

Solving matrix games with hesitant fuzzy pay-offs

M. R. Seikh¹, S. Karmakar² and M. Xia³

^{1,2}*Department of Mathematics, Kazi Nazrul University, Asansol-713 340, India*

³*School of Economics and Management, Beijing University, Beijing, China*

mrseikh@gmail.com, shuvasreekarmakar53@gmail.com, mmxia@bjtu.edu.cn

Abstract

The objective of this paper is to develop matrix games with pay-offs of triangular hesitant fuzzy elements (THFEs). To solve such games, a new methodology has been derived based on the notion of weighted average operator and score function of THFEs. Firstly, we formulate two non-linear programming problems with THFEs. Then applying the score function of THFEs, we transform these two problems into two non-linear multi-objective programming problems with triangular fuzzy numbers (TFNs). Finally, the Lexicographic method is used to solve these two multi-objective programming problems. A market share problem is considered to show the validity and applicability of the proposed methodology.

Keywords: Matrix Game, triangular hesitant fuzzy set, score function, multi-objective optimization, lexicographic method.

1 Introduction

A game is a mathematical tool that can conceive a conflicting circumstance, arises in the real world and to conclude such a situation. Actually, it is a study of some mathematical models which deal with the strategic interaction among the decision-makers. In reality, due to the lack of information and ambiguity of players, the decision may be imprecise. To handle such an environment, researchers use the notion of fuzzy sets and its various extensions.

Initially, the fuzzy set was introduced by Zadeh [44] in 1965, where every element of a set was defined along with their membership values. In 1980, Dubois and Prade [10] applied the concept of fuzzy set in game theory. Refining the model of Bector et al. [5] on game theory, Campos[9] proposed a fuzzy linear programming model to solve the matrix game. Li [13] solved the fuzzy matrix game through the fuzzy multi-objective approach. Vijay et al. [33] studied a generalized fuzzy relation approach to solve matrix games with fuzzy goals and fuzzy pay-offs. Li [14] implemented a lexicographic method to solve the matrix game with pay-offs represented by triangular fuzzy numbers. Seikh et al. [26] introduced an α -cut based approach to find the solution of matrix games with fuzzy pay-offs. Li [15, 16] also proposed different methodologies to compute the fuzzy value of matrix games with fuzzy pay-offs. Qiu et al. [21] solved a multi-objective fuzzy game using the lower limit of possibility degree. Xu et al. [39] discussed the possibility-necessity strategy of a matrix game with fuzzy pay-offs. Assuming the entries of the pay-off matrix as a trapezoidal fuzzy number, Krishnaveni [12] provided a new solution procedure to obtain the fuzzy optimal solution without converting to the classical version of the game. Jana and Roy [11] solved the matrix game with trapezoidal fuzzy pay-offs. Moradi and Limaei [20] utilized the fuzzy game-theoretical approaches for setting the balance between economic development and the environmental impact. Bigdeli et al. [8] proposed the solution procedure of a multi-objective game with fuzzy pay-offs. Alkahras [1] used the notion of fuzzy game theory to solve a green supply chain.

However, the fuzzy set uses only the membership degree, which measures the degree of belongingness, and the degree of non-belongingness is easily calculated as the complement of the belongingness to 1. But after introducing the intuitionistic fuzzy set (IFS) and intuitionistic fuzzy number (IFN) by Atanassov [2, 3], the mode of description of a fuzzy set experienced a little change with great significance. In IFS, the elements of the set are described along with

its degree of membership and non-membership, where the sum of the membership and non-membership value must be less than or equal to 1, if it is less than 1 then the remaining part is left as the hesitation degree. Li et al. [18] solved matrix games with intuitionistic fuzzy pay-offs through a non-linear programming approach. Li et al. [19] also provided the solution procedure of intuitionistic fuzzy game through a bi-objective programming approach. Seikh et al. [30, 24, 25, 28] provided different solution procedures of matrix games and bi-matrix games with IFN. Li [17] analyzed game theory in management using IFS. Seikh et al. [27] proposed a methodology to solve a game with pay-offs of IFN having the exponential membership function and quadratic non-membership function. Seikh et al. [29] also described a solution procedure of matrix games with intuitionistic fuzzy goals and intuitionistic fuzzy pay-offs using the aspiration level approach. Bhaumik et al. [7] made the solution of an intuitionistic fuzzy game using robust ranking. Basir et al. [4] brought in a methodology for solving zero-sum multi-criteria game with intuitionistic fuzzy goals. In the recent past, Xu et al. [38] utilized the zero-sum game approach with pay-offs of IFN in multi-criteria decision making. Roy and Bhaumik [23] discussed the multiple water attributes of fuzzy decision making.

Nevertheless, it can be seen that the degree of membership and non-membership values are not enough in some problems to assign an element correctly. Considering the decision-makers hesitancy, Torra[32] and Narukawa [31] extended the concept of fuzzy sets to hesitant fuzzy set (HFS), where the belongingness of an element is assigned by a set of possible membership values which must be lying on $[0, 1]$. When we have a margin of error, we can define the membership degree as the membership degree of an IFS; also, when some possibility distribution of possible values is supplied, we can construct a Type-2 fuzzy set. But when the decision-makers provide several possible values, then HFS is required to express such situations. Xia and Xu [37] described a clear concept between the relationship between HFS and Atanassov's IFS. Xu and Xia [40, 41] proposed a variety of distance measures and similarity measures for HFS and also defined the distance and correlation measures for hesitant fuzzy information. Rodriguez et al. [22] implemented hesitant fuzzy linguistic term sets in decision-making problems. Yu et al. [43] studied multi-criteria decision-making problems based on Choquet integral under hesitant fuzzy environment. Wei [34, 35] proposed the hesitant fuzzy prioritized weighted average and hesitant fuzzy prioritized weighted average geometric aggregation operator and successfully applied to multi-criteria decision-making problems.

However, in matrix games due to lack of information in data and lack of attention of a decision-maker, always exist some hesitancy. Therefore neither fuzzy set nor IFS are sufficient to describe pay-off values. This motivates us to imply the concept of HFS in matrix games. The elements of the pay-off matrix are represented by THFS, introduced by Yu[42].

The objective of this paper is to develop a matrix game considering the pay-offs as THFEs. Inspiring by Li [14], we study an effective method to solve the matrix game with THFE pay-offs. The game yields two multi-objective fuzzy linear programming problems. Utilizing the concept of score function, the THFEs are transformed into TFNs, and there exist several methods to solve a game with pay-offs of TFNs in literature. So one has an option to choose any of the proposed methods to solve the game. But here we choose the lexicographic method to make the solution of the game with pay-offs of TFN(proposed by Li [14]), because the techniques discussed by Campos [9] and Bector et al. [6] are mainly defuzzification approaches. They are not sufficient to obtain the membership functions of player's gain-floor and loss-ceiling. Also, these require some additional parameters and adequacies, which are difficult to be chosen for players.

Probably, this is the first attempt of hesitant fuzzy set in game theory. The gain of the maximizing player must be less than or equal to the loss of the minimizing player in the crisp or general fuzzy environment in general. But, this inequality is also preserved in a hesitant fuzzy environment, we prove this, which is the major contribution of our article.

This paper is organized in the following manner. Some basic definitions and the algebraic operations of the THFEs are described in Section 2. A matrix game with pay-offs of THFEs is constructed in Section 3; also, the solution procedure of such type of game is proposed here. The solution algorithm is also described in Section 3. Section 4 presents a numerical example with a brief discussion about the result of the example to illustrate the method, and finally, a short conclusion is drawn in Section 5.

2 Preliminaries

In this section, we recall some basic preliminaries.

Definition 2.1. (Triangular Fuzzy Number)[10] *A Fuzzy number $\tilde{\nu} = (\nu_L, \nu_M, \nu_U)$ defined on the set of real numbers \mathfrak{R} is said to be a Triangular Fuzzy Number (TFN), if the membership function of $\tilde{\nu}$, say $\mu_{\tilde{\nu}}(x)$ is defined as follows:*

$$\mu_{\tilde{\nu}}(x) = \begin{cases} 0, & \text{if } x < \nu_L; \\ \frac{x - \nu_L}{\nu_M - \nu_L}, & \text{if } \nu_L \leq x < \nu_M; \\ 1, & \text{if } x = \nu_M; \\ \frac{\nu_U - x}{\nu_U - \nu_M}, & \text{if } \nu_M < x \leq \nu_U; \\ 0, & \text{if } x > \nu_U \end{cases},$$

where ν_L and ν_U stand for the lower and upper values of $\tilde{\nu}$. Also TFN can be defined in another way. $\tilde{\nu} = (\nu_M - \underline{\gamma}, \nu_M, \nu_M + \overline{\gamma})$, where ν_M is the mode of the triangular fuzzy number $\tilde{\nu}$ and $\underline{\gamma}$ and $\overline{\gamma}$ are the left and right fuzzy extent, respectively.

Let $\tilde{\nu} = (\nu_L, \nu_M, \nu_U)$ and $\tilde{\xi} = (\xi_L, \xi_M, \xi_U)$ be two TFNs, then according to Li[14], $\tilde{\nu} \geq \tilde{\xi}$ iff $\nu_L \geq \xi_L$, $\nu_M \geq \xi_M$ and $\nu_U \geq \xi_U$. Also $\tilde{\nu} \leq \tilde{\xi}$ iff $\nu_L \leq \xi_L$, $\nu_M \leq \xi_M$ and $\nu_U \leq \xi_U$.

Definition 2.2. (Hesitant Fuzzy Set)[32, 31] Let U be the universe of discourse. A HFS on U is defined in the terms of a function that gives several possible values when applied on U . More clearly, Xia and Xu [37] expressed HFS as $E = \{ \langle x, h_E(x) \rangle | x \in U \}$ where each $h_E(x)$ is a set of some values in $[0, 1]$.

Definition 2.3. (Triangular Hesitant Fuzzy Set)[42] Let U be the universe of discourse. Then the THFS is defined in term of functions that give several possible TFNs. Symbolically, it is expressed as, $\tilde{E} = \{ \langle x, \tilde{h}_E(x) \rangle | x \in U \}$, where each $\tilde{h}_E(x)$ is a set of TFNs, which takes value from $[0, 1]$. Each $\tilde{h}_E(x)$ is called the triangular hesitant fuzzy element (THFE). Let \tilde{H} be the set of THFEs. and $\tilde{h}_E \in \tilde{H}$ then $h_E(x_i) = \{ \tilde{\nu} | \tilde{\nu} = (\nu_L, \nu_M, \nu_U) \}$.

Example 2.4. Suppose a new smart phone is launched in a targeted market. To get the feedback, the company chooses five customers arbitrarily and by some selected questionnaire the consumers give their opinions about that smart phone. If the most assurance of the first customer be 0.8, least assurance be 0.6 and the average assurance be 0.7, then it can be represented by a TFN (0.6, 0.7, 0.8). Similarly, if the assurance of the other customers be (0.55, 0.6, 0.65), (0.5, 0.55, 0.6), (0.55, 0.6, 0.65), (0.6, 0.75, 0.9), then the feedback of the smart phone can be summarized by a THFE (depicted in Figure 1) as $\tilde{h}_E(x) = \langle (0.6, 0.7, 0.8), (0.55, 0.6, 0.65), (0.5, 0.55, 0.6), (0.55, 0.6, 0.65), (0.6, 0.75, 0.9) \rangle$.

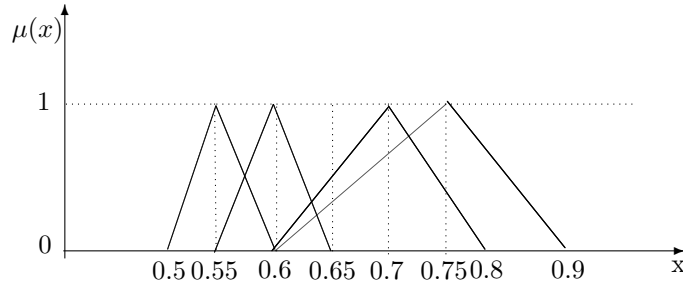


Figure 1: Triangular Hesitant Fuzzy Set

Definition 2.5. [35] Let $\tilde{\nu}_1 = (\nu_{1L}, \nu_{1M}, \nu_{1U})$, $\tilde{\nu}_2 = (\nu_{2L}, \nu_{2M}, \nu_{2U})$ and $\tilde{\nu} = (\nu_L, \nu_M, \nu_U)$ be three TFNs and \tilde{h}_{E1} , \tilde{h}_{E2} and \tilde{h}_E be three THFEs, where $\tilde{\nu}_1 \in \tilde{h}_{E1}$, $\tilde{\nu}_2 \in \tilde{h}_{E2}$ and $\tilde{\nu} \in \tilde{h}_E$, then the addition and multiplication are defined in the following manner:

Addition: $\tilde{h}_{E1} \uplus \tilde{h}_{E2} = \bigcup_{\tilde{\nu}_1 \in \tilde{h}_{E1}, \tilde{\nu}_2 \in \tilde{h}_{E2}} \{ (\nu_{1L} + \nu_{2L} - \nu_{1L}\nu_{2L}, \nu_{1M} + \nu_{2M} - \nu_{1M}\nu_{2M}, \nu_{1U} + \nu_{2U} - \nu_{1U}\nu_{2U}) \}$.

Multiplication: $\tilde{h}_{E1} \otimes \tilde{h}_{E2} = \bigcup_{\tilde{\nu}_1 \in \tilde{h}_{E1}, \tilde{\nu}_2 \in \tilde{h}_{E2}} \{ (\nu_{1L}\nu_{2L}, \nu_{1M}\nu_{2M}, \nu_{1U}\nu_{2U}) \}$.

Scalar Multiplication: For any positive quantity λ , $\lambda \tilde{h}_E = \bigcup_{\tilde{\nu} \in \tilde{h}_E} \{ (1 - (1 - \nu_L)^\lambda, 1 - (1 - \nu_M)^\lambda, 1 - (1 - \nu_U)^\lambda) \}$.

Definition 2.6. (Score Function)[42] For a THFE \tilde{h}_E , the score function is defined as, $S(\tilde{h}_E) = \frac{1}{N_h} \sum_{\tilde{\nu} \in \tilde{h}_E} \tilde{\nu}$, where N_h is the number of TFNs in \tilde{h}_E .

We have to remember that in the case of THFE, the score function, $S(\tilde{h}_E)$ is also a TFN. According to Wei[35], $\tilde{h}_{E1} \geq_H \tilde{h}_{E2}$ if $S(\tilde{h}_{E1}) \geq S(\tilde{h}_{E2})$ for two THFEs \tilde{h}_{E1} and \tilde{h}_{E2} . Here ' \geq_H ' stands for 'large than or equal to' in hesitant fuzzy sense.

Based on the operational principles of THFEs, Yu[42] gave the definitions of the Triangular Hesitant Fuzzy Weighted Average (THFWA) operator.

Definition 2.7. [42] Let \tilde{h}_i ($i=1,2,\dots,n$) be the collection of THFEs and w_i 's be the weight vectors of \tilde{h}_i with $w_i \in [0, 1]$ and $\sum_i w_i = 1$ then THFWA operator is a mapping $\tilde{H}^n \rightarrow \tilde{H}$ defined as,

$$THFWA(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \oplus_i w_i \tilde{h}_i = \bigcup_{\nu_1 \in \tilde{h}_1, \nu_2 \in \tilde{h}_2, \dots, \nu_n \in \tilde{h}_n} \left\langle \left(1 - \prod_{i=1}^n (1 - \nu_{iL})^{w_i}, 1 - \prod_{i=1}^n (1 - \nu_{iM})^{w_i}, 1 - \prod_{i=1}^n (1 - \nu_{iU})^{w_i} \right) \right\rangle.$$

3 The matrix game with THFE pay-offs

3.1 The implication of hesitant fuzzy elements in game theory

In reality, we face various conflicting situations, and game theory plays an active role in getting rid of this environment. But sometimes, due to the inadequate information, the situation becomes very complicated, and Players are bound to use fuzzy pay-offs in games instead of the crisp pay-offs. The fuzzy game has a huge application, not only in the field of mathematics but also in the field of biology, economics, medical science, political science, etc.

After introducing the IFS and IFN by Atanassov [2, 3], it is seen that IFS can manifest a situation more clearly. IFS(or IFN) accommodates the membership and non-membership function both.

But when player's hesitation character is considered, IFS fails to express the situation adequately, and hesitant fuzzy set appears to handle such a problem. In that case, one can use hesitant fuzzy pay-offs in games. For example, suppose a company is going to sell a particular commodity through retail and e-tail channel both. Initially, they are unaware of their profit, i.e., they do not know what will be the more profitable channel. So, they recruit four consultants who are divided into two groups, two of them choose the retail channel as more beneficial for the company, and the other two selects the e-tail channel. Studying the environment of the market, the consultants estimate the demand of the commodity, but due to the different tastes of people, the ability to afford for the product, availability of the product, the demand rate of the product fluctuates, which affects the profit. So, they have a little confusion and hesitancy on it. This situation can be appropriately expressed by a matrix game(considering the retail and e-tail channels as players) with the pay-offs of hesitant fuzzy elements.

3.2 The model formulation

Let us suppose, two Players A and B are involved in a matrix game with hesitant fuzzy pay-offs. Assume that X and Y are the sets of mixed strategies of Player A and Player B respectively. X and Y are stated as,

$$X = \{\mathbf{x} = (x_1, x_2, \dots, x_m)^T \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i = 1, 2, \dots, m\} \text{ and } Y = \{\mathbf{y} = (y_1, y_2, \dots, y_n)^T \mid \sum_{j=1}^n y_j = 1, y_j \geq 0, j = 1, 2, \dots, n\}.$$

Also we assume that, $S = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ and $S' = \{\beta_1, \beta_2, \dots, \beta_n\}$ are the set of pure strategies for Player A and Player B respectively. Actually, the mixed strategies x_i and y_j for Players A and B stand for the possibilities of which they take against their pure strategies $\alpha_i \in S$ ($i = 1, 2, \dots, m$) and $\beta_j \in S'$ ($j = 1, 2, \dots, n$). Without loss of generality, let us suppose that Player A be the maximizing player and Player B be the minimizing player in the game. If Player A takes the pure strategy $\alpha_i \in S$ to maximize his profit and Player B chooses the pure strategy $\beta_j \in S'$ to minimize his loss, then the outcome will be $\langle \alpha_i, \beta_j \rangle = \tilde{g}_{ij}$, the ij^{th} entry of the pay-off matrix \tilde{G} , which is considered as the gain of Player A , where \tilde{g}_{ij} is a THFE. The pay-off matrix can be written as, $\tilde{G} = (\tilde{g}_{ij})_{m \times n}$. Therefore, the game can be expressed by the triplet $\tilde{\Gamma} = (X, Y, \tilde{G})$. If Player A chooses the mixed strategy \mathbf{x} and Player B chooses \mathbf{y} , then the expected pay-off of Player A is calculated following Definition 2.5 as,

$$\tilde{E}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \tilde{G} \mathbf{y} = \sum_{i=1}^m \sum_{j=1}^n x_i \tilde{g}_{ij} y_j = \bigcup_{\nu_{ij} \in \tilde{g}_{ij}} \left\{ \bigoplus_{i=1}^m \bigoplus_{j=1}^n \left(1 - (1 - \nu_{ij}^L)^{x_i y_j}, 1 - (1 - \nu_{ij}^M)^{x_i y_j}, 1 - (1 - \nu_{ij}^U)^{x_i y_j} \right) \right\}.$$

From Definition 2.5, we see that the addition of two THFEs and the scalar multiplication of a THFE give a new THFE, hence the linear combination of two or more THFEs also form a THFE. As $\tilde{E}(\mathbf{x}, \mathbf{y})$ is the linear combinations of THFEs, thus these are two THFEs.

Definition 3.1. (Gain-floor) [16] Let us assume that Player A chooses the mixed strategy $\mathbf{x} \in X$, then the expected gain-floor of Player A is obtained as,

$$\tilde{v}_H(\mathbf{x}) = \min_{\mathbf{y} \in Y} \{\mathbf{x}^T \tilde{G} \mathbf{y}\}. \quad (1)$$

If this minimum is occurred for some pure strategy $\beta_j \in S'$ of Player B , then (1) can be rewritten as,

$$\tilde{v}_H(\mathbf{x}) = \min_j \left(\sum_{i=1}^m x_i \tilde{G}_j \right)$$

where $\tilde{G}_j = (\tilde{a}_{1j}, \tilde{a}_{2j}, \dots, \tilde{a}_{mj})^T$ be the j^{th} column of the pay-off matrix \tilde{G} . Now, Player A must take a $\mathbf{x}^* \in X$ to maximize the gain-floor and we obtain,

$$\tilde{v}_H^* = \tilde{v}_H(\mathbf{x}^*) = \max_{\mathbf{x} \in X} \min_j \left(\sum_{i=1}^m \tilde{g}_{ij} x_i \right) \quad (2)$$

Definition 3.2. (Loss-Ceiling) [16] *If Player B chooses the mixed strategy $\mathbf{y} \in Y$, then the expected loss-ceiling of Player B is obtained as,*

$$\tilde{w}_H(\mathbf{y}) = \max_{\mathbf{x} \in X} \{ \mathbf{x}^T \tilde{G} \mathbf{y} \}. \quad (3)$$

If this maximum value is occurred for some pure strategy $\alpha_i \in S$ of Player A, then (3) can be expressed as,

$$\tilde{w}_H(\mathbf{y}) = \max_i \left(\sum_{j=1}^n \tilde{G}_i y_j \right)$$

where $\tilde{G}_i = (\tilde{a}_{i1}, \tilde{a}_{i2}, \dots, \tilde{a}_{in})$ be the i^{th} row of \tilde{G} . Now, Player B takes a $\mathbf{y}^* \in Y$ to minimize the loss-ceiling and we obtain,

$$\tilde{w}_H^* = \tilde{w}_H(\mathbf{y}^*) = \min_{\mathbf{y} \in Y} \max_i \left(\sum_{j=1}^n \tilde{g}_{ij} y_j \right). \quad (4)$$

Computing the optimal strategies \mathbf{x}^* and \mathbf{y}^* is equivalent to calculating the following two hesitant fuzzy programming problems.

$$\begin{aligned} & \max \{ \tilde{v}_H \} \\ \text{s.t.} \quad & \sum_{i=1}^m \tilde{g}_{ij} x_i \geq_H \tilde{v}_H \\ & \sum_{i=1}^m x_i = 1 \\ & x_i \geq 0, i = 1, 2, \dots, m \\ & \tilde{v}_H \text{ is a THFE} \end{aligned} \quad (5)$$

and,

$$\begin{aligned} & \min \{ \tilde{w}_H \} \\ \text{s.t.} \quad & \sum_{j=1}^n \tilde{g}_{ij} y_j \leq_H \tilde{w}_H \\ & \sum_{j=1}^n y_j = 1 \\ & y_j \geq 0, j = 1, 2, \dots, n \\ & \tilde{w}_H \text{ is a THFE} \end{aligned} \quad (6)$$

Theorem 3.3. *Suppose that \mathbf{x}^* and \mathbf{y}^* are optimal solutions of problems (5) and (6) respectively. Also \tilde{v}_H^* and \tilde{w}_H^* are the corresponding expected pay-offs of Player A and the Player B, then $\tilde{v}_H^* \leq_H \tilde{w}_H^*$.*

Proof. As \mathbf{x}^* and \mathbf{y}^* , are the optimal solutions of the equation (5) and (6) respectively, then it is clear that they are feasible solutions also. According to Definitions 2.7, 3.1 and 3.2, \tilde{v}_H^* and \tilde{w}_H^* are defined as follows:

$$\begin{aligned} \tilde{v}_H^* &= (v_{HL}, v_{HM}, v_{HU}) = \max_{\mathbf{x} \in X} \min_j \left(\sum_{i=1}^m \tilde{g}_{ij} x_i \right) = \max_{\mathbf{x} \in X} \min_j \left\langle \left(1 - \prod_{i=1}^m (1 - g_{ijL})^{x_i}, 1 - \prod_{i=1}^m (1 - g_{ijM})^{x_i}, 1 - \prod_{i=1}^m (1 - g_{ijU})^{x_i} \right) \right\rangle = \\ & \left\langle \left(1 - \min_{\mathbf{x} \in X} \max_j \prod_{i=1}^m (1 - g_{ijL})^{x_i}, 1 - \min_{\mathbf{x} \in X} \max_j \prod_{i=1}^m (1 - g_{ijM})^{x_i}, 1 - \min_{\mathbf{x} \in X} \max_j \prod_{i=1}^m (1 - g_{ijU})^{x_i} \right) \right\rangle \text{ and} \\ \tilde{w}_H^* &= (w_{HL}, w_{HM}, w_{HU}) = \min_{\mathbf{y} \in Y} \max_i \left(\sum_{j=1}^n \tilde{g}_{ij} y_j \right) = \left\langle \left(1 - \max_{\mathbf{y} \in Y} \min_i \prod_{j=1}^n (1 - g_{ijL})^{y_j}, 1 - \max_{\mathbf{y} \in Y} \min_i \prod_{j=1}^n (1 - g_{ijM})^{y_j}, 1 - \max_{\mathbf{y} \in Y} \min_i \prod_{j=1}^n (1 - g_{ijU})^{y_j} \right) \right\rangle. \end{aligned}$$

Let us suppose that, there are k ($k = 1, 2, \dots, N$) numbers of TFNs in \tilde{v}_H^* and \tilde{w}_H^* , then the score functions of \tilde{v}_H^* and \tilde{w}_H^* are calculated as(as defined in 2.6),

$$S(\tilde{v}_H^*) = \left\langle \frac{\sum_{k=1}^N (1 - \min_{\mathbf{x} \in X} \max_j \prod_{i=1}^m (1 - g_{ijL})^{x_i})}{N}, \frac{\sum_{k=1}^N (1 - \min_{\mathbf{x} \in X} \max_j \prod_{i=1}^m (1 - g_{ijM})^{x_i})}{N}, \frac{\sum_{k=1}^N (1 - \min_{\mathbf{x} \in X} \max_j \prod_{i=1}^m (1 - g_{ijU})^{x_i})}{N} \right\rangle$$

and

$$S(\tilde{w}_H^*) = \left\langle \frac{\sum_{k=1}^N (1 - \max_{\mathbf{y} \in Y} \min_i \prod_{j=1}^n (1 - g_{ijL})^{y_j})}{N}, \frac{\sum_{k=1}^N (1 - \max_{\mathbf{y} \in Y} \min_i \prod_{j=1}^n (1 - g_{ijM})^{y_j})}{N}, \frac{\sum_{k=1}^N (1 - \max_{\mathbf{y} \in Y} \min_i \prod_{j=1}^n (1 - g_{ijU})^{y_j})}{N} \right\rangle.$$

Now, obviously, $\max_j \left(\sum_{i=1}^m x_i \ln(1 - g_{ijL}) \right) \geq \sum_{i=1}^m x_i \ln(1 - g_{ijL}) \geq \sum_{j=1}^n \left(\sum_{i=1}^m x_i \ln(1 - g_{ijL}) \right) y_j$ as $y_j \geq 0$ for $j = 1, 2, \dots, n$ and $\sum_{j=1}^n y_j = 1$.

Which shows that,

$$\max_j \left(\sum_{i=1}^m x_i \ln(1 - g_{ijL}) \right) \geq \sum_{i=1}^m \sum_{j=1}^n x_i y_j \ln(1 - g_{ijL}) \quad (7)$$

On the other hand, we can prove that,

$$\sum_{i=1}^m \sum_{j=1}^n x_i y_j \ln(1 - g_{ijL}) \geq \min_i \left(\sum_{j=1}^n y_j \ln(1 - g_{ijL}) \right) \quad (8)$$

Combining Equations (7) and (8), we have,

$$\max_j \left(\sum_{i=1}^m x_i \ln(1 - g_{ijL}) \right) \geq \sum_{i=1}^m \sum_{j=1}^n x_i y_j \ln(1 - g_{ijL}) \geq \min_i \left(\sum_{j=1}^n y_j \ln(1 - g_{ijL}) \right)$$

Hence, $\min_{\mathbf{x} \in X} \max_j \left(\sum_{i=1}^m x_i \ln(1 - g_{ijL}) \right) \geq \max_{\mathbf{y} \in Y} \min_i \left(\sum_{j=1}^n y_j \ln(1 - g_{ijL}) \right)$.

From which we obtain, $\min_{\mathbf{x} \in X} \max_j \left(\prod_{i=1}^m (1 - g_{ijL})^{x_i} \right) \geq \max_{\mathbf{y} \in Y} \min_i \left(\prod_{j=1}^n (1 - g_{ijL})^{y_j} \right)$.

Which implies, $1 - \min_{\mathbf{x} \in X} \max_j \left(\prod_{i=1}^m (1 - g_{ijL})^{x_i} \right) \leq 1 - \max_{\mathbf{y} \in Y} \min_i \left(\prod_{j=1}^n (1 - g_{ijL})^{y_j} \right)$.

Which shows that, $\frac{\sum_{k=1}^N \left(1 - \min_{\mathbf{x} \in X} \max_j \left(\prod_{i=1}^m (1 - g_{ijL})^{x_i} \right) \right)}{N} \leq \frac{\sum_{k=1}^N \left(1 - \max_{\mathbf{y} \in Y} \min_i \left(\prod_{j=1}^n (1 - g_{ijL})^{y_j} \right) \right)}{N}$.

Similarly, we can prove that, $\frac{\sum_{k=1}^N \left(1 - \min_{\mathbf{x} \in X} \max_j \left(\prod_{i=1}^m (1 - g_{ijM})^{x_i} \right) \right)}{N} \leq \frac{\sum_{k=1}^N \left(1 - \max_{\mathbf{y} \in Y} \min_i \left(\prod_{j=1}^n (1 - g_{ijM})^{y_j} \right) \right)}{N}$ and

$$\frac{\sum_{k=1}^N \left(1 - \min_{\mathbf{x} \in X} \max_j \left(\prod_{i=1}^m (1 - g_{ijU})^{x_i} \right) \right)}{N} \leq \frac{\sum_{k=1}^N \left(1 - \max_{\mathbf{y} \in Y} \min_i \left(\prod_{j=1}^n (1 - g_{ijU})^{y_j} \right) \right)}{N}.$$

Thus utilizing the ranking property of two TFNs(as defined in Definition 2.1) and score function of a THFE(as defined in Definition 2.6), we can conclude that, $S(\tilde{v}_H^*) \leq S(\tilde{w}_H^*)$ which is happened only when $\tilde{v}_H^* \leq_H \tilde{w}_H^*$. \square

This theorem states that the gain-floor of Player A cannot exceed the loss ceiling of Player B. According to Definition 2.7, problems (5) and (6) are converted to two fuzzy non-linear programming problems:

$$\begin{aligned} & \max\{\tilde{v}_H\} \\ \text{s.t.} & \quad \oplus_{i=1}^m x_i \tilde{g}_{ij} \geq_H \tilde{v}_H, \\ & \quad \sum_{i=1}^m x_i = 1 \\ & \quad x_i \geq 0, i = 1, 2, \dots, m \\ & \quad \tilde{v}_H \text{ is a THFE} \end{aligned} \quad (9)$$

$$\begin{aligned}
& \min\{\tilde{w}_H\} \\
\text{s.t.} \quad & \oplus_{j=1}^n y_j \tilde{g}_{ij} \leq_H \tilde{w}_H, \\
& \sum_{j=1}^n y_j = 1 \\
& y_j \geq 0, j = 1, 2, \dots, n \\
& \tilde{w}_H \text{ is a THFE}
\end{aligned} \tag{10}$$

Theorem 3.4. Let $(\mathbf{x}^*, \mathbf{y}^*)$ be the optimal solution of the pay-off matrix \tilde{G} with THFEs, then it is the solution of the pay-off matrix after using Score function to convert it into a matrix with pay-offs of TFNs and vice-versa.

Proof. Let $(\mathbf{x}^*, \mathbf{y}^*)$ be the optimal solution of the pay-off matrix \tilde{G} . Then,

$$\min_{\mathbf{y}} \{\max_{\mathbf{x}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\} = \tilde{E}(\mathbf{x}^*, \mathbf{y}^*) = \sum_i \sum_j \tilde{g}_{ij} x_i y_j = \sum_j \left(\sum_i \tilde{g}_{ij} x_i \right) y_j = \max_{\mathbf{x}} \{\min_{\mathbf{y}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}$$

Applying score function to the above relation, we have the following two equations

$$S(\min_{\mathbf{y}} \{\max_{\mathbf{x}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}) = S(\tilde{E}(\mathbf{x}^*, \mathbf{y}^*)) = S\left(\sum_i \sum_j \tilde{g}_{ij} x_i y_j\right) = S\left(\sum_j \left(\sum_i \tilde{g}_{ij} x_i\right) y_j\right) = \sum_i \sum_j S(\tilde{g}_{ij}) x_i y_j$$

$$\text{and } S(\max_{\mathbf{x}} \{\min_{\mathbf{y}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}) = S(\tilde{E}(\mathbf{x}^*, \mathbf{y}^*)) = S\left(\sum_i \sum_j \tilde{g}_{ij} x_i y_j\right) = S\left(\sum_j \left(\sum_i \tilde{g}_{ij} x_i\right) y_j\right) = \sum_i \sum_j S(\tilde{g}_{ij}) x_i y_j.$$

From the above two equations, we obtain

$$S(\min_{\mathbf{y}} \{\max_{\mathbf{x}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}) = S(\max_{\mathbf{x}} \{\min_{\mathbf{y}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}) = \sum_j \left(\sum_i S(\tilde{g}_{ij}) x_i \right) y_j = \tilde{E}(S(\tilde{g}_{ij})).$$

Hence, $(\mathbf{x}^*, \mathbf{y}^*)$ is also a solution of the pay-off matrix after applying the score function.

To prove the converse part, let us assume that $(\mathbf{x}^*, \mathbf{y}^*)$ is also a solution of the pay-off matrix after applying the score function, i.e.,

$$S(\min_{\mathbf{y}} \{\max_{\mathbf{x}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}) = S(\max_{\mathbf{x}} \{\min_{\mathbf{y}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}) = \tilde{E}(S(\tilde{g}_{ij})).$$

Now, we can write

$$S(\min_{\mathbf{y}} \{\max_{\mathbf{x}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}) = \tilde{E}(S(\tilde{g}_{ij})) = \sum_i \sum_j S(\tilde{g}_{ij}) x_i y_j = S\left(\sum_j \left(\sum_i \tilde{g}_{ij} x_i\right) y_j\right) = S(\tilde{E}(\mathbf{x}^*, \mathbf{y}^*))$$

and

$$S(\max_{\mathbf{x}} \{\min_{\mathbf{y}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}) = \tilde{E}(S(\tilde{g}_{ij})) = \sum_i \sum_j S(\tilde{g}_{ij}) x_i y_j = S\left(\sum_j \left(\sum_i \tilde{g}_{ij} x_i\right) y_j\right) = S\left(\sum_i \sum_j \tilde{g}_{ij} x_i y_j\right) = S(\tilde{E}(\mathbf{x}^*, \mathbf{y}^*))$$

Therefore, we can write $S(\min_{\mathbf{y}} \{\max_{\mathbf{x}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}) = S(\tilde{E}(\mathbf{x}^*, \mathbf{y}^*)) = S(\max_{\mathbf{x}} \{\min_{\mathbf{y}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\})$. Now, according to Definition 2.6, we have $\min_{\mathbf{y}} \{\max_{\mathbf{x}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\} = \tilde{E}(\mathbf{x}^*, \mathbf{y}^*) = \max_{\mathbf{x}} \{\min_{\mathbf{y}} \{\tilde{E}(\mathbf{x}, \mathbf{y})\}\}$. Hence $(\mathbf{x}^*, \mathbf{y}^*)$ is the optimal solution of the pay-off matrix \tilde{G} . \square

Applying score function, the problems (9) and (10) become,

$$\begin{aligned}
& \max\{S(\tilde{v}_H)\} \\
\text{s.t.} \quad & S(\oplus_i x_i \tilde{g}_{ij}) \geq S(\tilde{v}_H) \\
& \sum_{i=1}^m x_i = 1 \\
& x_i \geq 0, i = 1, 2, \dots, m \\
& \tilde{v}_H \text{ is a THFE}
\end{aligned} \tag{11}$$

$$\begin{aligned}
& \min\{S(\tilde{w}_H)\} \\
\text{s.t.} \quad & S(\oplus_j y_j \tilde{g}_{ij}) \leq S(\tilde{w}_H) \\
& \sum_{j=1}^n y_j = 1 \\
& y_j \geq 0, j = 1, 2, \dots, n \\
& \tilde{w}_H \text{ is a THFE}
\end{aligned} \tag{12}$$

Let us denote the score functions of \tilde{v}_H as $S(\tilde{v}_H) = (S_{v_{HM}} - \underline{\gamma}, S_{v_{HM}}, S_{v_{HM}} + \bar{\gamma})$ and the same of \tilde{w}_H as $S(\tilde{w}_H) = (S_{w_{HM}} - \underline{\gamma}, S_{w_{HM}}, S_{w_{HM}} + \bar{\gamma})$. Let us take,

$$S(\oplus_i x_i \tilde{g}_{ij}) = \left(1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijL})^{x_i}}{N}, 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijM})^{x_i}}{N}, 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijM})^{x_i}}{N}\right)$$

and

$$S(\oplus_j y_j \tilde{g}_{ij}) = \left(1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijL})^{y_j}}{N}, 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijM})^{y_j}}{N}, 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijM})^{y_j}}{N}\right).$$

Therefore, the LP problems (11) and (12) can be rewritten as,

$$\begin{aligned} & \max\{f_L = S_{v_{HM}} - \underline{\gamma}, f_M = S_{v_{HM}}, f_U = S_{v_{HM}} + \bar{\gamma}\} \\ \text{s.t.} \quad & \left(1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijL})^{x_i}}{N}, 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijM})^{x_i}}{N}, 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijM})^{x_i}}{N}\right) \geq (S_{v_{HM}} - \underline{\gamma}, S_{v_{HM}}, S_{v_{HM}} + \bar{\gamma}), \\ & \sum_{i=1}^m x_i = 1 \\ & x_i \geq 0, i = 1, 2, \dots, m \\ & \underline{\gamma} \geq 0, \bar{\gamma} \geq 0 \end{aligned} \tag{13}$$

and

$$\begin{aligned} & \min\{g_L = S_{w_{HM}} - \underline{\delta}, g_M = S_{w_{HM}}, g_U = S_{w_{HM}} + \bar{\delta}\} \\ \text{s.t.} \quad & \left(1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijL})^{y_j}}{N}, 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijM})^{y_j}}{N}, 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijM})^{y_j}}{N}\right) \leq (S_{w_{HM}} - \underline{\delta}, S_{w_{HM}}, S_{w_{HM}} + \bar{\delta}) \\ & \sum_{j=1}^n y_j = 1 \\ & y_j \geq 0, j = 1, 2, \dots, n \\ & \underline{\delta} \geq 0, \bar{\delta} \geq 0. \end{aligned} \tag{14}$$

Now, according to the ranking of TFNs, problems (13) and (14) transform into problems (15) and (16) respectively,

$$\begin{aligned} & \max\{f_L = S_{v_{HM}} - \underline{\gamma}, f_M = S_{v_{HM}}, f_U = S_{v_{HM}} + \bar{\gamma}\} \\ \text{s.t.} \quad & 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijL})^{x_i}}{N} \geq S_{v_{HM}} - \underline{\gamma} \\ & 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijM})^{x_i}}{N} \geq S_{v_{HM}} \\ & 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijU})^{x_i}}{N} \geq S_{v_{HM}} + \bar{\gamma} \\ & \sum_{i=1}^m x_i = 1 \\ & x_i \geq 0, i = 1, 2, \dots, m \\ & \underline{\gamma} \geq 0, \bar{\gamma} \geq 0 \end{aligned} \tag{15}$$

$$\begin{aligned} & \max\{g_L = S_{w_{HM}} - \underline{\delta}, g_M = S_{w_{HM}}, g_U = S_{w_{HM}} + \bar{\delta}\} \\ \text{s.t.} \quad & 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijL})^{y_j}}{N} \geq S_{w_{HM}} - \underline{\delta} \\ & 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijM})^{y_j}}{N} \geq S_{w_{HM}} \\ & 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijU})^{y_j}}{N} \geq S_{w_{HM}} + \bar{\delta} \end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^n y_j = 1 \\
& y_j \geq 0, j = 1, 2, \dots, n \\
& \underline{\delta} \geq 0, \bar{\delta} \geq 0.
\end{aligned} \tag{16}$$

The above two problems (15) and (16) are multi-objective programming problems. There are several methods to solve such problems, but here we use the Lexicographic method[14] to compute the optimal strategies. Our aim is to compute the vector $\mathbf{x} = (x_1, x_2, \dots, x_m)^T$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$ by using Lexicographic method.

In LP problem (15) the importance of f_L, f_M , and f_U should be different. Note that f_M should be superior to that of both objective functions f_L and f_U . Also, f_L and f_U have the same priority. First, we make a solution using the single objective function f_M and then to take a better approximation, we have to solve the non-linear programming problem with two objective functions f_L and f_U . Therefore, at first, the following problem is constructed:

$$\begin{aligned}
& \max\{f_M = S_{v_{HM}}\} \\
\text{s.t.} \quad & 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijL})^{x_i}}{N} \geq S_{v_{HM}} - \underline{\gamma} \\
& 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijM})^{x_i}}{N} \geq S_{v_{HM}} \\
& 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijU})^{x_i}}{N} \geq S_{v_{HM}} + \bar{\gamma} \\
& \sum_{i=1}^m x_i = 1 \\
& x_i \geq 0, i = 1, 2, \dots, m \\
& \underline{\gamma} \geq 0, \bar{\gamma} \geq 0
\end{aligned} \tag{17}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_m)^T, S_{v_{HM}}, \underline{\gamma}, \bar{\gamma}$ are decision variables. Solving the LP problem (17), we obtain $\hat{x}, \hat{S}_{v_{HM}}, \hat{\underline{\gamma}}, \hat{\bar{\gamma}}$ as optimal solution. Hence, to obtain better approximation, the second non-linear bi-objective programming problem is to be solved.

$$\begin{aligned}
& \max\{f_L = \hat{S}_{v_{HM}} - \underline{\gamma}, f_U = \hat{S}_{v_{HM}} + \bar{\gamma}\} \\
\text{s.t.} \quad & 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijL})^{x_i}}{N} \geq \hat{S}_{v_{HM}} - \underline{\gamma} \\
& 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijM})^{x_i}}{N} \geq \hat{S}_{v_{HM}} \\
& 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijU})^{x_i}}{N} \geq \hat{S}_{v_{HM}} + \bar{\gamma} \\
& \sum_{i=1}^m x_i = 1 \\
& \hat{S}_{v_{HM}} - \underline{\gamma} \geq \hat{S}_{v_{HM}} - \hat{\underline{\gamma}} \\
& \hat{S}_{v_{HM}} + \underline{\gamma} \geq \hat{S}_{v_{HM}} + \hat{\bar{\gamma}} \\
& x_i \geq 0, i = 1, 2, \dots, m \\
& \underline{\gamma} \geq 0, \bar{\gamma} \geq 0
\end{aligned} \tag{18}$$

To improve the values $\underline{\gamma}$ and $\bar{\gamma}$ we add two extra constraints $\hat{S}_{v_{HM}} - \underline{\gamma} \geq \hat{S}_{v_{HM}} - \hat{\underline{\gamma}}$ and $\hat{S}_{v_{HM}} + \underline{\gamma} \geq \hat{S}_{v_{HM}} + \hat{\bar{\gamma}}$ in the LP problem (18). This can be rewritten in the following form,

$$\begin{aligned}
& \max\{\bar{f}_L = -\underline{\gamma}, \bar{f}_U = \bar{\gamma}\} \\
\text{s.t.} \quad & \underline{\gamma} + 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijL})^{x_i}}{N} \geq \hat{S}_{v_{HM}} \\
& 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijM})^{x_i}}{N} \geq \hat{S}_{v_{HM}}
\end{aligned}$$

$$\begin{aligned}
& -\bar{\gamma} + 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijU})^{x_i}}{N} \geq \hat{S}_{v_{HM}} \\
& \sum_{i=1}^m x_i = 1 \\
& \underline{\gamma} \leq \hat{\gamma} \\
& \bar{\gamma} \geq \hat{\gamma} \\
& x_i \geq 0, i = 1, 2, \dots, m \\
& \underline{\gamma} \geq 0, \bar{\gamma} \geq 0
\end{aligned} \tag{19}$$

Since the objective functions $\bar{f}_L = -\underline{\gamma}$ and $\underline{f}_U = \bar{\gamma}$ have the same importance, we take average of these two and therefore the problem (19) is converted into,

$$\begin{aligned}
& \max \left\{ \frac{-\underline{\gamma} + \bar{\gamma}}{2} \right\} \\
\text{s.t.} \quad & \underline{\gamma} + 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijL})^{x_i}}{N} \geq \hat{S}_{v_{HM}} \\
& 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijM})^{x_i}}{N} \geq \hat{S}_{v_{HM}} \\
& -\bar{\gamma} + 1 - \sum_{k=1}^N \frac{\prod_i (1 - a_{ijU})^{x_i}}{N} \geq \hat{S}_{v_{HM}} \\
& \sum_{i=1}^m x_i = 1 \\
& \underline{\gamma} \leq \hat{\gamma} \\
& \bar{\gamma} \geq \hat{\gamma} \\
& x_i \geq 0, i = 1, 2, \dots, m \\
& \underline{\gamma} \geq 0, \bar{\gamma} \geq 0
\end{aligned} \tag{20}$$

Now, the problem (20) can be solved by using LINGO software and we obtain the optimal mixed strategy x^* for Player A.

A similar procedure is applied to the problem (16). Here also g_M has more importance than g_L and g_U . Also g_L and g_U should have the same importance. To calculate the optimal strategy y^* for Player B, two non-linear programming problems are constructed as follows:

$$\begin{aligned}
& \min \{g_M = S_{w_{HM}}\} \\
\text{s.t.} \quad & 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijL})^{y_j}}{N} \geq S_{w_{HM}} - \underline{\delta} \\
& 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijM})^{y_j}}{N} \geq S_{w_{HM}} \\
& 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijU})^{y_j}}{N} \geq S_{w_{HM}} + \bar{\delta} \\
& \sum_{j=1}^n y_j = 1 \\
& y_j \geq 0, j = 1, 2, \dots, n \\
& \underline{\delta} \geq 0, \bar{\delta} \geq 0
\end{aligned} \tag{21}$$

By solving this problem we obtain $\hat{y} = (y_1, y_2, \dots, y_n)^T, \hat{S}_{w_{HM}}, \hat{\underline{\delta}}, \hat{\bar{\delta}}$ which are also decision variables and optimal solutions of

problem (16). To get better approximation the following problem is to be solved which has two objective functions:

$$\begin{aligned}
& \min\{g_L = \hat{S}_{w_{HM}} - \underline{\delta}, g_U = \hat{S}_{w_{HM}} + \bar{\delta}\} \\
\text{s.t.} \quad & \underline{\delta} + 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijL})^{y_j}}{N} \leq \hat{S}_{w_{HM}} \\
& 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijM})^{y_j}}{N} \leq \hat{S}_{w_{HM}} \\
& -\bar{\delta} + 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijU})^{y_j}}{N} \leq \hat{S}_{w_{HM}} \\
& \sum_{j=1}^n y_j = 1 \\
& \hat{S}_{w_{HM}} - \underline{\delta} \leq \hat{S}_{w_{HM}} - \hat{\delta} \\
& \hat{S}_{w_{HM}} + \bar{\delta} \leq \hat{S}_{w_{HM}} + \hat{\delta} \\
& y_j \geq 0, j = 1, 2, \dots, n \\
& \underline{\delta} \geq 0, \bar{\delta} \geq 0
\end{aligned} \tag{22}$$

To obtain the better values of $\underline{\delta}$ and $\bar{\delta}$ we add two extra constraints $\hat{S}_{w_{HM}} - \underline{\delta} \leq \hat{S}_{w_{HM}} - \hat{\delta}$ and $\hat{S}_{w_{HM}} + \bar{\delta} \leq \hat{S}_{w_{HM}} + \hat{\delta}$ in the LP problem (22). This can be rewritten in the following form,

$$\begin{aligned}
& \min\{\bar{g}_L = -\underline{\delta}, \bar{g}_U = \bar{\delta}\} \\
\text{s.t.} \quad & \underline{\delta} + 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijL})^{y_j}}{N} \leq \hat{S}_{w_{HM}} \\
& 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijM})^{y_j}}{N} \leq \hat{S}_{w_{HM}} \\
& -\bar{\delta} + 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijU})^{y_j}}{N} \leq \hat{S}_{w_{HM}} \\
& \sum_{j=1}^n y_j = 1 \\
& \underline{\delta} \geq \hat{\delta} \\
& \bar{\delta} \leq \hat{\delta} \\
& y_j \geq 0, j = 1, 2, \dots, n \\
& \underline{\delta} \geq 0, \bar{\delta} \geq 0
\end{aligned} \tag{23}$$

As \bar{g}_L and \bar{g}_U have the same importance so by using the average operator, (23) can be rewritten as,

$$\begin{aligned}
& \min\left\{\frac{-\underline{\delta} + \bar{\delta}}{2}\right\} \\
\text{s.t.} \quad & \underline{\delta} + 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijL})^{y_j}}{N} \leq \hat{S}_{w_{HM}} \\
& 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijM})^{y_j}}{N} \leq \hat{S}_{w_{HM}} \\
& -\bar{\delta} + 1 - \sum_{k=1}^N \frac{\prod_j (1 - a_{ijU})^{y_j}}{N} \leq \hat{S}_{w_{HM}} \\
& \sum_{j=1}^n y_j = 1 \\
& \underline{\delta} \leq \hat{\delta} \\
& \bar{\delta} \leq \hat{\delta} \\
& y_j \geq 0, j = 1, 2, \dots, n \\
& \underline{\delta} \geq 0, \bar{\delta} \geq 0
\end{aligned} \tag{24}$$

Linguistic terms	THFE
Low	$((0.1,0.15,0.2),(0.1,0.2,0.3))$
Medium	$((0.3,0.4,0.5),(0.4,0.45,0.5))$
High	$((0.6,0.7,0.8))$
Very High	$((0.8,0.85,0.9))$

Table 1: Assigned THFS corresponding the Linguistic variables

Solving the above problem (24), we obtain the optimal strategy \mathbf{y}^* , for Player B .

The expected pay-off for Player A is now calculated by using the above mentioned definition

$$\tilde{E}(\tilde{A}) = \mathbf{x}^T \tilde{A} \mathbf{y} = \sum_i \sum_j x_i^* \tilde{g}_{ij} y_j^*$$

3.3 Algorithm

The algorithm of the solution procedure of a game with pay-offs of THFEs is described here.

Step 1: Considering a matrix game $\tilde{\Gamma}$, with pay-off of THFE.

Step 2: To solve the game we have to make two linear programming models depicted in problems (5) and (6).

Step 3: According to Wei [35], ranking of two THFE \tilde{h}_{E1} and \tilde{h}_{E2} are $\tilde{h}_{E1} \geq_H \tilde{h}_{E2}$ if, $S(\tilde{h}_{E1})$ and $S(\tilde{h}_{E2})$. Where, $S(\tilde{h}_{E1})$ and $S(\tilde{h}_{E2})$ are two TFN. Applying the score function to the both sides of the constraint and the restriction function of the problems (5) and (6), we obtain two non-linear multi-objective programming models, which are depicted in problems (13) and (14).

Step 4: Using the Lexicographic method, the NLP model (13) is converted into two problems (15) and (19). Similarly, the problem (14) is transformed into the problems (16) and (23).

Step 5: Using LINGO 13.0 software the problems (15) and (19) are solved and we obtain \mathbf{x}^* , the optimal strategy for Player A and by solving the problems (16) and (23) the optimal strategy \mathbf{y}^* for Player B is obtained.

Step 6: Using the optimal strategies \mathbf{x}^* and \mathbf{y}^* in the formula, $\mathbf{E}(\tilde{A}) = \sum_i \sum_j x_i^* \tilde{g}_{ij} y_j^*$, we can compute the expected pay-off of Player A .

4 The numerical example

This subsection provides a numerical example to illustrate the solution procedure of a matrix game with pay-offs of THFEs.

4.1 The market share problem

Suppose two companies C_1 and C_2 produce cell phone, in a targeted market. It is assumed that the market share of one company is increased while the other is decreased. To achieve their goal, they take some strategies, viz, Up-to-date technology, canvassing the product in digital media and reducing the cost of the product. Let company C_1 choose the first and the third strategies and label them as α_1 and α_2 . Also, company C_2 chooses the first and the second strategies and label them as β_1 and β_2 . Surveying the market the consultants of both of the companies collected some views of the retailers and the customers. According to them, the demand for the cell phone of the respected company must vary on the different tastes of the customers, availability of the product, price, capability of buying the product. So the consultants cannot estimate the number of products the company sale and a hesitation arise in the decision. Thus the pay-offs are assumed to be in the form of THFE here. If the company C_1 takes the strategy α_i and the company C_2 takes β_j (for $i=1,2$; $j=1,2$) then the outcome, i.e. the sales amount will be $\langle \alpha_i, \beta_j \rangle = \tilde{g}_{ij}$ (a THFE), the ij^{th} entry of the pay-off matrix. Suppose players assign the sales amount by some linguistic terms, namely, ‘Low’, ‘Medium’, ‘High’ and ‘Very High’ according to the score functions of the sales amount \tilde{g}_{ij} . When the score function of \tilde{g}_{ij} be $0.1 - 0.3$, then it is termed as ‘Low’. Similarly, ‘Medium’ assigns $0.4 - 0.5$, ‘High’ assigns $0.6 - 0.7$ and ‘Very High’ assigns $0.8 - 0.9$ i.e., if the sales amount be 10% to 30%, then it is termed as ‘Low’, when it stands between 40% and 50%, it is assigned as ‘medium’. 60% to 70% sell of the company implies the ‘High’ and 80% to 90% sell of the company implies the ‘Very high’ sell amount of the company. Regarding the decisions of Players, the outcomes are expressed by linguistic terms as follows:

$$\tilde{G} = \begin{pmatrix} \text{Medium} & \text{High} \\ \text{Very High} & \text{Low} \end{pmatrix}$$

The corresponding THFSs of these Linguistic variables are given in Table 1

Following Table 1, the pay-off matrix is transformed into the pay-off matrix with THFEs as,

$$\begin{pmatrix} \langle (0.3, 0.4, 0.5), (0.4, 0.45, 0.5) \rangle & \langle (0.6, 0.7, 0.8) \rangle \\ \langle (0.8, 0.85, 0.9) \rangle & \langle (0.1, 0.15, 0.2), (0.1, 0.2, 0.3) \rangle \end{pmatrix}$$

4.2 The solution procedure

Our aim is to calculate the optimal strategies $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{y} = (y_1, y_2)^T$.

According to problem (17), the following problem is constructed:

$$\begin{aligned}
 & \max\{f_M = S_{v_{HM}}\} \\
 \text{s.t.} \quad & 2 - (0.7^{x_1}0.2^{x_2} + 0.6^{x_1}0.2^{x_2}) \geq 2(S_{v_{HM}} - \underline{\gamma}) \\
 & 2 - (0.4^{x_1}0.9^{x_2} + 0.4^{x_1}0.9^{x_2}) \geq 2(S_{v_{HM}} - \underline{\gamma}) \\
 & 2 - (0.6^{x_1}0.15^{x_2} + 0.55^{x_1}0.15^{x_2}) \geq 2S_{v_{HM}} \\
 & 2 - (0.3^{x_1}0.85^{x_2} + 0.3^{x_1}0.8^{x_2}) \geq 2S_{v_{HM}} \\
 & 2 - (0.5^{x_1}0.1^{x_2} + 0.5^{x_1}0.1^{x_2}) \geq 2(S_{v_{HM}} + \bar{\gamma}) \\
 & 2 - (0.2^{x_1}0.8^{x_2} + 0.2^{x_1}0.7^{x_2}) \geq 2(S_{v_{HM}} + \bar{\gamma}) \\
 & \sum_{i=1}^m x_i = 1 \\
 & x_i \geq 0, i = 1, 2, \dots, m \\
 & \underline{\gamma} \geq 0, \bar{\gamma} \geq 0
 \end{aligned} \tag{25}$$

Solving this we obtain $\hat{x} = (0.7237, 0.2762)^T$, $\hat{S}_{v_{HM}} = 0.6033$, $\hat{\underline{\gamma}} = 1.2346$, $\hat{\bar{\gamma}} = 0.2250$. To obtain better approximation, we construct the following problem according to the problem (20).

$$\begin{aligned}
 & \max\left\{\frac{-\underline{\gamma} + \bar{\gamma}}{2}\right\} \\
 \text{s.t.} \quad & 2\underline{\gamma} + 2 - (0.7^{x_1}0.2^{x_2} + 0.6^{x_1}0.2^{x_2}) \geq 1.2066 \\
 & 2\underline{\gamma} + 2 - (0.4^{x_1}0.9^{x_2} + 0.4^{x_1}0.9^{x_2}) \geq 1.2066 \\
 & 2 - (0.6^{x_1}0.15^{x_2} + 0.55^{x_1}0.15^{x_2}) \geq 1.2066 \\
 & 2 - (0.3^{x_1}0.85^{x_2} + 0.3^{x_1}0.8^{x_2}) \geq 1.2066 \\
 & -2\bar{\gamma} + 2 - (0.5^{x_1}0.1^{x_2} + 0.5^{x_1}0.1^{x_2}) \geq 1.2066 \\
 & -2\bar{\gamma} + 2 - (0.2^{x_1}0.8^{x_2} + 0.2^{x_1}0.7^{x_2}) \geq 1.2066 \\
 & \sum_{i=1}^m x_i = 1 \\
 & x_i \geq 0, i = 1, 2, \dots, m
 \end{aligned} \tag{26}$$

and finally solving problem (26) the optimal strategy for company C_1 is calculated as $\mathbf{x}^* = (0.7237, 0.2762)^T$. Similarly, for company C_2 we construct the following problem according to (16),

$$\begin{aligned}
 & \min\{g_M = S_{w_{HM}}\} \\
 \text{s.t.} \quad & 2 - (0.7^{y_1}0.4^{y_2} + 0.6^{y_1}0.4^{y_2}) \leq 2(S_{w_{HM}}) - \underline{\delta} \\
 & 2 - (0.2^{y_1}0.9^{y_2} + 0.2^{y_1}0.9^{y_2}) \leq 2(S_{w_{HM}}) - \underline{\delta} \\
 & 2 - (0.6^{y_1}0.3^{y_2} + 0.55^{y_1}0.3^{y_2}) \leq 2S_{w_{HM}} \\
 & 2 - (0.15^{y_1}0.85^{y_2} + 0.15^{y_1}0.8^{y_2}) \leq 2S_{w_{HM}} \\
 & 2 - (0.5^{y_1}0.2^{y_2} + 0.5^{y_1}0.2^{y_2}) \leq 2(S_{w_{HM}} + \bar{\delta}) \\
 & 2 - (0.1^{y_1}0.8^{y_2} + 0.1^{y_1}0.7^{y_2}) \leq 2(S_{w_{HM}} + \bar{\delta}) \\
 & \sum_{j=1}^n y_j = 1 \\
 & y_j \geq 0, j = 1, 2, \dots, n \\
 & \underline{\delta} \geq 0, \bar{\delta} \geq 0
 \end{aligned} \tag{27}$$

Solving the above problem (27) we obtain, $\hat{y} = (0.4295, 0.5705)^T$, $\hat{S}_{w_{HM}} = 0.6033$, $\hat{\underline{\delta}} = 0.0000$, $\hat{\bar{\delta}} = 1.2346$. For obtaining better solution, we construct the following problem according to problem (23)

$$\begin{aligned}
& \min \left\{ \frac{-\gamma + \bar{\gamma}}{2} \right\} \\
\text{s.t.} \quad & 2\bar{\delta} + 2 - (0.7^{y_1} 0.4^{y_2} + 0.6^{y_1} 0.4^{y_2}) \leq 1.2066 \\
& 2\bar{\delta} + 2 - (0.2^{y_1} 0.9^{y_2} + 0.2^{y_1} 0.9^{y_2}) \leq 1.2066 \\
& 2 - (0.6^{y_1} 0.3^{y_2} + 0.55^{y_1} 0.3^{y_2}) \leq 1.2066 \\
& 2 - (0.15^{y_1} 0.85^{y_2} + 0.15^{y_1} 0.8^{y_2}) \leq 1.2066 \\
& -2\bar{\delta} + 2 - (0.5^{y_1} 0.2^{y_2} + 0.5^{y_1} 0.2^{y_2}) \leq 1.2066 \\
& -2\bar{\delta} + 2 - (0.1^{y_1} 0.8^{y_2} + 0.1^{y_1} 0.7^{y_2}) \leq 1.2066 \\
& \sum_{j=1}^n y_j = 1 \\
& y_j \geq 0, j = 1, 2, \dots, n
\end{aligned} \tag{28}$$

Solving the problem (28) and ignoring all other values, we take only the optimal strategy $y^* = (0.4295, 0.5705)^T$. Substituting the optimum strategies x^* and y^* in the expected pay-off for the company C_1 we obtain,

$$\tilde{E}(C_1) = \left\langle (0.4109, 0.5961, 0.6953), (0.4109, 0.5999, 0.7016), (0.5251, 0.6069, 0.6953), (0.5251, 0.6106, 0.7016) \right\rangle \tag{29}$$

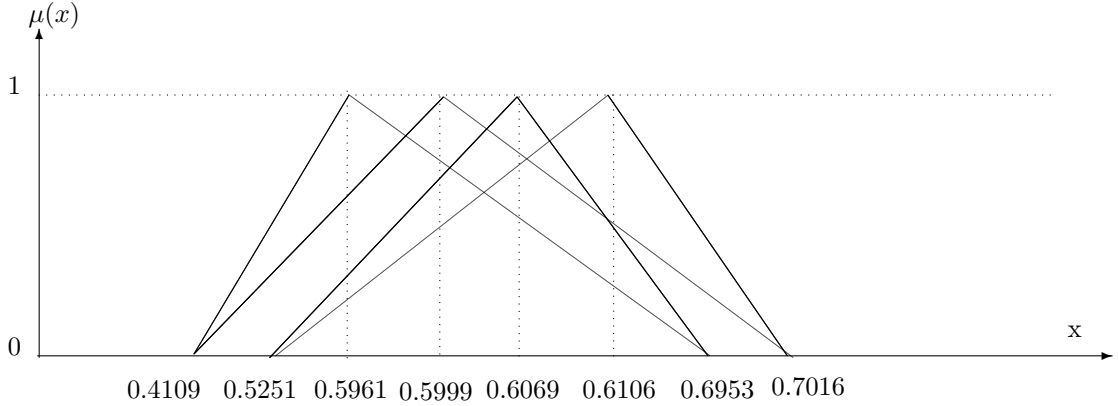


Figure 2: Expected pay-off for company C_1

4.3 The results and discussions

If company C_1 chooses the optimal strategy $x^* = (0.7237, 0.2762)^T$ and company C_2 chooses $y = (0.4295, 0.5705)$, then the expected pay-off of company C_1 is represented by (29), which is a THFS. The score function of the expected pay-off of company C_1 is $(0.4680, 0.6034, 0.6985)$ i.e., if the company C_1 prefers ‘up-to-date technology’ 72.37% and ‘reducing the cost’ of the product 27.62% as the strategies to keep to gain at a maximum scale and the company C_2 prefers ‘canvassing the product in digital media 42.95% and ‘reducing the product’ 57.05% to reduce their loss in a targeted market. The company C_1 can sell their product around 47% to 70%, which is assigned as ‘High’ rate of sell amount to decision-makers according to the linguistic variables provided by decision-makers.

5 Conclusions

In this paper, we have discussed the solution procedure of a matrix game with pay-offs of THFS. To find the optimal strategies, a pair of non-linear fuzzy programming problems are formulated using THFWA operator of THFEs. These problems are transformed into multi-objective programming problems by applying score functions of THFEs and the ranking order relation of the TFNs. Finally, the multi-objective programming problems are solved by using the Lexicographic method and obtain the optimal strategies for each player.

The contributions of the present article are,

- Our problem is concerned with the hesitancy of a decision-maker to predict a real-life situation.
- It is shown that the gain floor of Player A is less than or equal to the loss ceiling of Player B in a hesitant fuzzy environment.

- We can calculate the expected pay-off of the maximizing player in terms of a THFE, which is desirable.

The limitation of this proposed methodology is that it strictly depends on the score function of THFEs, and different types of score functions yield different kinds of solutions. Therefore, further study is needed to investigate a more general methodology.

Although the proposed methodology is illustrated with the market share problem, it can be applied to various decision-making areas such as economics, operation research, management, war science, etc.

Acknowledgement

We would like to thank the anonymous reviewers for their valuable suggestions, constructive comments, and cordial supports, which have been helpful for us to create a better version of our present work.

References

- [1] M. Alakhras, *Fuzzy system and game theory for green supply chain*, Computer Reviews Journal, **3** (2019), 65-87.
- [2] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87-96.
- [3] K. T. Atanassov, *Intuitionistic fuzzy sets*, Physica, Heidelberg, Springer-Verlag, 1999.
- [4] Z. Basir, I. Watresky, W. Salabum, J. Ali, *Intuitionistic fuzzy goals in zero-sum multi-criteria matrix games*, Symmetry, **9**(8) (2017), 158.
- [5] C. R. Bector, S. Chandra, *Fuzzy mathematical programming and fuzzy matrix games*, Springer Verlag, Berlin, Heidelberg, Germany, **169** (2005).
- [6] C. R. Bector, S. Chandra, V. Vijay, *Duality in linear programming with fuzzy parameters and matrix games with fuzzy pay-offs*, Fuzzy Sets and Systems, **146**(2) (2004), 253-269.
- [7] A. Bhaumik, S. K. Roy, D. F. Li, *Analysis of triangular intuitionistic fuzzy matrix games using robust ranking*, Journal of Intelligent and Fuzzy Systems, **33** (2017), 327-336.
- [8] H. Bigdeli, H. Hassanpour, J. Tayyebi, *Multi-objective security game with fuzzy pay-offs*, Iranian Journal of Fuzzy Systems, **16**(1) (2019), 89-101.
- [9] L. Campos, *Fuzzy linear programming models to solve fuzzy matrix games*, Fuzzy Sets and Systems, **32**(3) (1989), 275-289.
- [10] D. Dubois, H. Prade, *Fuzzy sets and systems: Theory and applications*, Mathematics in Science and Engineering, Academic Press, New York, 1980.
- [11] J. Jana, S. K. Roy, *Solution of matrix games with generalized trapezoidal fuzzy pay-offs*, Fuzzy Information and Engineering, **10**(2) (2018), 213-224.
- [12] G. Krishnaveni, G. Kandasamy, *A new approach for the solution of the fuzzy games*, Journal of Physics: Conference Series, (2018), DOI: <https://doi.org/10.1088/1742-6596/1000/1/012017>.
- [13] D. F. Li, *A fuzzy multi-objective approach to solve fuzzy matrix games*, The Journal of Fuzzy Mathematics, **7**(4) (1999), 907-912.
- [14] D. F. Li, *Lexicographic method for matrix games with pay-offs of triangular fuzzy numbers*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **16**(3) (2008), 371-389.
- [15] D. F. Li, *A fast approach to compute fuzzy values of matrix games with pay-offs of triangular fuzzy numbers*, European Journal of Operational Research, **223** (2012), 421-429.
- [16] D. F. Li, *An effective methodology for solving matrix games with fuzzy pay-offs*, IEEE Transactions on Cybernetics, **43**(2) (2013), 610-621.
- [17] D. F. Li, *Decision and game theory in management with intuitionistic fuzzy sets*, Springer-Verlag Berlin Heidelberg, **308** (2014).
- [18] D. F. Li, J. X. Nan, *A non-linear programming approach to matrix games with pay-offs of Atanassov's intuitionistic fuzzy sets*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **17**(4) (2009), 585-607.
- [19] D. F. Li, J. X. Nan, Z. P. Tang, K. J. Chen, X. D. Xiang, F. X. Hong, *A bi-objective programming approach to solve matrix games with pay-offs of Atanassov's triangular intuitionistic fuzzy numbers*, Iranian Journal of Fuzzy Systems, **9**(3) (2012), 93-110.
- [20] S. Moradi, S. M. Lamaei, *Multi-objective game theory model and fuzzy programming approach for sustainable watershed management*, Land Use Policy, **71** (2018), 363-371.
- [21] D. Qiu, X. Xing, S. Chen, *Solving multi-objective matrix games with fuzzy pay-offs through the lower limit of the possibility degree*, Symmetry, **9**(8) (2017), 130.

- [22] R. M. Rodriguez, L. Martinez, F. Herrera, *Hesitant fuzzy linguistic term sets for decision making*, IEEE Transactions on Fuzzy Systems, **20**(1) (2012), 109-119.
- [23] S. K. Roy, A. Bhaumik, *Intelligent water management: A triangular type-2 intuitionistic fuzzy matrix game approach*, Water Resources Management, **32**(3) (2018), 949-968.
- [24] M. R. Seikh, P. K. Nayak, M. Pal, *Notes on triangular intuitionistic fuzzy numbers*, International Journal of Mathematics in Operational Research, **5**(4) (2013), 446-465.
- [25] M. R. Seikh, P. K. Nayak, M. Pal, *Matrix games in intuitionistic fuzzy environment*, International Journal of Mathematics in Operational Research, **5**(6) (2013), 693-708.
- [26] M. R. Seikh, P. K. Nayak, M. Pal, *An alternative approach for solving fuzzy matrix games*, International Journal of Mathematics and Soft Computing, **5**(1) (2015), 79-92.
- [27] M. R. Seikh, P. K. Nayak, M. Pal, *Application of intuitionistic fuzzy mathematical programming with exponential membership and quadratic non-membership functions in matrix games*, Annals of Fuzzy Mathematics and Informatics, **9**(2) (2015), 183-195.
- [28] M. R. Seikh, P. K. Nayak, M. Pal, *Matrix game with intuitionistic fuzzy pay-offs*, Journal of Information and Optimization Sciences, **36**(1-2) (2015), 159-181.
- [29] M. R. Seikh, P. K. Nayak, M. Pal, *Aspiration level approach to solve matrix games with I-fuzzy goals and I-fuzzy pay-offs*, Pacific Science Review: A Natural Science and Engineering, **18**(1) (2016), 5-13.
- [30] M. R. Seikh, M. Pal, P. K. Nayak, *Application of triangular intuitionistic fuzzy numbers in bi-matrix games*, International Journal of Pure and Applied Mathematics, **79**(2) (2012), 235-247.
- [31] V. Torra, *Hesitant fuzzy sets*, International Journal of Intelligent Systems, **25**(6) (2010), 529-539.
- [32] V. Torra, Y. Narukawa, *On hesitant fuzzy sets and decision*, The 18th IEEE International Conference on Fuzzy Systems, Korea, (2009), 1378-1382.
- [33] V. Vijay, A. Mehra, S. Chandra, C. R. Bector, *Fuzzy matrix games via a fuzzy relation approach*, Fuzzy Optimization and Decision Making, **6**(4) (2007), 299-314.
- [34] G. Wei, *Hesitant fuzzy prioritized operators and their application to multiple attribute decision making*, Knowledge-Based Systems, **31** (2012), 176-182.
- [35] G. Wei, H. Wang, X. Zhao, R. Lin, *Hesitant triangular fuzzy information aggregation in multiple attribute decision making*, Journal of Intelligent and Fuzzy Systems, **26** (2014), 1201-1209.
- [36] M. M. Xia, *Interval-valued intuitionistic fuzzy matrix games based on archimedean t-conorm and t-norm*, International Journal of General System, **47**(3) (2018), 278-293.
- [37] M. M. Xia, Z. S. Xu, *Hesitant fuzzy information aggregation in decision making*, International Journal of Approximate Reasoning, **52**(3) (2011), 395-407.
- [38] J. Xu, J. Y. Dong, S. P. Wan, J. Gao, *Multiple attribute decision making with triangular intuitionistic fuzzy numbers based-on zero-sum game approach*, Iranian Journal of Fuzzy Systems, **16**(3) (2019), 97-112.
- [39] C. Xu, F. Meng, Q. Zhang, *PN equilibrium strategy for matrix games with fuzzy pay-offs*, Journal of Intelligence and Fuzzy Systems, **32**(3) (2017), 2195-2206.
- [40] Z. S. Xu, M. M. Xia, *Distance and similarity measures for hesitant fuzzy sets*, Information Sciences, **181**(11) (2011), 2128-2138.
- [41] Z. S. Xu, M. M. Xia, *On distance and correlation measures of hesitant fuzzy information*, International Journal of Intelligent Systems, **26**(5) (2011), 410-425.
- [42] D. Yu, *Triangular hesitant fuzzy set and its application to teaching quality evaluation*, Journal of Information and Computational Science, **10**(7) (2013), 1925-81934.
- [43] D. Yu, Y. Wu, W. Zhou, *Multi-criteria decision making based on Choquet integral under hesitant fuzzy environment*, Journal of Computational Information Systems, **7**(12) (2011), 4506-4513.
- [44] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(3) (1965), 338-352.