

## General definitions for the union and intersection of ordered fuzzy multisets

Á. Riesgo<sup>1</sup>, P. Alonso<sup>2</sup>, I. Díaz<sup>3</sup> and S. Montes<sup>4</sup>

<sup>1,4</sup>*Department of Statistics and O. R., University of Oviedo, Spain*

<sup>2</sup>*Department of Mathematics, University of Oviedo, Spain*

<sup>3</sup>*Department of Computer Science, University of Oviedo, Spain*

riesgoangel@uniovi.es, palonso@uniovi.es, sirene@uniovi.es, montes@uniovi.es

### Abstract

Since its original formulation, the theory of fuzzy sets has spawned a number of extensions where the role of membership values in the real unit interval  $[0, 1]$  is handed over to more complex mathematical entities. Amongst the many existing extensions, two similar ones, the fuzzy multisets and the hesitant fuzzy sets, rely on collections of several distinct values to represent fuzzy membership, the key difference being that the fuzzy multisets allow for repeated membership values whereas the hesitant fuzzy sets do not. But in neither case are these collections of values ordered, as they are simply represented through multisets or sets. In this paper, we study ordered fuzzy multisets, where the membership value can be an ordered  $n$ -tuple of values, thus accounting for both order and repetition. We present some basic definitions and results and explore the relation between these ordered fuzzy multisets and the fuzzy multisets and hesitant fuzzy sets.

*Keywords:* Fuzzy sets, fuzzy multisets, ordered fuzzy multisets.

## 1 Introduction

Fuzzy sets were originally introduced by Lotfi A. Zadeh as an extension of classical set theory [18], where the Boolean characteristic function of a set is replaced with a function into the real unit interval  $[0, 1]$ . Values other than 0 and 1 represent a situation where there is either limited knowledge or some sort of ambiguity about whether an element should be considered a member of a set. Thus, given a reference set  $X$ , Zadeh's theory of fuzzy sets makes it possible to state that an element  $x \in X$  is a member of a subset  $A$  with a membership value of, say, 0.9 (mostly a member), 0.1 (hardly a member) or 0.5 (half-way between membership and non-membership). This is particularly useful to model natural-language labels like, for instance, when individuals in a human group are characterised as being "old" or "young".

In the wake of the original (often called "ordinary") fuzzy sets, more sophisticated extensions have been proposed, where the membership value itself is made less precise by turning it into a collection of possible values rather than a single real number in  $[0, 1]$ . In fact, in many scenarios that call for a fuzzy approach, the treatment of the membership value as a precise real number between 0 and 1 is also a gross simplification since it may not always be possible to narrow the membership information down to an exact number. As an example, instead of claiming that a person is "young" with a membership value of 0.4, we might say that the membership value should be no less than 0.2 and no more than 0.4, so the extents of the acceptable membership values would be given by the real interval  $[0.2, 0.4]$ . This idea leads to a more general formulation of fuzzy sets where membership values are intervals, the interval-valued fuzzy sets [9]. An alternative generalisation that has been successful in the literature is the type-2 fuzzy sets [10], where the membership value is itself an ordinary fuzzy set, i.e. a function  $[0, 1] \rightarrow [0, 1]$ , typically a concave curve that spans an interval of uncertainty around a central membership value.

In all these extensions we have just mentioned, the ordinary fuzzy sets are extended by replacing the single-valued membership value with either an interval around a main value or a function that smoothly reaches a maximum at a central value. Thus, these more general types of fuzzy sets are to be interpreted as modelling a tolerance threshold of uncertainty around the membership value. For the purposes of our discussion, we will refer to them as smooth fuzzy sets.

Another generalisation of the ordinary fuzzy sets was proposed by R. R. Yager in 1986 under the name fuzzy bags, where the membership values are multisets (also called bags), collections of values where repetition is allowed (see [15]), in the real unit interval  $[0, 1]$ . The theory of fuzzy bags was further refined by S. Miyamoto [11], who provided the first definitions of the common operations such as the intersection and union that were consistent with the standard ones for the ordinary fuzzy sets. These generalised fuzzy sets are nowadays usually referred to as fuzzy multisets.

A similar approach to fuzzy multisets is used by hesitant fuzzy sets, which were proposed by V. Torra in 2010 [16], expanding on an idea first put forward by I. Grattan-Guinness in 1976 [7]. The hesitant theoretical framework broadens the range of the membership functions to encompass any subset of the  $[0, 1]$  interval. A multiset-based version was already proposed in Torra's original article, which brings the theory closer to that of fuzzy multisets. The difference lies in the fact that both theories have evolved using different definitions for some of the common operations. In fact, the intersection and union as conventionally defined for fuzzy multisets are not equivalent to the accepted definitions for hesitant fuzzy sets. In a recent paper [12], we introduced a family of definitions for the intersection and union of fuzzy multisets, where the hesitant definitions appear as a particular case.

The peculiarity of using sets or multisets as fuzzy membership values is that they do not represent a tolerance threshold around an ideal precise membership value, as in the smooth fuzzy set generalisations, but rather a collection of distinct possibilities. The real-life use case of fuzzy multisets and hesitant fuzzy sets emerges when the membership value can be determined through several alternative, but related, criteria or mechanisms that yield different values. In such cases, we have a multiplicity of possible membership values instead of one membership value.

We have to be particularly careful about the interpretation of the multiple membership values in multivalued fuzzy sets. For example, in a typical hesitant membership value like  $\{0.2, 0.3\}$ , we might be misled into thinking of the hesitancy as a lack of precision about the actual value, but that sort of situation would be better modelled through an interval-valued fuzzy set with a membership of  $[0.2, 0.3]$ . The important thing in a hesitant fuzzy set is that the different membership values do not represent an interval, but distinct possible values, even widely differing ones [13]. In a common intuitive interpretation of the fuzzy multisets and the hesitant fuzzy sets, each membership value is regarded as an independent verdict on membership that one particular "expert" or "decision maker" has produced. In this view, a hesitant membership value of  $\{0.1, 0.3\}$  would be regarded as the result of an expert considering that the set has a membership value of 0.1 and a second expert considering that the set has a membership value of 0.3. These "experts" can be human individuals or, more often, simply different criteria or methodologies that produce fuzzy membership values for a common phenomenon. Note that if the experts evaluate different phenomena, then it would make more sense to use separate fuzzy sets.

A problem with the experts' interpretation when we consider the hesitant membership value as simply a subset of  $[0, 1]$  is that these experts should be indistinguishable and even the number of experts involved is subject to variation. A more realistic scenario occurs when the number of experts is fixed and it may be possible to link the membership values to the expert that has produced it. In such a situation, the hesitant fuzzy sets are not a good model. On the one hand, it should be possible for a membership value to appear more than once. If both the first and the second expert produce a membership value of 0.2, then the hesitant membership value should be something like  $\langle 0.2, 0.2 \rangle$ , which is a multiset rather than a subset (we will use the angular bracket notation  $\langle \rangle$  for multisets). This example shows the advantage of using fuzzy multisets over hesitant fuzzy sets when repetition is meaningful. But accounting for repetition may not be enough if we consider the experts as being distinguishable. In such a case, which seems sensible for comparison purposes, then the membership multisets should be ordered  $n$ -tuples, so that  $(0.1, 0.2)$  is different from  $(0.2, 0.1)$  as a membership value. We will refer to these ordered vector-like membership values as ordered fuzzy multisets, which are the focus of this paper.

From this discussion, we can see that there are four distinct cases in the experts' model.

1. If the experts produce unrelated values for different phenomena, then the values for each expert should be treated as **independent fuzzy sets**, as there is no link between them.
2. If the experts produce related values that account for one phenomenon and there is a fixed number of them producing a membership value for each element in the universe, then the problem can be modelled with **ordered fuzzy multisets**.
3. If the experts produce related values that account for one phenomenon and there is a fixed number of them producing a membership value for each element in the universe and they are indistinguishable (as if in a secret

vote; we cannot know which expert produced which value for one element, but we can know that  $n$  experts chose the same value) then the problem should be modelled using **fuzzy multisets**.

4. If the experts produce related values that account for one phenomenon and there is a variable or unknown number of them producing membership values for each element in the universe and they are indistinguishable then the problem should be modelled using **hesitant fuzzy sets**.

As an example of a situation where the ordered fuzzy sets can be a better model than the others, let us consider the occasional use of fuzzy sets to represent greyscale images. In such proposals, the pixels in an image are regarded as the elements of the universe and the fuzzy membership value between 0 and 1 represents the greyscale range from black (0) to white (1). We may be tempted to use this model for colour images, and treat the three colour channels as if they were three experts evaluating the fuzzy membership of a pixel. In such a situation, it is clear that the use of hesitant fuzzy sets would result in a colour-blind model, where for example the six colours red, green, blue, yellow, cyan and magenta would all be represented with exactly the same membership value  $\{0, 1\}$ . Things get only slightly better with the hesitant fuzzy multisets where the red, green and blue colours would be represented as the same pixel value  $\langle 0, 0, 1 \rangle$  but differentiated from the yellow, cyan and magenta trio, which would be  $\langle 0, 1, 1 \rangle$ . Ordered fuzzy multisets would provide a better model since any two different colours would be represented with two different membership values like, for example,  $(1, 0, 0)$  for red and  $(0, 1, 0)$  for green.

These three types of multivalued fuzzy sets have received very disparate levels of attention in the fuzzy set research. In fact, there has been very little research so far concerning the ordered fuzzy multisets; they are pretty straightforward if we consider them as a Cartesian product, which may explain the scant interest such mathematical constructions have elicited. Still, it is possible to identify some formal definitions and results related to the multidimensional nature of these objects, which we will present in this paper. Two precedents of the ordered fuzzy multisets are the “vectorial fuzzy sets” introduced by L. Kóczy [8] and the “ $n$ -dimensional fuzzy sets”, which impose a sorting condition on the coordinates that we will ignore in our approach, and which have been studied by B. Bedregal et al. [1], citing previous work by Y. Shang, X. Yuan and E. S. Lee [14].

In this work, we are going to discuss how such situations can, and sometimes should, be modelled through these and other related fuzzy set extensions that also allow for multiple and different membership values. We will refer to these schemes, where for example two values like 0.1 and 0.3 may be valid membership values for an element  $x$  while 0.2 is not, as multivalued fuzzy sets. In particular, we will explore those extensions that represent membership through a finite collection of values, investigating the role played by order and repetition in such collections. Thus, we will focus on the ordered fuzzy multisets.

This paper is organised as follows: Section 2 introduces the fundamental results of the existing theories of ordinary fuzzy sets, hesitant fuzzy sets and fuzzy multisets. Section 3 introduces and studies the concept of ordered fuzzy multisets, as well as the more general definitions for the intersection and union of ordered fuzzy multisets. Finally, in Section 4 we sum up the main conclusions of our research.

## 2 Preliminaries

In this section, the basic and necessary definitions of fuzzy sets and their extensions are provided.

### 2.1 Fuzzy sets

In the following definitions, we assume that there is always an axiomatic reference set or universe, which we denote by  $X$ .

**Definition 2.1.** Let  $X$  be a universe. An **ordinary fuzzy set**  $A$  is a function  $A: X \rightarrow [0, 1]$ .

Given an element  $x \in X$ ,  $A(x)$  is called its **membership value** and the family of all the ordinary fuzzy sets over  $X$ ,  $\mathcal{F}(X)$ , is called the **ordinary fuzzy power set** over  $X$ .

We can see this function as a generalisation of the characteristic (or “indicator”) function for an ordinary set. Thus, in the particular case where  $A(x)$  is always 0 or 1, it can be regarded as a characteristic function into  $\{0, 1\}$ . The values in the image set of  $A$  are called the **membership values** of the fuzzy set.

Common set operations, such as the complement, intersection and union, have been defined for fuzzy sets in such a way that they reproduce the behaviour of the corresponding operations in ordinary sets (typically called **crisp sets** in fuzzy set theory). While there are whole families of acceptable definitions, here we will stick to the standard ones originally introduced by L. A. Zadeh, which are defined as follows [9].

**Definition 2.2.** Let  $X$  be a universe and let  $A$  and  $B$  be two ordinary fuzzy sets over  $X$ . The **complement** of  $A$ ,  $A^c$ , and the **intersection**,  $A \cap B$ , and **union**,  $A \cup B$ , of  $A$  and  $B$  are the ordinary fuzzy sets over  $X$  given by the following relations:

$$A^c(x) = 1 - A(x) \quad x \in X \quad (1)$$

$$(A \cap B)(x) = \min\{A(x), B(x)\} \quad x \in X \quad (2)$$

$$(A \cup B)(x) = \max\{A(x), B(x)\} \quad x \in X \quad (3)$$

While other operations and relations are also interesting, in this paper we will focus our analysis on the bare minimum of these three fundamental operations.

## 2.2 Hesitant fuzzy sets

As we commented in detail in the introduction, we will consider a generalisation of the hesitant fuzzy sets which is more appropriate in some cases. Thus, we are going to provide some basic notions about hesitant theory. A comprehensive introduction can be found in a recent book by Z. S. Xu [17]. The basic definition is as follows [16].

**Definition 2.3.** Let  $X$  be a universe. A **hesitant fuzzy set**  $\tilde{A}$  over  $X$  is a function  $\tilde{A}: X \rightarrow \mathcal{P}([0, 1])$ , where  $\mathcal{P}([0, 1])$  is the family of all the subsets of the real closed interval  $[0, 1]$ .

Given an element  $x \in X$ ,  $\tilde{A}(x)$  is called its **hesitant element** [17] and the family of all the hesitant fuzzy sets over  $X$ ,  $\mathcal{F}_H(X)$ , is called the **hesitant fuzzy power set** over  $X$ .

Let us now see the definitions for the three fundamental operations [16].

**Definition 2.4.** Let  $X$  be a universe and let  $\tilde{A}$  and  $\tilde{B} \in \mathcal{F}_H(X)$  be two hesitant fuzzy sets. The **hesitant complement**,  $\tilde{A}^c$ , of  $\tilde{A}$ , the **hesitant intersection**,  $\tilde{A} \cap \tilde{B}$  and the **hesitant union**,  $\tilde{A} \cup \tilde{B}$ , of  $\tilde{A}$  and  $\tilde{B}$  are the hesitant fuzzy sets defined by the following relations:

$$\tilde{A}^c(x) = \{t | 1 - t \in \tilde{A}(x)\} \quad (4)$$

$$(\tilde{A} \cap \tilde{B})(x) = \{\alpha \in \tilde{A}(x) \cup \tilde{B}(x) | \alpha \leq \min\{\sup\{\tilde{A}(x)\}, \sup\{\tilde{B}(x)\}\}\} \quad (5)$$

$$(\tilde{A} \cup \tilde{B})(x) = \{\alpha \in \tilde{A}(x) \cup \tilde{B}(x) | \alpha \geq \max\{\inf\{\tilde{A}(x)\}, \inf\{\tilde{B}(x)\}\}\} \quad (6)$$

where  $\sup\{\tilde{A}(x)\}$  is the supremum (least upper bound) of the hesitant element  $\tilde{A}(x)$  and  $\inf\{\tilde{A}(x)\}$  is its infimum (greatest lower bound).

The hesitant fuzzy sets include the ordinary fuzzy sets as the special case when all the hesitant elements have cardinality 1. In that particular case, the above definitions of complement, intersection and union behave like the ordinary ones [12].

In this paper, we will only consider **typical hesitant fuzzy sets** [2], where the hesitant elements are always finite, non-empty sets.

## 2.3 Fuzzy multisets

The fuzzy multisets associate each element in a universe with a multiset in the  $[0, 1]$  interval. In order to define these objects formally, we will first need to review the definition of multisets in a crisp sense [11, 15].

**Definition 2.5.** Let  $X$  be a universe. A **multiset**  $M$  is a function  $M: X \rightarrow \mathbb{N}$  (including zero).

Given an element  $x \in X$ ,  $M(x)$  is called its **multiplicity** and the family of all the multisets over  $X$ ,  $\mathbb{N}^X$ , is called the **power multiset** of  $X$ .

As in the case of fuzzy sets, using the function itself as the definition makes things simpler in terms of notation, even though a difference is often made between the multiset as a collection of elements and its **count** or **multiplicity function**.

A useful operation consists in extracting a set from the multiset by grouping together the repeated values [15]:

**Definition 2.6.** Let  $M$  be a multiset over a finite universe  $X$ . The **support** of  $M$  is the set  $\text{Supp}(M) \subseteq X$  defined by:

$$\text{Supp}(M) = \{x \in X | M(x) > 0\}$$

The concept of cardinality can be extended to those multisets that have a finite support in a natural way:

**Definition 2.7.** Let  $M$  be a multiset over a universe  $X$  such that its support is finite. The **cardinality** of  $M$  is the natural number defined [15] as:

$$|M| := \sum_{x \in X} M(x)$$

**Example 2.8.** Say we have a finite universe  $X = \{a, b\}$ . Then a multiset  $M \in \mathbb{N}^X$  is a function such as  $M(a) = 1$ ,  $M(b) = 2$  and its cardinality is  $|M| = 3$ . In set-like notation, this can be expressed as  $M = \langle a, b, b \rangle$ . The support of  $M$  is  $\text{Supp}(M) = \{a, b\}$ .

As we would expect, those multisets with the image set restricted to  $\{0, 1\}$  can be identified with the ordinary sets. So the multisets behave as an extension of the ordinary concept of a set, and in this regard share some characteristics with the fuzzy sets. In fact, multisets, when bounded by a maximum multiplicity  $m$ , can be regarded as a particular case of  $L$ -fuzzy sets [6] as the set of integer multiplicities  $\{0, 1, \dots, m\}$  can be identified with a discrete subset of the unit interval  $\{0, \frac{1}{m}, \dots, 1\}$  and form a complemented lattice [4]. This is not possible with unbounded multiplicities and multisets are also different from fuzzy sets in terms of their semantics, but it should be stressed that the two concepts are closely related and their basic operations are therefore very similar. Here we are going to define the union, which we will use later.

**Definition 2.9.** Let  $M, N \in \mathbb{N}^X$  be two multisets over a universe  $X$ . The **union** of  $M$  and  $N$  is the multiset  $M \cup N$  defined by the relation:

$$(M \cup N)(x) = \max\{M(x), N(x)\} \quad x \in X$$

The definition of intersection, which we will not use in this paper, is completely analogous, with min replacing max. An interesting recent study about multisets can be found in [5].

Now that we have defined the crisp multisets, we can define the fuzzy multisets as follows [11].

**Definition 2.10.** Let  $X$  be a universe. A **fuzzy multiset**  $\hat{A}$  over  $X$  is a function  $\hat{A}: X \rightarrow \mathbb{N}^{[0,1]}$ . Given an element  $x \in X$ ,  $\hat{A}(x)$  is called its **membership multiset** and the family of all the fuzzy multisets over  $X$ ,  $\mathcal{FM}(X)$ , is called the **fuzzy power multiset** over  $X$ .

**Example 2.11.** Say we have a single-element universe  $X = \{x\}$ . We can define a fuzzy multiset  $\hat{A}$  as  $\hat{A}(x) = \langle 0.1, 0.2, 0.2 \rangle$  in angular-bracket notation. Or in other words, using the formal definition above, the element  $x$  is being mapped into a function  $\hat{A}(x): [0, 1] \rightarrow \mathbb{N}$  defined as  $\hat{A}(x)(0.1) = 1$ ,  $\hat{A}(x)(0.2) = 2$  and  $\hat{A}(x)(t) = 0$  for any  $t \notin \{0.1, 0.2\}$ .

A fuzzy multiset thus associates each element in the universe with a crisp multiset over the real unit interval  $[0, 1]$ . By taking the supports of these membership multisets we can turn the fuzzy multiset into a hesitant fuzzy set. This idea prompted us to introduce the following definition in a recent article [12].

**Definition 2.12.** Let  $X$  be a universe and let  $\hat{A} \in \mathcal{FM}(X)$  be a fuzzy multiset. Its **hesitant fuzzy set support**  $\text{Supp}^h(\hat{A})$  is the hesitant fuzzy set such that, for any element  $x \in X$ ,  $(\text{Supp}^h(\hat{A}))(x) = \text{Supp}(\hat{A}(x))$ , where  $\text{Supp}$  is the support in the sense of Definition 2.6.

The complement of a fuzzy multiset can be defined in an intuitive and straightforward way as follows:

**Definition 2.13.** Let  $X$  be a universe and let  $\hat{A} \in \mathcal{FM}(X)$  be a fuzzy multiset. The **complement** of  $\hat{A}$  is the fuzzy multiset  $\hat{A}^c$  defined as:

$$(\hat{A}^c(x))(t) := (\hat{A}(x))(1 - t), \quad x \in X, \quad t \in [0, 1] \tag{7}$$

The intersection and union, however, are more problematic. The most common definitions in the literature on fuzzy multisets are those due to Miyamoto [11], which are based on the idea of arranging the elements of the multisets into ordered sequences and then performing a coordinatewise operation. But for this to be possible, the two operands must have the same cardinality, so we may have to extend one of them by repeating elements when the cardinalities differ. The procedures involved, although easy to explain in natural language, require some laborious workings in mathematical language [12]. The idea of a common cardinality, for each element in the universe, can be expressed through the following definition [12].

**Definition 2.14.** Let  $X$  be a universe. Given a function  $m: X \rightarrow \mathbb{N}$ , an  $m$ -**regular fuzzy multiset**  $\hat{A}$  over the universe  $X$  is a fuzzy multiset such that, for every  $x \in X$ ,  $\hat{A}(x)$  has a finite support and  $|\hat{A}(x)| = m(x)$ . We call  $m$  a **cardinality map**. The family of the  $m$ -**regular fuzzy multisets** over  $X$ ,  $\mathcal{FM}^m(X)$ , is called the  $m$ -**regular fuzzy power multiset**.

Any fuzzy multiset has its own cardinality map. The importance of this concept lies in the fact that some binary operations between fuzzy multisets can be defined in a straightforward way only when both operands share the same cardinality map.

We will also need to be able to map a multiset into a finite **sequence** of length  $n$  ( $\in \mathbb{N}$ ) or  $n$ -**tuple**, which for the real unit interval  $[0, 1]$  can be defined as an element of the  $n$ -dimensional real unit hypercube  $[0, 1]^n$ . Given a membership multiset  $M \in \mathbb{N}^{[0,1]}$  with cardinality  $n$ , a function mapping it to an  $n$ -tuple in  $[0, 1]^n$  is called an **ordering strategy**, with the family of all such functions being denoted by  $\mathcal{OS}(M)$ . The number of possible ordering strategies is the number of permutations of  $n$  elements with repetition and the two most common sorting strategies are the **ascending sort**  $s_\uparrow$  and the **descending sort**  $s_\downarrow$ , where the elements are sorted in ascending or descending order, respectively. We will use parentheses  $()$  for sequences; so we can write, for example,  $s_\downarrow(\langle 0.1, 0.2, 0.2 \rangle) = (0.2, 0.2, 0.1)$ .

It is now possible to define intersection and union by operating on sorted sequences [12], as follows.

**Definition 2.15.** *Let  $X$  be a universe and let  $m: X \rightarrow \mathbb{N}$  be a cardinality map. Given two  $m$ -regular fuzzy multisets  $\hat{A}$  and  $\hat{B}$  and two ordering strategies  $s_A$  and  $s_B$ , for each element  $x \in X$ , two new sequences  $\mu_{\hat{A} \cap_{(s_A, s_B)} \hat{B}}(x)$  and  $\mu_{\hat{A} \cup_{(s_A, s_B)} \hat{B}}(x)$  can be built with the pairwise minima and maxima:*

$$\begin{aligned} (\mu_{\hat{A} \cap_{(s_A, s_B)} \hat{B}}(x))_i &:= \min\{(s_A(\hat{A}(x)))_i, (s_B(\hat{B}(x)))_i\} \quad i \in \{1, \dots, m(x)\} \\ (\mu_{\hat{A} \cup_{(s_A, s_B)} \hat{B}}(x))_i &:= \max\{(s_A(\hat{A}(x)))_i, (s_B(\hat{B}(x)))_i\} \quad i \in \{1, \dots, m(x)\} \end{aligned}$$

The  $m$ -regular  $(s_A, s_B)$ -**ordered intersection**  $\hat{A} \cap_{(s_A, s_B)} \hat{B}$  and the  $m$ -regular  $(s_A, s_B)$ -**ordered union**  $\hat{A} \cup_{(s_A, s_B)} \hat{B}$  are the two  $m$ -regular fuzzy multisets defined by:

$$\begin{aligned} ((\hat{A} \cap_{(s_A, s_B)} \hat{B})(x))(t) &= |\{i | 1 \leq i \leq m(x), (\mu_{\hat{A} \cap_{(s_A, s_B)} \hat{B}}(x))_i = t\}| \\ ((\hat{A} \cup_{(s_A, s_B)} \hat{B})(x))(t) &= |\{i | 1 \leq i \leq m(x), (\mu_{\hat{A} \cup_{(s_A, s_B)} \hat{B}}(x))_i = t\}| \end{aligned}$$

For example, if two fuzzy multisets  $\hat{A}$  and  $\hat{B}$  over a universe  $X = \{x\}$  have membership multisets  $\hat{A}(x) = \langle 0.2, 0.3, 0.3 \rangle$  and  $\hat{B}(x) = \langle 0.4, 0.4, 0.1 \rangle$  at  $x$ , both with cardinality 3, then the choice of  $s_\uparrow$  as an ordering strategy would turn these multisets into two triplets  $(0.2, 0.3, 0.3)$  and  $(0.1, 0.4, 0.4)$ . The triplets with the pairwise minima and maxima would then be  $(0.1, 0.3, 0.3)$  and  $(0.2, 0.4, 0.4)$ , respectively, and the intersection and union (for this common cardinality and this given ordering strategy) at point  $x$  would be  $\langle 0.1, 0.3, 0.3 \rangle$  and  $\langle 0.2, 0.4, 0.4 \rangle$ , respectively.

Miyamoto's intersection and union are the particular case when both sorting strategies are chosen as  $s_A = s_B = s_\downarrow$ . It can easily be proved that choosing  $s_A = s_B = s_\uparrow$  produces the same results for both operations, so sorting in either ascending or descending order is simply a matter of convention. Other sorting strategies, however, lead to different results, so the choice of  $s_A$  and  $s_B$  does affect the behaviour of these operations.

If the two fuzzy multisets  $\hat{A}$  and  $\hat{B}$  do not share a common cardinality map, then it will be possible to apply the definition above through an additional mechanism, which we call **regularisation** [12], where the membership multiset with the shorter cardinality for each element  $x \in X$  is extended by increasing the multiplicity of either the lowest or the highest value. We omit the details here and will simply assume that definitions can always be extended to the general case by regularising multisets first.

As we have argued in a recent article [12], since we are working with finite sets we can make a definition that is independent of any particular sorting strategy by taking the multiset union of the ordered intersections and unions (Definition 2.15) resulting from all the combinations of possible sorting strategies  $(s_A, s_B)$ . This idea leads to the following definitions:

**Definition 2.16.** *Let  $X$  be a universe and let  $\hat{A}, \hat{B} \in \mathcal{FM}(X)$  be two fuzzy multisets. The **aggregate intersection** and the **aggregate union** of  $\hat{A}$  and  $\hat{B}$  are the fuzzy multisets  $\hat{A} \cap^\alpha \hat{B}$  and  $\hat{A} \cup^\alpha \hat{B}$  such that, for any element  $x \in X$ ,  $\hat{A} \cap^\alpha \hat{B}(x)$  is the multiset union of the  $(s_A, s_B)$ -ordered intersections and  $\hat{A} \cup^\alpha \hat{B}(x)$  is the multiset union of the  $(s_A, s_B)$ -ordered unions for all the possible pairs of ordering strategies  $(s_A, s_B)$ :*

$$\hat{A} \cap^\alpha \hat{B}(x) := \bigcup_{\substack{s_A \in \mathcal{OS}(\hat{A}) \\ s_B \in \mathcal{OS}(\hat{B})}} \hat{A} \cap_{(s_A, s_B)} \hat{B}(x) \quad x \in X \quad (8)$$

$$\hat{A} \cup^\alpha \hat{B}(x) := \bigcup_{\substack{s_A \in \mathcal{OS}(\hat{A}) \\ s_B \in \mathcal{OS}(\hat{B})}} \hat{A} \cup_{(s_A, s_B)} \hat{B}(x) \quad x \in X \quad (9)$$

Using the same example as in Definition 2.15, the fuzzy multisets over  $X = \{x\}$  given by  $\hat{A}(x) = \langle 0.2, 0.3, 0.3 \rangle$  and  $\hat{B}(x) = \langle 0.4, 0.4, 0.1 \rangle$  can lead to a regular ordered intersection  $\langle 0.2, 0.3, 0.3 \rangle$ , using the ascending sort, but also to the multiset  $\langle 0.1, 0.2, 0.3 \rangle$  by arranging the triplets in a different order, so the aggregate intersection will be the multiset union of both possibilities:  $\hat{A} \cap^a \hat{B}(x) = \langle 0.1, 0.2, 0.3, 0.3 \rangle$ .

All these forms of intersection and union that we have defined are consistent with the definitions for the ordinary fuzzy sets. But this aggregate form of intersection and union is in addition also consistent with the definitions for the hesitant fuzzy sets (Definition 2.4) in the sense that the operations commute with a bijection that can be established between  $\mathcal{F}_H(X)$  and a restriction of  $\mathcal{F}\mathcal{M}(X)$  that excludes multiplicities different from 0 and 1. There are basically two ways to determine this restricted version of  $\mathcal{F}\mathcal{M}(X)$ ; we can either take the subfamily that fulfils this cardinality condition or, alternatively, equalise any non-zero cardinalities through an equivalence relation, which is our preferred approach as we will do something similar to the ordered fuzzy sets later. This result is summarised in the next proposition [12].

**Proposition 2.17.** *Given a universe  $X$ , two fuzzy multisets  $\hat{A}, \hat{B} \in \mathcal{F}\mathcal{M}(X)$  are said to be **repetition-equivalent** when they have the same hesitant fuzzy set support (Definition 2.12); i.e.  $\text{Supp}^h(\hat{A}) = \text{Supp}^h(\hat{B})$ . It can easily be proved that this is an equivalence relation, which we denote by  $\hat{A} \sim_r \hat{B}$ .*

*With this equivalence relation, there is a bijection  $\phi_{MH}: \mathcal{F}\mathcal{M}(X)/\sim_r \leftrightarrow \mathcal{F}_H(X)$  defined by:*

$$\phi_{MH}([\hat{A}]) = \text{Supp}^h(\hat{A}) \quad [\hat{A}] \in \mathcal{F}\mathcal{M}(X)/\sim_r$$

*with  $\hat{A}$  being any representative of the equivalence class  $[\hat{A}]$ .*

Furthermore, the multiset complement (Definition 2.13) and the aggregate intersection and union (Definition 2.16) can be shown to be compatible with the equivalence relation  $\sim_r$  and with the bijection  $\phi_{MH}$  (see [12] for the proofs of these results).

### 3 Ordered fuzzy multisets

In the experts' model, fuzzy multisets can account for the repetitions in the values produced by the experts, but no distinction is made between, for example, the first expert producing the value 0.1 and the rest producing 0.2, or the second expert being the one who produced 0.1 with the others producing 0.2. This disregard for the order of the experts is a form of information loss, which may not be acceptable. In such situations, we may need to define a special type of multivalued fuzzy set where the membership values can appear multiple times and are labelled with a coordinate index. We can do this easily by replicating the notion of a fuzzy set over the various dimensions of the real unit hypercube  $[0, 1]^n$ .

**Definition 3.1.** *Let  $X$  be the universe and let  $n$  be a natural (non-zero) number. An  **$n$ -dimensional ordered fuzzy multiset**  $\vec{A}$  over  $X$  is a function  $\vec{A}: X \rightarrow [0, 1]^n$ .*

*Given an element  $x \in X$ ,  $\vec{A}(x)$  is called its **membership sequence** and the family of all the ordered fuzzy multisets over  $X$ ,  $\mathcal{F}^n(X)$ , is called the  **$n$ -dimensional ordered fuzzy power multiset** of  $X$ .*

The notion of ordered fuzzy multiset that we have defined in a formal way has already been used in fuzzy set theory under the name ‘‘vectorial fuzzy sets’’ (see [8], where it was used in image processing). Moreover, an ordered fuzzy multiset can be considered as a particular case of  $L$ -fuzzy set [6] with  $L$  being the  $[0, 1]^n$  unit hypercube. Here, however, we will focus on the study of its specific properties and its relationship to other extensions of classical fuzzy sets.

The restrictions of  $\vec{A}$  to the  $i$ -th coordinate in the image set,  $A_i$ , are by definition ordinary fuzzy sets. These  $n$  ordinary fuzzy sets  $A_1, \dots, A_n$  can be referred to as the **fuzzy coordinates** of  $\vec{A}$ .

The usual fuzzy set operations such as complement, union and intersection can be carried over to the  $[0, 1]^n$  space coordinatewise in a straightforward way, as we will see in the definitions that follow, where it is assumed that there is always a universe  $X$  and a natural number  $n$ .

**Definition 3.2.** *Let  $\vec{A}, \vec{B} \in \mathcal{F}^n(X)$  be two  $n$ -dimensional ordered fuzzy multisets over  $X$ . The **complement** of  $\vec{A}$ ,  $\vec{A}^c$ , and the **Cartesian intersection**,  $\vec{A} \cap \vec{B}$ , and **Cartesian union**,  $\vec{A} \cup \vec{B}$ , of  $\vec{A}$  and  $\vec{B}$  are the  $n$ -dimensional ordered fuzzy multisets over  $X$  given by the following relations:*

$$A_i^c(x) = 1 - A_i(x) \quad i = 1, \dots, n \quad x \in X \quad (10)$$

$$(A \cap B)_i(x) = \min\{A_i(x), B_i(x)\} \quad i = 1, \dots, n \quad x \in X \quad (11)$$

$$(A \cup B)_i(x) = \max\{A_i(x), B_i(x)\} \quad i = 1, \dots, n \quad x \in X \quad (12)$$

As these operations are defined in terms of the ones for the ordinary fuzzy sets, the properties of the latter are replicated in an obvious way. In particular, the Cartesian intersection and union are commutative and associative and the identity element is the ordered fuzzy multiset  $\vec{I}: X \rightarrow [0, 1]^n$ , defined as the unity membership  $\vec{I}(x) = (1, \dots, 1) \forall x \in X$ , for the intersection and the ordered fuzzy multiset  $\vec{0}: X \rightarrow [0, 1]^n$ , defined as the null membership  $\vec{0}(x) = (0, \dots, 0) \forall x \in X$ , for the union.

**Example 3.3.** *Let us suppose that we have a single-element universe  $X = \{x\}$ . A 3-dimensional ordered fuzzy multiset  $\vec{A}$  can be defined as  $\vec{A}(x) = (0.1, 0.3, 0.3)$ . Another 3-dimensional ordered fuzzy multiset  $\vec{B}$  would be  $\vec{B}(x) = (0.3, 0.1, 0.3)$ . Note that in this example the difference in the membership value for  $\vec{A}$  and  $\vec{B}$  is in the order of the coordinates, a distinction that could not be represented by multisets. The complement of  $\vec{A}$  would be given by  $\vec{A}^c(x) = (0.9, 0.7, 0.7)$  and the Cartesian intersection and union between  $\vec{A}$  and  $\vec{B}$  would be  $\vec{A} \cap \vec{B} = \{0.1, 0.1, 0.3\}$  and  $\vec{A} \cup \vec{B} = \{0.3, 0.3, 0.3\}$ .*

As the previous example shows, the membership values for the ordered fuzzy multisets are ordered  $n$ -tuples, so they carry more information content than the similar fuzzy multisets. Just like the fuzzy multisets extend the hesitant fuzzy sets with multiplicity information, we can think of the ordered fuzzy multisets as extending the fuzzy multisets with the order information.

We are now going to see how the fuzzy multisets can be identified with the ordered fuzzy multisets when the information about order is ignored. This is very similar to how the repetition information was handled when connecting the fuzzy multisets with the hesitant fuzzy sets. Following the same line of reasoning, we shall define an equivalence relation that lumps together the  $n$ -tuples that only differ in the order of their coordinates so that we can identify the equivalence classes for that relation, or their canonical representatives, with the fuzzy multisets that have a fixed cardinality  $n$  for every element in the universe.

Before tackling the problem for the fuzzy objects, we will consider the simpler relation between  $n$ -tuples and crisp multisets. In the definitions and results that follow, we assume that multisets and  $n$ -tuples are defined in a universe  $U$  with a natural number  $n$  being the dimension of the Cartesian space or the cardinality of the multisets.

First, we can relate an  $n$ -tuple to an underlying multiset by disregarding the order of the coordinates:

**Definition 3.4.** *Let  $a \in U^n$  be an  $n$ -tuple. The **multiset representation** of  $a$  is the multiset  $Mult(a)$  defined as:*

$$(Mult(a))(u) := |\{i \in [1, \dots, n] | a_i = u\}| \quad u \in U$$

This definition can be extended to the fuzzy multisets as follows.

**Definition 3.5.** *Let  $\vec{A} \in \mathcal{F}^n(X)$  be an  $n$ -dimensional ordered fuzzy multiset over  $X$ . The **fuzzy multiset representation** of  $\vec{A}$  is a fuzzy multiset  $Mult^f(\vec{A})$  defined as:*

$$Mult^f(\vec{A})(x) := Mult(\vec{A}(x)) \quad x \in X$$

So, the fuzzy multiset representation is simply built by taking the multiset representations of the membership sequence  $\vec{A}(x)$  at each element  $x$ . We can now define an equivalence relation that accounts for the permutation of coordinates in the ordered fuzzy multisets similar to the repetition-equivalence relation for fuzzy multisets that we introduced in Proposition 2.17.

**Definition 3.6.** *Let  $\vec{A}, \vec{B} \in \mathcal{F}^n(X)$  be two  $n$ -dimensional ordered fuzzy multisets over  $X$ .  $\vec{A}$  and  $\vec{B}$  are said to be **permutation-equivalent**,  $\vec{A} \sim_p \vec{B}$ , if their fuzzy multiset representations are the same:  $Mult^f(\vec{A}) = Mult^f(\vec{B})$ .*

It is trivial to prove that  $\sim_p$  is an equivalence relation.

In the quotient space  $\mathcal{F}^n(X)/\sim_p$ , we can designate a canonical representative by taking the representative such that for any  $x \in X$  the  $n$ -tuple has its coordinates sorted in ascending (or descending) order. Such a representative is obviously unique.

**Example 3.7.** *Let us suppose we have a single-element universe  $X = \{x\}$  and a 3-dimensional ordered fuzzy multiset  $\vec{A}$  defined over  $X$  as  $\vec{A}(x) = (0.1, 0.2, 0.2)$ . The multiset representation of this triplet is  $\langle 0.1, 0.2, 0.2 \rangle$ , so the fuzzy multiset representation is the fuzzy multiset defined by  $Mult^f(\vec{A})(x) = \langle 0.1, 0.2, 0.2 \rangle$ . Note that this fuzzy multiset is also the fuzzy multiset representation of, for example,  $\vec{B}(x) = (0.2, 0.1, 0.2)$  and  $\vec{C}(x) = (0.2, 0.2, 0.1)$ , so  $\vec{A} \sim_p \vec{B} \sim_p \vec{C}$ . The canonical representative of the equivalence class would be  $\vec{A}$ , as it is the one with the coordinates sorted in ascending order.*

The permutation equivalence relation makes it possible to define a bijection between its equivalence classes and the regular fuzzy multisets with a constant cardinality map  $m(x) = n$ . This intuitive idea can be stated as a proposition:

**Proposition 3.8.** *Let  $m: X \rightarrow \mathbb{N}$  be a constant cardinality map such that  $m(x) = n$  for all  $x \in X$ . There is a bijection  $\phi_{OM_n}: \mathcal{F}^n(X)/\sim_p \leftrightarrow \mathcal{F}\mathcal{M}_H^m(X)$  defined by:*

$$\phi_{OM_n}([\vec{A}]) = Mult^f(\vec{A}) \quad [\vec{A}] \in \mathcal{F}^n(X)/\sim_p$$

where  $\vec{A}$  is any representative of  $[\vec{A}]$ .

*Proof.* The function  $\phi_{OM_n}$  is defined for all equivalence classes  $[\vec{A}] \in \mathcal{F}^n(X)/\sim_p$  and it is trivial to prove that the function  $\phi_{OM_n}$  is injective. Given two ordered fuzzy multisets  $\vec{A}$  and  $\vec{B}$ , with equivalence classes  $[\vec{A}]$  and  $[\vec{B}]$  by  $\sim_p$ , their image by  $\phi_{OM_n}$  will be different if and only if  $Mult^f(\vec{A}) \neq Mult^f(\vec{B})$ , and then  $[\vec{A}] \neq [\vec{B}]$ .

It is also trivial to prove that  $\phi_{OM_n}$  is surjective. Given any fuzzy multiset  $\hat{A} \in \mathcal{F}\mathcal{M}_H^m(X)$ , we can build an ordered fuzzy multiset  $\vec{A}$  defined for each  $x \in X$  through an ordered sequence like  $(a_1(x), \dots, a_n(x))$ . The fuzzy multiset representation of  $\vec{A}$  is obviously  $\hat{A}$ , so  $\phi_{OM_n}([\vec{A}]) = \hat{A}$  for any fuzzy multiset  $\hat{A}$ .

Therefore,  $\phi_{OM_n}$  is a bijection.  $\square$

As in the previous cases we have discussed, the existence of this bijection  $\phi_{OM_n}$  backs the intuitive identification between the fuzzy multisets of fixed cardinality and the restricted ordered fuzzy multisets where order is ignored by considering the equivalence classes. Alternatively, we can ignore the order by working with the subset of  $\mathcal{F}^n(X)$  made up of the canonical representatives which, as we already mentioned, Bedregal et al. have studied under the name “ $n$ -dimensional fuzzy sets” [1].

But Proposition 3.8 is based on a fixed dimension  $n$  for the ordered fuzzy multisets in  $\mathcal{F}^n(X)$  and the same fixed cardinality for the fuzzy multisets in  $\mathcal{F}\mathcal{M}_H^m(X)$ . Since our final goal is to establish a connection between the ordered fuzzy multisets and the whole set of fuzzy multisets  $\mathcal{F}\mathcal{M}(X)$ , we will need to take into account the possibility of a variable number  $n$  of dimensions or multiset cardinalities and even allow for a peculiar hybrid version of ordered fuzzy multisets where different elements of the universe can be mapped to  $[0, 1]^n$  hypercubes of different dimensions. In fact, this generalisation will be essential in order to connect the ordered fuzzy multiset results with the bijection  $\phi_{MH}$  of Proposition 2.17 between the quotient space of the fuzzy multisets by the repetition equivalence relation and the hesitant fuzzy sets, since the aggregate intersection and union, like the hesitant intersection and union, often yield a result with a larger cardinality than the original sets. This leads to the need for an additional definition of a more general form of ordered fuzzy multiset, which may not be useful by itself, but which will bridge the gap between the ordered tuples and the multisets of any cardinality when used as fuzzy membership values.

First, we need to define the collection of all the  $[0, 1]^n$  hypercubes as a distinct mathematical space.

**Definition 3.9.** *The collection of all the real unit hypercubes is the union of all the  $[0, 1]^n$  spaces, which we will denote by  $[0, 1]^{1, \dots, \infty}$ .*

$$[0, 1]^{1, \dots, \infty} := \bigcup_{n \in \mathbb{N}} [0, 1]^n$$

**Definition 3.10.** *A variable-dimension ordered fuzzy multiset  $\vec{A}$  over  $X$  is a function  $\vec{A}: X \rightarrow [0, 1]^{1, \dots, \infty}$ .*

*The family of all the variable-dimension ordered fuzzy multisets over  $X$ ,  $\mathcal{F}^{1, \dots, \infty}(X)$ , is called the variable-dimension ordered fuzzy power multiset of  $X$ .*

With this definition, all the previous definitions based on a fixed dimension  $n$  can be adapted to a variable dimension trivially (with a regularisation step for the Cartesian intersection and union whenever there is a dimension mismatch). In particular, the multiset representation  $Mult^f$  (Definition 3.5) can be extended to  $\mathcal{F}^{1, \dots, \infty}(X)$  in a trivial way, and so can the permutation equivalence  $\sim_p$ . With this generalised  $\sim_p$  relation, we can take the quotient space  $\mathcal{F}^{1, \dots, \infty}(X)/\sim_p$  and restate Proposition 3.8 in a cardinality-independent way:

**Proposition 3.11.** *There is a bijection  $\phi_{OM}: \mathcal{F}^{1, \dots, \infty}(X)/\sim_p \leftrightarrow \mathcal{F}\mathcal{M}(X)$  defined by:*

$$\phi_{OM}([\vec{A}]) = Mult^f(\vec{A}) \quad [\vec{A}] \in \mathcal{F}^{1, \dots, \infty}(X)/\sim_p$$

where  $\vec{A}$  is any representative of  $[\vec{A}]$ .

*Proof.* This can be proved in the same way as Proposition 3.8, but allowing for different dimensions and cardinalities.  $\square$

This bijection  $\phi_{\text{OM}}$  allows us to identify the ordered fuzzy multisets, when order is ignored in the variable-dimension generalisation, with the fuzzy multisets. However, just as we have done for the  $\phi_{\text{MH}}$  bijection (see [12]), here we also have to prove that the common operations of complement, intersection and union commute with the new bijection. And as a first step, we need to prove that the three operations commute with the permutation equivalence relation.

This is quite straightforward for the complement:

**Proposition 3.12.** *For any two variable-dimensional ordered fuzzy multisets over  $X$   $\vec{A}, \vec{B} \in \mathcal{F}^{1, \dots, \infty}(X)$ , if  $\vec{A} \sim_p \vec{B}$  then  $\vec{A}^c \sim_p \vec{B}^c$ .*

*Proof.* If  $\vec{A} \sim_p \vec{B}$ , this means that their fuzzy multiset representations are the same, so for each  $x \in X$ , if  $\vec{A}(x)$  is the  $n$ -tuple  $(a_1, \dots, a_n)$ , then  $\vec{B}(x)$  must be an  $n$ -tuple  $(a_{\sigma(1)}, \dots, a_{\sigma(n)})$ , where  $\sigma$  is a permutation of the coordinate indices. Then their complements will be  $\vec{A}^c(x) = (1 - a_1, \dots, 1 - a_n)$  and  $\vec{B}^c(x) = (1 - a_{\sigma(1)}, \dots, 1 - a_{\sigma(n)})$ . And the multiset representations of these two  $n$ -tuples is the same multiset  $\langle 1 - a_1, \dots, 1 - a_n \rangle$ , so  $\text{Mult}(\vec{A}(x)) = \text{Mult}(\vec{B}(x))$  for all  $x \in X$ . Therefore, by definition of the fuzzy multiset representation (Definition 3.5),  $\text{Mult}^f(\vec{A}) = \text{Mult}^f(\vec{B})$ , which is the same as  $\vec{A}^c \sim_p \vec{B}^c$ .  $\square$

This means that the equivalence classes are preserved by the complement operation and so the definition of complement can be brought forward to the quotient space  $\mathcal{F}^{1, \dots, \infty}(X) / \sim_p$  by simply defining it in terms of any representative, as follows.

**Definition 3.13.** *Let  $[\vec{A}] \in \mathcal{F}^{1, \dots, \infty}(X) / \sim_p$  be a permutation equivalence class. The **complement** of  $[\vec{A}]$  is the permutation equivalence class  $[\vec{A}]^c$  defined by:*

$$[\vec{A}]^c := [\vec{A}^c]$$

*with  $\vec{A}^c$  being the complement, in the ordered fuzzy multiset sense, of any representative  $\vec{A}$  of the equivalence class  $[\vec{A}]$ .*

Unfortunately, the compatibility of the intersection and union operations for the ordered fuzzy multisets with the permutation equivalence relation is more complicated. The situation resembles the way Miyamoto's operations are not compatible with the repetition equivalence relation, which led us to resort to the aggregate versions in order to get the desired compatibility. Now it is the Cartesian intersection and union which are not compatible with the permutation equivalence relation.

**Example 3.14.** *As a very simple counterexample of this incompatibility, consider a single-element universe  $X = \{x\}$  and the three 2-dimensional ordered fuzzy multisets  $\vec{A}_1(x) = (0.1, 0.3)$ ,  $\vec{A}_2(x) = (0.3, 0.1)$  and  $\vec{B}(x) = (0.1, 0.2)$ . The Cartesian intersection of  $\vec{A}_1$  with  $\vec{B}$  is  $\vec{A}_1 \cap \vec{B}(x) = (0.1, 0.2)$  whereas the intersection of  $\vec{A}_2$  with  $\vec{B}$  is  $\vec{A}_2 \cap \vec{B}(x) = (0.1, 0.1)$ , so we have that  $\vec{A}_1 \sim_p \vec{A}_2$  but  $\vec{A}_1 \cap \vec{B} \not\sim_p \vec{A}_2 \cap \vec{B}$ . It is easy to check that a similar discrepancy occurs with the Cartesian union.*

So we will need to come up with an alternative definition of intersection and union that is compatible with the permutation equivalence relation. A naïve possibility would consist in defining a “sorted” version of the operations where the fuzzy coordinates of  $\vec{A}$  and  $\vec{B}$  are sorted in a first step. Such a sorting step would be tantamount to a mapping from the full  $[0, 1]^n$  hypercubes to the subset where the coordinates are in ascending (or descending) order. With such an approach we would be able to make the intersection and union commute under bijection  $\phi_{\text{OM}}$  with Miyamoto's intersection and union. But we want the ordered fuzzy multiset binary operations to commute with the aggregate intersection and union for the fuzzy multisets so that the bijection can also extend to the fuzzy sets through  $\phi_{\text{MH}}$ . In anticipation of this requirement, we are going to define an aggregate version of the intersection and union for the ordered fuzzy multisets that takes into account all the possible permutations, as follows.

**Definition 3.15.** *Let  $\vec{A}, \vec{B} \in \mathcal{F}^{1, \dots, \infty}(X)$  be two variable-dimensional ordered fuzzy multisets over  $X$  and let  $s_C$  be a sorting strategy. The **ordered aggregate intersection** of  $\vec{A}$  and  $\vec{B}$  and the **ordered aggregate union** of  $\vec{A}$  and  $\vec{B}$  are the variable-dimensional ordered fuzzy multisets  $\vec{A} \cap^{\text{oa}} \vec{B}$  and  $\vec{A} \cup^{\text{oa}} \vec{B}$  defined by:*

$$\vec{A} \cap^{\text{oa}} \vec{B} := s_C \left( \bigcup_{\substack{s_A \in \text{OS}(\vec{A}) \\ s_B \in \text{OS}(\vec{B})}} (s_A(\text{Mult}(\vec{A})) \cap s_B(\text{Mult}(\vec{B}))) \right) \quad (13)$$

$$\vec{A} \cup^{\text{oa}} \vec{B} := s_C \left( \bigcup_{\substack{s_A \in \text{OS}(\vec{A}) \\ s_B \in \text{OS}(\vec{B})}} (s_A(\text{Mult}(\vec{A})) \cup s_B(\text{Mult}(\vec{B}))) \right) \quad (14)$$

where the big  $\cup$  represents the union of fuzzy multisets, the  $\cap$  and  $\cup$  operations on the right-hand side are the Cartesian ones, and  $\vec{A}$  and  $\vec{B}$  have been regularised if necessary. Note that the word “aggregate” is here used to mean that the new operations are based on a multiset union of distinct possibilities under more basic types of operation, as we did in Definition 2.16, and not in a sense directly related to the aggregation operations commonly used with fuzzy membership values [3].

The multiset unions over all the permutations correspond to the aggregate operations for multisets, and so the previous relations can be restated more succinctly as:

$$\vec{A} \cap^{oa} \vec{B} = s_C(\text{Mult}(\vec{A}) \cap^a \text{Mult}(\vec{B})) \quad (15)$$

$$\vec{A} \cup^{oa} \vec{B} = s_C(\text{Mult}(\vec{A}) \cup^a \text{Mult}(\vec{B})) \quad (16)$$

These definitions are cumbersome in several respects. First, they are parameterised by an arbitrary ordering strategy  $s_C$ . We could of course designate a canonical one by choosing, for example, the ascending sort  $s_C = s^\uparrow$ , but the problem with any approach based on sorting is that some of the most fundamental properties of the intersection and union that we take for granted vanish, like symmetry or the existence of an identity element. For example, intersecting  $(0.2, 0.1)$  and  $(1, 1)$  with the ascending sort gives  $(0.1, 0.2)$ , so a re-arrangement comes about as a side-effect of an operation that should have no effect. We could get round these problems by choosing  $s_C$  based on the operands in each case. This is an area that may be worth further research but, in any case, these problems reveal that essential difficulties arise when trying to link the theories of ordered fuzzy multisets and fuzzy multisets as a manifestation of information loss while keeping operations compatible.

Despite these shortcomings, we can now prove that the ordered aggregate intersection and union we have just defined are compatible with the permutation equivalence relation. That will allow us to use these operations in order to define the intersection and union in the quotient space  $\mathcal{F}^{1, \dots, \infty}(X) / \sim_p$ .

**Proposition 3.16.** *The ordered aggregate intersection  $\cap^{oa}$ , as defined in Definition 3.15, is compatible with the permutation equivalence relation; i.e. given two pairs of variable-dimension ordered fuzzy multisets over  $X$   $\vec{A}_1, \vec{B}_1, \vec{A}_2, \vec{B}_2 \in \mathcal{F}^{1, \dots, \infty}(X)$ , if  $\vec{A}_1 \sim_p \vec{A}_2$  and  $\vec{B}_1 \sim_p \vec{B}_2$ , then  $\vec{A}_1 \cap^{oa} \vec{B}_1 \sim_p \vec{A}_2 \cap^{oa} \vec{B}_2$ .*

*Proof.*  $\vec{A}_1 \sim_p \vec{A}_2$  means that  $\text{Mult}^f(\vec{A}_1) = \text{Mult}^f(\vec{A}_2)$  and, similarly,  $\text{Mult}^f(\vec{B}_1) = \text{Mult}^f(\vec{B}_2)$ . Consequently,  $(s_A(\text{Mult}^f(\vec{A}_1)) \cap s_B(\text{Mult}^f(\vec{B}_1))) = (s_A(\text{Mult}^f(\vec{A}_2)) \cap s_B(\text{Mult}^f(\vec{B}_2)))$  and using the defining relation (13),  $\vec{A}_1 \cap_a \vec{B}_1 = \vec{A}_2 \cap_a \vec{B}_2$  if the same ordering strategy  $s_C$  is used in both cases and  $\vec{A}_1 \cap_a \vec{B}_1 \sim_p \vec{A}_2 \cap_a \vec{B}_2$  in any case.  $\square$

And the similar result (with the same type of proof, which we omit) for the ordered aggregate union:

**Proposition 3.17.** *The ordered aggregate union  $\cup^{oa}$ , as defined in Definition 3.15, is compatible with the permutation equivalence relation; i.e. given two pairs of variable-dimension ordered fuzzy multisets over  $X$   $\vec{A}_1, \vec{B}_1, \vec{A}_2, \vec{B}_2 \in \mathcal{F}^{1, \dots, \infty}(X)$ , if  $\vec{A}_1 \sim_p \vec{A}_2$  and  $\vec{B}_1 \sim_p \vec{B}_2$ , then  $\vec{A}_1 \cup^{oa} \vec{B}_1 \sim_p \vec{A}_2 \cup^{oa} \vec{B}_2$ .*

In a completely analogous way to our approach with the repetition equivalence relation when we established the connection between the hesitant fuzzy sets and the fuzzy multisets, the fact that the permutation equivalence classes are preserved by the ordered aggregate intersection and union means that these operations can be adapted to the quotient space  $\mathcal{F}^{1, \dots, \infty}(X) / \sim_p$  by simply defining them in terms of any representative, as follows.

**Definition 3.18.** *Let  $[\vec{A}], [\vec{B}] \in \mathcal{F}^{1, \dots, \infty}(X) / \sim_p$  be two permutation equivalence classes. The **intersection** and the **union** of  $[\vec{A}]$  and  $[\vec{B}]$  are the permutation equivalence classes  $[\vec{A}] \cap [\vec{B}]$  and  $[\vec{A}] \cup [\vec{B}]$  defined by:*

$$[\vec{A}] \cap [\vec{B}] := [\vec{A} \cap^{oa} \vec{B}] \quad (17)$$

$$[\vec{A}] \cup [\vec{B}] := [\vec{A} \cup^{oa} \vec{B}] \quad (18)$$

with  $\vec{A} \cap^{oa} \vec{B}$  and  $\vec{A} \cup^{oa} \vec{B}$  being the ordered aggregate intersection and union, respectively, of any two representatives  $\vec{A}$  and  $\vec{B}$  of the equivalence classes  $[\vec{A}]$  and  $[\vec{B}]$ .

Finally, we are now in a position to prove that these operations of complement, intersection and union commute with the bijection  $\phi_{OM}$ .

**Proposition 3.19.** *The complement on  $\mathcal{F}\mathcal{M}(X)$  and on  $\mathcal{F}^{1, \dots, \infty}(X) / \sim_p$  are related through the bijection  $\phi_{OM}$ :*

$$\phi_{OM}([\vec{A}]^c) = (\phi_{OM}([\vec{A}]))^c \quad [\vec{A}] \in \mathcal{F}^{1, \dots, \infty}(X) / \sim_p$$

*Proof.* This is an equality between fuzzy multisets. First, by Definition 3.13,  $[\vec{A}]^c = [\vec{A}^c]$  and so the left-hand side can be rearranged as  $\phi_{\text{OM}}([\vec{A}^c])$ , and, by the definition of  $\phi_{\text{OM}}$  (Proposition 3.11), this is the fuzzy multiset  $Mult^f(\vec{A}^c)$ , where  $\vec{A}^c$  is the canonical (or any other) representative of the equivalence class  $[\vec{A}^c]$  and also the complement of  $\vec{A}$ , a representative of the equivalence class  $[\vec{A}]$ . By definition of fuzzy multiset representation, for any  $x \in X$ ,  $Mult^f(\vec{A}^c)(x) = Mult(\vec{A}^c(x))$ , which is the crisp multiset defined as  $(Mult(\vec{A}^c(x)))(t) = |\{i \in 1, \dots, n | (A^c)_i(x) = t\}|$ . And by the definition of fuzzy set complement, this is the set  $|\{i \in 1, \dots, n | (A)_i(x) = 1 - t\}|$ .

On the other hand, in the right-hand side we have to take the complement of the fuzzy multiset  $\phi_{\text{OM}}([\vec{A}])$ , which is  $Mult^f(\vec{A})$  by the definition of  $\phi_{\text{OM}}$ , where  $\vec{A}$  is any representative of the equivalence class  $[\vec{A}]$ . And by definition of fuzzy multiset representation,  $Mult^f(\vec{A})$  is the fuzzy multiset defined as  $(Mult(\vec{A})(x))(t) = |\{i \in 1, \dots, n | A_i(x) = t\}|$  for all  $x \in X$ . And applying the formula (7) for the fuzzy multiset complement, we have that the complement of this fuzzy multiset will be defined as  $(Mult(\vec{A})^c(x))(t) = (Mult(\vec{A})(x))(1 - t) = |\{i \in 1, \dots, n | A_i(x) = 1 - t\}|$ , which is the same expression that we obtained for the left-hand side of the equality, thus proving the result.  $\square$

Having established the result that the bijection  $\phi_{\text{OM}}$  commutes with the complement, as defined for the fuzzy multisets and for the permutation equivalence classes of the ordered fuzzy multisets, we will now show that it also commutes with the (ordered aggregate) intersection and union.

**Proposition 3.20.** *The aggregate intersection on  $\mathcal{FM}(X)$  and the ordered aggregate intersection on  $\mathcal{F}^{1, \dots, \infty}(X) / \sim_p$  are related through the bijection  $\phi_{\text{OM}}$  as follows:*

$$\phi_{\text{OM}}([\vec{A}] \cap [\vec{B}]) = (\phi_{\text{OM}}([\vec{A}])) \cap^{\text{oa}} (\phi_{\text{OM}}([\vec{B}])) \quad [\vec{A}], [\vec{B}] \in \mathcal{F}^{1, \dots, \infty}(X) / \sim_p.$$

*Proof.* First, by Definition 3.18,  $[\vec{A}] \cap [\vec{B}] = [\vec{A} \cap^{\text{oa}} \vec{B}]$ , with  $\vec{A}$  and  $\vec{B}$  being two representatives of the equivalence classes  $[\vec{A}]$  and  $[\vec{B}]$ , respectively, so the left-hand side of the equality is  $\phi_{\text{OM}}([\vec{A} \cap^{\text{oa}} \vec{B}])$ . And by the definition of  $\phi_{\text{OM}}$  (Proposition 3.11), this is the fuzzy multiset  $Mult^f(\vec{A} \cap^{\text{oa}} \vec{B})$ . And, decomposing the Definition 13 in coordinates and evaluating it at each  $x \in X$ , this fuzzy multiset becomes the crisp multiset  $\bigcup_{\sigma_A, \sigma_B} \{\min\{a_{\sigma_A(i)}, b_{\sigma_B(i)} | i = 1, \dots, n(x)\}\}$  where  $\sigma_A$  and  $\sigma_B$  are all the possible permutations of the  $n(x)$  coordinate indices for  $\vec{A}$  and  $\vec{B}$ .

On the other hand, in the right-hand side we have to consider the two fuzzy multisets  $(\phi_{\text{OM}}([\vec{A}]))$  and  $(\phi_{\text{OM}}([\vec{B}]))$ , which are  $Mult^f(\vec{A})$  and  $Mult^f(\vec{B})$ , respectively, by the definition of  $\phi_{\text{OM}}$ . And if we take the ordered aggregate intersection of these two sets, based on equation (13), and evaluate it at  $x \in X$ , we get the crisp multiset  $\bigcup_{\sigma_A, \sigma_B} \{\min\{a_{\sigma_A(i)}, b_{\sigma_B(i)} | i = 1, \dots, n(x)\}\}$ , which is the same expression that we found for the left-hand side of the equality, thus proving the result.  $\square$

Similarly, we can prove an analogous result for the union:

**Proposition 3.21.** *The ordered aggregate union on  $\mathcal{FM}(X)$  and the union on  $\mathcal{F}^{1, \dots, \infty}(X) / \sim_p$  are related through the bijection  $\phi_{\text{OM}}$  as follows:*

$$\phi_{\text{OM}}([\vec{A}] \cup [\vec{B}]) = (\phi_{\text{OM}}([\vec{A}])) \cup^{\text{oa}} (\phi_{\text{OM}}([\vec{B}])) \quad [\vec{A}], [\vec{B}] \in \mathcal{F}^{1, \dots, \infty}(X) / \sim_p.$$

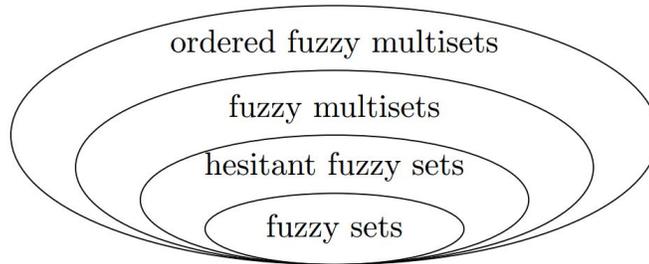
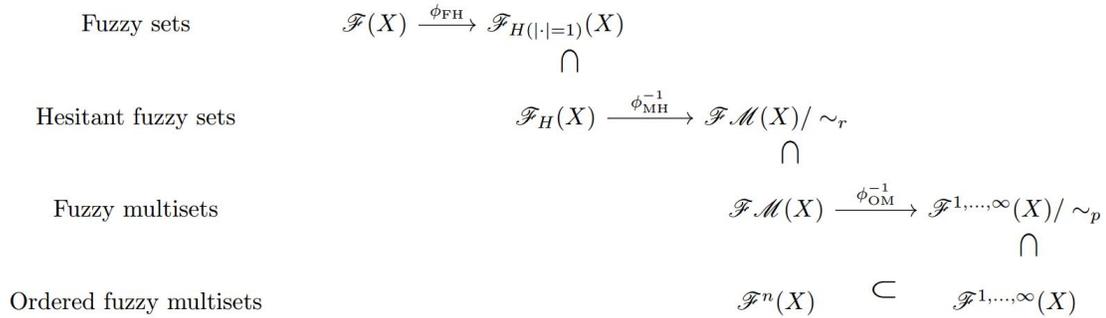
**Example 3.22.** *If we have a single-element universe  $X = \{x\}$  and two 3-dimensional ordered fuzzy multisets  $\vec{A}$  and  $\vec{B}$  defined by  $\vec{A}(x) = (0.1, 0.2, 0.3)$  and  $\vec{B} = (0.2, 0.3, 0.4)$ , their Cartesian intersection and union would be  $\vec{A} \cap \vec{B} = (0.1, 0.2, 0.3)$  and  $\vec{A} \cup \vec{B} = (0.2, 0.3, 0.4)$ , respectively. For the ordered aggregate operations, we would have to consider the distinct possible permutations, as in the multiset aggregate operations of (15) and (16). Using the ascending sort as the  $s_C$  ordering strategy, we would get:*

$$\begin{aligned} \vec{A} \cap^{\text{oa}} \vec{B} &= s_{\uparrow}(\langle 0.1, 0.2, 0.3 \rangle \cap^a \langle 0.2, 0.3, 0.4 \rangle) = s_{\uparrow}(\langle 0.1, 0.2, 0.2, 0.3 \rangle) = (0.1, 0.2, 0.2, 0.3) \\ \vec{A} \cup^{\text{oa}} \vec{B} &= s_{\uparrow}(\langle 0.1, 0.2, 0.3 \rangle \cup^a \langle 0.2, 0.3, 0.4 \rangle) = s_{\uparrow}(\langle 0.2, 0.3, 0.3, 0.4 \rangle) = (0.2, 0.3, 0.3, 0.4) \end{aligned}$$

*We see that the ordered aggregate operations are simply the aggregate operations for multisets with the result turned into an  $n$ -tuple through the specification of an ordering strategy.*

## 4 Conclusion

In this paper, we have stated the close relationship that exists between three types of multivalued fuzzy sets: the hesitant fuzzy sets, the fuzzy multisets and the ordered fuzzy multisets, continuing the work in a previous article [12]. We have identified some criteria about the actual situations that each type can model and introduced some new working definitions for the ordered ordinary fuzzy multisets. We have compared the definitions of the common operations of complement, intersection and union, and found some viable definitions that make it possible to identify the smaller sets with subsets or equivalence classes in the larger sets. The relations among these three types of fuzzy sets together with the ordinary fuzzy sets can be summed up in the following diagrams.



We have found that the use of ordered  $n$ -tuples as membership values gives rise to a form of multivalued fuzzy sets where the different membership criteria resulting in several values are associated with different coordinate axes. This makes the binary operations such as intersection and union dependent on the order of the coordinates and thus leads to a very different theory from both the hesitant fuzzy sets and the fuzzy multisets. Identifying these ordered fuzzy multisets with the fuzzy multisets with either Miyamoto's or Torra's operations is complicated and requires a somewhat cumbersome extension of the space to variable dimensions and the appropriate fine-tuning of the definitions of intersection and union that account for the loss of the information about order and repetition.

In future, we expect to investigate other relations such as subsethood and similitude measures in these types of multivalued fuzzy sets. Other research areas can include the difference in information content among the three types and finding new relations that link the three concepts.

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