if  $y_i^1 < y_i^2$  for  $i = 1, 2$ .

The orders  $\succeq$  and  $\succ$  can be defined similarly.

**Definition 2.1.** [9] For the problem (1), a feasible solution  $x^0$  is called:

- efficient if there is no  $x \in X$  such that  $Cx \preceq Cx^0$ .
- weakly efficient if there is no  $x \in X$  such that  $Cx \prec Cx^0$ .

We say  $x \in X$  (strictly) dominates  $x^0 \in X$  if  $(Cx \prec Cx^0) \vee (Cx \preceq Cx^0)$ .

Corresponding to the biobjective linear programming problem (1), a weighted sum linear programming problem has been formulated as follows:

$$\begin{aligned} \min \quad & \lambda_1 c_1 x + \lambda_2 c_2 x \\ \text{s.t.} \quad & x \in X. \end{aligned} \quad (2)$$

For the problem (1), it is well-known that [9]:

- $x^0 \in X$  is an efficient solution if and only if there exist  $\lambda_1, \lambda_2 > 0$  such that  $x^0$  is an optimal solution of the problem (2).
- $x^0 \in X$  is a weakly efficient solution if and only if there exist  $\lambda_1, \lambda_2 \geq 0$  with  $\lambda_1 + \lambda_2 > 0$  such that  $x^0$  is an optimal solution of the problem (2).
- $x^0 \in X$  is an efficient solution if there exist  $\lambda_1, \lambda_2 \geq 0$  with  $\lambda_1 + \lambda_2 > 0$  such that  $x^0$  is the unique optimal solution of the problem (2).

The vectors  $c_1$  and  $c_2$  in the problem (1) are assumed to be deterministic. However, in practical problems, they may be uncertain. An approach to deal with uncertain parameters is by using intervals. Thus, we consider the following interval biobjective linear programming problem:

$$\begin{aligned} \min \quad & Z(x) = \left( \sum_{j=1}^n [c_{1j}^l, c_{1j}^u] x_j, \sum_{j=1}^n [c_{2j}^l, c_{2j}^u] x_j \right)^t \\ \text{s.t.} \quad & x \in X. \end{aligned} \quad (3)$$

In fact, the exact coefficients  $c_{1j}$  and  $c_{2j}$  ( $j = 1, \dots, n$ ) in the problem (1) can vary in intervals  $[c_{1j}^l, c_{1j}^u]$  and  $[c_{2j}^l, c_{2j}^u]$  in the problem (3), respectively. In what follows:

$$\mathbf{C} = \{C = (c_1, c_2)^t \mid c_1 = (c_{11}, \dots, c_{1n}), c_2 = (c_{21}, \dots, c_{2n}), c_{ij} \in [c_{ij}^l, c_{ij}^u], i = 1, 2, j = 1, \dots, n\}.$$

Indeed,  $\mathbf{C}$  is the uncountable set of  $2 \times n$  matrices.

**Definition 2.2.** [16, 26] For the problem (3), a solution  $x^0 \in X$  is called:

- necessarily weak efficient if it is weak efficient to the problem (1) for all arbitrary chosen matrices  $C$  from  $\mathbf{C}$ .
- necessarily efficient if it is efficient to the problem (1) for all arbitrary chosen matrices  $C$  from  $\mathbf{C}$ .
- possibly efficient if it is efficient to the problem (1) for at least one chosen matrix  $C$  from  $\mathbf{C}$ .
- possibly weak efficient if it is weak efficient to the problem (1) for at least one chosen matrix  $C$  from  $\mathbf{C}$ .

After the well known Bellman and Zadeh's paper [3] in decision making under fuzzy environment, a widely used way to handle uncertain parameters in optimization problems is modeling them by fuzzy sets or fuzzy numbers.

**Definition 2.3.** [8] Let  $U$  denote a universal set. A fuzzy subset  $\tilde{A}$  of  $U$  is defined by its membership function  $\mu_{\tilde{A}} : U \rightarrow [0, 1]$ , which assigns to  $x \in U$  a real number  $\mu_{\tilde{A}}(x) \in [0, 1]$ .

**Definition 2.4.** [8, 29] An  $L-R$  fuzzy number is defined on the set of real numbers,  $\mathbb{R}$ , with the following membership function:

$$\mu(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right) & x \leq m \\ R\left(\frac{x-m}{\beta}\right) & x \geq m \end{cases} \quad (4)$$

where  $m$  is the mean value,  $\alpha > 0$  and  $\beta > 0$  are the left and the right spreads, respectively, and  $L(\cdot)$  is a left shape (reference) function such that it is nonincreasing on  $[0, \infty)$ ,  $L(0) = 1$  and  $L(x) = L(-x)$ .  $R(\cdot)$  is a right shape function similarly defined as  $L(\cdot)$ .

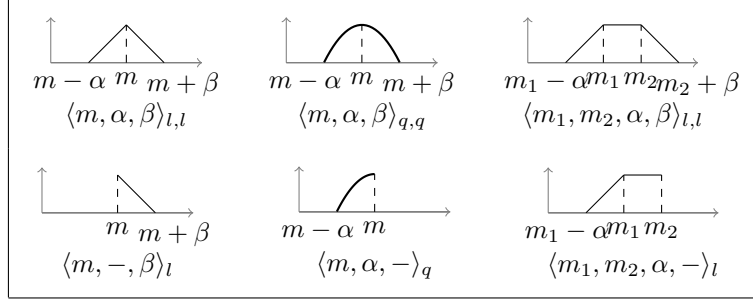


Figure 1: Some (flat)  $L - R$  and (flat) semi- $L - R$  fuzzy numbers.

We denote the above  $L - R$  fuzzy number by  $\langle m, \alpha, \beta \rangle_{L,R}$ . The popular shape functions are  $l(x) = \max\{0, 1 - |x|\}$ ,  $q(x) = \max\{0, 1 - x^2\}$ ,  $g(x) = e^{-x^2}$ , etc. [6,8,23]. If we consider  $L(x) = R(x) = l(x)$  or  $L(x) = R(x) = q(x)$  then the  $L - R$  fuzzy numbers  $\langle m, \alpha, \beta \rangle_{l,l}$  and  $\langle m, \alpha, \beta \rangle_{q,q}$  are called triangular and quadratic fuzzy numbers, respectively. Moreover, if the membership function (4) is considered as follows:

$$\mu(x) = \begin{cases} L\left(\frac{m_1-x}{\alpha}\right) & x \leq m_1 \\ 1 & m_1 \leq x \leq m_2 \\ R\left(\frac{x-m_2}{\beta}\right) & x \geq m_2 \end{cases} \quad (5)$$

where  $m_1, m_2 \in \mathbb{R}$ , then the fuzzy set is called a flat  $L - R$  fuzzy number. In the sequel, the flat  $L - R$  fuzzy number (5) is denoted by  $\langle m_1, m_2, \alpha, \beta \rangle_{L,R}$ . If we consider  $L(x) = R(x) = l(x)$  then the flat  $L - R$  fuzzy number  $\langle m, \alpha, \beta \rangle_{l,l}$  is the well-known trapezoidal fuzzy number. We can define a fuzzy set with using only some parts of  $L - R$  fuzzy numbers. Through this paper, we call these fuzzy sets semi- $L - R$  fuzzy numbers.

**Definition 2.5.** Let  $m, m_1, m_2 \in \mathbb{R}$ ,  $\alpha > 0$  and  $\beta > 0$  be arbitrary. Then:

- left and right semi- $L - R$  fuzzy numbers are denoted by  $\langle m, \alpha, - \rangle_L$  and  $\langle m, -, \beta \rangle_R$  with membership functions (6) and (7), respectively:

$$\mu(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right) & x \leq m \\ 0 & x > m, \end{cases} \quad (6)$$

$$\mu(x) = \begin{cases} 0 & x < m \\ R\left(\frac{x-m}{\beta}\right) & x \geq m. \end{cases} \quad (7)$$

- left and right flat semi- $L - R$  fuzzy numbers are denoted by  $\langle m_1, m_2, \alpha, - \rangle_L$  and  $\langle m_1, m_2, -, \beta \rangle_R$  with membership functions (8) and (9), respectively:

$$\mu(x) = \begin{cases} L\left(\frac{m_1-x}{\alpha}\right) & x \leq m_1 \\ 1 & m_1 \leq x \leq m_2 \\ 0 & x > m_2, \end{cases} \quad (8)$$

$$\mu(x) = \begin{cases} 0 & x < m_1 \\ 1 & m_1 \leq x \leq m_2 \\ R\left(\frac{x-m_2}{\beta}\right) & x \geq m_2. \end{cases} \quad (9)$$

Figure 1 shows some (flat)  $L - R$  and (flat) semi- $L - R$  fuzzy numbers. Another important concept in fuzzy set theory is  $\gamma$ -cut.

**Definition 2.6.** ([8]) Let  $\tilde{A}$  be a fuzzy subset of the universal set  $U$  with the membership function  $\mu_{\tilde{A}}$  and let  $\gamma \in (0, 1]$ . Then, the  $\gamma$ -cut of  $\tilde{A}$ , denoted by  $(\tilde{A})_\gamma$ , is the set  $\{x \in U \mid \mu_{\tilde{A}}(x) \geq \gamma\}$ .

In Definition 2.6, if we set  $\gamma = 0$ , we obtain  $(\tilde{A})_0 = U$  for all fuzzy sets  $\tilde{A}$ . So, some authors define  $(\tilde{A})_0 = cl(\{x \in U \mid \mu_{\tilde{A}}(x) > 0\})$ , where  $cl$  stands for the closure [5]. By using this definition, in the case of  $L(1) = 0$  and  $R(1) = 0$ , straightforward calculations imply that [6,8]:

- $(\langle m, \alpha, \beta \rangle_{L,R})_\gamma = [m - \alpha L^{-1}(\gamma), m + \beta R^{-1}(\gamma)], \quad (\langle m, \alpha, \beta \rangle_{L,R})_0 = [m - \alpha, m + \beta],$
- $(\langle m_1, m_2, \alpha, \beta \rangle_{L,R})_\gamma = [m_1 - \alpha L^{-1}(\gamma), m_2 + \beta R^{-1}(\gamma)], \quad (\langle m_1, m_2, \alpha, \beta \rangle_{L,R})_0 = [m_1 - \alpha, m_2 + \beta],$
- $(\langle m, \alpha, - \rangle_L)_\gamma = [m - \alpha L^{-1}(\gamma), m], \quad (\langle m, \alpha, - \rangle_L)_0 = [m - \alpha, m],$
- $(\langle m, -, \beta \rangle_R)_\gamma = [m, m + \beta R^{-1}(\gamma)], \quad (\langle m, -, \beta \rangle_R)_0 = [m, m + \beta],$
- $(\langle m_1, m_2, \alpha, - \rangle_L)_\gamma = [m_1 - \alpha L^{-1}(\gamma), m_2], \quad (\langle m_1, m_2, \alpha, - \rangle_L)_0 = [m_1 - \alpha, m_2],$
- $(\langle m_1, m_2, -, \beta \rangle_R)_\gamma = [m_1, m_2 + \beta R^{-1}(\gamma)], \quad (\langle m_1, m_2, -, \beta \rangle_R)_0 = [m_1, m_2 + \beta],$

where  $L^{-1}(\cdot)$  and  $R^{-1}(\cdot)$  are the inverse functions of  $L(\cdot)$  and  $R(\cdot)$  respectively.

**Remark 2.1.** In the case of  $L(1) \neq 0$  and/or  $R(1) \neq 0$ , we know that  $\lim_{x \rightarrow \infty} L(x) = 0$  and  $\lim_{x \rightarrow \infty} R(x) = 0$ , since  $L, R : [0, \infty) \rightarrow [0, 1]$  are nonincreasing functions [11]. In these instances, we consider  $(\tilde{A})_0 = cl(\{x \in U \mid \mu_{\tilde{A}}(x) > \epsilon\})$ , where  $\epsilon > 0$  is a sufficiently small number (for example,  $\epsilon = 0.001$ ). Thus, we consider 0-cuts of (flat)  $L - R$  and (flat) semi- $L - R$  fuzzy numbers, in the case of  $L(1) \neq 0$  and  $R(1) \neq 0$ , as follows:

- $(\langle m, \alpha, \beta \rangle_{L,R})_0 = [m - \alpha L^{-1}(\epsilon), m + \beta R^{-1}(\epsilon)],$
- $(\langle m_1, m_2, \alpha, \beta \rangle_{L,R})_0 = [m_1 - \alpha L^{-1}(\epsilon), m_2 + \beta R^{-1}(\epsilon)],$
- $(\langle m, \alpha, - \rangle_L)_0 = [m - \alpha L^{-1}(\epsilon), m],$
- $(\langle m, -, \beta \rangle_R)_0 = [m, m + \beta R^{-1}(\epsilon)],$
- $(\langle m_1, m_2, \alpha, - \rangle_L)_0 = [m_1 - \alpha L^{-1}(\epsilon), m_2],$
- $(\langle m_1, m_2, -, \beta \rangle_R)_0 = [m_1, m_2 + \beta R^{-1}(\epsilon)].$

The 0-cuts in other cases can be obtained similarly. For example, in the case of  $L(1) \neq 0$  and  $R(1) = 0$ ,  $(\langle m, \alpha, \beta \rangle_{L,R})_0$  is as follows:

$$(\langle m, \alpha, \beta \rangle_{L,R})_0 = [m - \alpha L^{-1}(\epsilon), m + \beta].$$

## 2.1 A fuzzy biobjective transportation problem

Transportation problem is a famous model which points to one of the vast application areas of linear programming. A classical model of transportation has  $m$  source (supply) nodes and  $n$  destination (demand) nodes. Source node  $i$ ,  $i = 1, \dots, m$ , has  $s_i > 0$  units of certain commodity and destination node  $j$ ,  $j = 1, \dots, n$ , needs  $d_j > 0$  units of that commodity. Moreover, we suppose that the problem is balanced. In other words, the total amount of supplies equals to the total amount of demands, i.e.  $\sum_{i=1}^m s_i = \sum_{j=1}^n d_j$ . Otherwise in order to balance the problem a dummy supply or demand node can be inserted [2]. Let  $x_{ij} \geq 0$  denote the number of commodities that should be transported from source node  $i$  to destination node  $j$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, n$ . Then, for every source node  $i$  and demand node  $j$ , we should have  $\sum_{j=1}^n x_{ij} = s_i$  and  $\sum_{i=1}^m x_{ij} = d_j$ , respectively.

In single objective transportation problem, there exists only one objective function which is usually cost or profit of transportation. However, in biobjective transportation problems, the decision maker is allowed to propose beside of the main objective function, another important objective function such as minimizing the delivery time, maximizing the delivery safety, minimizing the gas usage, etc. In this paper, we concentrate on a biobjective transportation problem. On the other hand, cost, profit, delivery time, safety of shipping commodities from sources to destinations are often uncertain in real world. As stated in the previous section, one way to model these uncertain parameters is using (flat)  $L - R$  or (flat) semi- $L - R$  fuzzy numbers. Let  $\tilde{c}_{ij}^1$  and  $\tilde{c}_{ij}^2$  denote the fuzzy coefficients of  $x_{ij}$  in the first and the second objective functions, respectively. Therefore, we consider the following fuzzy biobjective transportation problem:

$$\begin{aligned}
\min \quad & \tilde{c}_1 x = \sum_{i=1}^m \sum_{j=1}^n \tilde{c}_{ij}^1 x_{ij} \\
\min \quad & \tilde{c}_2 x = \sum_{i=1}^m \sum_{j=1}^n \tilde{c}_{ij}^2 x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^n x_{ij} = s_i \quad i = 1, \dots, m \\
& \sum_{i=1}^m x_{ij} = d_j \quad j = 1, \dots, n \\
& x_{ij} \geq 0 \quad i = 1, \dots, m, j = 1, \dots, n,
\end{aligned} \tag{10}$$

where  $\tilde{c}_{ij}^1$  and  $\tilde{c}_{ij}^2$  can be any arbitrary (flat)  $L - R$  or (flat) semi- $L - R$  fuzzy numbers.

It should be noted that in the general form of model (10), we have considered both objective functions of minimization type. If one of the objective functions is of maximization type, one can convert it to a minimization type by the fact that  $\max_x f(x) = -(\min_x -f(x))$  for any arbitrary function  $f$ . We denote the set of feasible solutions of the problem (10) by

$$X_T = \{(x_{11}, \dots, x_{mn})^t \mid \sum_{j=1}^n x_{ij} = s_i, \sum_{i=1}^m x_{ij} = d_j, x_{ij} \geq 0, i = 1, \dots, m, j = 1, \dots, n\}.$$

To solve the problem (10), we consider the following definitions.

**Definition 2.7.** Consider the problem (10) and  $\gamma \in [0, 1]$ . A solution  $x^o \in X_T$  is called  $\gamma$ -cut necessarily weak efficient ( $\gamma$ -CNWE for short), if it is necessarily weak efficient to its related interval biobjective transportation problem where the coefficients  $\tilde{c}_{ij}^1$  and  $\tilde{c}_{ij}^2$  are replaced by  $(\tilde{c}_{ij}^1)_\gamma$  and  $(\tilde{c}_{ij}^2)_\gamma$ , respectively.

**Definition 2.8.** A solution  $x^o \in X_T$  is called  $\gamma^*$ -necessarily weak efficient ( $\gamma^*$ -NWE for short, where  $\gamma^* \in [0, 1]$ ) to the problem (10), if it is  $\gamma^*$ -CNWE to the problem (10) and there is no other  $\gamma$ -CNWE solution  $x \in X_T$  with  $\gamma < \gamma^*$ .

In definitions 2.7 and 2.8, we need to find a necessarily weak efficient solution of interval biobjective transportation problems. The next section introduces an algorithm to this end.

### 3 Finding a necessarily weak efficient solution

In this section, we consider the interval biobjective linear programming problem (3). The goal of this section is finding an element of the set of necessarily weak efficient solutions of the problem (3). Of course, we conclude that it is an empty set when there is no such an element. We denote the set of weak efficient solutions for arbitrary  $C \in \mathbf{C}$  and the set of necessarily weak efficient solutions of the problem (3) by  $X_W(C)$  and  $X_{NW}$ , respectively.

**Lemma 3.1.** Consider the problem (3) and an arbitrary  $\bar{C} \in \mathbf{C}$ . If  $X_W(\bar{C}) \cap X_{NW} = \emptyset$  then  $X_{NW} = \emptyset$ .

*Proof.* By Definition 2.2,  $x^0 \in X_{NW}$  if and only if  $x^0 \in X_W(C)$  for all  $C \in \mathbf{C}$ . Thus,  $X_{NW} \subseteq X_W(C) \quad \forall C \in \mathbf{C}$  and  $X_{NW} \cap X_W(C) = X_{NW} \quad \forall C \in \mathbf{C}$ . This completes the proof.  $\square$

Based on Lemma 3.1, we select an arbitrary  $C \in \mathbf{C}$  (for example  $C = [(c_{11}^1, \dots, c_{1n}^1), (c_{21}^1, \dots, c_{2n}^1)]^t$ ) and obtain  $X_W(C)$  iteratively. For each  $x \in X_W(C)$ , we check whether  $x \in X_{NW}$  or  $x \notin X_{NW}$ . In the first case  $X_W(C) \cap X_{NW} \neq \emptyset$  and we obtain a necessarily weak efficient solution of the problem (3). If all solutions in  $X_W(C)$  are checked and none of them belongs to the set  $X_{NW}$  then  $X_W(C) \cap X_{NW} = \emptyset$  and, based on Lemma 3.1,  $X_{NW} = \emptyset$ . Therefore, we need a procedure to do the following tasks:

1. find an  $x \in X_W(C)$ ,
2. check whether  $x \in X_{NW}$  or  $x \notin X_{NW}$ ,
3. if  $x \notin X_{NW}$  then find another solution  $\hat{x}$  (if exists) such that  $\hat{x} \in X_W(C) \setminus \{x \mid x \text{ is checked in Step 2}\}$  and repeat Step 2 with  $x = \hat{x}$ .

Steps (1) and (3) can be done by using the well known Parametric Simplex Algorithm (PSA) [9] for biobjective linear programming problems which is based on the weighted sum problem (2). PSA finds all elements of  $X_W(C)$  by obtaining basic feasible solutions of the problem (2).

**Definition 3.2.** [9] Consider the feasible set of the problem (3)  $X = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$  where  $A$  is an  $m \times n$  matrix and  $b$  is an  $m$  vector. Assume that  $\text{rank}(A) = m$ . Let  $A_B$  be an  $m \times m$  invertible submatrix of  $A$  where  $B$  is the set of indices of the columns of  $A$  belonging to  $A_B$ . Then,  $B$  is called a basis. The solution  $x = (x_B = A_B^{-1}b, 0)$  is called a basic solution and a basic feasible solution (BFS) if  $x_B \geq 0$ . A variable  $x_j$  is called basic if  $j \in B$  and nonbasic if  $j \in N := \{1, \dots, n\} \setminus B$ .

Using PSA with tableau notation is common. Table 1 shows a general form of such tableaus for an arbitrary  $C$  to the problem (3). In Table 1:

- $y_j = A_B^{-1}a_j$ , where  $a_j$  is the  $j$ th column of  $A$ ,  $j = 1, \dots, n$ ,
- $\bar{b} = A_B^{-1}b$ ,

- $c_B^i = (c_{B_1}^i, \dots, c_{B_m}^i)$  is the vector of coefficients of basic variables in the vector  $(c_{i1}, c_{i2}, \dots, c_{in})$  for  $i = 1, 2$ ,
- $\hat{c}_j^i := c_j^i - z_j^i = c_{ij} - c_B^i y_j$ , for  $i = 1, 2$  and  $j = 1, \dots, n$ .

Table 1: A general form of PSA tableau.

Basis	$x_1$	...	$x_j$	...	$x_n$	RHS
$z_1(x)$	$\hat{c}_1^1 = c_1^1 - z_1^1$	...	$\hat{c}_j^1 = c_j^1 - z_j^1$	...	$\hat{c}_n^1 = c_n^1 - z_n^1$	$c_B^1 \bar{b}$
$z_2(x)$	$\hat{c}_1^2 = c_1^2 - z_1^2$	...	$\hat{c}_j^2 = c_j^2 - z_j^2$	...	$\hat{c}_n^2 = c_n^2 - z_n^2$	$c_B^2 \bar{b}$
B	$y_1 = \begin{pmatrix} y_{11} \\ \vdots \\ y_{m1} \end{pmatrix}$	...	$y_j = \begin{pmatrix} y_{1j} \\ \vdots \\ y_{mj} \end{pmatrix}$	...	$y_n = \begin{pmatrix} y_{1n} \\ \vdots \\ y_{mn} \end{pmatrix}$	$\bar{b} = \begin{pmatrix} b_1 \\ \vdots \\ \bar{b}_m \end{pmatrix}$

Corresponding to  $C \in \mathbf{C}$  in the problem (3), PSA considers the weighted sum problem (2) with  $(\lambda_1, \lambda_2) = (\lambda, 1 - \lambda)$  where  $0 \leq \lambda \leq 1$  and defines a parametric linear program as follows [9]:

$$\begin{aligned} \min \quad & C(\lambda) := \lambda c_1 x + (1 - \lambda) c_2 x \\ \text{s.t.} \quad & x \in X. \end{aligned} \quad (11)$$

Based on the properties of the problem (2), a BFS is a weak efficient solution of the problem (3) if it is an optimal solution of the problem (11). Indeed, the BFS in Table 1 is an optimal solution to the problem (11) if [9]:

$$\hat{C}(\lambda) := \lambda(\hat{c}_1^1, \dots, \hat{c}_n^1) + (1 - \lambda)(\hat{c}_1^2, \dots, \hat{c}_n^2) \geq 0.$$

By using PSA, first, we start with  $\lambda^p = 1$  ( $p = 1$ ) and obtain the optimal solution of the first objective function of the problem (3) with  $C = [(c_{11}^1, \dots, c_{1n}^1), (c_{21}^1, \dots, c_{2n}^1)]^t$ . The optimal solution (say  $\hat{x}^p$ ) is the first weak efficient solution of the problem (3) which is an extreme point of the polyhedron  $X$ . To find the next weak efficient solution of the problem (3), we do an iteration of PSA by entering a nonbasic variable to the basis and removing a basic variable. This can be done as follows [9]:

If  $I = \{i \in N \mid \hat{c}_i^1 \geq 0, \hat{c}_i^2 < 0\} \neq \emptyset$  then:

- (i)  $\lambda^{p+1} := \max_{i \in I} \frac{-\hat{c}_i^2}{\hat{c}_i^1 - \hat{c}_i^2}$ .
- (ii) select  $s$  such that  $\frac{-\hat{c}_s^2}{\hat{c}_s^1 - \hat{c}_s^2} = \min\{\frac{-\hat{c}_i^2}{\hat{c}_i^1 - \hat{c}_i^2} \mid i \in I\}$ .
- (iii) select  $r$  such that  $\frac{\bar{b}_r}{y_{rs}} = \min\{\frac{\bar{b}_j}{y_{js}} \mid y_{js} > 0, j = 1, \dots, m\}$ .
- (iv) update Table 1 by well known simplex pivoting at  $y_{rs}$  [2].

The solution  $\hat{x}^p$  is an optimal solution of the problem (11) for all  $\lambda^{p+1} \leq \lambda \leq \lambda^p$ . After doing Step (iv) a new BFS will be attained (say  $\hat{x}^{p+1}$ ) which is an optimal solution of the problem (11) for  $\lambda = \lambda^{p+1}$  (at least).  $\hat{x}^{p+1}$  is another extreme point of the polyhedron  $X$ , which is adjacent to  $\hat{x}^p$ . We suppose that the problem is not degenerate. Moreover, the whole solutions along the edge between  $\hat{x}^p$  and  $\hat{x}^{p+1}$  (i.e., the set  $\{\beta \hat{x}^p + (1 - \beta) \hat{x}^{p+1} \mid 0 \leq \beta \leq 1\}$ ) are also optimal solutions of the problem (11) and thus weak efficient to the problem (3). This process terminates when the set  $I$  becomes empty. In other words, the second objective function is optimal.

With the above process, we can find all weak efficient solutions to the problem (3). In what follows, we propose doing some new computations at each step of PSA to check the necessarily weak efficiency of a BFS. To this end, the next theorem has a principle role.

**Theorem 3.3.** Let  $\hat{x}^p = (\hat{x}_B^p = A_B^{-1}b, 0)^t$  be a BFS of the problem (3) with  $C \in \mathbf{C}$  as given in Table 1. Suppose that  $\hat{x}^p$  is weak efficient and

$$V = \min_{C \in \mathbf{C}} \max_{0 \leq \lambda \leq 1} \min_{j \in N} \hat{c}_j(\lambda) \quad (12)$$

where  $\hat{c}_j(\lambda) = \lambda \hat{c}_j^1 + (1 - \lambda) \hat{c}_j^2 = \lambda(c_j^1 - z_j^1) + (1 - \lambda)(c_j^2 - z_j^2)$ . Then,  $\hat{x}^p \in X_{NW}$  if  $V \geq 0$ . Otherwise,  $(V < 0)$   $\hat{x}^p \notin X_{NW}$ .

*Proof.* First, we assume that  $V \geq 0$ . Thus,  $\max_{0 \leq \lambda \leq 1} \min_{j \in N} \hat{c}_j(\lambda) \geq 0$  for all  $C \in \mathbf{C}$ . Therefore, there exists  $0 \leq \hat{\lambda} \leq 1$  such that  $\min_{j \in N} \hat{c}_j(\hat{\lambda}) \geq 0$  for all  $C \in \mathbf{C}$ . It implies that there exists  $0 \leq \hat{\lambda} \leq 1$  such that  $\hat{C}(\lambda) \geq 0$  for all  $C \in \mathbf{C}$ . Note that all components of the vector  $\hat{C}(\lambda)$ , related to the basic variables are zero. Thus,  $\hat{x}^p$  is an optimal solution of the problem (11) for all  $C \in \mathbf{C}$ . In other words,  $\hat{x}^p \in X_W(\mathbf{C})$  for all  $C \in \mathbf{C}$ . Hence,  $\hat{x}^p \in X_{NW}$ . Now, assume that  $V < 0$ . Then there exists a matrix  $\bar{C} \in \mathbf{C}$ , such that  $\max_{0 \leq \lambda \leq 1} \min_{j \in N} \hat{c}_j(\lambda) < 0$ . Thus, for  $\bar{C} \in \mathbf{C}$ ,  $\min_{j \in N} \hat{c}_j(\lambda) < 0, \quad \forall 0 \leq \lambda \leq 1$ . It means that, for  $\bar{C} \in \mathbf{C}$  and for every  $0 \leq \lambda \leq 1$  there exists and index  $j \in N$  such that  $\hat{c}_j(\lambda) < 0$ . Therefore, under non-degeneracy assumption,  $\hat{x}^p$  cannot be an optimal solution of the problem (11) for any parameter  $0 \leq \lambda \leq 1$ , when  $C = \bar{C}$ . Hence,  $\hat{x}^p \notin X_{NW}$ .  $\square$

The next theorem states another important result.

**Theorem 3.4.** *Suppose that  $\hat{x}^p$  and  $\hat{x}^{p+1}$  are adjacent BFSs which are weak efficient solutions to the problem (3) with an arbitrary  $C \in \mathbf{C}$ . If  $\hat{x}^p, \hat{x}^{p+1} \notin X_{NW}$  then no solution on the edge between  $\hat{x}^p$  and  $\hat{x}^{p+1}$  is a necessarily weak efficient solution to the problem (3).*

*Proof.* The edge between  $\hat{x}^p$  and  $\hat{x}^{p+1}$  is the set  $\{\beta\hat{x}^p + (1 - \beta)\hat{x}^{p+1} \mid 0 \leq \beta \leq 1\}$ . It is clear that  $\beta\hat{x}^p + (1 - \beta)\hat{x}^{p+1} \in X, \quad \forall 0 \leq \beta \leq 1$ . On the contrary, suppose that for  $0 < \bar{\beta} < 1$ , the point  $\bar{x} = \bar{\beta}\hat{x}^p + (1 - \bar{\beta})\hat{x}^{p+1}$  is necessarily weak efficient solution to the problem (3). Then,  $\bar{x} \in X_W(C)$  for all  $C \in \mathbf{C}$ . Thus, for arbitrary  $C \in \mathbf{C}$ , there exists  $0 \leq \lambda \leq 1$  such that  $\bar{x}$  is an optimal solution to the problem (11), i.e.  $C(\lambda)\bar{x} \leq C(\lambda)x, \quad \forall x \in X$ . Then, we have:

$$\begin{aligned} C(\lambda)\bar{x} = \bar{\beta}C(\lambda)\hat{x}^p + (1 - \bar{\beta})C(\lambda)\hat{x}^{p+1} &\leq C(\lambda)x, \quad \forall x \in X, \\ C(\lambda)\bar{x} &\leq C(\lambda)\hat{x}^p, \\ C(\lambda)\bar{x} &\leq C(\lambda)\hat{x}^{p+1}. \end{aligned}$$

If  $C(\lambda)\hat{x}^p > C(\lambda)\bar{x}$  or  $C(\lambda)\hat{x}^{p+1} > C(\lambda)\bar{x}$  then  $C(\lambda)\bar{x} < \bar{\beta}C(\lambda)\hat{x}^p + (1 - \bar{\beta})C(\lambda)\hat{x}^{p+1}$  which is a contradiction. Therefore,  $C(\lambda)\hat{x}^p = C(\lambda)\hat{x}^{p+1} = C(\lambda)\bar{x} \leq C(\lambda)x, \quad \forall x \in X$ . It means that  $\hat{x}^p, \hat{x}^{p+1} \in X_{NW}$  (a contradiction with the fact that  $\hat{x}^p, \hat{x}^{p+1} \notin X_{NW}$ ). Hence,  $\bar{x} \notin X_{NW}$  and the proof is complete.  $\square$

**Corollary 3.5.** *Suppose that after completion of the PSA, the weak efficient BFSs,  $\hat{x}^1, \dots, \hat{x}^q$  are obtained where the value  $V$  in Theorem 3.1 is negative for all of them. Then,  $X_{NW} = \emptyset$ .*

*Proof.* By Theorems 3.1 and 3.2, the proof is clear.  $\square$

Based on the above results, we suggest obtaining the value  $V$  for each BFS in the PSA. If  $V \geq 0$  then we obtain an element of  $X_{NW}$  and we stop. Otherwise, we will continue the PSA. When PSA terminates and  $V < 0$  for all BFSs, we conclude, by Corollary 3.1, that  $X_{NW} = \emptyset$ . In the sequel, we explain a method for finding the value  $V$ .

The value  $V$  in (12) can be obtained by solving the following optimization problem:

$$\begin{aligned} \min \quad &\sigma \\ \text{s.t.} \quad &\min_{j \in N} \hat{c}_j(\lambda) \leq \sigma \quad \forall 0 \leq \lambda \leq 1, \\ &C \in \mathbf{C}, \quad \sigma \in \mathbb{R}. \end{aligned} \tag{13}$$

For every  $0 \leq \lambda \leq 1$ , the constraint  $\min_{j \in N} \hat{c}_j(\lambda) \leq \sigma$  can be replaced with the following constraints:

$$\begin{aligned} \hat{c}_j(\lambda) &\leq \sigma + My_j, \quad \forall j \in N \\ \sum_{j \in N} y_j &\leq |N| - 1 \\ y_j &\in \{0, 1\}, \quad \forall j \in N, \end{aligned}$$

where  $|N|$  denotes the cardinality of the set  $N$  and  $M$  is a sufficiently large positive number. Thus, we need to solve the following mixed integer optimization problem:

$$\begin{aligned} \min \quad &\sigma \\ \text{s.t.} \quad &\hat{c}_j(\lambda) \leq \sigma + My_j, \quad \forall j \in N, \quad \forall 0 \leq \lambda \leq 1 \\ &\sum_{j \in N} y_j \leq |N| - 1 \\ &C \in \mathbf{C}, \quad \sigma \in \mathbb{R}, \quad y_j \in \{0, 1\} \quad \forall j \in N. \end{aligned} \tag{14}$$

The next proposition shows the important role of zero-one variables and the second constraint of the problem (14).



**Proposition 3.6.** *Suppose that  $(\sigma^*, C^*, y^*)$  is an optimal solution of the problem (14). Then  $\min_{j \in N} \hat{c}_j^*(\lambda) \leq \sigma^*$ , for all  $0 \leq \lambda \leq 1$ .*

*Proof.* Let  $0 \leq \lambda \leq 1$ . Suppose that  $\min_{j \in N} \hat{c}_j^*(\lambda) = \hat{c}_{t_1}^*(\lambda) = \dots = \hat{c}_{t_r}^*(\lambda)$  where  $t_1, \dots, t_r \in N$ . Note that the cardinality of the set  $\{t_1, \dots, t_r\}$  is at least one. The constraint  $\sum_{j \in N} y_j \leq |N| - 1$  implies that at least one of  $y_j$ s must be zero. We show that one of the values  $y_{t_1}^*, \dots, y_{t_r}^*$  must be zero. Otherwise, assume that  $y_s^* = 0$  where  $s \neq t_1, \dots, s \neq t_r$ . Then,  $\hat{c}_s^*(\lambda) > \hat{c}_{t_i}^*(\lambda)$  for all  $i = 1, \dots, r$ . Now, by the first constraints of the problem (14), we have:

$$\hat{c}_s^*(\lambda) \leq \sigma^*, \quad \& \quad \hat{c}_j^*(\lambda) \leq \sigma^* + M y_j^* \quad \forall j \in N \setminus \{s\}.$$

Thus,  $\sigma^* \geq \hat{c}_s^*(\lambda) > \hat{c}_{t_1}^*(\lambda)$  which contradicts the optimality of the value of  $\sigma^*$ . Therefore, one of the values  $y_{t_1}^*, \dots, y_{t_r}^*$  must be zero. Without loss of generality, suppose that  $y_1^* = 0$ . Then,  $\sigma^* \geq \hat{c}_{t_1}^*(\lambda) = \min_{j \in N} \hat{c}_j^*(\lambda)$ . This completes the proof.  $\square$

**Proposition 3.7.** *The problem (14) is always feasible and bounded.*

*Proof.* It is easy to check that  $y_j = 0, \forall j \in N$ ,  $C = ((c_{11}^l, \dots, c_{1n}^l), (c_{21}^l, \dots, c_{2n}^l))^t$ , and  $\sigma = \max_{j \in N, 0 \leq \lambda \leq 1} \hat{c}_j(\lambda)$  satisfy all constraints of the problem (14). Thus, it is always feasible. Now, assume that  $N = \{j_1, \dots, j_{|N|}\}$ . By the second constraint of the problem (14),  $\sigma$  must satisfies the following inequalities:

$$\begin{aligned} \sigma &\geq \hat{c}_{j_1}(\lambda), \quad \forall \quad 0 \leq \lambda \leq 1, \quad \text{or} \\ \sigma &\geq \hat{c}_{j_2}(\lambda), \quad \forall \quad 0 \leq \lambda \leq 1, \quad \text{or} \\ &\quad \vdots \quad \quad \quad \text{or} \\ \sigma &\geq \hat{c}_{j_{|N|}}(\lambda), \quad \forall \quad 0 \leq \lambda \leq 1. \end{aligned}$$

Thus, we have:

$$\begin{aligned} \sigma &\geq \max_{0 \leq \lambda \leq 1} \hat{c}_{j_1}(\lambda), \quad \text{or} \\ \sigma &\geq \max_{0 \leq \lambda \leq 1} \hat{c}_{j_2}(\lambda), \quad \text{or} \\ &\quad \vdots \quad \quad \quad \text{or} \\ \sigma &\geq \max_{0 \leq \lambda \leq 1} \hat{c}_{j_{|N|}}(\lambda). \end{aligned}$$

Since  $C \in \mathbf{C}$  in the problem (14) then, for obtaining the minimum value of  $\sigma$ , we should have:

$$\begin{aligned} \sigma &\geq \sigma_1 := \min_{C \in \mathbf{C}} \max_{0 \leq \lambda \leq 1} \hat{c}_{j_1}(\lambda), \quad \text{or} \\ \sigma &\geq \sigma_2 := \min_{C \in \mathbf{C}} \max_{0 \leq \lambda \leq 1} \hat{c}_{j_2}(\lambda), \quad \text{or} \\ &\quad \vdots \quad \quad \quad \text{or} \\ \sigma &\geq \sigma_{|N|} := \min_{C \in \mathbf{C}} \max_{0 \leq \lambda \leq 1} \hat{c}_{j_{|N|}}(\lambda). \end{aligned}$$

The sets  $\{\lambda \mid 0 \leq \lambda \leq 1\}$  and  $\mathbf{C}$  are compact sets. Moreover,  $\hat{c}_j(\lambda)$  is a continuous function for all  $j \in N$ . Therefore,  $\sigma_1, \dots, \sigma_{|N|}$  are finite numbers. Hence,  $\sigma \geq \min\{\sigma_1, \dots, \sigma_{|N|}\}$ . It means that  $\sigma$  is bounded below and, thus, the problem (14) is always bounded.  $\square$

The problem (14) has infinitely many constraints. We solve it by the relaxation procedure proposed by Shimizu and Aiyoshi [31] by the following steps:

### Algorithm 3.1

**Input:** The problem (14) and  $\epsilon > 0$  (a predefined tolerance).

**Step 1.** Let  $k = 1$  and  $\lambda^k = 1$ .

**Step 2.** Solve the following mixed integer linear programming problem:

$$\begin{aligned} \min \quad & \sigma \\ \text{s.t.} \quad & \hat{c}_j(\lambda^h) \leq \sigma + My_j, \quad \forall j \in N, \quad h = 1, 2, \dots, k \\ & \sum_{j \in N} y_j \leq |N| - 1 \\ & C \in \mathbf{C}, \quad \sigma \in \mathbb{R}, \quad y_j \in \{0, 1\} \quad \forall j \in N. \end{aligned} \quad (15)$$

Let  $(C^k, \sigma^k)$  be an optimal solution of the problem (15).

**Step 3.** Substitute  $C^k$  in  $\hat{c}_j(\lambda)$  and solve the following linear programming problem:

$$\begin{aligned} \max \quad & \theta \\ \text{s.t.} \quad & \theta \leq \hat{c}_j(\lambda), \quad \forall j \in N \\ & 0 \leq \lambda \leq 1, \quad \theta \in \mathbb{R}. \end{aligned} \quad (16)$$

Let  $(\theta^*, \lambda^{k+1})$  be an optimal solution of the problem (16).

**Step 4.** If  $\theta^* \leq \sigma^k + \epsilon$  then stop,  $V = \sigma^k$ . Otherwise, let  $k = k + 1$  and go to Step 2.

**Output:** The value  $V$ .

Indeed, the problem (15) is a discretization of the problem (14) (discrete values  $\lambda^1, \dots, \lambda^k$  for  $0 \leq \lambda \leq 1$ ). Moreover, the problem (16) seeks whether there is  $0 \leq \lambda \leq 1$  such that violates the constraints of the problem (15) or not. If there is such a  $\lambda$  then it will be added to the problem (15) and the process will be repeated. Otherwise, the algorithm terminates. The following proposition shows that the problem (16) always has an optimal solution.

**Proposition 3.8.** *The linear programming problem (16) is always feasible and bounded.*

*Proof.* Indeed,  $\lambda = 0$  and  $\theta = \min_{j \in N} \hat{c}_j(0)$  satisfy the constraints of the problem (16). Now, assume that  $N = \{j_1, \dots, j_{|N|}\}$ . The constraints imply that  $\theta \leq \min_{j \in N} \max_{0 \leq \lambda \leq 1} \hat{c}_j(\lambda)$ . The continuity of  $\hat{c}_j(\lambda)$  and compactness of the set  $\{\lambda \mid 0 \leq \lambda \leq 1\}$  imply that  $\min_{j \in N} \max_{0 \leq \lambda \leq 1} \hat{c}_j(\lambda)$  is a finite number and an upper bound for the variable  $\theta$ . Hence, the problem (16) is always bounded.  $\square$

Note that the problem (16) is a linear program with only two variables  $\lambda$  and  $\theta$ . It can be solved easily by linear programming solvers. However, the only time consuming step of Algorithm 3.1 is Step 2, where the problem (15) should be solved. It can be also solved by using commercial softwares effectively. In fact, the suggested method of this section solves an interval biobjective linear programming problem. To see the quality of the final solution of it, we solve the following two examples.

**Example 3.9.** Consider the following interval biobjective linear program taken from [17]:

$$\begin{aligned} \max \quad & [1, 2]x_1 + [3, 4]x_2 \\ \max \quad & [2, 3]x_1 + [1.5, 2.5]x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 19, \\ & 5x_1 + 2x_2 \leq 47, \\ & 0 \leq x_1 \leq 9, \\ & 0 \leq x_2 \leq 8. \end{aligned} \quad (17)$$

Inuiguchi and Sakawa [17] have shown that  $x^1 = (3, 8)^t$  and  $x^2 = (7, 6)^t$  are possibly efficient solutions to the problem (17). Solving the problem (17) by PSA and Algorithm 3.1 leads to the point  $x^* = (7, 6)^t$  as a necessarily weak efficient solution. It can be understood from Definition 2.2 that necessarily weak efficient solutions are better solutions than possibly efficient ones. In fact, if we consider  $\bar{C} \in \mathbf{C}$  as:

$$\bar{C} = \begin{pmatrix} 2 & 3 \\ 2 & 1.5 \end{pmatrix},$$

then  $\bar{C}(7, 6)^t = (32, 23)^t \succeq (30, 18)^t = \bar{C}(3, 8)^t$ . Thus, the point  $(3, 8)^t$  is not as good as  $(7, 6)^t$ . The details of computations are as follows. Note that we first convert the objective functions of the problem (17) to minimization. The first weak efficient solution in the PSA is  $\hat{x}^p = (3, 8)^t$  ( $p = 1$  and  $\lambda^p = 1$ ) with the following simplex tableau.

Table 2: The first weak efficient solution in the PSA.

Basis	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	RHS
$z_1(x)$	0	0	2	0	0	0	38
$z_2(x)$	0	0	3	0	0	-3.5	29
$x_1$	1	0	1	0	0	-2	3
$x_4$	0	0	-5	1	0	8	16
$x_5$	0	0	-1	0	1	2	6
$x_2$	0	1	0	0	0	1	8

In Table 2,  $x_3$ ,  $x_4$ ,  $x_5$ , and  $x_6$  are slack variables. Now, we should check the necessarily weak efficiency of the BFS given in Table 2 by Algorithm 3.1.

**Input:** We choose  $\epsilon = 0.01$ . The problem (14) related to Table (2) is:

$$\begin{aligned}
& \min \quad \sigma \\
& s.t. \quad -\lambda c_{11} - (1 - \lambda)c_{21} \leq \sigma + My_3, \quad \forall 0 \leq \lambda \leq 1, \\
& \quad \quad -\lambda(-2c_{11} + c_{12}) - (1 - \lambda)(-2c_{21} + c_{22}) \leq \sigma + My_6, \quad \forall 0 \leq \lambda \leq 1, \\
& \quad \quad y_3 + y_6 \leq 1, \\
& \quad \quad c_{11} \in [-2, -1], c_{12} \in [-4, -3], c_{21} \in [-3, -2], c_{22} \in [-2.5, -1.5], \quad \sigma \in \mathbb{R}, \quad y_3, y_6 \in \{0, 1\}.
\end{aligned}$$

Note that all coefficients of slack variables in both objective functions are zero.

**Step 1.**  $k = 1$  and  $\lambda^k = 1$ .

**1st iteration:**

**Step 2.**  $C^1 = \begin{pmatrix} -2 & -3 \\ -3 & -2.5 \end{pmatrix}$ ,  $\sigma^1 = -1$ .

**Step 3.**  $\theta^* = -1$ ,  $\lambda^2 = 1$ .

**Step 4.**  $\theta^* \leq \sigma^1 + \epsilon$ , then stop.  $V = -1$ .

**Output:**  $V = -1$ .

Since  $V < 0$  then  $\hat{x}^1 = (3, 8)^t \notin X_{NW}$ . Now, since  $I = \{6\} \neq \emptyset$  then  $s = 6$  ( $x_6$  enters the basis) and  $r = 2$  ( $x_4$  leaves the basis). The simplex pivoting at  $y_{26}$  leads to the new BFS  $\hat{x}^2 = (7, 6)^t$  with the following simplex tableau. Now, we

Table 3: The second weak efficient solution in the PSA.

Basis	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	RHS
$z_1(x)$	0	0	2	0	0	0	25
$z_2(x)$	0	0	6.5/8	3.5/8	0	0	26
$x_1$	1	0	-2/8	2/8	0	0	7
$x_6$	0	0	-5/8	1/8	0	1	2
$x_5$	0	0	2/8	-2/8	1	0	2
$x_2$	0	1	5/8	-1/8	0	0	6

should check the necessarily weak efficiency of the BFS given in Table 3 by Algorithm 3.1.

**Input:** We choose  $\epsilon = 0.01$ . The problem (14) related to Table 3 is:

$$\begin{aligned}
& \min \quad \sigma \\
& s.t. \quad -\lambda(-0.25c_{11} + 0.625c_{12}) - (1 - \lambda)(-0.25c_{21} + 0.625c_{22}) \leq \sigma + My_3, \quad \forall 0 \leq \lambda \leq 1, \\
& \quad \quad -\lambda(0.25c_{11} - 0.125c_{12}) - (1 - \lambda)(0.25c_{21} - 0.125c_{22}) \leq \sigma + My_4, \quad \forall 0 \leq \lambda \leq 1, \\
& \quad \quad y_3 + y_4 \leq 1, \\
& \quad \quad c_{11} \in [-2, -1], c_{12} \in [-4, -3], c_{21} \in [-3, -2], c_{22} \in [-2.5, -1.5], \quad \sigma \in \mathbb{R}, \quad y_3, y_4 \in \{0, 1\}.
\end{aligned}$$

**Step 1.**  $k = 1$  and  $\lambda^k = 1$ .

**1st iteration:**

**Step 2.**  $C^1 = \begin{pmatrix} -1 & -4 \\ -3 & -2.5 \end{pmatrix}$ ,  $\sigma^1 = -0.25$ .

**Step 3.**  $\theta^* = 0.4375$ ,  $\lambda^2 = 0$ .

**Step 4.**  $\theta^* \not\leq \sigma^1 + \epsilon$ ,  $k = 2$ , go to Step 2.

**2nd iteration:**

**Step 2.**  $C^2 = \begin{pmatrix} -2 & -3 \\ -2 & -2.5 \end{pmatrix}$ ,  $\sigma^2 = 0.187$ .

**Step 3.**  $\theta^* = 0.187$ ,  $\lambda^3 = 0$ .

**Step 4.**  $\theta^* \leq \sigma^2 + \epsilon$ , then stop.  $V = 0.187$ .

**Output:**  $V = 0.187$ .

Since  $V > 0$ , then  $\hat{x}^2 = (7, 6)^t \in X_{NW}$ . Moreover, the optimality condition for the second objective function in Table 3 is satisfied. Thus, the PSA terminates.

MATLAB routines were used to implement Algorithm 3.1. The run time for solving the problem (17) was 1.08 seconds.

**Remark 3.1.** We solved all the numerical examples of this paper by MATLAB 9.6.0.1072779. We used the MATLAB function "intlinprog" for solving mixed integer linear programming problems. Moreover, we ran the MATLAB codes on a laptop with a 2.2 GHz Intel(R) Core(TM) i7-3632QM processor and 8 GB of RAM.

**Example 3.10.** The following interval biobjective linear program is taken from [17]:

$$\begin{aligned} \max \quad & [2, 3]x_1 + [1.5, 2.5]x_2 \\ \max \quad & [3, 4]x_1 + [0.5, 0.8]x_2 \\ \text{s.t.} \quad & 3x_1 + 4x_2 \leq 42, \\ & 3x_1 + x_2 \leq 24, \\ & 0 \leq x_2 \leq 9, \\ & x_1 \geq 0. \end{aligned} \tag{18}$$

In [17], the point  $x = (6, 6)^t$  is obtained as a necessarily efficient solution and thus necessarily weak efficient solution. Solving (18) by the PSA and Algorithm 3.1 yields  $\hat{x}^1 = (6, 6)^t$  and  $\hat{x}^2 = (8, 0)^t$  as necessarily weak efficient solutions.

**Remark 3.2.** By Corollary 3.1, if for all attained weak efficient BFSs,  $\hat{x}^1, \dots, \hat{x}^q$ , after completion of the PSA, the value  $V$  in Theorem 3.1 is negative then  $X_{NW} = \emptyset$ . In this case, for suggesting a solution to the problem (3), we choose one of the solutions  $\hat{x}^1, \dots, \hat{x}^q$  with the largest value  $V$  in Algorithm 3.1. Indeed, in this case, it is one of the possibly weak efficient solutions to the problem (3).

## 4 Solving fuzzy biobjective transportation problem

In this section, we focus on the problem (10). We propose a new algorithm for solving it which is based on the popular bisection method [29] and the method of Section 3. Initially by obtaining the  $\gamma$ -cuts of fuzzy coefficients, the problem (10) is converted to an interval biobjective linear program with two objective functions  $\sum_{i=1}^m \sum_{j=1}^n (\tilde{c}_{ij}^1)_\gamma x_{ij}$  and  $\sum_{i=1}^m \sum_{j=1}^n (\tilde{c}_{ij}^2)_\gamma x_{ij}$ . The parametric objective function coefficients  $(\tilde{c}_{ij}^1)_\gamma$  and  $(\tilde{c}_{ij}^2)_\gamma$  are closed intervals which depend on parameter  $\gamma$  where  $\gamma \in [0, 1]$ . Therefore, for a fixed value of  $\gamma$ , we have an instance of the problem (3).

We first set  $\gamma = \gamma_1 = 0$  (the smallest value of  $\gamma$ ). Then, the yielded interval biobjective linear program is solved by the method of Section 3. If we obtain a necessarily weak efficient solution then the desired  $\gamma^*$ -NWE solution to the problem (10) is achieved where  $\gamma^* = 0$ . Otherwise, the problem (10) has no  $\gamma^*$ -NWE solution at level  $\gamma^* = 0$  and we should check positive values of  $\gamma^*$ . To do that, we set  $\gamma = \gamma_2 = 1$  (the largest value of  $\gamma$ ) and solve the yielded interval biobjective linear program by the PSA and Algorithm 3.1 (note that the real number  $a$  can be considered as the interval  $[a, a]$ ). If the set of necessarily weak efficient solutions is empty then we conclude that the problem (10) has no  $\gamma^*$ -NWE solution for  $0 \leq \gamma^* \leq 1$ . However, in this case, for suggesting a solution to the problem (10), we propose one of the attained BFSs with the largest value  $V$  based on Remark 3.2. Otherwise, we obtain a  $\gamma$ -CNWE optimal solution to the problem (10) with  $\gamma = 1$ . To find a  $\gamma^*$ -NWE solution, based on bisection method, we set  $\gamma = \frac{\gamma_1 + \gamma_2}{2}$  and update the interval of parameter  $\gamma$  as  $[\gamma_1, \frac{\gamma_1 + \gamma_2}{2}]$ .

The above process will be repeated with the new interval of parameter  $\gamma$ . In this section, since the set of necessarily weak efficient solutions depends on a value of parameter  $\gamma$ , we denote it by  $X_{NW}(\gamma)$  instead of  $X_{NW}$ . Without loss of generality, suppose that in the  $k$ th iteration of the above process, the interval of parameter  $\gamma$  is  $[\gamma_1, \gamma_2]$ . We obtain the new interval of  $\gamma$  as follows:

- If  $X_{NW}(\frac{\gamma_1 + \gamma_2}{2}) \neq \emptyset$  then  $[\gamma_1^{new}, \gamma_2^{new}] = [\gamma_1, \frac{\gamma_1 + \gamma_2}{2}]$ .
- If  $X_{NW}(\frac{\gamma_1 + \gamma_2}{2}) = \emptyset$  then  $[\gamma_1^{new}, \gamma_2^{new}] = [\frac{\gamma_1 + \gamma_2}{2}, \gamma_2]$ .

The above explained process is summarized in a step by step algorithm as follows:

**Algorithm 4.1**

**Input:** The problem (10) and  $\epsilon > 0$  (a predefined tolerance).

**Step 1.** By using  $\gamma$ -cuts of fuzzy coefficients  $\tilde{c}_{ij}^1$  and  $\tilde{c}_{ij}^2$  ( $i = 1, \dots, m$ ,  $j = 1, \dots, n$ ) convert the problem (10) to an interval biobjective linear program with objective functions  $\sum_{i=1}^m \sum_{j=1}^n (\tilde{c}_{ij}^1)_\gamma x_{ij}$  and  $\sum_{i=1}^m \sum_{j=1}^n (\tilde{c}_{ij}^2)_\gamma x_{ij}$ . We call this problem as the problem (10) $_\gamma$ .

**Step 2.** Let  $\gamma_1 = 0$  and run the PSA with Algorithm 3.1 to the problem (10) $_{\gamma_1}$ .

**2.1.** If  $X_{NW}(\gamma_1) \neq \emptyset$  then stop. The attained solution in  $X_{NW}(\gamma_1)$  is a  $\gamma^*$ -NWE solution to the problem (10) where  $\gamma^* = 0$ . Otherwise, go to Step 3.

**Step 3.** Let  $\gamma_2 = 1$  and run the PSA with Algorithm 3.1 to the problem (10) $_{\gamma_2}$ .

**3.1.** If  $X_{NW}(\gamma_2) = \emptyset$  then stop. The problem (10) has no  $\gamma^*$ -NWE solution for  $0 \leq \gamma^* \leq 1$ . Choose one of the attained BFSs with the largest value  $V$  as a solution to the problem (10). If  $X_{NW}(\gamma_2) \neq \emptyset$  then go to Step 4.

**Step 4.** Let  $\gamma = \frac{\gamma_1 + \gamma_2}{2}$  and run the PSA with Algorithm 3.1 to the problem (10) $_\gamma$ . Two cases are possible:

**4.1.** Case 1,  $X_{NW}(\gamma) \neq \emptyset$ .

If  $|\gamma_2 - \gamma_1| < \epsilon$  (a termination criterion) then stop. The attained solution in  $X_{NW}(\gamma)$  is a  $\gamma^*$ -NWE solution to the problem (10) where  $\gamma^* = \gamma$ . Otherwise, update  $\gamma_2 = \gamma$  and go to Step 4.

**4.2.** Case 2,  $X_{NW}(\gamma) = \emptyset$ .

If  $|\gamma_2 - \gamma_1| < \epsilon$  then stop. The attained solution in  $X_{NW}(\gamma_2)$  is a  $\gamma^*$ -NWE solution to the problem (10) where  $\gamma^* = \gamma_2$ . Otherwise, update  $\gamma_1 = \gamma$  and go to Step 4.

**Output:**  $\gamma^*$ -NWE solution to the problem (10) (if there is such a solution).

The following theorem shows that Algorithm 4.1 is convergent.

**Theorem 4.1.** *Algorithm 4.1 terminates after a finite number of iterations for any arbitrary positive  $\epsilon$ .*

*Proof.* At first, the interval of parameter  $\gamma$  is  $[\gamma_1 = 0, \gamma_2 = 1]$ . Suppose that Algorithm 4.1 does not terminate in iteration one. Then, at the beginning of the second iteration the interval of parameter  $\gamma$  is  $[\gamma_1 = 0, \gamma_2 = \frac{1}{2}]$  or  $[\gamma_1 = \frac{1}{2}, \gamma_2 = 1]$ . Thus, the length of the new interval is  $\frac{1}{2}$ . Similarly, the length of the interval of parameter  $\gamma$  at the end of iteration  $k$  is  $(\frac{1}{2})^k$ . Therefore, Algorithm 4.1 terminates at most after  $[\frac{-\ln \epsilon}{\ln 2}] + 1$  iterations where  $[x]$  denotes the largest integer number smaller than or equal to  $x$ .  $\square$

Note that Algorithm 4.1 does not change the constraint set of the fuzzy biobjective transportation problem (10). In other words, all interval biobjective linear programs that should be solved are interval biobjective transportation problems. In fact, we never change  $X_T$  in our computations. Therefore, the well known network simplex algorithm for transportation problems can be used [2]. Indeed, all computations on the parametric simplex tableaus (Table 1) can be done on parametric transportation tableaus similarly [2]. Hence, we can benefit from the good structure of a transportation problem [2]. However, many other methods for solving a fuzzy biobjective transportation problem change the constraints set [18, 19, 32, 33]. Thus, they need to solve the new yielded problems by only linear or nonlinear programming techniques. Moreover, the following examples show the good performance of Algorithm 4.1 and its comparison with the other methods.

**Example 4.2.** By this example, we show that Algorithm 4.1 solves a fuzzy biobjective transportation problem with various fuzzy objective functions coefficients. So, consider the following fuzzy biobjective problem:

$$\begin{aligned}
\min \quad & \tilde{c}_{11}^1 x_{11} + \tilde{c}_{12}^1 x_{12} + \tilde{c}_{13}^1 x_{13} + \tilde{c}_{14}^1 x_{14} + \tilde{c}_{21}^1 x_{21} + \tilde{c}_{22}^1 x_{22} + \tilde{c}_{23}^1 x_{23} + \tilde{c}_{24}^1 x_{24} + \\
& \tilde{c}_{31}^1 x_{31} + \tilde{c}_{32}^1 x_{32} + \tilde{c}_{33}^1 x_{33} + \tilde{c}_{34}^1 x_{34} \\
\min \quad & \tilde{c}_{11}^2 x_{11} + \tilde{c}_{12}^2 x_{12} + \tilde{c}_{13}^2 x_{13} + \tilde{c}_{14}^2 x_{14} + \tilde{c}_{21}^2 x_{21} + \tilde{c}_{22}^2 x_{22} + \tilde{c}_{23}^2 x_{23} + \tilde{c}_{24}^2 x_{24} + \\
& \tilde{c}_{31}^2 x_{31} + \tilde{c}_{32}^2 x_{32} + \tilde{c}_{33}^2 x_{33} + \tilde{c}_{34}^2 x_{34} \\
s.t. \quad & x_{11} + x_{12} + x_{13} + x_{14} = 17 \\
& x_{21} + x_{22} + x_{23} + x_{24} = 12 \\
& x_{31} + x_{32} + x_{33} + x_{34} = 21 \\
& x_{11} + x_{21} + x_{31} = 16 \\
& x_{12} + x_{22} + x_{32} = 9 \\
& x_{13} + x_{23} + x_{33} = 14 \\
& x_{14} + x_{24} + x_{34} = 11 \\
& x_{ij} \geq 0, \quad i = 1, 2, 3, \quad j = 1, 2, 3, 4,
\end{aligned} \tag{19}$$

where the fuzzy coefficients for the first and the second objective functions are given in tables 4 and 5, respectively.

Tables 6 and 7 involve the  $\gamma$ -cuts of fuzzy coefficients of tables 4 and 5, respectively. Moreover, figures 2 and 3 show the various graphs of fuzzy coefficients of tables 4 and 5, respectively.

Table 4: The fuzzy coefficients  $\tilde{c}_{ij}^1$  in Example 4.1.

$\tilde{c}_{11}^1 = \langle 1.5, 0.5, - \rangle_l$	$\tilde{c}_{21}^1 = \langle 2.5, 0.5, 0.5 \rangle_{l,l}$	$\tilde{c}_{31}^1 = \langle 1.5, 0.5, 0.5 \rangle_{q,q}$
$\tilde{c}_{12}^1 = \langle 3.5, 4, 0.5, 0.5 \rangle_{l,l}$	$\tilde{c}_{22}^1 = \langle 4.5, 0.5, 0.5 \rangle_{q,q}$	$\tilde{c}_{32}^1 = \langle 4.5, 0.5, - \rangle_q$
$\tilde{c}_{13}^1 = \langle 2.5, -, 0.5 \rangle_l$	$\tilde{c}_{23}^1 = \langle 2, 2.5, 0.5, - \rangle_l$	$\tilde{c}_{33}^1 = \langle 2, 1, - \rangle_l$
$\tilde{c}_{14}^1 = \langle 1, -, 1.5 \rangle_q$	$\tilde{c}_{24}^1 = \langle 4, 1, - \rangle_l$	$\tilde{c}_{34}^1 = \langle 3.5, 0.5, 0.5 \rangle_{l,l}$

Table 5: The fuzzy coefficients  $\tilde{c}_{ij}^2$  in Example 4.1.

$\tilde{c}_{11}^2 = \langle 2, 2.5, -, 0.5 \rangle_{l,l}$	$\tilde{c}_{21}^2 = \langle 3, 3.5, 0.5, 0.5 \rangle_{l,l}$	$\tilde{c}_{31}^2 = \langle 3.5, 0.5, - \rangle_q$
$\tilde{c}_{12}^2 = \langle 1.5, 0.5, 0.5 \rangle_{q,q}$	$\tilde{c}_{22}^2 = \langle 1, 0.5, 0.5 \rangle_{l,l}$	$\tilde{c}_{32}^2 = \langle 2, -, 1 \rangle_q$
$\tilde{c}_{13}^2 = \langle 4, 0.5, - \rangle_l$	$\tilde{c}_{23}^2 = \langle 0.5, -, 1 \rangle_q$	$\tilde{c}_{33}^2 = \langle 1.5, 0.5, 0.5 \rangle_{l,l}$
$\tilde{c}_{14}^2 = \langle 3, -, 0.5 \rangle_l$	$\tilde{c}_{24}^2 = \langle 3.5, -, 1 \rangle_l$	$\tilde{c}_{34}^2 = \langle 3, 1.5, - \rangle_l$

Table 6:  $\gamma$ -cuts of fuzzy coefficients  $\tilde{c}_{ij}^1$ .

$(\tilde{c}_{11}^1)_\gamma = [1 + 0.5\gamma, 1.5]$	$(\tilde{c}_{21}^1)_\gamma = [2 + 0.5\gamma, 3 - 0.5\gamma]$	$(\tilde{c}_{31}^1)_\gamma = [1.5 - 0.5\sqrt{1 - \gamma^2}, 1.5 + 0.5\sqrt{1 - \gamma^2}]$
$(\tilde{c}_{12}^1)_\gamma = [3 + 0.5\gamma, 4.5 - 0.5\gamma]$	$(\tilde{c}_{22}^1)_\gamma = [4.5 - 0.5\sqrt{1 - \gamma^2}, 4.5 + 0.5\sqrt{1 - \gamma^2}]$	$(\tilde{c}_{32}^1)_\gamma = [3 - 0.5\sqrt{1 - \gamma^2}, 3]$
$(\tilde{c}_{13}^1)_\gamma = [2.5, 3 - 0.5\gamma]$	$(\tilde{c}_{23}^1)_\gamma = [1.5 + 0.5\gamma, 2.5]$	$(\tilde{c}_{33}^1)_\gamma = [1 + \gamma, 2]$
$(\tilde{c}_{14}^1)_\gamma = [1, 1 + 1.5\sqrt{1 - \gamma^2}]$	$(\tilde{c}_{24}^1)_\gamma = [3 + \gamma, 4]$	$(\tilde{c}_{34}^1)_\gamma = [3 + 0.5\gamma, 4 - 0.5\gamma]$

Table 7:  $\gamma$ -cuts of fuzzy coefficients  $\tilde{c}_{ij}^2$ .

$(\tilde{c}_{11}^2)_\gamma = [2, 3 - 0.5\gamma]$	$(\tilde{c}_{21}^2)_\gamma = [2.5 + 0.5\gamma, 4 - 0.5\gamma]$	$(\tilde{c}_{31}^2)_\gamma = [3.5 - 0.5\sqrt{1 - \gamma^2}, 3.5]$
$(\tilde{c}_{12}^2)_\gamma = [1.5 - 0.5\sqrt{1 - \gamma^2}, 1.5 + 0.5\sqrt{1 - \gamma^2}]$	$(\tilde{c}_{22}^2)_\gamma = [0.5 + 0.5\gamma, 1.5 - 0.5\gamma]$	$(\tilde{c}_{32}^2)_\gamma = [2, 2 + \sqrt{1 - \gamma^2}]$
$(\tilde{c}_{13}^2)_\gamma = [3.5 + 0.5\gamma, 4]$	$(\tilde{c}_{23}^2)_\gamma = [0.5, 0.5 + \sqrt{1 - \gamma^2}]$	$(\tilde{c}_{33}^2)_\gamma = [1 + 0.5\gamma, 2 - 0.5\gamma]$
$(\tilde{c}_{14}^2)_\gamma = [3, 3.5 - 0.5\gamma]$	$(\tilde{c}_{24}^2)_\gamma = [3.5, 4.5 - \gamma]$	$(\tilde{c}_{34}^2)_\gamma = [1.5 + 1.5\gamma, 3]$

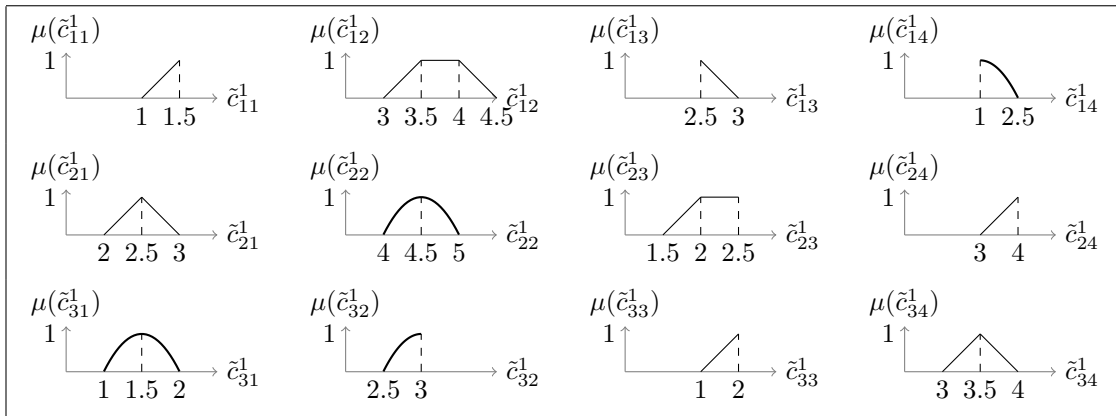


Figure 2: Graphs of membership functions  $\tilde{c}_{ij}^1$ .

Now, by using Algorithm 4.1, we have the following results:

**Input:** The problem (19) and  $\epsilon = 0.02$ .

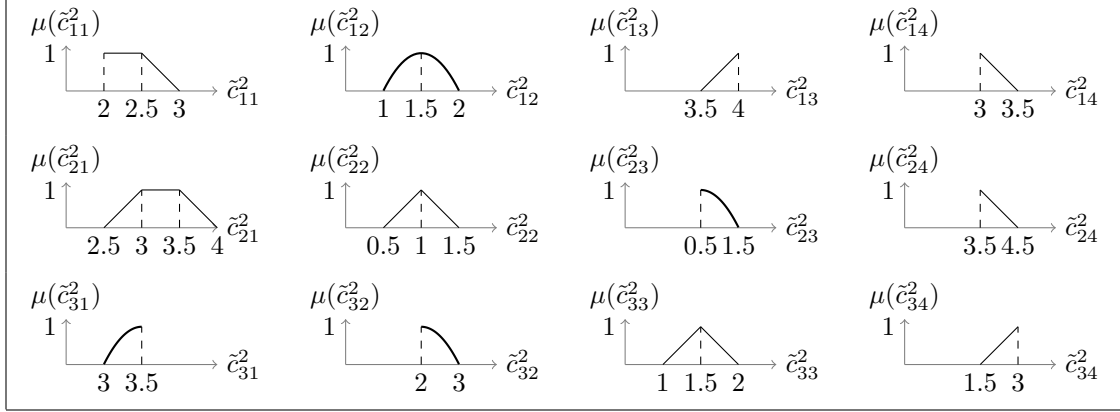
**1st iteration:**

**Step 2.**  $\gamma_1 = 0$  and  $X_{NW}(\gamma_1) = \emptyset$ .

**Step 3.**  $\gamma_2 = 1$  and  $X_{NW}(\gamma_2) \neq \emptyset$ .

**Step 4.**  $\gamma = 0.5$  and  $X_{NW}(\gamma) = \emptyset$ . Since  $|\gamma_2 - \gamma_1| = 0.5 > 0.02$  then  $\gamma_1 = 0.5, \gamma_2 = 1$  and next iteration should be run. After 5 iterations, we met termination criterion  $|\gamma_2 - \gamma_1| = 0.015 < 0.02$  where  $\gamma^* = 0.859375$  and the attained solution in  $X_{NW}(\gamma)$  was:

$$x_{11} = 6, x_{14} = 11, x_{23} = 12, x_{31} = 10, x_{32} = 9, x_{33} = 2, \tag{20}$$


 Figure 3: Graphs of membership functions  $\tilde{c}_{ij}^2$ .

with zero value for the other variables.

**Output.** The given  $\gamma^*$ -NWE solution in (20).

If we substitute  $\gamma^* \approx 0.86$  in the intervals presented in tables 6 and 7 then the output solution is weak efficient for all objective coefficients within those intervals. Indeed, the yielded intervals are corresponding to the large values of the membership functions, which are desirable to the decision maker. Moreover, the run time of the MATLAB code for solving the problem (19) was 20.79 seconds.

**Example 4.3.** We have taken this example from [18, 19]. Consider a fuzzy biobjective transportation problem with three sources and four destinations. The supplies of the sources 1, 2, and 3 are 10, 20, and 40, respectively. Further, the destinations 1, 2, 3, and 4 have 20, 10, 15, and 25 demands, respectively. Furthermore, the first objective function is minimization of delivery time while the second objective function is maximization of transportation profit. The fuzzy coefficients (fuzzy times and fuzzy profits) of the two objective functions with the notations of this paper are presented in tables 8 and 9, respectively.

 Table 8: The fuzzy coefficients  $\tilde{c}_{ij}^1$  in Example 4.2.

Fuzzy times		Destinations			
		1	2	3	4
Sources	1	$\langle 10, 5, - \rangle_l$	$\langle 9, 5, - \rangle_l$	$\langle 8, 3, - \rangle_l$	$\langle 7, 1, - \rangle_l$
	2	$\langle 9, 6, - \rangle_l$	$\langle 6, 3, - \rangle_l$	$\langle 10, 4, - \rangle_l$	$\langle 15, 9, - \rangle_l$
	3	$\langle 12, 10, - \rangle_l$	$\langle 10, 5, - \rangle_l$	$\langle 9, 2, - \rangle_l$	$\langle 10, 2, - \rangle_l$

 Table 9: The fuzzy coefficients  $\tilde{c}_{ij}^2$  in Example 4.2.

Fuzzy profits		Destinations			
		1	2	3	4
Sources	1	$\langle 2, -, 5 \rangle_l$	$\langle 3, -, 5 \rangle_l$	$\langle 6, -, 3 \rangle_l$	$\langle 3, -, 2 \rangle_l$
	2	$\langle 8, -, 3 \rangle_l$	$\langle 7, -, 2 \rangle_l$	$\langle 6, -, 6 \rangle_l$	$\langle 4, -, 6 \rangle_l$
	3	$\langle 7, -, 5 \rangle_l$	$\langle 5, -, 6 \rangle_l$	$\langle 7, -, 0 \rangle_l$	$\langle 3, -, 1 \rangle_l$

Solving this fuzzy biobjective transportation problem by Algorithm 4.1 leads to the following necessarily weak efficient solution after 4 iterations with  $\gamma^* = 0.9375$ ,

$$x^* = (x_{11} = 0, x_{12} = 0, x_{13} = 0, x_{14} = 10, x_{21} = 10, x_{22} = 10, x_{23} = 0, x_{24} = 0, x_{31} = 10, x_{32} = 0, x_{33} = 15, x_{34} = 15)^t.$$

Table 10 shows some details of computations in Algorithm 4.1. The run times of the MATLAB code in Table 10 are in seconds. The algorithm of [18] gives the following solution to this problem:

$$\bar{x} = (x_{11} = 0, x_{12} = 0, x_{13} = 5, x_{14} = 5, x_{21} = 0, x_{22} = 0, x_{23} = 0, x_{24} = 20, x_{31} = 20, x_{32} = 10, x_{33} = 10, x_{34} = 0)^t.$$

Choose  $\bar{C} = (\bar{c}_1, \bar{c}_2)^t$  as follows:

$$\bar{c}_1 = (c_{11}^1 = 10, c_{12}^1 = 9, c_{13}^1 = 8, c_{14}^1 = 7, c_{21}^1 = 9, c_{22}^1 = 6, c_{23}^1 = 10, c_{24}^1 = 15, c_{31}^1 = 12, c_{32}^1 = 10, c_{33}^1 = 9, c_{34}^1 = 10)^t,$$

$$\bar{c}_2 = (c_{11}^2 = 2, c_{12}^2 = 3, c_{13}^2 = 6, c_{14}^2 = 3, c_{21}^2 = 8, c_{22}^2 = 7, c_{23}^2 = 6, c_{24}^2 = 4, c_{31}^2 = 7, c_{32}^2 = 5, c_{33}^2 = 7, c_{34}^2 = 3)^t.$$

Table 10: Some results of Algorithm 4.1.

Iteration	$\gamma$	$X_{NW}(\gamma)$	$\gamma_1$	$\gamma_2$	$ \gamma_2 - \gamma_1 $	Run times
1	0.5	empty	0.5	1	0.5	1.38
2	0.75	empty	0.75	1	0.25	1.39
3	0.875	empty	0.875	1	0.125	0.87
4	0.9375	nonempty	0.875	0.9375	0.0625	0.37

Then,  $\bar{c}_1 x^* = 625 < \bar{c}_1 \bar{x} = 805$  and  $-\bar{c}_2 x^* = -400 < -\bar{c}_2 \bar{x} = -385$ . Note that the second objective function is multiplied by  $-1$ , since it is of maximization type. Thus,  $\bar{C}x^* \preceq \bar{C}\bar{x}$  and the solution  $x^*$  (achieved by Algorithm 4.1) dominates the solution  $\bar{x}$ . Moreover, the revised algorithm of [19] yields the following solution after 10 iterations:

$$\hat{x} = (x_{11} = 0, x_{12} = 0, x_{13} = 0, x_{14} = 10, x_{21} = 0, x_{22} = 3, x_{23} = 3, x_{24} = 14, x_{31} = 20, x_{32} = 7, x_{33} = 12, x_{34} = 1)^t.$$

We also have:

$$\bar{c}_1 \hat{x} = 756 > \bar{c}_1 x^* = 625 \text{ and } -\bar{c}_2 \hat{x} = -387 > -\bar{c}_2 x^* = -400.$$

Therefore,  $\bar{C}\hat{x} \preceq \bar{C}x^*$  and the solution  $x^*$  dominates the solution  $\hat{x}$ .

It should be emphasized that although the most usefulness of Algorithm 4.1 is solving fuzzy biobjective transportation problems, since it does not change the structure of the constraints, but it is also implementable to any fuzzy biobjective linear programming problems. The next example shows this feature of Algorithm 4.1.

**Example 4.4.** Consider the following fuzzy biobjective linear program, presented in [21]:

$$\begin{aligned} \min \quad & c_{11}^1 x_1 + c_{12}^1 x_2 \\ \min \quad & c_{21}^2 x_1 + c_{22}^2 x_2 \\ \text{s.t.} \quad & -x_1 - x_2 \leq -6, \\ & -2x_1 - x_2 \leq -9, \\ & x_1, x_2 \geq 0, \end{aligned} \tag{21}$$

where  $c_{11}^1 = \langle 1, 0.5, 0.5 \rangle_{l,l}$ ,  $c_{12}^1 = \langle 2, 0.2, 1 \rangle_{l,l}$ ,  $c_{21}^2 = \langle 2, 2, 1 \rangle_{l,l}$ , and  $c_{22}^2 = \langle 1, 1, 2 \rangle_{l,l}$ .

The method proposed by Luhandjula and Rangoaga [21] obtains the point  $x = (3, 3)^t$  as the satisficing solution to the problem (21). If we solve the problem (21) by Algorithm 4.1, we obtain  $x^* = (6, 0)^t$  as a  $\gamma^*$ -NWE solution where  $\gamma^* = 0$ . In addition, if we choose

$$C = \begin{pmatrix} 1 & 2 \\ 2 & 2.5 \end{pmatrix},$$

then  $Cx^* = (6, 12)^t < Cx = (9, 13.5)^t$ . Thus,  $Cx^* \preceq Cx$  and the solution  $x^*$  dominates the solution  $x$ .

## 5 Concluding remarks

In this paper, we were concentrated on a biobjective transportation problem with fuzzy coefficients in the objective functions. To solve those problems a kind of optimal solution based on necessarily weak efficient solutions is defined. A new algorithm is proposed to find such a solution. The new algorithm can handle various kinds of fuzzy numbers while most of the existing methods only deal with a special type of fuzzy numbers such as triangular, trapezoidal, or L-R fuzzy numbers. Another significant feature of the new algorithm is that it does not change the good structure of the constraints of the transportation problem. Therefore, it helps us to benefit from the famous network simplex algorithm. In fact, the new algorithm converts the main problem to an interval biobjective linear programming problem. Thus, we suggested a new method for finding a necessarily weak efficient solution of an interval biobjective linear program. Since necessarily weak efficient solutions are weak efficient for all interval data, our attained solutions dominate some of the solutions of the existing methods. This is shown by several numerical examples. Developing other formulations for checking necessarily (weak) efficiency is a good subject for research. In this paper, we only considered fuzzy coefficients in the objective functions. Extending the method for solving a problem with fully fuzzy parameters is another subject for further research. Moreover, another direction for research is considering a problem with more than two objective functions.



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