Iranian Journal of Fuzzy Systems

Volume 17, Number 5, (2020), pp. 13-22

University of Sistan and Baluchestan

Individual ergodic theorem for intuitionistic fuzzy observables using intuitionistic fuzzy state

K. Čunderlíková[1]

[1] *Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia*

cunderlikova.lendelova@gmail.com

Abstract

The classical ergodic theory has been built on σ-algebras. Later the Individual ergodic theorem was studied on more general structures like MV-algebras and quantum structures. The aim of this paper is to formulate the Individual ergodic theorem for intuitionistic fuzzy observables using **m**-almost everywhere convergence, where **m** is an intuitionistic fuzzy state. We show the Kolmogorov construction for intuitionistic fuzzy observables, too.

Keywords: The intuitionistic fuzzy event, the intuitionistic fuzzy observable, the intuitionistic fuzzy state, the product, the upper limit, the lower limit, the **m**-almost everywhere convergence, the **m**-preserving transformation, the individual ergodic theorem, the Kolmogorov construction.

1 Introduction

In [1, 2] K.T. Atanassov introduced the notion of intuitionistic fuzzy sets. Then in [8] B. Riečan defined the intuitionistic fuzzy state on the family of intuitionistic fuzzy events $\mathcal{F} = \{(\mu_A, \nu_A)\,; \mu_A + \nu_A \leq 1_\Omega\}$, where μ_A, ν_A are $\mathcal{S}$-measurable functions, $\mu_A, \nu_A : \Omega \to [0, 1]$, as a mapping $\mathbf{m}$ from the family $\mathcal{F}$ to the set R by the formula

$$\mathbf{m}\big((\mu_A, \nu_A)\big) = (1 - \alpha)\int_\Omega \mu_A dP + \alpha\left(1 - \int_\Omega \nu_A dP\right),$$

where $P : \mathcal{S} \to [0, 1]$ is a probability measure and $\alpha \in [0, 1]$.

In paper [4] we defined the upper and the lower limits for sequence of intuitionistic fuzzy observables. We used an intuitionistic fuzzy state **m** to define the notion of almost everywhere convergence. We compared two concepts of **m**-almost everywhere convergence.

In paper [5] we studied the **m**-almost everywhere convergence of sequence of intuitionistic fuzzy observables $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ given by $g_n(x_1, \ldots, x_n) = h_n \circ g_n^{-1}$, where $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ is the joint intuitionistic fuzzy observable of intuitionistic fuzzy observables $x_1, \ldots, x_n$ and $g_n : R^n \to R$ is a Borel measurable function. We showed the connection between **m**-almost everywhere convergence of this sequence of intuitionistic fuzzy observables and P-almost everywhere convergence of random variables in classical probability space induced by Kolmogorov construction. This connection is a start point for proving the Individual ergodic theorem for intuitionistic fuzzy observables using **m**-almost everywhere convergence.

Recall that the formulation of the Individual ergodic theorem for intuitionistic fuzzy events with product first appeared in the paper [3]. There we used $\mathcal{P}$-almost everywhere convergence, where $\mathcal{P}$ was a separating intuitionistic fuzzy probability. Since the intuitionistic fuzzy probability $\mathcal{P}$ can be decomposited to two intuitionistic fuzzy states, it is usefull to study **m**-almost everywhere convergence, where **m** is an intuitionistic fuzzy state. In this paper we formulate the Individual ergodic theorem for intuitionistic fuzzy observables using **m**-almost everywhere convergence. We show the Kolmogorov construction for intuitionistic fuzzy observables, too.

Corresponding Author: K. Čunderlíková

Received: November 2019; Revised: February 2020; Accepted: April 2020.

Remark that in a whole text we use a notation IF as an abbreviation for intuitionistic fuzzy.

2 IF-events, IF-states, IF-observables and IF-mean value

First we start with definitions of basic notions (see [1, 2, 9]).

Definition 2.1. *Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.*

Definition 2.2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable.*

The family of all IF-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$, $\mu_A : \Omega \longrightarrow [0,1]$ will will called the membership function, $\nu_A : \Omega \longrightarrow [0,1]$ be called the non-membership function.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Lukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$\mathbf{A} \oplus \mathbf{B} = ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)), \quad \text{and} \quad \mathbf{A} \odot \mathbf{B} = ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))$$

and the partial ordering is given by $\mathbf{A} \leq \mathbf{B}$ if and only if $\mu_A \leq \mu_B, \nu_A \geq \nu_B$. In paper we use max-min connectives defined by

$$\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B), \quad \text{and} \quad \mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$$

and the de Morgan rules $(a \vee b)^* = a^* \wedge b^*$, and $(a \wedge b)^* = a^* \vee b^*$, where $a^* = 1 - a$.

Example 2.3. *Fuzzy set $f : \Omega \longrightarrow [0,1]$ can be regarded as IF-set, if we put $\mathbf{A} = (f, 1_\Omega - f)$. If $f = \chi_A$, then the corresponding IF-set has the form $\mathbf{A} = (\chi_A, 1_\Omega - \chi_A) = (\chi_A, \chi_{A'})$. In this case $\mathbf{A} \oplus \mathbf{B}$ corresponds to the union of sets, $\mathbf{A} \odot \mathbf{B}$ to the intersection of sets and $\leq$ to the set inclusion.*

In the IF-probability theory [10] instead of the notion of probability we use the notion of state.

Definition 2.4. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ is called an IF-state, if the following conditions are satisfied:*

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ *and* $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, *then* $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ *(i.e.* $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$*), then* $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Probably the most useful result in the IF-state theory is the following representation theorem [8]:

Theorem 2.5. *To each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists exactly one probability measure $P : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$,*

$$\mathbf{m}(\mathbf{A}) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega \nu_A dP\right).$$

Proof. In [8] *Theorem.* □

The third basic notion in the probability theory is the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form $[a, b) = \{x \in R : a \leq x < b\}$. Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets.

Definition 2.6. *By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \emptyset$, *then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$, *then* $x(A_n) \nearrow x(A)$.

If we denote $x(A) = \big(x^\flat(A), 1_\Omega - x^\sharp(A)\big)$ for each $A \in \mathcal{B}(R)$, then $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables, where $\mathcal{T} = \{f : \Omega \to [0,1]; f$ is $\mathcal{S} - measurable\}$.

Remark 2.7. *Sometimes we need to work with n-dimensional IF-observable $x : \mathcal{B}(R^n) \to \mathcal{F}$ defined as a mapping with the following conditions:*

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *if* $A \cap B = \emptyset$, $A, B \in \mathcal{B}(R^n)$, *then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$;

(iii) *if* $A_n \nearrow A$, *then* $x(A_n) \nearrow x(A)$ *for each* $A, A_n \in \mathcal{B}(R^n)$.

If $n = 1$, then we simply say that x is an IF-observable.

Similarly as in the classical case the following theorem can be proved [7, 10].

Theorem 2.8. *Let $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be an IF-observable, $\mathbf{m} : \mathcal{F} \longrightarrow [0,1]$ be an IF-state. Define the mapping $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ by the formula $\mathbf{m}_x(C) = \mathbf{m}(x(C))$. Then $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ is a probability measure.*

Proof. In [7] *Proposition 3.1.* □

Since now $\mathbf{m}_x : \mathcal{B}(R) \to [0,1]$ plays an analogous role as $P_\xi : \mathcal{B}(R) \to [0,1]$, we can define **IF-expected value** $\mathbf{E}(x)$ by the same formula (see [7]).

Definition 2.9. *We say that an IF-observable x is an integrable IF-observable, if the integral $\int_R t \; d\mathbf{m}_x(t)$ exists. In this case we define IF-expected value $\mathbf{E}(x) = \int_R t \; d\mathbf{m}_x(t)$. If the integral $\int_R t^2 \; d\mathbf{m}_x(t)$ exists, then we define IF-dispersion $\mathbf{D}^2(x)$ by the formula*

$$\mathbf{D}^2(x) = \int_R t^2 \; d\mathbf{m}_x(t) - \big(\mathbf{E}(x)\big)^2 = \int_R (t - \mathbf{E}(x))^2 \; d\mathbf{m}_x(t).$$

3 Product operation, joint IF-observable and function of several IF-observables

In [6] we introduced the notion of product operation on the family of IF-events $\mathcal{F}$ and showed an example of this operation.

Definition 3.1. *We say that a binary operation $\cdot$ on $\mathcal{F}$ is product if it satisfies the following conditions:*

(i) $(1_\Omega, 0_\Omega) \cdot (a_1, a_2) = (a_1, a_2)$ *for each* $(a_1, a_2) \in \mathcal{F}$;

(ii) *the operation $\cdot$ is commutative and associative;*

(iii) *if* $(a_1, a_2) \odot (b_1, b_2) = (0_\Omega, 1_\Omega)$ *and* $(a_1, a_2), (b_1, b_2) \in \mathcal{F}$, *then*

$$(c_1, c_2) \cdot \big((a_1, a_2) \oplus (b_1, b_2)\big) = \big((c_1, c_2) \cdot (a_1, a_2)\big) \oplus \big((c_1, c_2) \cdot (b_1, b_2)\big)$$

and $\big((c_1, c_2) \cdot (a_1, a_2)\big) \odot \big((c_1, c_2) \cdot (b_1, b_2)\big) = (0_\Omega, 1_\Omega)$, *for each* $(c_1, c_2) \in \mathcal{F}$;

(iv) *if* $(a_{1n}, a_{2n}) \searrow (0_\Omega, 1_\Omega)$, $(b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$ *and* $(a_{1n}, a_{2n}), (b_{1n}, b_{2n}) \in \mathcal{F}$, *then* $(a_{1n}, a_{2n}) \cdot (b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$.

In the following theorem is the example of product operation for IF-events.

Theorem 3.2. *The operation $\cdot$ defined by $(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2)$ for each $(x_1, y_1), (x_2, y_2) \in \mathcal{F}$ is product operation on $\mathcal{F}$.*

Proof. In [6] *Theorem 1.* □

In [9] B. Riečan defined the notion of a joint IF-observable and he proved its existence.

Definition 3.3. *Let $x, y : \mathcal{B}(R) \to \mathcal{F}$ be two IF-observables. The joint IF-observable of the IF-observables x, y is a mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:*

(i) $h(R^2) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *if* $A, B \in \mathcal{B}(R^2)$ *and* $A \cap B = \emptyset$, *then* $h(A \cup B) = h(A) \oplus h(B)$ *and* $h(A) \odot h(B) = (0_\Omega, 1_\Omega)$;

(iii) *if* $A, A_1, \ldots \in \mathcal{B}(R^2)$ *and* $A_n \nearrow A$, *then* $h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C) \cdot y(D)$ *for each* $C, D \in \mathcal{B}(R)$.

Theorem 3.4. *For each two IF-observables* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *there exists their joint IF-observable.*

Proof. In [9] *Theorem 3.3.* □

Remark 3.5. *The joint IF-observable of IF-observables* x, y *from Definition 3.3 is two-dimensional IF-observable.*

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is the function of several IF-observables. About this says the following definition.

Definition 3.6. *Let* $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ *be IF-observables,* h_n *their joint IF-observable and* $g_n : R^n \to R$ *a Borel measurable function. Then we define the IF-observable* $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ *by the formula* $g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big)$, *for each* $A \in \mathcal{B}(R)$.

4 Kolmogorov construction

In this section we introduce the notion of compatibility of intuitionistic fuzzy observables as follows.

Definition 4.1. *Let* $(\mathcal{F}, \cdot)$ *be a family of IF-events with product and* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *be the IF-observables on* $\mathcal{F}$. *We say that IF-observables* x, y *are compatible, if there exists their joint IF-observable* h.

We can generalize the notion of compatibility for k IF-observables $x_{i_1}, \ldots, x_{i_k}$.

Definition 4.2. *Let* $J \subset N$, $J = \{i_1, \ldots, i_k\}$ *and* $x_{i_1}, \ldots, x_{i_k}$ *be the IF-observables on* $\mathcal{F}$. *We say that the IF-observables* $x_{i_1}, \ldots, x_{i_k}$ *are compatible, if there exists a mapping* $h_J : \mathcal{B}(R^{|J|}) \to \mathcal{F}$ *satisfying the following conditions:*

(i) $h_J(R^{|J|}) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$

(ii) *if* $A, B \in \mathcal{B}(R^{|J|})$ *and* $A \cap B = \emptyset$, *then* $h_J(A \cup B) = h_J(A) \oplus h_J(B)$ *and* $h_J(A) \odot h_J(B) = (0_\Omega, 1_\Omega)$;

(iii) *if* $A, A_1, \ldots \in \mathcal{B}(R^{|J|})$ *and* $A_n \nearrow A$, *then* $h_J(A_n) \nearrow h_J(A)$;

(iv) $h_J(A_{i_1} \times \ldots \times A_{i_k}) = x_{i_1}(A_{i_1}) \cdot \ldots \cdot x_{i_k}(A_{i_k})$ *for each* $A_{i_1}, \ldots, A_{i_k} \in \mathcal{B}(R)$.

By *Definition 4.2* to every compatible IF-observables $x_1, \ldots, x_n$ there exists a morphism $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ (i.e. $h_n(R^n) = (1_\Omega, 0_\Omega)$, h_n is additive and continuous) such that $h_n(A_1 \times \ldots \times A_n) = x_1(A_1) \cdot \ldots \cdot x_n(A_n)$, for each $A_1, \ldots, A_n \in \mathcal{B}(R)$.

Lemma 4.3. *The mapping* $h_J : \mathcal{B}(R^{|J|}) \to \mathcal{F}$ *from Definition 4.2 satisfies the following conditions:*

(v) *if* $A \in \mathcal{B}(R)$, *then* $h_J(\{(t_1, \ldots, t_i, \ldots, t_k) \mid (t_1, \ldots, t_i, \ldots, t_k) \in R^{|J|}, t_i \in A\}) = x_i(A)$;

(vi) *if* $J_1 \subset J_2 \subset N$, *then* $h_{J_2}(\pi^{-1}_{J_2,J_1}(A)) = h_{J_1}(A)$ *for each* $A \in \mathcal{B}(R^{|J_1|})$, *where* $\pi_{J_2,J_1} : \mathcal{B}(R^{|J_2|}) \to \mathcal{B}(R^{|J_1|})$ *is the projection.*

Proof. (v)

$$\begin{aligned} h_J(\{(t_1, \ldots, t_i, \ldots, t_k) | (t_1, \ldots, t_i, \ldots, t_k) \in R^{|J|}, t_i \in A\}) &= h_J(R \times \ldots \times R \times A \times R \times \ldots \times R) \\ &= x_1(R) \cdot \ldots \cdot x_{i-1}(R) \cdot x_i(A) \cdot x_{i+1}(R) \cdot \ldots \cdot x_k(R) \\ &= (1_\Omega, 0_\Omega) \cdot \ldots \cdot (1_\Omega, 0_\Omega) \cdot x_i(A) \cdot (1_\Omega, 0_\Omega) \cdot \ldots \cdot (1_\Omega, 0_\Omega) \\ &= (1_\Omega, 0_\Omega) \cdot x_i(A) \cdot (1_\Omega, 0_\Omega) = x_i(A) \end{aligned}$$

(vi) Let $J_1 \subset J_2 \subset N$; $A = A_{t_1} \times \ldots \times A_{t_k} \in \mathcal{B}(R^{|J_1|})$,

$$\pi^{-1}_{J_1,J_2}(A) = R \times \ldots \times R \times A_{t_1} \times \ldots \times A_{t_k} \times R \times \ldots \times R \in \mathcal{B}(R^{|J_2|}).$$

Then

$$\begin{aligned} h_{J_2}(\pi^{-1}_{J_2,J_1}(A)) &= x_{s_1}(R) \cdot \ldots \cdot x_{s_i}(R) \cdot x_{t_1}(A_{t_1}) \cdot \ldots \cdot x_{t_k}(A_{t_k}) \cdot x_{s_j}(R) \cdot \ldots \cdot x_{s_n}(R) \\ &= (1_\Omega, 0_\Omega) \cdot \ldots \cdot (1_\Omega, 0_\Omega) \cdot x_{t_1}(A_{t_1}) \cdot \ldots \cdot x_{t_k}(A_{t_k}) \cdot (1_\Omega, 0_\Omega) \cdot \ldots \cdot (1_\Omega, 0_\Omega) \end{aligned}$$

$$
\begin{aligned}
&= (1_\Omega, 0_\Omega) \cdot x_{t_1}(A_{t_1}) \cdot \ldots \cdot x_{t_k}(A_{t_k}) \cdot (1_\Omega, 0_\Omega) \\
&= x_{t_1}(A_{t_1}) \cdot \ldots \cdot x_{t_k}(A_{t_k}) \\
&= h_{J_1}(A).
\end{aligned}
$$

Let we consider the set $\mathcal{L} = \{A \in \mathcal{B}(R^{|J|}),\ h_{J_2}(\pi^{-1}_{J_2,J_1}(A)) = h_{J_1}(A)\}$ and denote $\mathcal{D}$ the set of all rectangles $A_{t_1} \times \ldots \times A_{t_k};\ A_{t_1}, \ldots, A_{t_k} \in \mathcal{B}(R)$.

Evidently $\mathcal{L} \supset \mathcal{D}$. The properties of mapping h_J imply that $\mathcal{L}$ is $q - \sigma$-algebra over ring $s(\mathcal{D})$ generated by set $\mathcal{D}$. Therefore

$$\mathcal{L} \supset q - \sigma(s(\mathcal{D})) = \sigma(s(\mathcal{D})) = \mathcal{B}(R^{|J_1|})$$

that implies $h_{J_2}(\pi^{-1}_{J_2,J_1}(A)) = h_{J_1}(A)$, where $A \in \mathcal{B}(R^{|J_1|})$. □

Proposition 4.4. *Let* $\mathbf{m}$ *be an IF-state on a family of IF-events with product* $(\mathcal{F}, \cdot)$*. Define* $P_n : \mathcal{B}(R^n) \to [0,1]$ *by the formula*

$$P_n(A) = \mathbf{m}(h_n(A)), \quad A \in \mathcal{B}(R).$$

Then P_n *is a probability measure such that*

$$P_n(\{(t_1, \ldots, t_i, \ldots, t_n) \mid (t_1, \ldots, t_i, \ldots, t_n) \in R^n,\ t_i \in A\}) = \mathbf{m}(x_i(A)) = \mathbf{m}_{x_i}(A).$$

Proof. The first assertion is clear. Further

$$
\begin{aligned}
P_n(\{(t_1, \ldots, t_i, \ldots, t_n) | (t_1, \ldots, t_i, \ldots, t_n) \in R^n, t_i \in A\}) &= \mathbf{m}(h_n(R \times \ldots \times R \times A \times R \times \ldots \times R)) \\
&= \mathbf{m}(x_1(R) \cdot \ldots \cdot x_{i-1}(R) \cdot x_i(A) \cdot x_{i+1}(R) \cdot \ldots \cdot x_n(R)) \\
&= \mathbf{m}((1_\Omega, 0_\Omega) \cdot x_i(A) \cdot (1_\Omega, 0_\Omega)) = \mathbf{m}(x_i(A)) = \mathbf{m}_{x_i}(A).
\end{aligned}
$$

□

Proposition 4.5. *Let* $\emptyset \neq J \subset N$*,* J *be finite,* $J = \{t_1, \ldots, t_k\}$*. Then there exists exactly one probability measure* $P_J : \mathcal{B}(R^k) \to [0,1]$ *such that*

$$P_J(A_1 \times \ldots \times A_k) = \mathbf{m}(x_{t_1}(A_1) \cdot \ldots \cdot x_{t_k}(A_k))$$

for each $A_1, \ldots, A_k \in \mathcal{B}(R)$*.*

Proof. Let $I = \{1, \ldots, t_k\} \supset J$, $\pi_{I,J}$ be the projection from R^{t_k} to R^k. Then

$$\pi^{-1}_{I,J}(A_1 \times \ldots \times A_k) = B_1 \times \ldots \times B_{t_k},$$

where $B_{t_i} = A_i$ $(i = 1, 2, \ldots, k)$; $B_j = R$ if $j \notin J$. Therefore

$$
\begin{aligned}
P_{t_k}(\pi^{-1}_{I,J}(A_1 \times \ldots \times A_k)) &= P_{t_k}(B_1 \times \ldots \times B_{t_k}) \\
&= \mathbf{m}(h_{t_k}(B_1 \times \ldots \times B_{t_k})) \\
&= \mathbf{m}(x_1(B_1) \cdot \ldots \cdot x_{t_k}(B_{t_k})) \\
&= \mathbf{m}(x_{t_1}(A_1) \cdot \ldots \cdot x_{t_k}(A_k)).
\end{aligned}
$$

Put $P_J = P_{t_k} \circ \pi^{-1}_{I,J} : \mathcal{B}(R^k) \to [0,1]$. Then P_J is a probability measure with the property stated in *Proposition 4.5.*

If μ is other measure with this property, then P_J coincides with μ on each rectangles and therefore they coincide on $\mathcal{B}(R^k)$. □

By property (vi) we obtained a family of probability measures $\{P_J \mid \emptyset \neq J \subset N, J \text{ finite}\}$ given by

$$P_J(A) = \mathbf{m}\big(h_J(A)\big),\ A \in \mathcal{B}(R).$$

The family satisfies the Kolmogorov consistency condition. E.g., if $J_2 = \{1, 2, 3\}$, $J_1 = \{1, 3\}$ and $\pi_{J_2,J_1} : R^3 \to R^2$ is the projection (assigning to a triple (t_1, t_2, t_3) a pair (t_1, t_3)) then

$$
\begin{aligned}
P_{J_2}(\pi^{-1}_{J_2,J_1}(A \times B)) &= P_{J_2}(\{(t_1, t_2, t_3) : (t_1, t_3) \in A \times B\}) \\
&= P_{J_2}(A \times R \times B) \\
&= \mathbf{m}(x_1(A) \cdot x_2(R) \cdot x_3(B)) \\
&= \mathbf{m}(x_1(A) \cdot x_3(B)) \\
&= P_{J_1}(A \times B).
\end{aligned}
$$

$$P_{J_2} \circ \pi^{-1}_{J_2,J_1} = P_{J_1}.$$

Proposition 4.6. *The family $\{P_J \mid \emptyset \neq J \subset N, J \text{ finite}\}$ satisfies the Kolmogorov consistency condition, i.e.*

$$P_{J_2}(\pi^{-1}_{J_2,J_1}(A)) = P_{J_1}(A)$$

whenever $J_1 \subset J_2$, $A \in \mathcal{B}(R^{|J_1|})$, where $\pi_{J_2,J_1} : R^{|J_2|} \to R^{|J_1|}$ is the projection.

Proof. P_{J_1} and $P_{J_2} \circ \pi^{-1}_{J_2,J_1}$ are two measures on $\mathcal{B}(R^{|J_1|})$ coinciding on the family of all rectangles. □

At this point we may use the Kolmogorov consistency theorem.

Proposition 4.7. *Let $\mathcal{C}$ be the family of all cylinders in R^N, i.e.*

$$\mathcal{C} = \{\pi^{-1}_J(A) \mid \emptyset \neq J \subset N, J \text{ finite}, A \in \mathcal{B}(R^{|J|})\}.$$

Then there exists exactly one probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that $P(\pi^{-1}_J(A)) = P_J(A)$, for each cylinders $\pi^{-1}_J(A)$. Particularly

$$P(\{(t_n)_1^\infty : t_i \in A_i, i = 1, 2, \ldots, n\}) = \mathbf{m}(h_n(A_1 \times \ldots \times A_n)) = \mathbf{m}(x_1(A_1) \cdot \ldots \cdot x_n(A_n)).$$

Proof. It follows by the Kolmogorov theorem and *Proposition 4.6.* □

Proposition 4.8. *Define the coordinate function $\xi_n : R^N \to R$ by the formula $\xi_n((t_i)_1^\infty) = t_n$. Then ξ_n is a random variable with respect to $\sigma(\mathcal{C})$ such that $P_{\xi_n} = \mathbf{m} \circ x_n = \mathbf{m}_{x_n}$.*

Proof. If $A \in \mathcal{B}(R)$, then $\xi_n^{-1}(A) = \{(t_i)_1^\infty ; t_n \in A\} = \pi^{-1}_{\{n\}}(A) \in \mathcal{C}$. Moreover

$$P_{\xi_n}(A) = P(\xi_n^{-1}(A)) = P(\pi^{-1}_{\{n\}}(A)) = P_{\{n\}}(A) = \mathbf{m}(x_n(A)) = \mathbf{m}_{x_n}(A).$$

□

Remark 4.9. *By the preceding procedure to each sequence $(x_n)_n$ we can construct a sequence $(\xi_n)_n$ of a random variables.*

5 Lower and upper limits, m-almost everywhere convergence

In [4] we defined the notions of lower and upper limits for a sequence of IF-observables and showed the connection between two kinds of $\mathbf{m}$-almost everywhere convergence.

Definition 5.1. *We shall say that a sequence $(x_n)_n$ of IF-observables has $\limsup\limits_{n\to\infty}$, if there exists an IF-observable $\overline{x} : \mathcal{B}(R) \to \mathcal{F}$ such that*

$$\overline{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)$$

for every $t \in R$. We write $\overline{x} = \limsup\limits_{n\to\infty} x_n$.

Note that if another IF-observable y satisfies the above condition, then $\mathbf{m} \circ y = \mathbf{m} \circ \overline{x}$.

Definition 5.2. *A sequence $(x_n)_n$ of IF-observables has $\liminf\limits_{n\to\infty}$, if there exists an IF-observable $\underline{x}$ such that*

$$\underline{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)$$

for all $t \in R$. Notation: $\underline{x} = \liminf\limits_{n\to\infty} x_n$.

Proposition 5.3. *A sequence $(x_n)_n$ of an IF-observables converges $\mathbf{m}$-almost everywhere to 0 if and only if*

$$\mathbf{m}\Big(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)\Big) = \mathbf{m}\Big(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)\Big) = \mathbf{m}\big(0_{\mathcal{F}}((-\infty, t))\big),$$

for every $t \in R$.

Proof. In [4] *Proposition 4.1.* □

In accordance with *Proposition 5.3* we can extend the notion of **m**-almost everywhere convergence by the following way.

Definition 5.4. *A sequence $(x_n)_n$ of an IF-observables converges* **m***-almost everywhere to an IF-observable x, if*

$$\mathbf{m}\Big(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) = \mathbf{m}\Big(\bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) = \mathbf{m}\big(x((-\infty,t))\big),$$

for every $t \in R$.

The next theorem is important for the proof of the Individual ergodic theorem in intuitionistic fuzzy case, where we work with the sequence of several IF-observables induced by the Borel function.

Theorem 5.5. *Let $(x_n)_n$ be a sequence of IF-observables, $(\xi_n)_n$ be the sequence of corresponding projections, $(g_n)_n$ be a sequence of Borel measurable functions $g_n : R^n \to R$. If the sequence $\big(g_n(\xi_1,\ldots,\xi_n)\big)_n$ converges P-almost everywhere, then the sequence $\big(g_n(x_1,\ldots,x_n)\big)_n$ converges* **m***-almost everywhere and*

$$\mathbf{m}\Big(\limsup_{n\to\infty} g_n(x_1,\ldots,x_n)\big((-\infty,t)\big)\Big) = \mathbf{m}\Big(\liminf_{n\to\infty} g_n(x_1,\ldots,x_n)\big((-\infty,t)\big)\Big)$$

for each $t \in R$. Moreover

$$P\Big(\{u \in R^N : \limsup_{n\to\infty} g_n\big(\xi_1(u),\ldots,\xi_n(u)\big) < t\}\Big) = \mathbf{m}\Big(\limsup_{n\to\infty} g_n(x_1,\ldots,x_n)\big((-\infty,t)\big)\Big)$$

for each $t \in R$.

Proof. In [5] *Theorem 5.1.* □

6 Individual ergodic theorem

First we recall the classical Individual ergodic theorem. Let (X,σ,P) be a probability space, $T : X \to X$ be a measure preserving transformation (i.e. $A \in \sigma$ implies $T^{-1}(A) \in \sigma$ and $P\big(T^{-1}(A)\big) = P(A)$), $\xi : X \to R$ be an integrable random variable. Then there exists an integrable random variable ξ^* such that the following conditions are satisfied:

(i) $E(\xi) = E(\xi^*)$,

(ii) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}(\xi \circ T^i) = \xi^*$ P-almost everywhere,

(iii) $\xi^* = \xi \circ T$ P-almost everywhere.

We defined the IF-mean value of an IF-observable and **m**-almost everywhere convergence in the previous sections. Now we must define a transformation preserving an IF-state **m**.

Definition 6.1. *Let $(\mathcal{F},\cdot)$ be a family of IF-events with product,* **m** *be an IF-state. Then a mapping $\tau : \mathcal{F} \to \mathcal{F}$ is said to be a* **m***-preserving transformation, if the following conditions are satisfied:*

(i) $\tau\big((1_\Omega, 0_\Omega)\big) = (1_\Omega, 0_\Omega)$;

(ii) *if $\mathbf{A},\mathbf{B} \in \mathcal{F}$ and $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, then $\tau(\mathbf{A}) \odot \tau(\mathbf{B}) = (0_\Omega, 1_\Omega)$ and $\tau(\mathbf{A} \oplus \mathbf{B}) = \tau(\mathbf{A}) \oplus \tau(\mathbf{B})$;*

(iii) *if $\mathbf{A}_n \nearrow \mathbf{A}$, $\mathbf{A}_n, \mathbf{A} \in \mathcal{F}$, $n \in \mathbf{N}$, then $\tau(\mathbf{A}_n) \nearrow \tau(\mathbf{A})$;*

(iv) $\mathbf{m}\big(\tau(\mathbf{A}) \cdot \tau(\mathbf{B})\big) = \mathbf{m}(\mathbf{A} \cdot \mathbf{B})$ *for each $\mathbf{A},\mathbf{B} \in \mathcal{F}$.*

The next theorem says about a representation of **m**-preserving transformation.

Theorem 6.2. *Let $\tau : \mathcal{F} \to \mathcal{F}$ be a* **m**-*preserving transformation, where* **m** *is an IF-state. For each* $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$ *denote*

$$\tau(\mathbf{A}) = \big(\tau^\flat(\mu_A), 1_\Omega - \tau^\sharp(1_\Omega - \nu_A)\big).$$

Then the mappings $\tau^\flat, \tau^\sharp : \mathcal{T} \to \mathcal{T}$ are the measure preserving transformations in a tribe $\mathcal{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathcal{S}-$ measurable$\}$.

Proof. Let $\tau : \mathcal{F} \to \mathcal{F}$ be a **m**-preserving transformation. Then from *Definition 6.1* we obtain that the mapping τ is satisfying the four conditions:

(i) Let $\tau\big((1_\Omega, 0_\Omega)\big) = (1_\Omega, 0_\Omega)$. Then $(1_\Omega, 0_\Omega) = \tau\big((1_\Omega, 0_\Omega)\big) = \big(\tau^\flat(1_\Omega), 1_\Omega - \tau^\sharp(1_\Omega)\big)$ and therefore we have $1_\Omega = \tau^\flat(1_\Omega), 1_\Omega = \tau^\sharp(1_\Omega)$.

(ii) Let $\mathbf{A}, \mathbf{B} \in \mathcal{F}$ and $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$. Then $(0_\Omega, 1_\Omega) = (\mu_A \odot \mu_B, \nu_A \oplus \nu_B) = \big((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega\big)$.

$$0_\Omega = (\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega \quad \text{and} \quad 1_\Omega = (\nu_A + \nu_B) \wedge 1_\Omega.$$

Therefore,

$$\begin{aligned} \mu_A + \mu_B &\leq 1_\Omega && (1) \\ \nu_A + \nu_B &\geq 1_\Omega. && (2) \end{aligned}$$

By (2) we obtain

$$(1_\Omega - \nu_A) \odot (1_\Omega - \nu_B) = \big(1_\Omega - (\nu_A + \nu_B)\big) \vee 0_\Omega = 0_\Omega.$$

Since $\tau(\mathbf{A}) \odot \tau(\mathbf{B}) = (0_\Omega, 1_\Omega)$, we get

$$\begin{aligned} (0_\Omega, 1_\Omega) &= \big(\tau^\flat(\mu_A), 1_\Omega - \tau^\sharp(1_\Omega - \nu_A)\big) \odot \big(\tau^\flat(\mu_B), 1_\Omega - \tau^\sharp(1_\Omega - \nu_B)\big) \\ &= \big(\tau^\flat(\mu_A) \odot \tau^\flat(\mu_B), (1_\Omega - \tau^\sharp(1_\Omega - \nu_A)) \oplus (1_\Omega - \tau^\sharp(1_\Omega - \nu_B))\big) \end{aligned}$$

$$\begin{aligned} 0_\Omega &= \tau^\flat(\mu_A) \odot \tau^\flat(\mu_B) \\ 1_\Omega &= (1_\Omega - \tau^\sharp(1_\Omega - \nu_A)) \oplus (1_\Omega - \tau^\sharp(1_\Omega - \nu_B)) \\ &= \big(1_\Omega - \tau^\sharp(1_\Omega - \nu_A) + 1_\Omega - \tau^\sharp(1_\Omega - \nu_B)\big) \wedge 1_\Omega. \end{aligned}$$

Therefore $\tau^\sharp(1_\Omega - \nu_A) + \tau^\sharp(1_\Omega - \nu_B) \leq 1_\Omega$ and

$$\tau^\sharp(1_\Omega - \nu_A) \odot \tau^\sharp(1_\Omega - \nu_B) = \big(\tau^\sharp(1_\Omega - \nu_A) + \tau^\sharp(1_\Omega - \nu_B) - 1_\Omega\big) \vee 0_\Omega = 0_\Omega.$$

Finally since $\tau(\mathbf{A} \oplus \mathbf{B}) = \tau(\mathbf{A}) \oplus \tau(\mathbf{B})$, then

$$\begin{aligned} \tau\big((\mu_A \oplus \mu_B, \nu_A \odot \nu_B)\big) &= \big(\tau^\flat(\mu_A), 1_\Omega - \tau^\sharp(1_\Omega - \nu_A)\big) \oplus \big(\tau^\flat(\mu_B), 1_\Omega - \tau^\sharp(1_\Omega - \nu_B)\big), \\ \big(\tau^\flat(\mu_A \oplus \mu_B), 1_\Omega - \tau^\sharp(1_\Omega - \nu_A \odot \nu_B)\big) &= \big(\tau^\flat(\mu_A) \oplus \tau^\flat(\mu_B), (1_\Omega - \tau^\sharp(1_\Omega - \nu_A)) \odot (1_\Omega - \tau^\sharp(1_\Omega - \nu_B))\big). \end{aligned}$$

Hence $\tau^\flat(\mu_A \oplus \mu_B) = \tau^\flat(\mu_A) \oplus \tau^\flat(\mu_B)$, and

$$1_\Omega - \tau^\sharp(1_\Omega - \nu_A \odot \nu_B) = (1_\Omega - \tau^\sharp(1_\Omega - \nu_A)) \odot (1_\Omega - \tau^\sharp(1_\Omega - \nu_B))$$

and using de Morgan rule: $f \odot g = 1_\Omega - (1_\Omega - f) \oplus (1_\Omega - g)$ on the second equality we obtain

$$1_\Omega - \tau^\sharp\big((1_\Omega - \nu_A) \oplus (1_\Omega - \nu_B)\big) = 1_\Omega - \tau^\sharp(1_\Omega - \nu_A) \oplus \tau^\sharp(1_\Omega - \nu_B).$$

Therefore

$$\tau^\sharp\big((1_\Omega - \nu_A) \oplus (1_\Omega - \nu_B)\big) = \tau^\sharp(1_\Omega - \nu_A) \oplus \tau^\sharp(1_\Omega - \nu_B).$$

(iii) If $\mathbf{A}_n = (\mu_{A_n}, \nu_{A_n}) \nearrow \mathbf{A} = (\mu_A, \nu_A)$, i.e. $\mu_{A_n} \nearrow \mu_A$ and $\nu_{A_n} \searrow \nu_A$, then $\tau(\mathbf{A}_n) \nearrow \tau(\mathbf{A})$.
Hence

$$\begin{aligned} \big(\tau^\flat(\mu_{A_n}), 1_\Omega - \tau^\sharp(1_\Omega - \nu_{A_n})\big) &\nearrow \big(\tau^\flat(\mu_A), 1_\Omega - \tau^\sharp(1_\Omega - \nu_A)\big) \\ \tau^\flat(\mu_{A_n}) &\nearrow \tau^\flat(\mu_A) \\ 1_\Omega - \tau^\sharp(1_\Omega - \nu_{A_n}) &\searrow 1_\Omega - \tau^\sharp(1_\Omega - \nu_A) \\ \tau^\sharp(1_\Omega - \nu_{A_n}) &\nearrow \tau^\sharp(1_\Omega - \nu_A). \end{aligned}$$

(iv) Let $\mathbf{m}\big(\tau(\mathbf{A})\cdot\tau(\mathbf{B})\big) = \mathbf{m}(\mathbf{A}\cdot\mathbf{B})$ for each $\mathbf{A},\mathbf{B}\in\mathcal{F}$. Then

$$\begin{aligned}
\tau(\mathbf{A})\cdot\tau(\mathbf{B}) &= \mathbf{A}\cdot\mathbf{B},\\
\big(\tau^\flat(\mu_A), 1_\Omega - \tau^\sharp(1_\Omega-\nu_A)\big)\cdot\big(\tau^\flat(\mu_B), 1_\Omega - \tau^\sharp(1_\Omega-\nu_B)\big) &= (\mu_A,\nu_A)\cdot(\mu_B,\nu_B),\\
\big(\tau^\flat(\mu_A)\cdot\tau^\flat(\mu_B), 1_\Omega - \tau^\sharp(1_\Omega-\nu_A)\cdot\tau^\sharp(1_\Omega-\nu_B)\big) &= (\mu_A\cdot\mu_B, 1_\Omega-(1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B)).
\end{aligned}$$

Hence

$$\begin{aligned}
\tau^\flat(\mu_A)\cdot\tau^\flat(\mu_B) &= \mu_A\cdot\mu_B,\\
1_\Omega - \tau^\sharp(1_\Omega-\nu_A)\cdot\tau^\sharp(1_\Omega-\nu_B) &= 1_\Omega-(1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B),\\
\tau^\sharp(1_\Omega-\nu_A)\cdot\tau^\sharp(1_\Omega-\nu_B) &= (1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B).
\end{aligned}$$

□

Theorem 6.3. (Individual ergodic theorem) *Let $(\mathcal{F},\cdot)$ be a family of IF-events with product, $\mathbf{m}$ be an IF-state. Let x be an integrable IF-observable and τ be an $\mathbf{m}$-preserving transformation. Then there exists an integrable IF-observable x^* such that*

(i) $\mathbf{E}(x) = \mathbf{E}(x^*)$,

(ii) $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(\tau^i\circ x) = x^*$ $\mathbf{m}$*-almost everywhere.*

Proof. Let $x_n = \tau^{n-1}\circ x$ $(n=1,2,\ldots)$, i.e

$$x_1 = x, x_2 = \tau\circ x, x_3 = \tau^2\circ x,\ldots. \tag{3}$$

Let us return to the Kolmogorov extension process (see *Proposition 4.7*). Let us consider the probability space $(R^N,\sigma(\mathcal{C}),P)$ such that

$$P\big(\{(t_i)_{i=1}^\infty : t_1\in A_1,\ldots,t_n\in A_n\}\big) = \mathbf{m}\big(x_1(A_1)\cdot\ldots\cdot x_n(A_n)\big) \tag{4}$$

for each $n\in N$ and $A_i\in\mathcal{B}(R)$.

Let $T: R^N\to R^N$ be the shift defined by the formula $T\big((t_n)_n\big) = (s_n)_n$, where $s_n = t_{n+1}$ $(n=1,2,\ldots)$.

Let $A = \{(t_i)_{i=1}^\infty : t_1\in A_1,\ldots,t_n\in A_n\}$ is the cylinder. In this case

$$T^{-1}(A) = \{(t_i)_{i=1}^\infty : T\big((t_i)_{i=1}^\infty\big)\in A\} = \{(t_i)_{i=1}^\infty : t_{1+1}\in A_1,\ldots,t_{n+1}\in A_n\}.$$

Therefore using (4), (3) and (iv) from *Definition 6.1* we have

$$\begin{aligned}
P\big(T^{-1}(A)\big) &= \mathbf{m}\big(x_{1+1}(A_1)\cdot\ldots\cdot x_{n+1}(A_n)\big)\\
&= \mathbf{m}\big(\tau\circ x(A_1)\cdot\ldots\cdot\tau^n\circ x(A_n)\big)\\
&= \mathbf{m}\Big(\tau\big(x(A_1)\big)\cdot\ldots\cdot\tau\big(\tau^{n-1}\circ x(A_n)\big)\Big)\\
&= \mathbf{m}\Big(\tau\big(x_1(A_1)\big)\cdot\ldots\cdot\tau\big(x_n(A_n)\big)\Big)\\
&= \mathbf{m}\big(x_1(A_1)\cdot\ldots\cdot x_n(A_n)\big) = P(A).
\end{aligned}$$

Hence the mapping T preserves the probability measure P, i.e. $P\big(T^{-1}(A)\big) = P(A)$.

Since the IF-observable $x = x_1$ is integrable, the first coordinate function ξ_1 defined by $\xi_1\big((t_i)_{i=1}^\infty\big) = t_1$ is integrable, too (see *Proposition 4.8* and *Definition 2.9*). Therefore by the Individual ergodic theorem there exists an integrable random variable ξ^* such that $E(\xi^*) = E(\xi)$ and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(\xi\circ T^i) = \xi^* \qquad P-\text{almost everywhere.}$$

Of course $\xi\circ T^i = \xi_{i+1}$, hence

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\xi_j = \xi^* \qquad P-\text{almost everywhere.}$$

Put $g_n(u_1, \dots, u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i$. By *Theorem 5.5* the sequence of IF-observables

$$\Big(g_n(x_1, \dots, x_n)\Big)_n = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)_n = \left(\frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x\right)_n$$

is convergent **m**-almost everywhere to the IF-observable $x^* = \limsup_{n\to\infty} g_n(x_1, \dots, x_n)$ and

$$P\big(\xi^*(-\infty, t)\big) = \mathbf{m}\big(x^*\big((-\infty, t)\big)\big)$$

for each $t \in R$. Since $P_{\xi^*} = \mathbf{m}_{x^*}$ and $P_{\xi_1} = \mathbf{m}_{x_1} = \mathbf{m}_x$, we obtain $\mathbf{E}(x) = E(\xi_1) = E(\xi^*) = \mathbf{E}(x^*)$. □

7 Conclusion

The paper is concerned with ergodic theory for family of intuitionistic fuzzy events. We proved the Individual ergodic theorem for intuitionistic fuzzy observables using **m**-almost everywhere convergence, where **m** is an intuitinistic fuzzy state. Since the intuitionistic fuzzy probability $\mathcal{P}$ can be decomposited to two intuitionistic fuzzy states, then this results can be applied to $\mathcal{P}$-almost everywhere convergence, too.

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets: Theory and applications*, Physica Verlag, New York, 1999.

[2] K. T. Atanassov, *On intuitionistic fuzzy sets*, Springer-Verlag Berlin Heidelberg, 2012.

[3] K. Čunderlíková, *The individual ergodic theorem on the IF-events with product*, Soft Computing - A Fusion of Foundations, Methodologies and Applications, **14**(3) (2010), 229-234.

[4] K. Čunderlíková, *Upper and lower limits and* **m**-*almost everywhere convergence of intuitionistic fuzzy observables*, Notes on Intuitionistic Fuzzy Sets, **24**(4) (2018), 40-49.

[5] K. Čunderlíková, **m**-*almost everywhere convergence of intuitionistic fuzzy observables induced by Borel measurable function*, Notes on Intuitionistic Fuzzy Sets, **25**(2) (2019), 29-40.

[6] K. Lendelová, *Conditional IF-probability*, Lawry J. et al. (eds), Advances in Soft Computing: Soft Methods for Integrated Uncertainty Modelling, **37** (2006), 275-283.

[7] K. Lendelová, B. Riečan, *Weak law of large numbers for IF-events*, B. De Baets, et al. (eds), Current Issues in Data and Knowledge Engineering, Warszawa, EXIT, (2004), 309-314.

[8] B. Riečan, *On a problem of Radko Mesiar: General form of IF-probabilities*, Fuzzy Sets and Systems, **152** (2006), 1485-1490.

[9] B. Riečan, *On the probability and random variables on IF events*, D. Ruan, et al. (eds), Applied Artifical Intelligence, Proc. 7th FLINS Conf. Genova, World Scientific, (2006), 138-145.

[10] B. Riečan, *Analysis of fuzzy logic models*, V. M. Koleshko, (ed), Intelligent Systems, INTECH, (2012), 219-244.