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A modification of probabilistic hesitant fuzzy sets and its application to multiple criteria decision making

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Abstract

Probabilistic hesitant fuzzy set (PHFS) is a fruitful concept that adds to hesitant fuzzy set (HFS) the term of probability which is able to retain more information than the usual HFS. Here, we demonstrate that the existing definitions of PHFS are not still reasonable, and therefore, we first improve the PHFS definition. By endowing the set and algebraic operations with a new re-definition of PHFS, we propose a class of T-norm-based and S-norm-based operations for PHFSs together with a number of aggregation operators. Eventually, on the basis of the new operators, the effectiveness and practicality of re-defined PHFS will be tested using three multiple criteria decision making (MCDM) problems concerning the automotive industry safety evaluation, the evaluation of Chines hospitals and the evaluation of alternatives in an investment company.

Keywords: Hesitant fuzzy set, probabilistic hesitant fuzzy set, aggregation operation, multiple criteria decision making.

1 Introduction

Hesitant fuzzy set (HFS) is a more successful concept in real-life applications, and it has been widely used in many decision making problems [10, 28]. In recent years, the concept of HFS has been developed to the other kinds of concepts [7, 8, 22, 23] whose most widely used extensions are interval-valued hesitant fuzzy set [2], higher order hesitant fuzzy set [7], dual hesitant fuzzy set [32], hesitant fuzzy linguistic term set [9], etc.

Although, a HFS [25] is able to consider possible membership degrees of an element into a set, but those possible values being provided by experts may not have the same weight. To more explanation, we assume that a decision maker should provide the evaluation of alternatives with respect to some criteria in a multiple criteria decision making (MCDM) problem [3, 5, 11, 17, 18, 21, 26] in which the opinion of four experts are returned. In this regard, the decision maker may return the opinion of four experts in the form of a hesitant fuzzy element (HFE) which is denoted by $\{0.4, 0.5\}$. Here, the resulted HFE only returns the opinion of experts anonymously, and this is while, the original information has not been captured. In fact, by the help of the current description of the HFE $\{0.4, 0.5\}$, we are not able to judge reasonably in situations where (i) one expert assigns 0.4 in order to study the preferences for a scheme, and the other three experts assign 0.5; (ii) two experts assign 0.4 and the other preferences are 0.5; (iii) three experts study the preferences by the help of value 0.4 and the remaining expert assigns 0.5. In such a case, the decision maker can not correctly derive that decision being in accordance with the opinions of the four experts. Therefore, in order to represent sufficiently the original information in modelling the real and exact evaluations, the concept of HFS has been extended to that which is known as probabilistic hesitant fuzzy set (PHFS) [31]. Using the concept of PHFS, the above-mentioned situations can be described respectively by the probabilistic hesitant fuzzy elements (PHFEs) in the forms of (i): ${}^{\wp}h_1 = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1} \{\langle \hbar_1, \wp_1 \rangle = \langle 0.4, 0.25 \rangle, \langle 0.5, 0.75 \rangle\}$, (ii): ${}^{\wp}h_2 = \bigcup_{\langle \hbar_2, \wp_2 \rangle \in {}^{\wp}h_2} \{\langle \hbar_2, \wp_2 \rangle = \langle 0.4, 0.50 \rangle, \langle 0.5, 0.50 \rangle\}$ and (iii): ${}^{\wp}h_3 = \bigcup_{\langle \hbar_3, \wp_3 \rangle \in {}^{\wp}h_3} \{\langle \hbar_3, \wp_3 \rangle = \langle 0.4, 0.75 \rangle, \langle 0.5, 0.25 \rangle\}$.

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Here, what should be pointed out is that the concept of PHFS is different from the concepts of hesitant L-fuzzy set [4] and hesitant fuzzy multiset [14]. The former concept extends the HFS concept without simulating the probability of occurrence of each element, and the latter concept roles play as an extension form of HFS whose elements are in the form of fixed-length vectors without concerning on the probability of occurrence.

Up to now, a number of researchers have tried to define and improve the concept of PHFS. Zhu [31] applied the concept of probability to the definition of HFS and tried to overcome the defect of HFS by the use of PHFS. Zhang et al. [30] by considering this fact that the sum of all probabilities involved in a PHFE needs not to be always 1, attempted to further developed the operations and integrations of the PHFSs. Zhang and Wu [29] then improved and further perfected the PHFS theory, and also presented a number of operations of PHFSs by emphasising on the application of PHFSs in MCDM. Li and Wang [16] firstly introduced the concept of Hausdorff distance measure between two PHFEs, and then extended a QUALIFLEX[1] technique regarding the selection of green suppliers by the use of probability hesitant fuzzy information. Ding et al. [6] proposed a number of PHFE distance measures to apply them in an interactive method to deal with a probabilistic hesitant fuzzy multiple criteria group decision making involved incomplete weight information. Jiang and Ma [13] presented some operations on the PHFEs by taking only Frank T-norm and S-norm into account, and then investigated a technique to multiple criteria group decision making based on the proposed operators.
Here, it should be mentioned that not only we consider Frank T-norm and S-norm of PHFEs, but also we extend the results to the other kinds of T-norms and S-norms.

By the way, the above-mentioned contributions on PHFSs have not been succeed in formulating how a binary and multiple-component operation should be defined, specifically, the operations of addition (or union) and multiplication (or intersection) on the PHFSs. We will demonstrate that not only the probability of *union of independent events* should be considered in defining the probability part of a PHFS, but also the probability of *intersection of independence events* has to be considered in the definition of binary operations for PHFSs. This is the main superiority of the re-visited definition of operations in comparison with the existing ones including Farhadinia and Xu's [12], Zhang et al.'s [30] and Pang et al.'s [20] definitions.

Finally, in the section on experimental results, we concentrate on the use of VIKOR[2] technique in MCDM because of some advantages compared to the other algorithms. Some of the superiority of VIKOR technique over the others are:
(i) The comparison of VIKOR technique with TOPSIS[3] approach [19] shows that the first one is a relative distance-based method while the second one is a rank index-based one, and this is while, both of them are compromise ranking approaches. Moreover, the attitude of decision maker is considered as a key task in forming the VIKOR technique, meanwhile, such a task is a missing item in the TOPSIS approach.
(ii) The ranking orders that come from PROMETHEE[4] and ELECTRE[5] [19] are those determined respectively by the parameters S and R in the VIKOR technique.
Only these reasons are enough to show that the VIKOR technique performs better than the latter mentioned techniques.

The organization of this paper is as follows: we firstly present some preliminaries and needed background on HFSs in Section 2. Then in Section 3, we improve the exiting definitions of PHFS, and propose a number of aggregation operations on PHFSs. Section 4 is devoted to the comparison of the existing concepts with the modified PHFS by re-considering three discussed MCDM problems. Section 5 concludes this contribution, and it outlines some possible future research directions.

2 Preliminaries

In the following section, we describe briefly the basic definitions and operations of hesitant fuzzy set (HFS) [25] and its generalization which is referred here to as the probabilistic hesitant fuzzy set (PHFS) [31].

Definition 2.1. [25, 28]s *Let X be the universe of discourse. A hesitant fuzzy set (HFS) H on X is defined in terms of a function $h(x)$ when applied to X, it returns a finite subset of* $[0,1]$*, i.e.,* $H=\{\langle x,h(x)\rangle : x\in X\}$*, where $h(x)$, referred to as the hesitant fuzzy element (HFE), is a set of some values in* $[0,1]$ *denoting the possible membership degree of the element $x\in X$ to the set H.*

In view of the above definition, the HFS H can be denoted by, $H=\{\langle x,\bigcup_{\hbar\in h(x)}\{\hbar\}\rangle : x\in X\}$.

[1]QUALItative FLEXible multiple
[2]VIsekriterijumska optimizacijai KOmpromisno Resenje
[3]Technique for Order Preference by Similarity to Ideal Solution
[4]Preference Ranking Organization METhod for Enrichment Evaluation
[5]ELimination Et Choix Traduisant la REalite

Suppose that the three HFEs $h=\bigcup_{\hbar\in h}\{\hbar\}$, $h_1=\bigcup_{\hbar_1\in h_1}\{\hbar_1\}$ and $h_2=\bigcup_{\hbar_2\in h_2}\{\hbar_2\}$ are to be considered. Then, a number of operations on the HFEs which are also HFEs are described as the following: (see e.g. [25, 28])

$$h^c=\bigcup_{\hbar\in h}\{1-\hbar\}; \tag{1}$$

$$h_1\cup h_2=\bigcup_{\hbar_1\in h_1,\hbar_2\in h_2}\max\{\hbar_1,\hbar_2\}; \qquad h_1\cap h_2=\bigcup_{\hbar_1\in h_1,\hbar_2\in h_2}\min\{\hbar_1,\hbar_2\}; \tag{2}$$

$$h^\lambda=\bigcup_{\hbar\in h}\{\hbar^\lambda\}; \qquad \lambda h=\bigcup_{\hbar\in h}\{1-(1-\hbar)^\lambda\}, \quad \lambda>0; \tag{3}$$

$$h_1\oplus h_2=\bigcup_{\hbar_1\in h_1,\hbar_2\in h_2}\{\hbar_1+\hbar_2-\hbar_1\hbar_2\}; \qquad h_1\otimes h_2=\bigcup_{\hbar_1\in h_1,\hbar_2\in h_2}\{\hbar_1\hbar_2\}. \tag{4}$$

It is interesting to note that the definitions (1)-(4) are not only possible expressions of HFE operations. The great verity of expressions that describe the complement, union and intersection operations are respectively referred to as the negations, S-norms (or T-conorms) and T-norms [15]. Furthermore, by employing Characterization Theorem [15], any T-norm (or S-norm) is able to produce the following Archimedean T-norm (or S-norm) using the non-unique *additive generator* F $\top(x,y)=F^{-1}(F(x)+F(y))$. Similarly, the following Archimedean S-norm can be produced by the use of non-unique additive generator G $\perp(x,y)=G^{-1}(G(x)+G(y))$. In the above formulas, $F:[0,1]\to[0,\infty)$ stands for a strictly decreasing function with the properties $F(1)=0$, and $G(x)=F(1-x)$.

Needless to say that the above-mentioned Archimedean T-norm might be converted to a non-unique *multiplicative generator* $\hat{F}$ by the rule of $\top(x,y)=\hat{F}^{-1}(\hat{F}(x)\hat{F}(y))$. Here, $\hat{F}(x)=exp(-F(x))$ is satisfied for all $x\in[0,1]$.

Now, we are in a position to recall a number of T-norms (or S-norms) which are called Algebraic, Einstein, Hamacher, and Frank T-norms (or S-norms). The latter T-norms (or S-norms) are produced by the help of different additive generators F and G (see [1]) as the following:

- If $F_1(x)=-log(x)$ and $G_1(x)=-log(1-x)$, then $F_1^{-1}(x)=e^{-x}$ and $G_1^{-1}(x)=1-e^{-x}$, and moreover, Algebraic T-norm and Algebraic S-norm are resulted in

$$\top_1(x,y)=xy, \qquad \perp_1(x,y)=x+y-xy; \tag{5}$$

- If we consider $F_2(x)=log(\frac{2-x}{x})$ and $G_2(x)=log(\frac{2-(1-x)}{1-x})$ together with $F_2^{-1}(x)=\frac{2}{e^x+1}$, $G_2^{-1}(x)=1-\frac{2}{e^x+1}$, then we can conclude that Einstein T-norm and Einstein S-norm are in the forms of

$$\top_2(x,y)=\frac{xy}{1+(1-x)(1-y)}, \qquad \perp_2(x,y)=\frac{x+y}{1+xy}; \tag{6}$$

- By $F_3(x)=log(\frac{\epsilon+(1-\epsilon)x}{x})$, $\epsilon>0$ and $G_3(x)=log(\frac{\epsilon+(1-\epsilon)(1-x)}{1-x})$ together with $F_3^{-1}(x)=\frac{\epsilon}{e^x+\epsilon-1}$, $G_3^{-1}(x)=1-\frac{\epsilon}{e^x+\epsilon-1}$, we then define Hamacher T-norm and Hamacher S-norm as the following

$$\top_3^\epsilon(x,y)=\frac{xy}{\epsilon+(1-\epsilon)(x+y-xy)}, \qquad \perp_3^\epsilon(x,y)=\frac{x+y-xy-(1-\epsilon)xy}{1-(1-\epsilon)xy}, \quad \epsilon>0; \tag{7}$$

- If $F_4(x)=log(\frac{\epsilon-1}{\epsilon^x-1})$, $\epsilon>1$ and $G_4(x)=log(\frac{\epsilon-1}{\epsilon^{1-x}-1})$, and moreover, $F_4^{-1}(x)=\frac{log(\frac{\epsilon-1+e^x}{e^x})}{log(\epsilon)}$, $G_4^{-1}(x)=1-\frac{log(\frac{\epsilon-1+e^x}{e^x})}{log(\epsilon)}$, then we may get Frank T-norm and Frank S-norm as follows

$$\top_4^\epsilon(x,y)=log_\epsilon(1+\frac{(\epsilon^x-1)(\epsilon^y-1)}{\epsilon-1}), \qquad \perp_4^\epsilon(x,y)=1-log_\epsilon(1+\frac{(\epsilon^{1-x}-1)(\epsilon^{1-y}-1)}{\epsilon-1}),$$
$$\epsilon>1. \tag{8}$$

3 Probabilistic hesitant fuzzy sets (PHFSs)

In this portion, to resolve the deficiency of the existing definitions of a probabilistic hesitant fuzzy set (PHFS), we are going to modify that was first proposed by Jiang and Ma [13] as an extension of a HFS. Then, a number of generalized operations of probabilistic hesitant fuzzy elements (PHFEs) are introduced on the basis of some T-norms and S-norms.

Recently, due to the fact that the membership degrees of a HFS are not able to describe adequately the uncertainty in the decision judgement, Farhadinia and Xu [12] modified Zhang and Wu's [29] definition of weighted hesitant fuzzy set (WHFS), and recalled it as the ordered weighted hesitant fuzzy set (OWHFS). This modification lets the membership degrees of an element be determined by some possible values together with their corresponding importance weights.

Definition 3.1. [12] *Let X be the universe of discourse. An ordered weighted hesitant fuzzy set (OWHFS) on X is defined as:*

$$^{\wp}H = \{\langle x, {}^{\wp}h(x)\rangle : x \in X\} = \{\langle x, \bigcup_{\langle \hbar(x), \wp(x)\rangle \in {}^{\wp}h(x)} \{\langle \hbar(x), \wp(x)\rangle\}\rangle : x \in X\}, \tag{9}$$

where $^{\wp}h(x)$, referred to as the ordered weighted hesitant fuzzy element (OWHFE), is a set of some different values in $[0,1]$. It denotes all possible membership degrees of the element $x \in X$ to the set $^{\wp}H$, and $\wp(x) \in [0,1]$ is the weight of $\hbar(x)$ such that $\sum_{\wp(x)\in|^{\wp}h(x)|} \wp(x) = 1$ for any $x \in X$.

We should say that if all $\wp(x) \in |^{\wp}h(x)|$ are taken the same for any $x \in X$, then the OWHFS $^{\wp}H$ is reduced to a typical HFS. Farhadinia and Xu [12] implemented the unification procedure for OWHFEs in which each OWHFE has the same length compared to the other considered OWHFEs, and then introduced the following operations to OWHFEs:

Let $^{\wp}h_1 = \bigcup_{\langle \hbar_1, \wp_1\rangle \in {}^{\wp}h_1}\{\langle \hbar_1, \wp_1\rangle\}$ and $^{\wp}h_2 = \bigcup_{\langle \hbar_2, \wp_2\rangle \in {}^{\wp}h_2}\{\langle \hbar_2, \wp_2\rangle\}$ be two OWHFEs. Then, some operations on the OWHFEs $^{\wp}h_1$ and $^{\wp}h_2$ are defined as follows (see [12]):

$$^{\wp}h_1^{\lambda} = \bigcup_{\langle \hbar_1, \wp_1\rangle \in {}^{\wp}h_1} \{\langle {\hbar_1}^{\lambda}, \wp_1\rangle\}; \tag{10}$$

$$\lambda {}^{\wp}h_1 = \bigcup_{\langle \hbar_1, \wp_1\rangle \in {}^{\wp}h_1} \{\langle 1-(1-\hbar_1)^{\lambda}, \wp_1\rangle\}; \tag{11}$$

$$^{\wp}h_1 \oplus {}^{\wp}h_2 = \bigcup_{\langle \hbar_1, \wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2, \wp_2\rangle \in {}^{\wp}h_2} \{\langle 1-(1-\hbar_1)(1-\hbar_2), \overline{\wp_1+\wp_2}\rangle\}; \tag{12}$$

$$^{\wp}h_1 \otimes {}^{\wp}h_2 = \bigcup_{\langle \hbar_1, \wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2, \wp_2\rangle \in {}^{\wp}h_2} \{\langle \hbar_1\hbar_2, \overline{\wp_1+\wp_2}\rangle\}, \tag{13}$$

where the normalized weight of the above binary operations is defined in the form of

$$\overline{\wp_1+\wp_2} = \frac{\wp_1+\wp_2}{\sum(\wp_1+\wp_2)}, \qquad \langle \hbar_1, \wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2, \wp_2\rangle \in {}^{\wp}h_2. \tag{14}$$

It seems that Farhadinia and Xu's [12] motivation for defining such operations comes from their incorrect interpretation of $\lambda^{\wp}h_1$ and $^{\wp}h_1 \oplus {}^{\wp}h_1$ where they supposed that $\lambda(=2)^{\wp}h_1 = {}^{\wp}h_1 \oplus {}^{\wp}h_1$. This is while, the left-hand side of the equality is a single operation, but the right-hand side is a binary operation. Taking this consideration into account, Farhadinia and Xu [12] modified the definition of operations by criticizing Zhang et al.'s [30] and Pang et al.'s [20] definitions.
However Farhadinia and Xu's [12] apparently ignored the fact that not only the limited version of probability of *union of independent events*, that is, $P(P_1 \bigcup P_2) = P(P_1 = \wp_1) \bigcup P(P_2 = \wp_2) = P(P_1 = \wp_1) + P(P_2 = \wp_2)$ where $P(P_1 \bigcap P_2) = P(P_1 = \wp_1) \bigcap P(P_2 = \wp_2) = \emptyset$ is not able to describe comprehensively and reasonably all the corresponding binary operations, but also the probability of *intersection of independence events*, that is, $P(P_1 \bigcap P_2) = P(P_1 = \wp_1) \bigcap P(P_2 = \wp_2) = P(P_1 = \wp_1).P(P_2 = \wp_2)$ should be considered in the definition of related operations.
By the way, we should consider both the probability of *union of independent events* and the probability of *intersection of independence events* respectively in defining the *addition* and the *multiplication* of PHFEs.
This is the superiority of the next re-visited definitions compared to the existing ones including Farhadinia and Xu's [12], Zhang et al.'s [30] and Pang et al.'s [20] definitions.

Let $^{\wp}h_1 = \bigcup_{\langle \hbar_1, \wp_1\rangle \in {}^{\wp}h_1}\{\langle \hbar_1, \wp_1\rangle\}$ and $^{\wp}h_2 = \bigcup_{\langle \hbar_2, \wp_2\rangle \in {}^{\wp}h_2}\{\langle \hbar_2, \wp_2\rangle\}$ be two PHFEs, and T-norm $\top$ and S-norm $\perp$ are those given in previous section. Then,

- Algebraic product and algebraic sum

$$\begin{aligned} ^{\wp}h_1 \oplus {}^{\wp}h_2 &= \bigcup_{\langle \hbar_1, \wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2, \wp_2\rangle \in {}^{\wp}h_2} \{\langle \perp_1(\hbar_1, \hbar_2), 1-(1-\wp_1)(1-\wp_2)\rangle\} \\ &= \bigcup_{\langle \hbar_1, \wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2, \wp_2\rangle \in {}^{\wp}h_2} \{\langle 1-(1-\hbar_1)(1-\hbar_2), 1-(1-\wp_1)(1-\wp_2)\rangle\}, \end{aligned}$$

$$
\begin{aligned}
{}^{\wp}h_1 \otimes {}^{\wp}h_2 &= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \top_1(\hbar_1,\hbar_2), \wp_1\wp_2\rangle\} \\
&= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \hbar_1\hbar_2, \wp_1\wp_2\rangle\};
\end{aligned}
$$

- Einstein product and Einstein sum

$$
\begin{aligned}
{}^{\wp}h_1 \oplus {}^{\wp}h_2 &= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \bot_2(\hbar_1,\hbar_2), 1-(1-\wp_1)(1-\wp_2)\rangle\} \\
&= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \frac{\hbar_1+\hbar_2}{1+\hbar_1\hbar_2}, 1-(1-\wp_1)(1-\wp_2)\rangle\}, \\
{}^{\wp}h_1 \otimes {}^{\wp}h_2 &= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \top_2(\hbar_1,\hbar_2), \wp_1\wp_2\rangle\} \\
&= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \frac{\hbar_1\hbar_2}{1+(1-\hbar_1)(1-\hbar_2)}, \wp_1\wp_2\rangle\};
\end{aligned}
$$

- Hamacher product and Hamacher sum ($\epsilon > 0$)

$$
\begin{aligned}
{}^{\wp}h_1 \oplus {}^{\wp}h_2 &= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \bot_3(\hbar_1,\hbar_2), 1-(1-\wp_1)(1-\wp_2)\rangle\} \\
&= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \frac{\hbar_1+\hbar_2-\hbar_1\hbar_2-(1-\epsilon)\hbar_1\hbar_2}{1-(1-\epsilon)\hbar_1\hbar_2}, 1-(1-\wp_1)(1-\wp_2)\rangle\}, \\
{}^{\wp}h_1 \otimes {}^{\wp}h_2 &= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \top_3(\hbar_1,\hbar_2), \wp_1\wp_2\rangle\} \\
&= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \frac{\hbar_1\hbar_2}{\epsilon+(1-\epsilon)(\hbar_1+\hbar_2-\hbar_1\hbar_2)}, \wp_1\wp_2\rangle\};
\end{aligned}
$$

- Frank product and Frank sum ($\epsilon > 1$)

$$
\begin{aligned}
{}^{\wp}h_1 \oplus {}^{\wp}h_2 &= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \bot_4(\hbar_1,\hbar_2), 1-(1-\wp_1)(1-\wp_2)\rangle\} \\
&= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle 1-log_{\epsilon}(1+\frac{(\epsilon^{1-\hbar_1}-1)(\epsilon^{1-\hbar_2}-1)}{\epsilon-1}), 1-(1-\wp_1)(1-\wp_2)\rangle\}, \\
{}^{\wp}h_1 \otimes {}^{\wp}h_2 &= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle \top_4(\hbar_1,\hbar_2), \wp_1\wp_2\rangle\} \\
&= \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1, \langle \hbar_2,\wp_2\rangle \in {}^{\wp}h_2} \{\langle log_{\epsilon}(1+\frac{(\epsilon^{\hbar_1}-1)(\epsilon^{\hbar_2}-1)}{\epsilon-1}), \wp_1\wp_2\rangle\}.
\end{aligned}
$$

Analogously, we may define

$$
\lambda^{\wp}h_1 = \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1} \{\langle G^{-1}(\lambda G(\hbar_1)), \wp_1\rangle\} \quad \text{and} \quad ({}^{\wp}h_1)^{\lambda} = \bigcup_{\langle \hbar_1,\wp_1\rangle \in {}^{\wp}h_1} \{\langle F^{-1}(\lambda F(\hbar_1)), \wp_1\rangle\},
$$

where $\lambda > 0$, $F : [0,1] \to [0,\infty)$ stands for a strictly decreasing function with the properties $F(1) = 0$, and $G(x) = F(1-x)$.

In addition to the above-mentioned relations, the *neutral elements* of addition and multiplication operators are respectively specified as

$$
{}^{\wp}h^{\mathbf{0}} = \bigcup_{\langle \hbar,\wp\rangle \in {}^{\wp}h} \{\langle 0,0\rangle\}, \tag{15}
$$

$$
{}^{\wp}h^{\mathbf{1}} = \bigcup_{\langle \hbar,\wp\rangle \in {}^{\wp}h} \{\langle 1,1\rangle\}. \tag{16}
$$

Keeping the above *algebraical operations* into mind, we are now in a position to define the *set operations* as the following

$$
\begin{aligned}
{}^{\wp}h_1^c &= \bigcup_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1} \{\langle \aleph(\hbar_1),\wp_1\rangle\} = \bigcup_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1} \{\langle 1-\hbar_1,\wp_1\rangle\}; \qquad (17)\\
{}^{\wp}h_1 \cup {}^{\wp}h_2 &= \bigcup_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1,\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2} \{\langle \max\{\hbar_1,\hbar_2\}, 1-(1-\wp_1)(1-\wp_2)\rangle\}; \qquad (18)\\
{}^{\wp}h_1 \cap {}^{\wp}h_2 &= \bigcup_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1,\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2} \{\langle \min\{\hbar_1,\hbar_2\}, \wp_1\wp_2\rangle\}. \qquad (19)
\end{aligned}
$$

Theorem 3.2. *Let* ${}^{\wp}h_1 = \bigcup_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1}\{\langle \hbar_1,\wp_1\rangle\}$ *and* ${}^{\wp}h_2 = \bigcup_{\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2}\{\langle \hbar_2,\wp_2\rangle\}$ *be two PHFEs. Then, all the above-mentioned operations* ${}^{\wp}h_1 \oplus {}^{\wp}h_2$, ${}^{\wp}h_1 \otimes {}^{\wp}h_2$, $\lambda{}^{\wp}h_1$, $({}^{\wp}h_1)^{\lambda}$, ${}^{\wp}h_1^c$, ${}^{\wp}h_1 \cup {}^{\wp}h_2$ *and* ${}^{\wp}h_1 \cap {}^{\wp}h_2$ *are also PHFEs.*

Proof. It is sufficient to prove that the definition of probabilities in both cases of addition (or union) and multiplication (or intersection) result in the unity property.
Addition (or union) case. On the basis of assumptions $\sum_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1}\wp_1 = 1$ and $\sum_{\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2}\wp_2 = 1$, we conclude that

$$
\begin{aligned}
\sum_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1,\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2} 1-(1-\wp_1)(1-\wp_2) &= \sum_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1,\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2} (\wp_1+\wp_2-\wp_1\wp_2)\\
&= \sum_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1}\wp_1 + \sum_{\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2}\wp_2 - \sum_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1,\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2}\wp_1\wp_2\\
&= 1+1-\sum_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1}\wp_1(\sum_{\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2}\wp_2)\\
&= 1+1-\sum_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1}\wp_1(1) = 1.
\end{aligned}
$$

Multiplication (or intersection) case. In the same manner, we result in

$$
\sum_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1,\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2}\wp_1\wp_2 = \sum_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1}\wp_1(\sum_{\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2}\wp_2) = \sum_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1}\wp_1(1) = 1.
$$

□

Theorem 3.3. *Let* ${}^{\wp}h_1 = \bigcup_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1}\{\langle \hbar_1,\wp_1\rangle\}$ *and* ${}^{\wp}h_2 = \bigcup_{\langle \hbar_2,\wp_2\rangle\in {}^{\wp}h_2}\{\langle \hbar_2,\wp_2\rangle\}$ *be two PHFEs. Then, the following properties hold true*

$$
{}^{\wp}h_1 \oplus {}^{\wp}h_2 = {}^{\wp}h_2 \oplus {}^{\wp}h_1, \quad \lambda({}^{\wp}h_1 \oplus {}^{\wp}h_2) = \lambda{}^{\wp}h_1 \oplus \lambda{}^{\wp}h_2, \quad (\lambda_1+\lambda_2){}^{\wp}h_1 = \lambda_1{}^{\wp}h_1 \oplus \lambda_2{}^{\wp}h_1, \qquad (20)
$$

$$
{}^{\wp}h_1 \otimes {}^{\wp}h_2 = {}^{\wp}h_2 \otimes {}^{\wp}h_1, \quad ({}^{\wp}h_1 \otimes {}^{\wp}h_2)^{\lambda} = ({}^{\wp}h_1)^{\lambda} \otimes ({}^{\wp}h_2)^{\lambda}, \quad {}^{\wp}h_1^{(\lambda_1+\lambda_2)} = {}^{\wp}h_1^{\lambda_1} \otimes {}^{\wp}h_1^{\lambda_2}, \qquad (21)
$$

for any $\lambda, \lambda_1, \lambda_2 > 0$.

Proof. The proof is straightforward, and therefore, it will be omitted. □

In simple terms, we can say that if a T-norm of two PHFSs needs to be defined, then the probability part of that T-norm should be stated as the algebraic sum of probabilities instead of just their multiplication, and if a S-norm of two PHFSs is going to be defined, then the probability part of that S-norm must be as the algebraic product of probabilities instead of again their multiplication.

Consider $\{{}^{\wp}h_1, {}^{\wp}h_2, ..., {}^{\wp}h_n\}$ as a collection of PHFEs, and $w = (w_1, w_2, ..., w_n)$ denotes the weight vector of the PHFEs such that $0 \le w_i \le 1$ for $i = 1, 2, ..., n$ and $\sum_{i=1}^{n} w_i = 1$. Then, we conclude that

Theorem 3.4. *The probabilistic hesitant fuzzy weighted average (PHFWA) operator*

$$
\begin{aligned}
PHFWA({}^{\wp}h_1, {}^{\wp}h_2, ..., {}^{\wp}h_n) &= w_1{}^{\wp}h_1 \oplus w_2{}^{\wp}h_2 \oplus ... \oplus w_n{}^{\wp}h_n\\
&= \bigcup_{\langle \hbar_1,\wp_1\rangle\in {}^{\wp}h_1,...,\langle \hbar_n,\wp_n\rangle\in {}^{\wp}h_n} \{\langle G^{-1}(\sum_{i=1}^{n} w_i G(\hbar_i)), 1-\Pi_{i=1}^{n}(1-\wp_i)\rangle\} \qquad (22)
\end{aligned}
$$

is a PHFE.
Here, G *denotes an additive generator with the property* $\perp(x, y) = G^{-1}(G(x) + G(y))$.

Proof. To simplify the exposition, we only prove that the equality (22) hold true for $\perp_1$ where $G_1(x) = -log(1-x)$ and $G_1^{-1}(x) = 1 - e^{-x}$. Therefore, we should show that

$$\begin{aligned} & PHFWA({}^{\wp}h_1, {}^{\wp}h_2, ..., {}^{\wp}h_n) = w_1{}^{\wp}h_1 \oplus w_2{}^{\wp}h_2 \oplus ... \oplus w_n{}^{\wp}h_n \\ & = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_n, \wp_n \rangle \in {}^{\wp}h_n} \{\langle 1 - \Pi_{i=1}^{n}(1-\hbar_i)^{w_i}, 1 - \Pi_{i=1}^{n}(1-\wp_i) \rangle\}. \end{aligned}$$

We can prove the theorem using mathematical induction on the parameter n. It is assumed that the equation (22) holds true for $n = k$, that is

$$\begin{aligned} & PHFWA({}^{\wp}h_1, {}^{\wp}h_2, ..., {}^{\wp}h_k) = w_1{}^{\wp}h_1 \oplus w_2{}^{\wp}h_2 \oplus ... \oplus w_k{}^{\wp}h_k \\ & = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_k, \wp_k \rangle \in {}^{\wp}h_k} \{\langle G_1^{-1}(\sum_{i=1}^{k} w_i G_1(\hbar_i)), \sum_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_k, \wp_k \rangle \in {}^{\wp}h_k} 1 - \Pi_{i=1}^{k}(1-\wp_i) \rangle\} \\ & = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_k, \wp_k \rangle \in {}^{\wp}h_k} \{\langle 1 - \Pi_{i=1}^{k}(1-\hbar_i)^{w_i}, 1 - \Pi_{i=1}^{k}(1-\wp_i) \rangle\}. \end{aligned} \tag{23}$$

Then, in the case where $n = k + 1$, we conclude that

$$\begin{aligned} & PHFWA({}^{\wp}h_1, {}^{\wp}h_2, ..., {}^{\wp}h_{k+1}) = PHFWA({}^{\wp}h_1, {}^{\wp}h_2, ..., {}^{\wp}h_k) \oplus w_{k+1}{}^{\wp}h_{k+1} \\ & = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_k, \wp_k \rangle \in {}^{\wp}h_k} \{\langle G_1^{-1}(\sum_{i=1}^{k} w_i G_1(\hbar_i)), \sum_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_k, \wp_k \rangle \in {}^{\wp}h_k} 1 - \Pi_{i=1}^{k}(1-\wp_i) \rangle\} \\ & \oplus w_{k+1}{}^{\wp}h_{k+1} \\ & = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_k, \wp_k \rangle \in {}^{\wp}h_k} \{\langle 1 - (1-\hbar_1)^{w_1} ... (1-\hbar_k)^{w_k}, 1 - (1-\wp_1)...(1-\wp_k) \rangle\} \oplus w_{k+1}{}^{\wp}h_{k+1} \\ & = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_{k+1}, \wp_{k+1} \rangle \in {}^{\wp}h_{k+1}} \{\langle 1 - (1-\hbar_1)^{w_1} ... (1-\hbar_{k+1})^{w_{k+1}}, 1 - (1-\wp_1)...(1-\wp_{k+1}) \rangle\} \\ & = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_{k+1}, \wp_{k+1} \rangle \in {}^{\wp}h_{k+1}} \{\langle 1 - \Pi_{i=1}^{k+1}(1-\hbar_i)^{w_i}, 1 - \Pi_{i=1}^{k+1}(1-\wp_i) \rangle\}. \end{aligned}$$

This means that the equation (22) holds true for $n = k + 1$ which results in satisfaction of (22) for any n. □

Now we suppose that $\{{}^{\wp}h_1, {}^{\wp}h_2, ..., {}^{\wp}h_n\}$ is to be a collection of PHFEs, and moreover, $w = (w_1, w_2, ..., w_n)$ stand for the weight vector of the PHFEs such that $0 \leq w_i \leq 1$ for $i = 1, 2, ..., n$ and $\sum_{i=1}^{n} w_i = 1$. In this regard, we may define

Theorem 3.5. *The probabilistic hesitant fuzzy weighted geometric (PHFWG) operator*

$$\begin{aligned} PHFWG({}^{\wp}h_1, {}^{\wp}h_2, ..., {}^{\wp}h_n) & = {}^{\wp}h_1^{w_1} \otimes {}^{\wp}h_2^{w_2} \otimes ... \otimes {}^{\wp}h_n^{w_n} \\ & = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_n, \wp_n \rangle \in {}^{\wp}h_n} \{\langle F^{-1}(\sum_{i=1}^{n} w_i F(\hbar_i)), \Pi_{i=1}^{n}(\wp_i) \rangle\} \end{aligned} \tag{24}$$

is a PHFE.
Here, F stands for an additive generator with the property $\top(x, y) = F^{-1}(F(x) + F(y))$.

Proof. The proof is analogous to that of Theorem 3.4, and therefore is omitted. □

As a corollary of Theorem 3.5 together with the relation (5) where $F_1(x) = -log(x)$ with $F_1^{-1}(x) = e^{-x}$, we can result

$$\begin{aligned} PHFWG({}^{\wp}h_1, {}^{\wp}h_2, ..., {}^{\wp}h_n) & = {}^{\wp}h_1^{w_1} \otimes {}^{\wp}h_2^{w_2} \otimes ... \otimes {}^{\wp}h_n^{w_n} \\ & = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_n, \wp_n \rangle \in {}^{\wp}h_n} \{\langle F^{-1}(\sum_{i=1}^{n} w_i F(\hbar_i)), \Pi_{i=1}^{n}(\wp_i) \rangle\} \\ & = \bigcup_{\langle \hbar_1, \wp_1 \rangle \in {}^{\wp}h_1, ..., \langle \hbar_n, \wp_n \rangle \in {}^{\wp}h_n} \{\langle \Pi_{i=1}^{n}(\hbar_i)^{w_i}, \Pi_{i=1}^{n}(\wp_i) \rangle\}. \end{aligned} \tag{25}$$

4 PHFS multiple criteria decision making

In this section, three different multiple criteria decision making (MCDM) problems that were investigated previously by Zhang et al. [30], by Song et al. [24] and by Wang and Li [27] are considered to outline the actual comparative procedures between the proposed concept of PHFE and the existing ones.

4.1 Case study I

Recently, Zhang et al. [30] investigated a practical application of PHFEs in evaluating the safety of automobiles as a main transportation for long trips. The safety system is divided in two parts: active and passive safety systems. The active safety system including brake, anti-lock brake and vehicle stability systems which are respectively denoted by C_1, C_2 and C_3 is used for increasing the stability of automobiles and decreasing the deviation of manipulation. Moreover, the passive safety system including supplementary restraint system and automobile body sheet which are respectively denoted by C_4 and C_5 is used for minimizing the impact damages specially in the case where the automobiles are not able to avoid the collision. Furthermore, among five brands including Buick, Ford, Toyota, Audi and Tesla being respectively denoted by A_1, A_2, A_3, A_4 and A_5, a consumer may chose one of them according to their safety system. Three experts are invited to provide their evaluations as that given in the forms of Tables 1-3. Furthermore, the weights of the experts are considered to be respectively as 0.4, 0.4, and 0.2, and the weights of five attributes C_1, C_2, C_3, C_4 and C_5 are all taken into consideration as 0.2.

Table 1. The probabilistic hesitant fuzzy decision matrix provided by expert 1.

	Brake (C_1)	Antilock brake (C_2)	Vehicle stability (C_3)	Supplementary restraint (C_4)	Automobile body sheet (C_5)
Buick (A_1)	$\{\langle 0.6,0.25\rangle, \langle 0.7,0.5\rangle, \langle 0.8,0.25\rangle\}$	$\{\langle 0.5,0.6\rangle, \langle 0.6,0.4\rangle\}$	$\{\langle 0.6,0.8\rangle, \langle 0.7,0.2\rangle\}$	$\{\langle 0.3,0.4\rangle, \langle 0.4,0.6\rangle\}$	$\{\langle 0.6,0.3\rangle, \langle\langle 0.7,0.3\rangle, \langle 0.8,0.4\rangle\}$
Toyota (A_2)	$\{\langle 0.5,0.5\rangle, \langle 0.6,0.5\rangle\}$	$\{\langle 0.4,0.3\rangle, \langle 0.5,0.3\rangle, \langle 0.6,0.4\rangle\}$	$\{\langle 0.5,0.2\rangle, \langle 0.6,0.3\rangle, \langle 0.7,0.5\rangle\}$	$\{\langle 0.6,0.6\rangle, \langle 0.7,0.2\rangle\}$	$\{\langle 0.7,0.5\rangle, \langle 0.8,0.5\rangle\}$
Ford (A_3)	$\{\langle 0.7,0.5\rangle, \langle 0.8,0.3\rangle\}$	$\{\langle 0.3,0.2\rangle, \langle 0.4,0.4\rangle, \langle 0.5,0.4\rangle\}$	$\{\langle 0.5,0.2\rangle, \langle 0.6,0.8\rangle\}$	$\{\langle 0.5,0.4\rangle, \langle 0.6,0.6\rangle\}$	$\{\langle 0.6,0.3\rangle, \langle 0.7,0.7\rangle\}$
Audi (A_4)	$\{\langle 0.8,0.5\rangle, \langle 0.85,0.3\rangle, \langle 0.9,0.2\rangle\}$	$\{\langle 0.7,0.6\rangle, \langle 0.8,0.2\rangle\}$	$\{\langle 0.6,0.4\rangle, \langle 0.7,0.3\rangle, \langle 0.8,0.3\rangle\}$	$\{\langle 0.5,0.5\rangle, \langle 0.6,0.5\rangle\}$	$\{\langle 0.7,0.7\rangle, \langle 0.8,0.3\rangle\}$
Tesla (A_5)	$\{\langle 0.65,0.5\rangle, \langle 0.75,0.5\rangle\}$	$\{\langle 0.5,0.5\rangle, \langle 0.6,0.5\rangle\}$	$\{\langle 0.7,0.3\rangle, \langle 0.8,0.7\rangle\}$ height	$\{\langle 0.5,0.3\rangle, \langle 0.6,0.3\rangle, \langle 0.7,0.4\rangle\}$	$\{\langle 0.7,0.2\rangle, \langle 0.8,0.2\rangle, \langle 0.9,0.4\rangle\}$ height

Table 2. The probabilistic hesitant fuzzy decision matrix provided by expert 2.

	Brake (C_1)	Antilock brake (C_2)	Vehicle stability (C_3)	Supplementary restraint (C_4)	Automobile body sheet (C_5)
Buick (A_1)	$\{\langle 0.5,0.5\rangle, \langle 0.7,0.5\rangle\}$	$\{\langle 0.7,0.7\rangle, \langle 0.8,0.2\rangle, \langle 0.9,0.1\rangle\}$	$\{\langle 0.4,0.3\rangle, \langle 0.6,0.5\rangle\}$	$\{\langle 0.5,0.3\rangle, \langle 0.6,0.4\rangle, \langle 0.7,0.3\rangle\}$	$\{\langle 0.7,0.4\rangle, \langle 0.8,0.6\rangle\}$
Toyota (A_2)	$\{\langle 0.5,0.4\rangle, \langle 0.6,0.6\rangle\}$	$\{\langle 0.6,0.7\rangle, \langle 0.7,0.3\rangle\}$	$\{\langle 0.6,0.3\rangle, \langle 0.7,0.4\rangle, \langle 0.8,0.3\rangle\}$	$\{\langle 0.6,0.5\rangle, \langle 0.7,0.5\rangle\}$	$\{\langle 0.5,0.7\rangle, \langle 0.6,0.2\rangle, \langle 0.8,0.1\rangle\}$
Ford (A_3)	$\{\langle 0.5,0.2\rangle, \langle 0.6,0.3\rangle, \langle 0.7,0.5\rangle\}$	$\{\langle 0.7,0.6\rangle, \langle 0.8,0.2\rangle\}$	$\{\langle 0.3,0.4\rangle, \langle 0.5,0.6\rangle\}$	$\{\langle 0.6,0.5\rangle, \langle 0.7,0.2\rangle, \langle 0.8,0.3\rangle\}$	$\{\langle 0.6,0.5\rangle, \langle 0.7,0.5\rangle\}$
Audi (A_4)	$\{\langle 0.7,0.4\rangle, \langle 0.8,0.4\rangle\}$	$\{\langle 0.6,0.5\rangle, \langle 0.7,0.1\rangle, \langle 0.8,0.4\rangle\}$	$\{\langle 0.7,0.5\rangle, \langle 0.8,0.3\rangle, \langle 0.9,0.2\rangle\}$	$\{\langle 0.5,0.6\rangle, \langle 0.6,0.6\rangle\}$	$\{\langle 0.7,0.2\rangle, \langle 0.8,0.6\rangle\}$
Tesla (A_5)	$\{\langle 0.6,0.5\rangle, \langle 0.7,0.4\rangle, \langle 0.8,0.1\rangle\}$	$\{\langle 0.6,0.4\rangle, \langle 0.7,0.6\rangle\}$	$\{\langle 0.4,0.5\rangle, \langle 0.5,0.5\rangle\}$ height	$\{\langle 0.7,0.4\rangle, \langle 0.8,0.4\rangle\}$	$\{\langle 0.6,0.2\rangle, \langle 0.7,0.4\rangle, \langle 0.8,0.4\rangle\}$

Table 3. The probabilistic hesitant fuzzy decision matrix provided by expert 3.

	Brake (C_1)	Antilock brake (C_2)	Vehicle stability (C_3)	Supplementary restraint (C_4)	Automobile body sheet (C_5)
Buick (A_1)	$\{\langle 0.6,0.3\rangle, \langle 0.7,0.5\rangle, \langle 0.8,0.2\rangle\}$	$\{\langle 0.6,0.4\rangle, \langle 0.7,0.4\rangle\}$	$\{\langle 0.7,0.6\rangle, \langle 0.8,0.4\rangle\}$	$\{\langle 0.65,0.4\rangle, \langle 0.7,0.2\rangle, \langle 0.75,0.4\rangle\}$	$\{\langle 0.6,0.2\rangle, \langle\langle 0.7,0.8\rangle\}$
Toyota (A_2)	$\{\langle 0.6,0.5\rangle, \langle 0.7,0.3\rangle\}$	$\{\langle 0.5,0.3\rangle, \langle 0.6,0.4\rangle, \langle 0.7,0.3\rangle\}$	$\{\langle 0.5,0.3\rangle, \langle 0.6,0.7\rangle\}$	$\{\langle 0.6,0.8\rangle, \langle 0.8,0.2\rangle\}$	$\{\langle 0.55,0.4\rangle, \langle 0.6,0.3\rangle, \langle 0.65,0.3\rangle\}$
Ford (A_3)	$\{\langle 0.5,0.3\rangle, \langle 0.6,0.7\rangle\}$	$\{\langle 0.5,0.5\rangle, \langle 0.7,0.5\rangle\}$	$\{\langle 0.6,0.5\rangle, \langle 0.7,0.3\rangle, \langle 0.8,0.2\rangle\}$	$\{\langle 0.6,0.2\rangle, \langle 0.7,0.4\rangle, \langle 0.8,0.4\rangle\}$	$\{\langle 0.7,0.4\rangle, \langle 0.8,0.4\rangle\}$
Audi (A_4)	$\{\langle 0.5,0.3\rangle, \langle 0.6,0.4\rangle, \langle 0.7,0.3\rangle\}$	$\{\langle 0.5,0.3\rangle, \langle 0.6,0.3\rangle, \langle 0.7,0.4\rangle\}$	$\{\langle 0.7,0.2\rangle, \langle 0.8,0.6\rangle\}$	$\{\langle 0.6,0.4\rangle, \langle 0.7,0.6\rangle\}$	$\{\langle 0.7,0.6\rangle, \langle 0.8,0.4\rangle\}$
Tesla (A_5)	$\{\langle 0.7,0.5\rangle, \langle 0.8,0.5\rangle\}$	$\{\langle 0.6,0.6\rangle, \langle 0.7,0.4\rangle\}$	$\{\langle 0.5,0.2\rangle, \langle 0.6,0.4\rangle, \langle 0.7,0.4\rangle\}$ height	$\{\langle 0.7,0.8\rangle, \langle 0.8,0.2\rangle\}$	$\{\langle 0.6,0.3\rangle, \langle 0.7,0.7\rangle\}$ height

In order to do integration all the experts' information with respect to the attributes, we implement here the PHFWA operator (22) instead of that were utilized by Zhang et al. [30]. By the way, to save more spaces, we only give the weighted average scores of A_i $(i = 1, 2, ..., 5)$ as seen in Table 4.

Table 4. The score values of car brands.

	Score of (C_1)	Score of (C_2)	Score of (C_3)	Score of (C_4)	Score of (C_5)
Buick (A_1)	0.2749	0.3124	0.3306	0.2240	0.3328
Toyota (A_2)	0.2903	0.2447	0.2467	0.3514	0.2845
Ford (A_3)	0.3012	0.2791	0.2738	0.2691	0.3461
Audi (A_4)	0.3177	0.3019	0.2941	0.2982	0.3979
Tesla (A_5)	0.3153	0.3107	0.3050	0.3216	0.2888 height

Table 5. The ranking results of car brands.

	Ranking order	Optimal alternative
Zhang et al.'s [30] method	$A_4 > A_5 > A_1 > A_3 > A_2$	A_4
The proposed method	$A_4 > A_5 > A_1 > A_3 > A_2$	A_4 height

As can be seen from Table 5, both Zhang et al.'s [30] results and the results of the proposed method are in accordance with the usual cognitions where the safety performance of Audi is higher than the remaining brands, and this conclusion is clearly still meaningful for European and American cars than the Japanese one. Indeed, such a result indicates that the proposed method is adequately reasonable and practical.

4.2 Case study II

In this part, we are going to extend the VIKOR method [19] which was first introduced for finding a compromise solution of a MCDM problem accompanying the non-commensurable and conflicting criteria to that with the PHFE information. The interpretation of compromise solution is related to the agreement that is established by mutual concessions. In fact, the compromise solution defines a multiple criteria ranking index which is based on the L_p-metric measurement of *closeness* to an *ideal* solution.

Now, we consider a decision matrix containing elements of the form of PHFEs as

$$D = \begin{bmatrix} & \mathrm{C}_1 & \mathrm{C}_2 & \cdots & C_n \\ \mathrm{A}_1 & {}^{\wp}h_{11} & {}^{\wp}h_{12} & & {}^{\wp}h_{1n} \\ \mathrm{A}_2 & {}^{\wp}h_{21} & {}^{\wp}h_{22} & & {}^{\wp}h_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathrm{A}_m & {}^{\wp}h_{m1} & {}^{\wp}h_{m2} & & {}^{\wp}h_{mn} \end{bmatrix},$$

in which ${}^{\wp}h_{ij}$ $(i = 1, ..., m,\ j = 1, ..., n)$ denotes the rating of alternative A_i with respect to the criterion C_j whose weight is w_j.

Before going further with a more explicit description of the algorithm, we assume that the included PHFEs contain elements whose positions may be moved arbitrarily.
Adopting from Zhang et al. [30], we suppose hereafter ${}^{\wp}h = \bigcup_{\langle \hbar, \wp\rangle \in {}^{\wp}h} \{\langle \hbar, \wp\rangle\}$ stands for an *ordered-PHFE* whose elements are arranged in an *increasing order* of real values $\hbar\wp$.
In this regard, we introduce the best and worst values of A_i over the criterion C_j as follows:

$${}^{\wp}h_{ij}^{\max} := \langle \hbar_{ij}^{\max}, \wp_{ij}^{\max}\rangle = \max_{\langle \hbar_{ij}, \wp_{ij}\rangle \in {}^{\wp}h_{ij}} \{\langle \hbar_{ij}, \wp_{ij}\rangle\}, \tag{26}$$

$${}^{\wp}h_{ij}^{\min} := \langle \hbar_{ij}^{\min}, \wp_{ij}^{\min}\rangle = \min_{\langle \hbar_{ij}, \wp_{ij}\rangle \in {}^{\wp}h_{ij}} \{\langle \hbar_{ij}, \wp_{ij}\rangle\}. \tag{27}$$

Now, we are in a position to represent the PHFE-based VIKOR method consisting of the following steps:

Step 1. Specify the positive ideal solution (PIS) and also the negative ideal solution (NIS) as the following:

$${}^{\wp}h_j^+ := \langle \hbar_j^+, \wp_j^+\rangle = \max_{1\le i\le m} \{\langle \hbar_{ij}^{\max}, \wp_{ij}^{\max}\rangle\}, \tag{28}$$

$${}^{\wp}h_j^- := \langle \hbar_j^-, \wp_j^-\rangle = \min_{1\le i\le m} \{\langle \hbar_{ij}^{\min}, \wp_{ij}^{\min}\rangle\}. \tag{29}$$

Step 2. Compute the overall satisfactory for the majority of criteria

$$S_i = \sum_{j=1}^{n} w_j \frac{\|{}^{\wp}h_j^+ - {}^{\wp}h_{ij}\|}{\|{}^{\wp}h_j^+ - {}^{\wp}h_j^-\|} = \sum_{j=1}^{n} w_j \frac{\|\langle \hbar_j^+, \wp_j^+\rangle - \langle \hbar_{ij}, \wp_{ij}\rangle\|}{\|\langle \hbar_j^+, \wp_j^+\rangle - \langle \hbar_j^-, \wp_j^-\rangle\|}, \quad i = 1, ..., m, \tag{30}$$

and for the sacrifice of each individual criterion

$$R_i = \max_{1\le j\le n} \{w_j \frac{\|{}^{\wp}h_j^+ - {}^{\wp}h_{ij}\|}{\|{}^{\wp}h_j^+ - {}^{\wp}h_j^-\|}\} = \max_{1\le j\le n} \{w_j \frac{\|\langle \hbar_j^+, \wp_j^+\rangle - \langle \hbar_{ij}, \wp_{ij}\rangle\|}{\|\langle \hbar_j^+, \wp_j^+\rangle - \langle \hbar_j^-, \wp_j^-\rangle\|}\}, \quad i = 1, ..., m. \tag{31}$$

In the above relations, the distance measure $\|.\|$ is considered the Hausdorff distance between two PHFEs given by (see [16])

$$\|{}^{\wp}h_{ij}^{(1)} - {}^{\wp}h_{ij}^{(2)}\| = \frac{1}{2}\{\frac{1}{l_{(1)}} \sum_{i=1}^{l_{(1)}} \min_{1\le j\le l_{(2)}} |\hbar_{ij}^{(1)}\wp_{ij}^{(1)} - \hbar_{ij}^{(2)}\wp_{ij}^{(2)}| + \frac{1}{l_{(2)}} \sum_{j=1}^{l_{(2)}} \min_{1\le i\le l_{(1)}} |\hbar_{ij}^{(1)}\wp_{ij}^{(1)} - \hbar_{ij}^{(2)}\wp_{ij}^{(2)}|\}, \tag{32}$$

in which the notation $l_{(k)}$ $(k = 1, 2)$ indicates the number of elements in the PHFE ${}^{\wp}h_{ij}^{(k)}$ $(k = 1, 2)$.

By considering $\min_{1\le i\le m}\{S_i\}$, we are intended to find a solution with the maximum group utility, and by considering $\min_{1\le i\le m}\{R_i\}$, we are willing to minimize the individual regret of the opponent. By considering both these optimization problems simultaneously, we are indeed intended to find a compromise solution. Therefore, if α stands for the weight of strategy of criteria majority (or maximum group utility), then we do as the following:

Step 3. Calculate

$$Q_i = \alpha \frac{S_i - S^+}{S^- - S^+} + (1-\alpha)\frac{R_i - R^+}{R^- - R^+}, \tag{33}$$

where $S^+ := \min_{1\le i\le m}\{S_i\}$, $S^- := \max_{1\le i\le m}\{S_i\}$, $R^+ := \min_{1\le i\le m}\{R_i\}$ and $R^- := \max_{1\le i\le m}\{R_i\}$.

Taking the above parameters into account, we find that the smaller value of Q_i indicates the better alternative A_i.

Step 4. By sorting the values S_i, R_i and Q_i in a decreasing order, we may rank the alternatives A_i's.

Step 5. The alternative $A^{(1)}$ sands for a compromise solution if the following conditions are to be held:

C1. $Q(A^{(2)}) - Q(A^{(1)}) \ge \frac{1}{m-1}$, where $A^{(1)}$ and $A^{(2)}$ denote those alternatives with the first and second positions in the ranking list, respectively;

C2. The alternative $A^{(1)}$ must have the best position in the both rankings of S_i and R_i.

In the case where one of the above-mentioned conditions is not held, a set of compromise solutions are then suggested:

If the first condition is not held, all the alternatives $A^{(i)}$ $(i = 1, 2, ..., MAX)$ are considered as the compromise solutions where $Q(A^{(MAX)}) - Q(A^{(1)}) \le \frac{1}{m-1}$.

If the second condition is not held, the alternatives $A^{(1)}$ and $A^{(2)}$ are then the compromise solutions.

In the sequel, in order to show the applicability of the proposed concept, we illustrate the efficiency of this concept in solving a MCDM problem being investigated previously by song et al. [24].

Here, the tendency is to find which Chines hospital has to be selected in the case where the medical resources are limited and also the target population is old-age. Three criteria are mainly considered: C_1: environment of health service; C_2: treatment optimization; C_3: social resource allocation and also health services. Related to the latter criteria, the weight vector is assumed to be $w = (w_1 = 0.2, w_2 = 0.1, w_3 = 0.7)$. With respect to these criteria, four candidate hospitals are taken into consideration: A_1: West China Hospital of Sichuan University, A_2: Huashan Hospital of Fudan University, A_3: Union Medical College Hospital; A_4: Chinese PLA General Hospital.

Since the influence factors cannot be described just by one option, the experts are asked to express their preferences for the above-mentioned hospitals according to the criteria by the help of PHFEs which are demonstrated in the form of the following probabilistic hesitant fuzzy decision matrix (see Table 6).

Table 6. The probabilistic hesitant fuzzy decision matrix.

	Environment of health service (C_1)	Treatment optimization (C_2)	Social resource allocation (C_3)
West China Hospital (A_1)	$\{\langle 0.5, 0.4\rangle, \langle 0.7, 0.6\rangle\}$	$\{\langle 0.9, 1.0\rangle\}$	$\{\langle 0.3, 0.2\rangle, \langle 0.5, 0.8\rangle\}$
Huashan Hospital (A_2)	$\{\langle 0.8, 0.3\rangle, \langle 0.9, 0.7\rangle\}$	$\{\langle 0.5, 1.0\rangle\}$	$\{\langle 0.8, 0.4\rangle, \langle 0.9, 0.6\rangle\}$
Union Medical College Hospital (A_3)	$\{\langle 0.5, 1.0\rangle\}$	$\{\langle 0.7, 0.5\rangle, \langle 0.9, 0.5\rangle\}$	$\{\langle 0.8, 0.6\rangle, \langle 0.9, 0.4\rangle\}$
PLA General Hospital (A_4)	$\{\langle 0.8, 0.5\rangle, \langle 0.9, 0.5\rangle\}$	$\{\langle 0.3, 0.5\rangle, \langle 0.6, 0.5\rangle\}$ height	$\{\langle 0.7, 1.0\rangle\}$ height

Remark 4.1. *Before going further, we should note that all the PHFEs of Table 6 have not been arranged in the increasing order of real values* $\hbar \times \wp$*. This means that all the data in Table 6 are not ordered-PHFEs. To eliminate this problem, we replace the* $(3, 3)$*-th entry with* $\{\langle 0.9, 0.4\rangle\ \langle 0.8, 0.6\rangle\}$.

In what follows, we implement the PHFE-based VIKOR algorithm to obtain the optimal project. Therefore, it needs to go through the following steps:

Step 1. The PIS and NIS are determined in the forms of

$$\{{}^{\wp}h_j^+ := \langle \hbar_j^+, \wp_j^+\rangle\}_{j=1}^3 = \{\langle 0.9, 0.7\rangle, \langle 0.9, 1.0\rangle, \langle 0.9, 0.6\rangle\},$$
$$\{{}^{\wp}h_j^- := \langle \hbar_j^-, \wp_j^-\rangle\}_{j=1}^3 = \{\langle 0.5, 0.4\rangle, \langle 0.3, 0.5\rangle, \langle 0.3, 0.2\rangle\}.$$

Step 2. We calculate the overall satisfactory for the majority of criteria as the following

$$\begin{aligned} S_1 \quad &= \sum_{j=1}^{3} w_j \frac{\|{}^{\wp}h_j^+ - {}^{\wp}h_{1j}\|}{\|{}^{\wp}h_j^+ - {}^{\wp}h_j^-\|} = \sum_{j=1}^{3} w_j \frac{\|\langle \hbar_j^+, \wp_j^+\rangle - \langle \hbar_{1j}, \wp_{1j}\rangle\|}{\|\langle \hbar_j^+, \wp_j^+\rangle - \langle \hbar_j^-, \wp_j^-\rangle\|} \\ &= w_1 \frac{\|\langle \hbar_1^+, \wp_1^+\rangle - \langle \hbar_{11}, \wp_{11}\rangle\|}{\|\langle \hbar_1^+, \wp_1^+\rangle - \langle \hbar_1^-, \wp_1^-\rangle\|} + w_2 \frac{\|\langle \hbar_2^+, \wp_2^+\rangle - \langle \hbar_{12}, \wp_{12}\rangle\|}{\|\langle \hbar_2^+, \wp_2^+\rangle - \langle \hbar_2^-, \wp_2^-\rangle\|} + w_3 \frac{\|\langle \hbar_3^+, \wp_3^+\rangle - \langle \hbar_{13}, \wp_{13}\rangle\|}{\|\langle \hbar_3^+, \wp_3^+\rangle - \langle \hbar_3^-, \wp_3^-\rangle\|} \\ &\bar{0}.2 \frac{0.105}{0.43} + 0.1 \frac{0.0}{0.75} + 0.7 \frac{0.070}{0.48} = 0.1509, \end{aligned}$$

and furthermore, $S_2 = 0.0533$, $\quad S_3 = 0.1342$, $\quad S_4 = 0.3152$, which means that $S_4 < S_1 < S_3 < S_2$. Moreover, we obtain the sacrifice of each individual criterion as

$$\begin{aligned} R_1 \quad &= \max_{1 \le j \le 3} \{w_j \frac{\|{}^{\wp}h_j^+ - {}^{\wp}h_{1j}\|}{\|{}^{\wp}h_j^+ - {}^{\wp}h_j^-\|}\} = \max_{1 \le j \le 3} \{w_j \frac{\|\langle \hbar_j^+, \wp_j^+\rangle - \langle \hbar_{1j}, \wp_{1j}\rangle\|}{\|\langle \hbar_j^+, \wp_j^+\rangle - \langle \hbar_j^-, \wp_j^-\rangle\|}\} \\ &= \max\{w_1 \frac{\|\langle \hbar_1^+, \wp_1^+\rangle - \langle \hbar_{11}, \wp_{11}\rangle\|}{\|\langle \hbar_1^+, \wp_1^+\rangle - \langle \hbar_1^-, \wp_1^-\rangle\|}, w_2 \frac{\|\langle \hbar_2^+, \wp_2^+\rangle - \langle \hbar_{12}, \wp_{12}\rangle\|}{\|\langle \hbar_2^+, \wp_2^+\rangle - \langle \hbar_2^-, \wp_2^-\rangle\|}, w_3 \frac{\|\langle \hbar_3^+, \wp_3^+\rangle - \langle \hbar_{13}, \wp_{13}\rangle\|}{\|\langle \hbar_3^+, \wp_3^+\rangle - \langle \hbar_3^-, \wp_3^-\rangle\|}\} \\ &= \max\{0.2 \frac{0.105}{0.43}, 0.1 \frac{0.0}{0.75}, 0.7 \frac{0.070}{0.48}\} = 0.1021 \end{aligned}$$

and furthermore, $R_2 = 0.0533$, $\quad R_3 = 0.0605$, $\quad R_4 = 0.2333$. which means that $R_4 < R_1 < R_3 < R_2$. We obtain that

$$S^+ := \min_{1 \le i \le 4} \{S_i\} = 0.0533, \quad S^- := \max_{1 \le i \le 4} \{S_i\} = 0.3152,$$
$$R^+ := \min_{1 \le i \le 4} \{R_i\} = 0.0533, \quad R^- := \max_{1 \le i \le 4} \{R_i\} = 0.2333.$$

Step 3. Keeping the above results in mind, we find for $\alpha = \frac{1}{2}$ that

$$Q_1 = \frac{1}{2}(\frac{S_1 - S^+}{S^- - S^+}) + \frac{1}{2}(\frac{R_1 - R^+}{R^- - R^+}) = \frac{1}{2}(\frac{0.1509 - 0.0533}{0.3152 - 0.0533}) + \frac{1}{2}(\frac{0.1021 - 0.0533}{0.2333 - 0.0533}) = 0.3219,$$

and furthermore, $Q_2 = 0.0$, $\quad Q_3 = 0.1744$, $\quad Q_4 = 1.0$. which means that $Q_4 < Q_1 < Q_3 < Q_2$.

Step 4. Since all the values S_i, R_i and Q_i suggest the same ordering, we are now able to rank the alternatives as

$$A_4 < A_1 < A_3 < A_2.$$

From the results summarized in Table 7, one can observe that although the optimal alternative attained by the three methods is A_2 (Huashan Hospital of Fudan University), but Zhang et al.'s [30] method is based on the score and deviation values of PHFEs, that is, the PHFEs are sorted by taking the absolute priorities into account which is not very logical and accurate. In a sense, such a sorting technique of PHFEs is not actually precise.
On the other hand, although Song et al's [24] possibility degree-based method may reduce effectively the complexity of computation, but it is not sensitive enough to un-ordered-PHFEs (see Remark 4.1).

Table 7. The ranking results of Chines hospitals.

	Ranking order	Optimal alternative
Zhang et al.'s [30] method	$A_1 < A_4 < A_3 < A_2$	A_2
Song et al.'s [24] method	$A_1 < A_4 < A_3 < A_2$	A_2
The proposed method	$A_4 < A_1 < A_3 < A_2$	A_2

4.3 Case study III

In order to further investigate the capability of the modified PHFE concept, we resolve the problem of PHFE-based MCDM which was considered by Wang and Li in [27]. There, Wang and Li [27] took the case where there are a number of decision makers $\mathcal{D} = \{D_1, D_2, ..., D_r\}$, each with a preference ordering over a set of alternatives A_i $(i = 1, ..., m)$ with respect to the criteria C_j $(j = 1, ..., n)$ in the form of decision matrices consisting of PHFE elements as:

$$D_k = \begin{bmatrix} & \mathrm{C}_1 & \mathrm{C}_2 & \cdots & C_n \\ \mathrm{A}_1 & {}^{\wp}h_{11}^k & {}^{\wp}h_{12}^k & & {}^{\wp}h_{1n}^k \\ \mathrm{A}_2 & {}^{\wp}h_{21}^k & {}^{\wp}h_{22}^k & & {}^{\wp}h_{2n}^k \\ \vdots & \vdots & \vdots & & \vdots \\ \mathrm{A}_m & {}^{\wp}h_{m1}^k & {}^{\wp}h_{m2}^k & & {}^{\wp}h_{mn}^k \end{bmatrix}, \quad (k = 1, 2, ..., r).$$

Here, all criteria are supposed to be independent to each other, and their weights w_j $(j = 1, 2, ..., n)$ are assumed to be completely unknown.
In the remainder of this section, we present the MCDM technique of Wang and Li [27] with some minor modifications. Wang and Li [27]'s MCDM method

Step 1. With the individual probabilistic hesitant fuzzy decision matrices at hand, we integrate them into an overall decision matrix by the use of aggregation ${}^{\wp}h := PHFWA({}^{\wp}h^1, {}^{\wp}h^2, ..., {}^{\wp}h^r)$ given by (22).

Step 2. We derive the weight of the criteria as

$$w_i = \frac{\sum_{j=1}^{n} E({}^{\wp}h_{ij})}{\sum_{i=1}^{m}\sum_{j=1}^{n} E({}^{\wp}h_{ij})}, \quad i = 1, ..., m, \tag{34}$$

where $E({}^{\wp}h_{ij}) := E(\langle \hbar_{ij}, \wp_{ij}\rangle) = \sum_{\langle \hbar_{ij}, \wp_{ij}\rangle \in {}^{\wp}h_{ij}} \hbar_{ij} \times \wp_{ij}$ for $i = 1, ..., m$, $j = 1, ..., n$.

Step 3. By supposing the ideal alternative A_i^* under all criterion C_j $(j = 1, 2, ..., n)$ to be defined by

$$\langle \hbar_i^*, \wp_i^*\rangle = \max_{1\leq j\leq n}\{\langle \hbar_{ij}, \wp_{ij}\rangle\}, \quad i = 1, ..., m, \tag{35}$$

where $\max_{1\leq j\leq n}\{\langle \hbar_{ij}, \wp_{ij}\rangle\} = Arg(\max_{1\leq j\leq n}\langle \hbar_{ij} \times \wp_{ij}\rangle)$, we compute the weighted correlation coefficient between A_i and A_i^* as

$$\rho_\omega(A_i, A_i^*) = \frac{Cov_\omega(A_i, A_i^*)}{(Cov_\omega(A_i, A_i) \times Cov_\omega(A_i^*, A_i^*))^{\frac{1}{2}}}, \quad i = 1, ..., m, \tag{36}$$

in which

$$Cov_\omega(A_i, A_i^*) = E[\langle \hbar_i, \wp_i\rangle - E(\omega_i\langle \hbar_i, \wp_i\rangle)][\langle \hbar_i^*, \wp_i^*\rangle - E(\omega_i\langle \hbar_i^*, \wp_i^*\rangle)]. \tag{37}$$

Here, the notation $\omega_i\langle \hbar_i, \wp_i\rangle$ indicates the term of $\langle \omega_i\hbar_i, \wp_i\rangle$.

Step 4. Taking the values of $\rho_\omega(A_i, A_i^*)$ for $(i = 1, 2, ..., m)$ into account, we are able to rank all alternatives A_i by the help of increasing order of

$$R_i = \sum_{j=1}^{n} \rho_\omega(A_i, A_i^*)|_{C_j}, \quad i = 1, 2, ..., m. \tag{38}$$

where $\rho_\omega(A_i, A_i^*)|_{C_j}$ stands for the computation value of $\rho_\omega(A_i, A_i^*)$ with respect to the criterion C_j.

Now, we are in a position to state the main result of this section concerning the MCDM problem being investigated priorly by Wang and Li [27].

Wang and Li [27] considered a problem in which an investment company is going to invest the sum of money by taking four alternatives: A_1 (car company), A_2 (food company), A_3 (computer company) and A_4 (arms company) into account. These alternatives are evaluated based on the following five criteria: C_1 (productivity), C_2 (technological innovation capability), C_3 (marketing capability), C_4 (management) and C_5 (risk avoidance). In this regard, the company has invited four equal-weighted experts D_r $(r = 1, 2, 3, 4)$ to provide their preference evaluations on the above-mentioned alternatives. From each strategic decision area, the experts provide the preference evaluations in the form of PHFEs which are obtained by consulting ten non-similar persons in the same industry using an online questionnaire. These date are respectively shown in Tables 8-11.

Table 8. The probabilistic hesitant fuzzy decision matrix provided by D_1.

	C_1	C_2	C_3	C_4	C_5
A_1	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.7, 1\rangle\}$	$\{\langle 0.4, 0.4\rangle, \langle 0.3, 0.6\rangle\}$	$\{\langle 0.6, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$
A_2	$\{\langle 0.3, 1\rangle\}$	$\{\langle 0.4, 1\rangle\}$	$\{\langle 0.6, 0.5\rangle, \langle 0.5, 0.5\rangle\}$	$\{\langle 0.7, 1\rangle\}$	$\{\langle 0.6, 1\rangle\}$
A_3	$\{\langle 0.6, 0.4\rangle, \langle 0.7, 0.6\rangle\}$	$\{\langle 0.4, 0.5\rangle, \langle 0.5, 0.5\rangle\}$	$\{\langle 0.3, 1\rangle\}$	$\{\langle 0.3, 1\rangle\}$	$\{\langle 0.4, 1\rangle\}$
A_4	$\{\langle 0.4, 1\rangle\}$	$\{\langle 0.3, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.4, 0.5\rangle, \langle 0.5, 0.5\rangle\}$

Table 9. The probabilistic hesitant fuzzy decision matrix provided by D_2.

	C_1	C_2	C_3	C_4	C_5
A_1	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.6, 1\rangle\}$	$\{\langle 0.4, 1\rangle\}$	$\{\langle 0.7, 1\rangle\}$	$\{\langle 0.3, 1\rangle\}$
A_2	$\{\langle 0.4, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.4, 1\rangle\}$	$\{\langle 0.6, 1\rangle\}$	$\{\langle 0.6, 1\rangle\}$
A_3	$\{\langle 0.6, 1\rangle\}$	$\{\langle 0.6, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.3, 1\rangle\}$	$\{\langle 0.4, 1\rangle\}$
A_4	$\{\langle 0.3, 1\rangle\}$	$\{\langle 0.4, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.7, 1\rangle\}$

Table 10. The probabilistic hesitant fuzzy decision matrix provided by D_3.

	C_1	C_2	C_3	C_4	C_5
A_1	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.8, 0.4\rangle, \langle 0.7, 0.6\rangle\}$	$\{\langle 0.4, 1\rangle\}$	$\{\langle 0.7, 1\rangle\}$	$\{\langle 0.4, 1\rangle\}$
A_2	$\{\langle 0.2, 0.4\rangle, \langle 0.3, 0.6\rangle\}$	$\{\langle 0.4, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.4, 1\rangle\}$	$\{\langle 0.6, 1\rangle\}$
A_3	$\{\langle 0.6, 0.4\rangle, \langle 0.7, 0.6\rangle\}$	$\{\langle 0.4, 0.5\rangle, \langle 0.5, 0.5\rangle\}$	$\{\langle 0.6, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.7, 1\rangle\}$
A_4	$\{\langle 0.6, 1\rangle\}$	$\{\langle 0.3, 1\rangle\}$	$\{\langle 0.4, 0.5\rangle, \langle 0.3, 0.5\rangle\}$	$\{\langle 0.4, 0.7\rangle, \langle 0.5, 0.3\rangle\}$	$\{\langle 0.2, 1\rangle\}$

Table 11. The probabilistic hesitant fuzzy decision matrix provided by D_4.

	C_1	C_2	C_3	C_4	C_5
A_1	$\{\langle 0.7, 1\rangle\}$	$\{\langle 0.7, 1\rangle\}$	$\{\langle 0.6, 1\rangle\}$	$\{\langle 0.4, 0.5\rangle, \langle 0.3, 0.5\rangle\}$	$\{\langle 0.8, 1\rangle\}$
A_2	$\{\langle 0.3, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.8, 1\rangle\}$	$\{\langle 0.7, 0.5\rangle, \langle 0.6, 0.5\rangle\}$	$\{\langle 0.4, 1\rangle\}$
A_3	$\{\langle 0.4, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$	$\{\langle 0.2, 1\rangle\}$	$\{\langle 0.1, 1\rangle\}$	$\{\langle 0.7, 0.5\rangle, \langle 0.6, 0.5\rangle\}$
A_4	$\{\langle 0.2, 1\rangle\}$	$\{\langle 0.6, 1\rangle\}$	$\{\langle 0.4, 0.5\rangle, \langle 0.3, 0.5\rangle\}$	$\{\langle 0.6, 1\rangle\}$	$\{\langle 0.5, 1\rangle\}$

Remark 4.2. *Before going further, we should note that all the PHFEs of Tables 8-11 have not been arranged in the increasing order of real values $\hbar \times \wp$. This means that all the data in Tables 8-11 are not ordered-PHFEs. To eliminate this problem, for instance, we replace the $(2,3)$-th entry in Table 8 with $\{\langle 0.5, 0.5\rangle\ \langle 0.6, 0.5\rangle\}$.*

Now, by the help of Step 1 of Algorithm 4.3, we are able to achieve the overall decision matrix of four individual probabilistic hesitant fuzzy decision matrices D_r $(r = 1, 2, 3, 4)$ as that given in Table 12.

Table 12. The proposed aggregated probabilistic hesitant fuzzy decision matrix.

	C_1	C_2	C_3	C_4	C_5
A_1	$\{\langle 0.9625, 1\rangle\}$	$\{\langle 0.9928, 0.5\rangle, \langle 0.9892, 0.5\rangle\}$	$\{\langle 0.9136, 0.25\rangle, \langle 0.9137, 0.25\rangle, \langle 0.8992, 0.25\rangle, \langle 0.8993, 0.25\rangle\}$	$\{\langle 0.9784, 0.5\rangle, \langle 0.9748, 0.5\rangle\}$	$\{\langle 0.9580, 1\rangle\}$
A_2	$\{\langle 0.7648, 0.5\rangle, \langle 0.7942, 0.5\rangle\}$	$\{\langle 0.8740, 1\rangle\}$	$\{\langle 0.9760, 0.5\rangle, \langle 0.9700, 0.5\rangle\}$	$\{\langle 0.9784, 0.5\rangle, \langle 0.9712, 0.5\rangle\}$	$\{\langle 0.9616, 1\rangle\}$
A_3	$\{\langle 0.9616, 0.25\rangle, \langle 0.9712, 0.25\rangle, \langle 0.9713, 0.25\rangle, \langle 0.9784, 0.25\rangle\}$	$\{\langle 0.9280, 0.25\rangle, \langle 0.9400, 0.25\rangle, \langle 0.9401, 0.25\rangle, \langle 0.9500, 0.25\rangle\}$	$\{\langle 0.8880, 1\rangle\}$	$\{\langle 0.7795, 1\rangle\}$	$\{\langle 0.9676, 0.5\rangle, \langle 0.9568, 0.5\rangle\}$
A_4	$\{\langle 0.8656, 1\rangle\}$	$\{\langle 0.8824, 1\rangle\}$	$\{\langle 0.9100, 0.25\rangle, \langle 0.8950, 0.25\rangle, \langle 0.8950, 0.25\rangle, \langle 0.8775, 0.25\rangle\}$	$\{\langle 0.9400, 0.5\rangle, \langle 0.9500, 0.5\rangle\}$	$\{\langle 0.9280, 0.5\rangle, \langle 0.9400, 0.5\rangle\}$

Table 13. The Zhang et al.'s [30] and Song et al.'s [24] aggregated probabilistic hesitant fuzzy decision matrix.

	C_1	C_2	C_3	C_4	C_5
A_1	$\{\langle 0.9625, 1\rangle\}$	$\{\langle 0.9928, 0.4\rangle, \langle 0.9892, 0.6\rangle\}$	$\{\langle 0.9136, 0.16\rangle, \langle 0.9137, 0.24\rangle, \langle 0.8992, 0.24\rangle, \langle 0.8993, 0.36\rangle\}$	$\{\langle 0.9784, 0.5\rangle, \langle 0.9748, 0.5\rangle\}$	$\{\langle 0.9580, 1\rangle\}$
A_2	$\{\langle 0.7648, 0.4\rangle, \langle 0.7942, 0.6\rangle\}$	$\{\langle 0.8740, 1\rangle\}$	$\{\langle 0.9760, 0.5\rangle, \langle 0.9700, 0.5\rangle\}$	$\{\langle 0.9784, 0.5\rangle, \langle 0.9712, 0.5\rangle\}$	$\{\langle 0.9616, 1\rangle\}$
A_3	$\{\langle 0.9616, 0.16\rangle, \langle 0.9712, 0.24\rangle, \langle 0.9713, 0.24\rangle, \langle 0.9784, 0.36\rangle\}$	$\{\langle 0.9280, 0.25\rangle, \langle 0.9400, 0.25\rangle, \langle 0.9401, 0.25\rangle, \langle 0.9500, 0.25\rangle\}$	$\{\langle 0.8880, 1\rangle\}$	$\{\langle 0.7795, 1\rangle\}$	$\{\langle 0.9676, 0.5\rangle, \langle 0.9568, 0.5\rangle\}$
A_4	$\{\langle 0.8656, 1\rangle\}$	$\{\langle 0.8824, 1\rangle\}$	$\{\langle 0.9100, 0.25\rangle, \langle 0.8950, 0.25\rangle, \langle 0.8950, 0.25\rangle, \langle 0.8775, 0.25\rangle\}$	$\{\langle 0.9400, 0.7\rangle, \langle 0.9500, 0.3\rangle\}$	$\{\langle 0.9280, 0.5\rangle, \langle 0.9400, 0.5\rangle\}$

Table 14. The Wang and Li's [27] aggregated probabilistic hesitant fuzzy decision matrix.

	C_1	C_2	C_3	C_4	C_5
A_1	$\{\langle 0.5, 0.75\rangle, \langle 0.75, 0.25\rangle\}$	$\{\langle 0.6, 0.25\rangle, \langle 0.7, 0.65\rangle, \langle 0.8, 0.1\rangle\}$	$\{\langle 0.3, 0.15\rangle, \langle 0.4, 0.6\rangle, \langle 0.6, 0.25\rangle\}$	$\{\langle 0.3, 0.125\rangle, \langle 0.4, 0.125\rangle, \langle 0.6, 0.25\rangle, \langle 0.7, 0.5\rangle\}$	$\{\langle 0.3, 0.25\rangle, \langle 0.4, 0.25\rangle, \langle 0.5, 0.25\rangle, \langle 0.8, 0.25\rangle\}$
A_2	$\{\langle 0.2, 0.1\rangle, \langle 0.3, 0.65\rangle, \langle 0.4, 0.25\rangle\}$	$\{\langle 0.4, 0.5\rangle, \langle 0.5, 0.5\rangle\}$	$\{\langle 0.4, 0.25\rangle, \langle 0.5, 0.375\rangle, \langle 0.6, 0.125\rangle, \langle 0.8, 0.25\rangle\}$	$\{\langle 0.4, 0.25\rangle, \langle 0.6, 0.375\rangle, \langle 0.7, 0.375\rangle\}$	$\{\langle 0.6, 0.75\rangle, \langle 0.4, 0.25\rangle\}$
A_3	$\{\langle 0.4, 0.25\rangle, \langle 0.6, 0.45\rangle, \langle 0.7, 0.3\rangle\}$	$\{\langle 0.4, 0.25\rangle, \langle 0.5, 0.5\rangle, \langle 0.6, 0.25\rangle\}$	$\{\langle 0.2, 0.25\rangle, \langle 0.3, 0.25\rangle, \langle 0.5, 0.25\rangle, \langle 0.6, 0.25\rangle\}$	$\{\langle 0.1, 0.25\rangle, \langle 0.3, 0.5\rangle, \langle 0.5, 0.25\rangle\}$	$\{\langle 0.4, 0.5\rangle, \langle 0.6, 0.125\rangle, \langle 0.7, 0.375\rangle\}$
A_4	$\{\langle 0.2, 0.25\rangle, \langle 0.3, 0.25\rangle, \langle 0.4, 0.25\rangle, \langle 0.6, 0.25\rangle\}$	$\{\langle 0.3, 0.5\rangle, \langle 0.4, 0.25\rangle, \langle 0.6, 0.25\rangle\}$	$\{\langle 0.3, 0.25\rangle, \langle 0.4, 0.25\rangle, \langle 0.5, 0.5\rangle\}$	$\{\langle 0.4, 0.175\rangle, \langle 0.6, 0.25\rangle, \langle 0.5, 0.575\rangle\}$	$\{\langle 0.2, 0.25\rangle, \langle 0.5, 0.325\rangle, \langle 0.4, 0.125\rangle, \langle 0.7, 0.25\rangle\}$

Following Zhang et al.'s [30], Song et al.'s [24] and Wang and Li's [27] methods, the corresponding aggregated probabilistic hesitant fuzzy decision matrices are represented in the forms of Tables 13 and 14.
What should be noticed here in regard of the data in Table 14 is that the aggregated values of PHFEs, which are indeed HFEs, do not remain in the form of HFEs. For instance, the $(1, 1)$-th entry in Table 14 must be in the form of a HFE because it is aggregated value of four corresponding HFEs in Tables 8-11. But this is not the case, and the existing $(1, 1)$-th entry in Table 14 is not a HFE.

If we apply Step 2 of Algorithm 4.3 by considering the proposed method, then the weight of criteria will be calculated as those given in the last row of Table 15. The results of Wang and Li's [27], Zhang et al.'s [30], and Song et al.'s [24] methods have been summarized in the first rows of Table 15.
Table 15 implies that Wang and Li's [27] method is not able to distinguish between some weight values as well as the other methods.

Table 15. The weights of criteria.

	Weight of criterion	Order of weights
Wang and Li.'s [27] method	$w_1 = 0.1900,\ w_2 = 0.2100,\ w_3 = 0.1900,\ w_4 = 0.2000,\ w_5 = 0.2100$	$w_1 = w_3 < w_4 < w_2 = w_5$
Zhang et al.'s [30] method	$w_1 = 0.2075,\ w_2 = 0.2095,\ w_3 = 0.1537,\ w_4 = 0.1876,\ w_5 = 0.2416$	$w_3 < w_4 < w_1 < w_2 < w_5$
Song et al.'s [24] method	$w_1 = 0.2075,\ w_2 = 0.2095,\ w_3 = 0.1537,\ w_4 = 0.1876,\ w_5 = 0.2416$	$w_3 < w_4 < w_1 < w_2 < w_5$
The proposed method	$w_1 = 0.2073,\ w_2 = 0.2095,\ w_3 = 0.1538,\ w_4 = 0.1877,\ w_5 = 0.2416$	$w_3 < w_4 < w_1 < w_2 < w_5$

By applying Step 3 of Algorithm 4.3, we conclude that all the proposed method, Zhang et al.'s [30] and Song et al.'s [24] methods give rise to

$$A_1^* = \langle \hbar_1^*, \wp_1^* \rangle = \{\langle 0.9625, 1\rangle\},\ A_2^* = \langle \hbar_2^*, \wp_2^* \rangle = \{\langle 0.9616, 1\rangle\},$$
$$A_3^* = \langle \hbar_3^*, \wp_3^* \rangle = \{\langle 0.8880, 1\rangle\},\ A_4^* = \langle \hbar_4^*, \wp_4^* \rangle = \{\langle 0.8824, 1\rangle\}.$$

Moreover, by keeping the values of weighted correlation coefficients $\rho_\omega(A_i, A_i^*)$ for $(i = 1, 2, 3, 4)$ in the mind, Step 4 gives rise to the ranking results which are represented in Table 16.

Table 16. The ranking results of various investments.

	Ranking order	Optimal alternative
Wang and Li.'s [27] method	$A_4 < A_1 < A_3 < A_2$ $0.4200 < 2.5800 < 3.0000 < 4.3600$	A_2
Zhang et al.'s [30] method	$A_3 < A_4 < A_2 < A_1$ $-0.0013 < -0.0002 < 0.0016 < 0.0010$	A_1
Song et al.'s [24] method	$A_3 < A_4 < A_2 < A_1$ $-0.0013 < -0.0002 < 0.0016 < 0.0010$	A_1
The proposed method	$A_3 < A_4 < A_1 < A_2$ $-0.1271 < -0.0083 < 0.0353 < 0.1190$	A_2

According to Table 16, we can observe that the ranking order of Zhang et al.'s [30] and Song et al.'s [24] methods do not agree well with the ranking order of the proposed method. Such a fair outcome is due to the reasons explained before. The former method [30] is based on the score and deviation values of PHFEs, and the latter method [24] is not sensitive enough to un-ordered-PHFEs. Furthermore, Wang and Li's [27] method suffers from some of the drawbacks which were encountered priorly in Remark 4.2 and the paragraphs following that.

5 Conclusions and future works

In this contribution, we first indicated that how the existing definitions of PHFS can be improved. Then, a verity of set and algebraic operations for the modified PHFS were proposed based on T-norms and S-norms together with a number of aggregation operators. By re-considering two MCDM problems, which are implemented to evaluate the automotive industry safety and the priority of Chines hospitals, we demonstrated that not only the re-defined PHFS can produce precise results, but also it is more efficacious compared to the existing ones. In the future, we will focus on the definition of division and subtraction operations on PHFSs to enhance the theory with new insights, and moreover, some other potential applications of MCDM problems involving PHFSs will be explored.

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