The cross-migrativity equation with respect to semi-t-operators

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Abstract

The cross-migrativity has been investigated for families of certain aggregation operators, such as t-norms, t-subnorms and uninorms. In this paper, we aim to study the cross-migrativity property for semi-t-operators, which are generalizations of t-operators by omitting commutativity. Specifically, we give all solutions of the cross-migrativity equation for all possible combinations of semi-t-operators. Moreover, it is shown that if a semi-t-operator \(F\) is \(\alpha\)-cross-migrative over another semi-t-operator \(G\), then \(G\) must be a semi-nullnorm except one case. Finally, it is pointed out that the cross-migrativity property between two semi-t-operators is always determined by their underlying operators except a few cases.

Keywords: Fuzzy connectives, cross-migrativity, semi-t-operators, semi-nullnorms.

1 Introduction

The aggregation function is an essential tool in many fields like mathematics, computer sciences, economics and social sciences [2, 4, 10, 11]. And many researchers focus on analysis of interesting properties of aggregation functions [1], like idempotency, modularity, Frank and Alsina equations [3], migrativity [9, 16, 18, 20] and distributivity. Thereinto, migrativity is an important property of binary operations defined on the unit interval because it plays a key role in decision making processes and image processing. Hence, Fodor and Rudas [7] considered the migrative functional equation

\[
T(\alpha x, y) = T(x, \alpha y), \quad \text{for all } (x, y) \in [0, 1]^2
\]  

(1)

and fully characterized all continuous Archimedean t-norms satisfying Eq.(1). And then they [8] extended Eq.(1) into

\[
T(T_0(\alpha x), y) = T(x, T_0(\alpha y)), \quad \text{for all } (x, y) \in [0, 1]^2
\]  

(2)

where \(T_0\) is a t-norm. Moreover, when Fodor et al. [6] studied the computing of aggregation operators, they found the classical commuting equation over the t-norms

\[
T(T_0(x, y), T_0(u, v)) = T_0(T(x, u), T(y, v))
\]  

(3)

has only solution \(T = T_0\). By fixing \(u = 1\) and writing \(y = \alpha, v = y\) in Eq.(3), we can obtain a weaker functional equation

\[
T(T_0(\alpha x), y) = T_0(x, T(\alpha, y)), \quad \text{for all } (x, y) \in [0, 1]^2.
\]  

(4)

The t-norm \(T\) satisfying Eq.(4) is said to be \(\alpha\)-cross-migrative with respect to \(T_0\) (or shortly \(T\) is \((\alpha, T_0)\)-cross-migrative) [6]. The cross-migrativity property has been studied for t-norms [6, 12], t-subnorms [19] and uninorms [21, 22]. The paper will focus on studying the cross-migrativity with respect to semi-t-operators and give the sufficient and necessary conditions of the cross-migrativity equations for all of combinations of semi-t-operators.
The remainder of this paper is organized as follows. In Section 2 we will recall some results and structures related to basic fuzzy logic connectives used in this paper and define the cross-migrativity with respect to semi-t-operators. In Sections 3 4 5 6 we will respectively characterize all solutions of the cross-migrativity equations for four kinds of possible combinations of semi-t-operators. In Section 7 we give our conclusions and suggestions for further research.

2 Preliminaries

In this section, to make this work self-contained, we recall some definitions and results employed in the paper.

Definition 2.1. [13]

(i) A binary operator $T : [0, 1]^2 \to [0, 1]$ is called a semi-t-norm if it is increasing, associative and it has a neutral element $1$,

(ii) A binary operator $S : [0, 1]^2 \to [0, 1]$ is called a semi-t-conorm if it is increasing, associative and it has a neutral element $0$.

Clearly, a t-norm is a commutative semi-t-norm while a t-conorm is a commutative semi-t-conorm.

Definition 2.2. [14] The operation $F : [0, 1]^2 \to [0, 1]$ is called a t-operator if it is commutative, associative, increasing and such that $F(0, 0) = 0$, $F(1, 0) = 0$, $F(1, 1) = 1$ and the functions $F_0$ and $F_1$ are continuous, where $F_0(x) = F(0, x)$ and $F_1(x) = F(1, x)$.

In fact, T. Calvo [3] introduced the notion of nullnorms in order to generalize the functional equations of Frank and Alsina into uninorms and nullnorms. In [15], M. Mas has proven that t-operators and nullnorms are equivalent. But it is interesting that their own generalizations semi-nullnorms and semi-t-operators are not equivalent. Now, let us introduce definitions of semi-nullnorms.

Definition 2.3. [3] The operation $F : [0, 1]^2 \to [0, 1]$ is called a semi-nullnorm if it is increasing, associative, has an absorbing element $k \in [0, 1]$ and satisfies

(i) $F(0, x) = F(x, 0) = x$, for all $x \leq k$.

(ii) $F(1, x) = F(x, 1) = x$, for all $x \geq k$.

Note that definitions of semi-nullnorms in this paper is slightly different with ones in [5], since we require the associativity. Thus the set of all semi-nullnorms is a proper subset of the set of all semi-t-operators.

Theorem 2.4. [5] A binary operation $F$ is a semi-nullnorm with an absorbing element $k \in (0, 1)$ if and only if there exists a semi-t-norm $T$ and a semi-t-conorm $S$ such that $F$ is given by

$$F(x, y) = \begin{cases} kS(\frac{x}{k}, \frac{y}{k}) & \text{if } (x, y) \in [0, k]^2, \\ k & \text{if } (x, y) \in [k, 1]^2, \\ (5) & \text{otherwise.} \\ 
\end{cases}$$

Definition 2.5. [5] An binary operation $F : [0, 1]^2 \to [0, 1]$ is called a semi-t-operator if it is associative, non-decreasing, fulfills $F(0, 0) = 0, F(1, 0) = 0, F(1, 1) = 1$ and such that the functions $F_0, F_1, F^0, F^1$ are continuous, where $F_0(x) = F(0, x), F_1(x) = F(1, x), F^0(x) = F(x, 0)$ and $F^1(x) = F(x, 1)$.

Let $F_{a,b}$ denote the family of all semi-t-operators such that $F(0, 0) = a$ and $F(1, 0) = b$. Then we obtain the following theorem.

Theorem 2.6. [17] Let $F : [0, 1]^2 \to [0, 1]$, write $F(0, 1) = a$, $F(1, 0) = b$. Operation $F \in F_{a,b}$ if and only if there exists a semi-t-norm $T_F$ and a semi-t-conorm $S_F$ such that

$$F(x, y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\ b + (1 - b)T_F(\frac{x-b}{1-b}, \frac{y-b}{1-b}) & \text{if } (x, y) \in [b, 1]^2, \\ a & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b & \text{if } (x, y) \in [b, 1] \times [0, b], \\ x & \text{otherwise,} \\ (6) & \end{cases}$$
for $a \leq b$ and
\[
F(x, y) = \begin{cases} 
b S_F\left(\frac{x}{b}, \frac{y}{b}\right) & \text{if } (x, y) \in [0, b]^2, \\
a + (1 - a) F\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x, y) \in [a, 1]^2, \\
a & \text{if } (x, y) \in [0, a] \times [a, 1], \\
b & \text{if } (x, y) \in [b, 1] \times [0, b], \\
y & \text{otherwise,}
\end{cases}
\tag{7}
\]
for $a \geq b$.

Now, let us define the cross-migrativity of semi-t-operators.

**Definition 2.7.** Let $\alpha \in [0, 1]$ and $F, G : [0, 1]^2 \rightarrow [0, 1]$ two semi-t-operators. $F$ is said to be $\alpha$-cross-migrative with respect to $G$ or $F$ is $(\alpha, G)$-cross-migrative or $(F, G)$ is $\alpha$-cross-migrative if
\[
F(G(\alpha, x), y) = G(x, F(\alpha, y)) \text{ for all } x, y \in [0, 1].
\tag{8}
\]

According to the order relationship between $a, b$ and $c, d$, we need to consider the following four cases:

1. the cross-migrativity of $F \in F_{a,b}$ with $a \geq b$ over $G \in F_{c,d}$ with $c \geq d$;
2. the cross-migrativity of $F \in F_{a,b}$ with $a \geq b$ over $G \in F_{c,d}$ with $c \leq d$;
3. the cross-migrativity of $F \in F_{a,b}$ with $a \leq b$ over $G \in F_{c,d}$ with $c \geq d$;
4. the cross-migrativity of $F \in F_{a,b}$ with $a \leq b$ over $G \in F_{c,d}$ with $c \leq d$.

Next, let us study them in turn.

### 3 Cross-migrativity of $F \in F_{a,b}$ with $a \geq b$ over $G \in F_{c,d}$ with $c \geq d$

In this section, depending on the order relationship among $\alpha, a, b, c, d$, we need consider the following three different cases: (1) $\alpha \leq \min(b, d)$, (2) $\alpha \geq \max(a, c)$, (3) $\min(b, d) < \alpha < \max(a, c)$. Now, let us consider the first subcase.

#### 3.1 $\alpha \leq \min(b, d)$

**Lemma 3.1.** Let $\alpha \leq \min(b, d), F \in F_{a,b}$ with $a \geq b$ and $G \in F_{c,d}$ with $c \geq d$. If $F$ is $\alpha$-cross-migrative over $G$, then $a = c \geq b = d \geq \alpha$.

**Proof.** Assume that $a > c$. Taking $x = 0, y = 1$, then we have that $a = F(\alpha, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = c$. It contradicts with $a > c$. Hence $a \leq c$.

Taking $x = 1, y = 0$, then we have that $b = F(c, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, a) = d$.

Taking $x = 1, y = 1$, then we have that $c = F(c, 1) = F(G(\alpha, 1), 1) = G(1, F(\alpha, 1)) = G(1, a) = a$.

Therefore, $\alpha \leq b = d \leq a = c$. \hfill $\Box$

**Theorem 3.2.** Let $\alpha \leq \min(b, d), F \in F_{a,b}$ with $a \geq b$ and $G \in F_{c,d}$ with $c \geq d$. Then $F$ is $\alpha$-cross-migrative over $G$ if and only if

1. $\alpha \leq b = d \leq a = c$,
2. $S_F$ is $\frac{\alpha}{b}$-cross-migrative over $S_G$,

where $S_F$ and $S_G$ are the underlying semi-t-conorms of $F$ and $G$, respectively.

**Proof.** $(\Rightarrow)$ Firstly, we can directly get from Lemma 3.1 that (i). To prove (ii), consider $x, y \in [0, b]$ in Eq. (8), we have that the left-hand side of Eq. (8) is
\[
F(G(\alpha, x), y) = F(b S_F\left(\frac{\alpha}{b}, \frac{x}{b}\right), y) = b S_F(S_F\left(\frac{\alpha}{b}, \frac{x}{b}\right), \frac{y}{b}),
\tag{9}
\]
while the right-hand side of Eq. (8) is
\[
G(x, F(\alpha, y)) = G(x, b S_F\left(\frac{\alpha}{b}, \frac{y}{b}\right)) = b S_G\left(\frac{x}{b}, S_F\left(\frac{\alpha}{b}, \frac{y}{b}\right)\right).
\tag{10}
\]

Note that $F$ is $\alpha$-cross-migrative over $G$, then $S_F(S_F\left(\frac{\alpha}{b}, \frac{x}{b}\right), \frac{y}{b}) = S_G\left(\frac{x}{b}, S_F\left(\frac{\alpha}{b}, \frac{y}{b}\right)\right)$. Thus we get that $S_F$ is $\frac{\alpha}{b}$-cross-migrative over $S_G$ because of $\frac{\alpha}{b}, \frac{y}{b} \in [0, 1]$. Hence (ii) is proved.
Suppose that $x, y \in [0, 1]$, then we have the following cases to be checked.

Assume that $y \geq a$, from structures of $F, G$ and $\alpha \leq b = d \leq a = c$, then we have that $G(\alpha, x) \leq G(\alpha, 1) = a, G(x, a) = a$, and then $F(G(\alpha, x), y) = a = G(x, a) = G(x, F(\alpha, y))$.

Assume that $b \leq x \leq a$, from structures of $F, G$ and $\alpha \leq b = d \leq a = c$, then we have that $F(G(\alpha, x), y) = y = G(x, F(\alpha, y))$.

Assume that $y < b$, from structures of $F, G$, $\alpha \leq b = d \leq a = c$ and $S_F$ is $\frac{2}{b}$-cross-migrative over $S_G$, then we have that $bS_F(\frac{2}{b}, \frac{2}{b}) \leq b$. For $x \in [0, 1]$, there exist the following subcases to consider.

- If $x \geq a$, then $F(G(\alpha, x), y) = F(\alpha, y) = b = G(x, bS_F(\frac{2}{b}, \frac{2}{b})) = G(x, F(\alpha, y))$.
- If $x \in [b, a]$, then $F(G(\alpha, x), y) = F(x, y) = b = G(x, bS_F(\frac{2}{b}, \frac{2}{b})) = G(x, F(\alpha, y))$.
- If $x \leq b$, then $F(G(\alpha, x), y) = F(bS_F(\frac{2}{b}, \frac{2}{b}), y) = bS_F(S_G(\frac{2}{b}, \frac{2}{b}), \frac{2}{b}) = bS_F(\frac{2}{b}, S_F(\frac{2}{b}, \frac{2}{b})) = G(x, bS_F(\frac{2}{b}, \frac{2}{b})) = G(x, F(\alpha, y))$.

\[\Box\]

3.2 $\alpha \geq \max(a, c)$

Lemma 3.3. Let $\alpha \geq \max(a, c)$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If $F$ is $\alpha$-cross-migrative over $G$, then $\alpha \geq c \geq a \geq b = d$.

Proof. Assume that $b < d$. Taking $x = 1, y = 0$, then we have that $b = F(\alpha, 0) = F(\alpha, 1, 0) = G(1, F(\alpha, 0)) = G(1, b) = d$. It contradicts with $b < d$. Thus $b \geq d$.

Taking $x = 0, y = 1$, then we have that $a = F(d, 1) = F(\alpha, 0, 1) = G(0, F(\alpha, 1)) = G(0, a) = c$.

Taking $x = d, y = 0$, then we have that $d = F(d, 0) = F(\alpha, d, 0) = G(d, F(\alpha, 0)) = G(d, b) = b$.

Therefore, $d = d \leq a = c \leq \alpha$.

\[\Box\]

Theorem 3.4. Let $\alpha \geq \max(a, c)$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. Then $F$ is $\alpha$-cross-migrative over $G$ if and only if

(i) $b \leq d \leq a = c \leq \alpha$.

(ii) $T_F$ is $\frac{\alpha-a}{1-a}$-cross-migrative over $T_G$.

Proof. ($\Rightarrow$) We can directly obtain from Lemma 3.3 that (i). To prove (ii), similarly, consider $x, y \in [a, 1]$ in Eq. (8), according to $b \leq d \leq a = c \leq \alpha$, we have that the left-hand side of Eq. (8) is

\[F(G(\alpha, x), y) = F(a + (1-a)T_G(\frac{\alpha-a}{1-a}, \frac{x-a}{1-a}), y) = a + (1-a)T_G(T_F(\frac{\alpha-a}{1-a}, \frac{x-a}{1-a}), \frac{y-a}{1-a}), \tag{11}\]

while the right-hand side of Eq. (8) is

\[G(x, F(\alpha, y)) = G(a, a + (1-a)T_F(\frac{\alpha-a}{1-a}, \frac{y-a}{1-a}), y) = a + (1-a)T_G(T_F(\frac{\alpha-a}{1-a}, \frac{y-a}{1-a}), \frac{x-a}{1-a}). \tag{12}\]

Note that $F$ is $\alpha$-cross-migrative over $G$, then $T_F(T_G(\frac{\alpha-a}{1-a}, \frac{y-a}{1-a}), \frac{y-a}{1-a}) = T_G(T_F(\frac{\alpha-a}{1-a}, \frac{y-a}{1-a}), \frac{y-a}{1-a})$. Thus we get that $T_F$ is $\frac{\alpha-a}{1-a}$-cross-migrative over $T_G$ because of $\frac{x-a}{1-a}, \frac{y-a}{1-a} \in [0, 1]$. Hence (ii) is proved.

($\Leftarrow$) Suppose that $x, y \in [0, 1]$, then we have the following cases to be checked.

Assume that $y \leq b$, from structure of $T, G$ and $b = d \leq a = c \leq \alpha$, then we have that $G(\alpha, x) \geq G(\alpha, 0) = b, F(\alpha, y) = G(x, y)$, then $G(x, F(\alpha, y) = b = G(x, b) = G(x, F(\alpha, y))$.

Assume that $y \in [b, a]$, then we have that $G(x, F(\alpha, y) = G(x, y) = y = F(G(\alpha, x), y)$.

Assume that $y \geq a$, then we need to consider two subcases: $x \leq a$ and $x \geq a$.

- If $x \leq a$, then $G(x, F(\alpha, y)) = G(x, a + (1-a)T_F(\alpha-a, \frac{y-a}{1-a})) = a$ because of $a + (1-a)T_F(\alpha-a, \frac{y-a}{1-a}) \geq a$ and $F(G(\alpha, x), y) = a$ because of $G(\alpha, x) \leq G(\alpha, a) = a$. Thus $G(x, F(\alpha, y)) = a = F(G(\alpha, x), y)$.

- If $x \geq a$, then $G(x, F(\alpha, y) = G(a + (1-a)T_G(\frac{\alpha-a}{1-a}, \frac{x-a}{1-a}), y) = a + (1-a)T_G(T_F(\frac{\alpha-a}{1-a}, \frac{x-a}{1-a}), \frac{y-a}{1-a}) = a + (1-a)T_G(\frac{x-a}{1-a}, T_F(\frac{\alpha-a}{1-a}, \frac{x-a}{1-a})) = G(x, a + (1-a)T_F(\frac{\alpha-a}{1-a}, \frac{x-a}{1-a})) = G(x, F(\alpha, y))$ because of $T_F$ is $\frac{\alpha-a}{1-a}$-cross-migrative over $T_G$.

\[\Box\]

3.3 $\min(b, d) < \alpha < \max(a, c)$

For this case, there are four different subcases to be considered: (1) $a \leq c$ and $b \leq d$, (2) $a \leq c$ and $b \geq d$, (3) $a \geq c$ and $b \leq d$, (4) $a \geq c$ and $b \geq d$. But it is interesting that the above four cases have the same results. Now we start Subcase (1).
Lemma 3.5. Let \( b = \min(b, d) < \alpha < \max(a, c) = c \), \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( a = c > \alpha > b = d \).

**Proof.** Taking \( x = 1, y = 0 \), we have that \( b = F(c, 0) = F(\alpha, 1) = G(1, F(\alpha, 0)) = G(1, b) = d \).

Assume that \( b \leq a \leq \alpha < c \). Taking \( x = 1 \) and \( y = 1 \), then we have that \( c = F(c, 1) = F(\alpha, 1) = G(1, F(\alpha, 1)) = G(1, a) = a \).

Therefore it follows that \( b = d < \alpha < a = c \).

For Subcase (2), we have the following lemma.

Lemma 3.6. Let \( d = \min(b, d) < \alpha < \max(a, c) = c \), \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( b = d < \alpha < a = c \).

**Proof.** Assume that \( a < \alpha < c \). Taking \( x = 0 \) and \( y = 1 \), then we have that \( a = F(d, 1) = F(\alpha, 0, 1) = G(0, F(\alpha, 1)) = G(0, a) = c \).

Assume that \( d < \alpha < b \). Taking \( x = 1 \) and \( y = 0 \), then we have that \( b = F(c, 0) = F(\alpha, 1, 0) = G(1, F(\alpha, 0)) = G(1, a) = a \).

Therefore it follows that \( a = c > \alpha > b = d \).

For Subcase (3), there is the following lemma.

Lemma 3.7. Let \( b = \min(b, d) < \alpha < \max(a, c) = a \), \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( b = d < \alpha < a = c \).

**Proof.** Taking \( x = 0 \) and \( y = 1 \), then we have that \( G(\alpha, 0) \leq d \leq c \leq a \), and then it holds that \( a = F(\alpha, 0, 1) = G(0, F(\alpha, 1)) = G(0, a) = c \).

Assume that \( d < \alpha < b \leq a = c \). Taking \( x = 1 \) and \( y = 0 \), then we have that \( b = F(c, 0) = F(\alpha, 1, 0) = G(1, F(\alpha, 0)) = G(1, a) = a \).

Therefore it follows that \( a = c > \alpha > b = d \).

For Subcase (4), we have the following lemma.

Lemma 3.8. Let \( d = \min(b, d) < \alpha < \max(a, c) = a \), \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( b = d < \alpha < a = c \).

**Proof.** Taking \( x = 0 \) and \( y = 1 \), then we have that \( a = F(d, 0) = F(\alpha, d, 0) = G(d, F(\alpha, 0)) = G(d, b) = b \).

Assume that \( d < \alpha < b \leq a = c \). Taking \( x = 1 \) and \( y = 0 \), then we have that \( b = F(c, 0) = F(\alpha, 1, 0) = G(1, F(\alpha, 0)) = G(1, a) = a \).

Therefore it follows that \( a = c > \alpha > b = d \).

Theorem 3.9. Let \( \min(b, d) < \alpha < \max(a, c) \), \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \), then \( F \) is \( \alpha \)-cross-migrative over \( G \) if and only if \( b = d < \alpha < a = c \).

**Proof.** (\( \Rightarrow \)) From Lemmas 3.7 and 3.8 we can directly get the result.

(\( \Leftarrow \)) Suppose that \( x, y \in [0, 1] \), we consider the following cases.

Assume that \( y \leq b \), there exist the following subcases to be checked.

- If \( x \geq a \), then \( F(\alpha, x, y) = F(\alpha, y) = b = G(x, b) = G(\alpha, y) \).
- If \( x \in [b, a] \), then \( F(\alpha, x, y) = F(x, y) = b = G(\alpha, x) = G(x, F(\alpha, y)) \).
- If \( x \leq b \), then \( F(\alpha, x, y) = F(b, y) = b = G(x, b) = G(\alpha, y) \).

Assume that \( y \in [b, a] \), we have that \( F(\alpha, x, y) = y = G(x, y) = G(\alpha, y) \).

Assume that \( y \geq a \), we have the following subcases to be checked.

- If \( x \geq a \), then \( F(\alpha, x, y) = F(\alpha, y) = a = G(x, a) = G(\alpha, y) \).
- If \( x \in [b, a] \), then \( F(\alpha, x, y) = F(x, y) = a = G(x, a) = G(\alpha, y) \).
- If \( x \leq b \), then \( F(\alpha, x, y) = F(b, y) = a = G(x, a) = G(\alpha, y) \).

Remark 3.10. By means of Theorems 3.2, 3.3 and 3.4, we know that \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) is not \( \alpha \)-cross-migrativity over \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \) except \( F, G \in \mathcal{F}_{a,b} \). Moreover, if \( F, G \in \mathcal{F}_{a,b} \), we find the fact as follow:
(i) For \( \alpha \leq b \), \((\alpha, G)\)-cross-migrativity of \( F \) is completely determined by the underlying semi-t-conorms \( S_F \) and \( S_G \), which indicates that the underlying semi-t-conorms \( S_F \) and \( S_G \) have corresponding cross-migrativity, that is, the cross-migrative property of two semi-t-operators is completely ascribed to the cross-migrative property of their corresponding underlying semi-t-conorms.

(ii) For \( \alpha \geq a \), \((\alpha, G)\)-cross-migrativity of \( F \) is completely determined by the underlying semi-t-norms \( T_F \) and \( T_G \), which indicates that the underlying semi-t-norms \( T_F \) and \( T_G \) have corresponding cross-migrativity, that is, the cross-migrative property of two semi-t-operators is completely ascribed to the cross-migrative property of their corresponding underlying semi-t-norms.

(iii) For \( b < \alpha < a \), \( F \) is always \((\alpha, G)\)-cross-migrative.

**Example 3.11.** Let \( F, G \in \mathcal{F}_{0.6,0.4} \) be two semi-t-operators as follows:

\[
F(x, y) = \begin{cases} 
\max(x, y) & \text{if } (x, y) \in [0, 0.4] \times [0, 0.4], \\
x + y - 5xy & \text{if } (x, y) \in [0, 0.2] \cup [0.2, 0.4] \times [0, 0.4], \\
2.5xy - 1.5x - 1.5y + 1.5 & \text{if } (x, y) \in [0.2, 0.4] \times [0.2, 0.4], \\
0.6 & \text{if } (x, y) \in [0.6, 0.6] \times [0, 0.4], \\
0.4 & \text{if } (x, y) \in [0.4, 1] \times [0, 0.4], \\
y & \text{otherwise},
\end{cases}
\]

\[
G(x, y) = \begin{cases} 
\max(x, y) & \text{if } (x, y) \in [0, 0.4] \times [0, 0.4], \\
2.5xy - 1.5x - 1.5y + 1.5 & \text{if } (x, y) \in [0.2, 0.4] \times [0.2, 0.4], \\
0.6 & \text{if } (x, y) \in [0.6, 0.6] \times [0, 0.4], \\
0.4 & \text{if } (x, y) \in [0.4, 1] \times [0, 0.4], \\
y & \text{otherwise},
\end{cases}
\]  

(i) If \( \alpha \leq 0.4 \), then \((S_F, S_G)\) is 0.5-cross-migrative, and then \((F, G)\) is 0.2-cross-migrative.

(ii) If \( \alpha \in (0.4, 0.6) \), then \((F, G)\) is 0.5-cross-migrative.

(iii) If \( \alpha \geq 0.6 \), then \((T_F, T_G)\) is 0.6-cross-migrative, and then \((F, G)\) is 0.84-cross-migrative.

![Fig.1. Structures of \( F \) (left) and \( G \) (right) in Section 3 when \( a \geq b \).](image)

4 **Cross-migrativity of \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) over \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \)**

In this section, depending on the order relationship among \( a, b, c, d \), we need to consider three different cases: (1) \( \alpha \leq \min(b, c) \), (2) \( \alpha \geq \max(a, d) \), (3) \( \min(b, c) < \alpha < \max(a, d) \). Now, let us consider the first case.
4.1 \( \alpha \leq \min(b, c) \)

**Lemma 4.1.** Let \( \alpha \leq \min(b, c) \), \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( \alpha \leq a = b = c = d \).

**Proof.** Assume that \( a > c \). Taking \( x = 0, y = 1 \) in Eq. (8), then we have that \( a = F(a, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = c \), which contradicts with \( a > c \). Thus \( a \leq c \), i.e. \( \alpha \leq b \leq a \leq c \leq d \).

Next, taking \( x = 1, y = 0 \) in Eq. (8), then we get that \( b = F(c, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, \alpha) = d \).

Because of \( b \leq a \leq c \leq d \) and \( b = d \), we obtain that \( a = b = c = d \). \( \square \)

**Theorem 4.2.** Let \( \alpha \leq \min(b, c) \), \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). Then \( F \) is \( \alpha \)-cross-migrative over \( G \) if and only if

(i) \( \alpha \leq a = b = c = d \).

(ii) \( S_F \) is \( \frac{a}{a} \)-cross-migrative over \( S_G \), where \( S_F \) and \( S_G \) are the underlying semi-t-conorms of \( F \) and \( G \), respectively.

**Proof.** (\( \Rightarrow \)) Clearly, we can directly get (i) from Lemma 4.1. To prove (ii), consider \( x, y \in [0, a] \) in Eq. (8), according to \( \alpha \leq b = d = a = c \), we know that the left-hand side of Eq. (8) is

\[
F(G(\alpha, x), y) = F(aS_G(\frac{\alpha}{a}, \frac{x}{a}), y) = aS_F(S_G(\frac{\alpha}{a}, \frac{x}{a}), \frac{y}{a}),
\]

while the right-hand side of Eq. (8) is

\[
G(x, F(\alpha, y)) = G(x, aS_F(\frac{\alpha}{a}, \frac{y}{a})) = aS_G(\frac{x}{a}, \frac{y}{a}).
\]

Hence it follows that \( S_F(S_G(\frac{\alpha}{a}, \frac{x}{a}), \frac{y}{a}) = S_G(\frac{x}{a}, S_F(\frac{\alpha}{a}, \frac{y}{a})) \). Note that \( \frac{x}{a}, \frac{y}{a} \in [0, 1] \), then we get that \( S_F \) is \( \frac{a}{a} \)-cross-migrative over \( S_G \). Thus we have proven (ii).

(\( \Leftarrow \)) Suppose that \( x, y \in [0, 1] \), we consider the following cases:

Assume that \( x, y \in [a, 1] \), from structures of \( F, G \) and \( \alpha \leq b = d = a = c \), we have that \( F(G(\alpha, x), y) = F(a, y) = a = G(x, F(\alpha, y)) \).

Assume that \( x, y \in [0, a] \), from structures of \( T, G, \alpha \leq b = d = a = c \), \( S_F \) is \( \frac{a}{a} \)-cross-migrative over \( S_G \), that is, \( S_F(S_G(\frac{\alpha}{a}, \frac{x}{a}), \frac{y}{a}) = S_G(\frac{x}{a}, S_F(\frac{\alpha}{a}, \frac{y}{a})) \), then it follows that \( F(G(\alpha, x), y) = F(aS_G(\frac{\alpha}{a}, \frac{x}{a}), y) = aS_F(S_G(\frac{\alpha}{a}, \frac{x}{a}), \frac{y}{a}) = \alpha, x \), \( \alpha \), \( \frac{y}{a} \) \) in Eq. (8), according to \( \alpha \geq \min(a, d) \), we get that \( x, y \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a] \). Without loss of generality, we further assume that \( x \geq y \), then it holds that \( F(G(\alpha, x), y) = F(a, y) = a = G(x, aS_F(\frac{\alpha}{a}, \frac{y}{a})) = G(x, F(\alpha, y)) \). \( \square \)

4.2 \( \alpha \geq \max(a, d) \)

**Lemma 4.3.** Let \( \alpha \geq \max(a, d) \), \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( \alpha \geq a = c = b = d \).

**Proof.** Assume that \( b < d \). Taking \( x = 1, y = 0 \), then we have that \( b = F(a, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = d \), which contradicts with \( b < d \). Thus \( b \geq d \), that is, \( \alpha \geq a \geq b \geq d \geq c \).

Next, taking \( x = 0, y = 1 \), then we have that \( a = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = c \).

Because of \( a \geq b \geq d \geq c \) and \( a = c \), we obtain that \( a = b = c = d \).

Therefore it follows that \( \alpha \geq b = d = a = c \). \( \square \)

**Theorem 4.4.** Let \( \alpha \geq \max(a, d) \), \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). Then \( F \) is \( \alpha \)-cross-migrative over \( G \) if and only if

(i) \( \alpha \geq b = d = a = c \).

(ii) \( T_F \) is \( \frac{a}{1-a} \)-cross-migrative over \( T_G \).

**Proof.** (\( \Rightarrow \)) (i) is clearly established from Lemma 4.3. To prove (ii), consider \( x, y \in [a, 1] \) in Eq. (8), we have from \( b = d = a = c \leq \alpha \) that the left-hand side of Eq. (8) is

\[
F(G(\alpha, x), y) = F(a + (1 - a)T_G(\frac{\alpha - a}{1 - a}, \frac{x - a}{1 - a}), y) = a + (1 - a)T_F(T_G(\frac{\alpha - a}{1 - a}, \frac{x - a}{1 - a}), \frac{y - a}{1 - a}),
\]

(17)
while the right-hand side of Eq. (8) is
\[ G(x, F(a, y)) = G(x, a + (1 - a)T_F(\frac{a - a}{1 - a}, \frac{y - a}{1 - a})) = a + (1 - a)T_G(\frac{a - a}{1 - a}, T_F(\frac{a - a}{1 - a}, \frac{y - a}{1 - a})). \] (18)

Thus it follows that
\[ T_F(T_G(\frac{\gamma - a}{1 - a}, \frac{\gamma - a}{1 - a}), \frac{\gamma - a}{1 - a}) = T_G(\frac{\gamma - a}{1 - a}, T_F(\frac{\gamma - a}{1 - a}, \frac{\gamma - a}{1 - a})). \]

(\iff) Suppose that \( x, y \in [0, 1] \), we consider the following cases:

Assume that \( x, y \in [0, a] \), from structures of \( F, G \) and \( a \sim b = d = a = c \), we have that \( F(G(a, x), y) = F(a, y) = a = G(x, a) = G(x, F(a, y)) \).

Assume that \( x, y \in [a, 1] \), from structures of \( F \) and \( G \), \( G(a, x) = G(a, x) \), \( T_F \) is \( \alpha - \frac{1}{\alpha} \)-cross-migrative over \( T_G \), namely,
\[ T_F(T_G(\frac{\gamma - a}{1 - a}, \frac{\gamma - a}{1 - a}), \frac{\gamma - a}{1 - a}) = T_G(\frac{\gamma - a}{1 - a}, T_F(\frac{\gamma - a}{1 - a}, \frac{\gamma - a}{1 - a})), \]
then we have that \( F(G(a, x), y) = F(a + (1 - a)T_F(\frac{\gamma - a}{1 - a}, \frac{\gamma - a}{1 - a}), y) = a + (1 - a)T_F(T_F(\frac{\gamma - a}{1 - a}, \frac{\gamma - a}{1 - a}), \frac{\gamma - a}{1 - a}) = G(x, a + (1 - a)T_F(\frac{\gamma - a}{1 - a}, \frac{\gamma - a}{1 - a})) = G(x, F(a, y)). \)

Assume that \( (x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times [0, a] \). Without loss of generality, we further assume that \( x \geq y \), then it holds that \( F(G(a, x), y) = F(a + (1 - a)T_G(\frac{\gamma - a}{1 - a}, \frac{\gamma - a}{1 - a}), y) = a = G(x, a) = G(x, F(a, y)) \). \( \square \)

**Remark 4.5.** For cases \( a \equiv \min(b, c) \) and \( \alpha \geq \max(a, d) \), if \( F \) is the \( (\alpha, G) \)-cross-migrativity, then both \( F \) and \( G \) degenerate into semi-nullnorms shown in the Fig.2. Furthermore, from Theorems 4.2 and 4.4, the cross-migrativity between semi-t-operators is completely determined by their own underlying operators.

**Example 4.6.**

\[
F(x, y) = \begin{cases} 
\min(x + y, 0, 4) & \text{if } (x, y) \in [0, 0.4]^2, \\
\max(x, y) & \text{if } (x, y) \in [0.4, 0.5] \times [0, 0.5] \cup [0, 0.4] \times [0.4, 0.5], \\
5xy + 4 - 4x - 4y & \text{if } (x, y) \in [0.8, 1]^2, \\
\min(x, y) & \text{if } (x, y) \in [0.5, 0.8] \times [0.8, 1] \cup [0.8, 1] \times [0.5, 0.8], \\
0.5 & \text{otherwise.}
\end{cases}
\] (19)

\[
G(x, y) = \begin{cases} 
\max(x, y) & \text{if } (x, y) \in [0, 0.5]^2, \\
\min(x, y) & \text{if } (x, y) \in [0.5, 1]^2, \\
0.5 & \text{otherwise.}
\end{cases}
\] (20)

(i) If \( \alpha \leq 0.5 \), then \( (S_T, S_G) \) is \( 0.8 \)-cross-migrative, and then \( (F, G) \) is \( 0.4 \)-cross-migrative.

(ii) If \( \alpha \geq 0.5 \), then \( (T_F, T_G) \) is \( 0.6 \)-cross-migrative, and then \( (F, G) \) is \( 0.8 \)-cross-migrative.

\[ \begin{array}{c|c|c|c|c|}
 & a & T_F & & \\
\hline
S_F & & & & \\
\hline
a & & & & \\
\hline
0 & & & & \\
\end{array} \] Fig.2. Structure of \( F \) (left) and \( G \) (right) in Subsection 4.1 (4.2)

**4.3** \( \min(b, c) < \alpha < \max(a, d) \)

For this case, there are four different subcases to be considered: (1) \( a \geq d, b \geq c \), (2) \( a \geq d, b \leq c \), (3) \( a \leq d, b \geq c \), (4) \( a \leq d, b \leq c \). But it is interesting that the above four cases have the same result that \( F \) is not \( \alpha \)-cross-migrative over \( G \). Now we start Subcase (1).
Lemma 4.7. Let \( c = \min(b, c) < \alpha < \max(a, d) = a, F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). Then \( F \) is not \( \alpha \)-cross-migrative over \( G \).

Proof. Taking \( x = 0 \) and \( y = 1 \), from \( G(\alpha, 0) \leq G(\alpha, 1) \leq G(a, 1) = a \), then we have that \( a = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = c \), which contradicts with \( c < \alpha < a \).

For Subcase (2), there is the following lemma.

Lemma 4.8. Let \( b = \min(b, c) < \alpha < \max(a, d) = a, F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). Then \( F \) is not \( \alpha \)-cross-migrative over \( G \).

Proof. Taking \( x = 0, y = 1 \), then we have that \( a = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = c \). Note that \( a \geq d \geq c \) and \( a = c \), we get that \( a = c = d \).

Next, taking \( x = 1, y = 0 \), we have that \( b = F(a, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = a \). Then we can get that \( a = b = c = d \). But it contradicts with \( b < \alpha < a \).

For Subcase (3), we have the following lemma.

Lemma 4.9. Let \( c = \min(b, c) < \alpha < \max(a, d) = d, F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). Then \( F \) is not \( \alpha \)-cross-migrative over \( G \).

Proof. Assume that \( \alpha < b \), taking \( x = 1, y = 0 \), then we have that \( \alpha = F(\alpha, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = d \). But it contradicts \( \alpha < \alpha \). Thus it holds that \( b \leq \alpha < d \).

Next, taking \( x = 1, y = 0 \), then we have that \( b = F(\alpha, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = d \). But it contradicts \( b \leq \alpha < d \).

For Subcase (4), we have the following lemma.

Lemma 4.10. Let \( b = \min(b, c) < \alpha < \max(a, d) = d, F \in \mathcal{F}_{a,b} \) with \( a \geq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). Then \( F \) is not \( \alpha \)-cross-migrative over \( G \).

Proof. Taking \( x = 1, y = 0 \), then from \( G(\alpha, 1) \geq c \geq b \), it follows that \( b = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = d \). But it contradicts \( b < \alpha < d \).

Remark 4.11. From Theorems 4.2, 4.7, Lemmas 4.8, 4.9, 4.10 if \( F \in \mathcal{F}_{a,b} \) with \( a \geq b \) over \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \) is \( \alpha \)-cross-migrative, then we can boldly assert these two semi-t-operators must be semi-nullnorm.

5 Cross-migrativity of \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) over \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \)

In this section, depending on the order relationship among \( \alpha, a, b, c, d \), we will consider three different cases: (1) \( \alpha \leq \min(a, c) \), (2) \( \alpha \geq \max(b, d) \), (3) \( \min(a, c) < \alpha < \max(b, d) \). Next, let us consider the first case.

5.1 \( \alpha \leq \min(a, c) \)

Lemma 5.1. Let \( \alpha \leq \min(a, c) \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( \alpha \leq a \leq c = d \leq b \).

Proof. Assume that \( \alpha > c \), taking \( x = 0, y = 1 \), then we have that \( a = F(\alpha, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = c \), which contradicts \( \alpha > c \). Thus \( a \leq c \).

Next, taking \( x = 1, y = 1 \), then we have that \( c = F(c, 1) = F(G(\alpha, 1), 1) = G(1, F(\alpha, 1)) = G(1, a) = d \).

Assume that \( d > b \), taking \( x = 1, y = 0 \), then we obtain that \( b = F(c, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, a, \alpha) = d \), which contradicts \( d > b \). Therefore \( \alpha \leq a \leq c = d \leq b \).

Theorem 5.2. Let \( \alpha \leq \min(a, c) \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). Then \( F \) is \( \alpha \)-cross-migrative over \( G \) if and only if one of the following two cases holds.

(i) If \( \alpha \leq a = c = d \leq b \), then \( S_F \) is \( \alpha \)-cross-migrative over \( S_G \).

(ii) If \( \alpha \leq a < c = d \leq b \), and let \( x_0 = \sup\{x \in [0, 1]|G(\alpha, x) \leq a\} \), then one of the following two subcases holds.
Theorem 5.5. Let \( G(x, y) = a \), then \( G(x, a) = G(a, x) \) when \( x \leq x_0, y \leq a \).

(b) If \( G(x, x_0) < a \), then \( G(x, a) = G(a, x) \) when \( x < x_0, y \leq a \).

Proof. For (i), clearly, it follows from Lemma 5.1 that \( a \leq a \leq c = d \leq b \). Further, if \( a \leq a = c = d \leq b \) and \( x \leq a, y \leq a \), then, from structures of \( F, G \), it follows that \( a \leq a \leq c = d \leq b \) and \( x \leq a, y \leq a \).

Conversely, suppose that \( x, y \in [0, 1] \), we consider the following cases:

Assume that \( x, y \in [0, 1] \), from structures of \( F, G \) and \( a \leq d = a \leq b \), we have that \( F(G(a, x), y) = F(a, y) = a = G(x, a) = G(x, F(a, y)) \).

Assume that \( x, y \in [0, 1] \), from structures of \( F, G \) and \( a \leq d = a \leq b \) and \( a \leq a \leq d \leq b \), we have that \( S_F(G(x, a), y) = F(a, y) = a = G(x, a) = G(x, F(a, y)) \).

Assume that \( x, y \in [0, 1] \), from structures of \( F, G \) and \( a \leq d = a \leq b \) and \( a \leq a \leq d \leq b \), we have that \( F(G(x, a), y) = F(a, y) = a = G(x, a) = G(x, F(a, y)) \).

For (ii), we only prove (ii) because (ii)(b) is similar. By Lemma 5.1, we can assume that \( a \leq a < c = d \leq b \).

Specially, we know from Eq. 5.1 that \( G(a, x, y) = G(x, F(a, y)) \) when \( x \leq x_0, y \leq a \). Further, it follows from structure of \( G \) and definition of \( x_0 \) that \( G(x, a) = a \) when \( x \leq x_0 \). Next, we prove the last result in (ii)(a), that is, \( G(x, a) = G(x, x) = G(x, a) \) holds when \( x > x_0 \). Note that \( a \leq G(x, a) \leq c \) when \( x > x_0 \), then we have from structure of \( F \) and Eq. 5.1 that

\[
G(x, x) = F(G(a, x), y) = G(x, F(a, y)).
\]

Specially, take \( y = 0 \) and \( y = 1 \) in Eq. 5.1 respectively, then we can obtain the required result.

Conversely, suppose that \( x, y \in [0, 1] \), we consider the following cases:

Assume that \( x \leq x_0, y \leq a \), then it holds that \( G(x, a) \) since the assumption \( F(G(a, x), y) = G(x, F(a, y)) \) when \( x \leq x_0, y \leq a \).

Assume that \( x \leq x_0, y \geq a \), then it follows from structures of \( F, G \) and assumption \( G(x, a) = a \) when \( x \leq x_0 \) that \( F(G(x, a), y) = a = G(x, a) = G(x, F(a, y)) \).

Assume that \( x > x_0 \), then it follows from structures of \( F, G \) that \( G(x, a) \leq G(x, F(a, y)) \) \( G(x, a) \). Further, using assumption \( G(x, a) = G(x, a) = G(x, a) \) when \( x > x_0 \), we can obtain that \( G(x, F(a, y)) = G(x, a) \). Therefore, it holds that \( F(G(x, a), y) = G(x, a) = G(x, F(a, y)) \).

Remark 5.3. (i) For Case (i) in Theorem 5.2, \( F \) is a cross-migrative over \( G \), then \( G \) must be a semi-nullnorm and the cross-migrativity is completely determined by their underlying semi-t-conorms \( S_F \) and \( S_G \).

(ii) For Case (ii) in Theorem 5.2, \( F \) is a cross-migrative over \( G \), then \( G \) must be a semi-nullnorm and the cross-migrativity is completely determined by the cross-migrativity of \( F, G \) on the domain \( x \leq x_0, y \leq a \) or \( x < x_0, y \leq a \), and by the values of \( G(x, a) \). These results are different from the above other obtained results.

\[ \alpha \geq \max(b, d) \]

Lemma 5.4. Let \( \alpha \geq \max(b, d) \), \( F \in F_{a,b} \) with \( a \leq b \) and \( G \in F_{c,d} \) with \( c \leq d \). If \( F \) is a cross-migrative over \( G \), then \( a \leq c = d \leq b \leq a \).

Proof. Assume that \( d < a \). Taking \( x = 0, y = 1 \), then we can get that \( a = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, c) = c \), then \( a = c > d \), which contradicts \( c \leq d \). Thus \( d \geq a \).

Assume that \( b < d \). Taking \( x = 1, y = 0 \), then we get that \( b = F(\alpha, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = d \), which contradicts \( b < d \). Thus \( d \geq b \).

Next, taking \( x = 0, y = 0 \), then we have that \( d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = G(0, b) = c \).

Thus it follows that \( \alpha \geq b \geq c = d \geq a \).

Theorem 5.5. Let \( \alpha \geq \max(b, d) \), \( F \in F_{a,b} \) with \( a \leq b \) and \( G \in F_{c,d} \) with \( c \leq d \). Then \( F \) is a cross-migrative over \( G \) if and only if one of the following results is true:
(i) If \( a \leq c = d = b \leq \alpha \), then \( T_F \) is \( \frac{a-b}{\alpha} \)-cross-migrative over \( T_G \).

(ii) If \( a \leq c = d < b \leq \alpha \), and denote \( x_0 = \inf\{x \in [0,1] | G(x, x) \geq b\} \), then one of the following two subcases holds.

(a) If \( G(\alpha, x_0) = b \), then \( G(x, y) = \begin{cases} G(\alpha, x) & \text{if } x < x_0, \\ b & \text{if } x \geq x_0, \end{cases} \) and \( F(\alpha, x, y) = G(x, F(\alpha, y)) \text{ when } x \geq x_0, y \geq b \).

(b) If \( G(\alpha, x_0) < b \), then \( G(x, y) = \begin{cases} G(\alpha, x) & \text{if } x \leq x_0, \\ b & \text{if } x > x_0, \end{cases} \) and \( F(\alpha, x, y) = G(x, F(\alpha, y)) \text{ when } x > x_0, y \geq b \).

**Proof.** For (i), clearly, it follows from Lemma 5.4 that \( \alpha \leq a \leq c = d = b \leq \alpha \). Further, if \( a \leq c = d \leq b \leq \alpha \) and \( x, y \in [a, 1] \) in Eq. (8), since \( F \) is \( \alpha \)-cross-migrative over \( G \), then we have that \( b + (1-b)T_F(T_G(\frac{a-b}{\alpha}, \frac{x-y}{\alpha}), \frac{y-b}{\alpha}) = b + (1-b)T_G(T_F(\frac{a-b}{\alpha}, \frac{x-y}{\alpha}), \frac{y-b}{\alpha}) \), and thus \( T_F(T_G(\frac{a-b}{\alpha}, \frac{x-y}{\alpha}), \frac{y-b}{\alpha}) = T_G(T_F(\frac{a-b}{\alpha}, \frac{x-y}{\alpha}), \frac{y-b}{\alpha}) \).

Conversely, suppose that \( x, y \in [0, 1] \), we consider the following cases:

Assume that \( x, y \in [0, 1] \), from structure of \( F, G \) and \( a \leq c = d = b \leq \alpha \), we have that \( F(\alpha, x) = F(d, y) = d = G(x, d) = G(x, F(\alpha, y)) \).

Assume that \( x, y \in [1, b] \), from structure of \( F, G \), \( a \leq c = d = b \leq \alpha \) and \( T_F \) is \( \frac{a-b}{\alpha} \)-cross-migrative over \( T_G \), that is, \( T_F(T_G(\frac{a-b}{\alpha}, \frac{x-y}{\alpha}), \frac{y-b}{\alpha}) = T_G(T_F(\frac{a-b}{\alpha}, \frac{x-y}{\alpha}), \frac{y-b}{\alpha}) \), then we have that \( F(\alpha, x, y) = F(b + (1-b)T_G(T_F(\frac{a-b}{\alpha}, \frac{x-y}{\alpha}), \frac{y-b}{\alpha}) = b + (1-b)T_G(T_F(\frac{a-b}{\alpha}, \frac{x-y}{\alpha}), \frac{y-b}{\alpha}) = G(x, F(\alpha, y)) \).

For (ii), we only prove (ii)(a) because (ii)(b) is similar. By Lemma 5.4, we can assume that \( a \leq c = d \leq b \leq \alpha \). Specially, we know from Eq. (8) that \( F(\alpha, x) = G(x, F(\alpha, y)) \) when \( x \geq x_0, y \geq b \). Further, it follows from structure of \( G \) and definition of \( x_0 \) that \( G(x, b) = b \) when \( x \geq x_0 \). Next, we prove the remaining result in (ii)(a), that is, \( G(x, b) = G(\alpha, x) \) holds when \( x < x_0 \). Note that \( c \leq G(\alpha, x) < b \) when \( x < x_0 \), then we have from structure of \( F \) and Eq. (8) that

\[
G(\alpha, x) = G(x, F(\alpha, y)).
\]

Specially, take \( y = 0 \) and \( y = 1 \) in Eq. (8) respectively, then we can obtain the required result.

Conversely, suppose that \( x, y \in [0, 1] \), we consider the following cases:

Assume that \( x \geq x_0, y \geq b \), then it holds that Eq. (8) since the assumption \( F(\alpha, x) = G(x, F(\alpha, y)) \) when \( x \geq x_0, y \geq b \).

Assume that \( x \geq x_0, y \leq b \), then it follows from structures of \( F, G \) and assumption \( G(x, b) = b \) when \( x \geq x_0 \) that \( F(\alpha, x, y) = b = G(x, b) = G(x, F(\alpha, y)) \).

Assume that \( x < x_0 \), then it follows from structures of \( F, G \) that \( G(x, b) \leq G(x, F(\alpha, y)) \leq G(x, \alpha) \). Further, using assumption \( G(x, b) = G(\alpha, x) = G(x, \alpha) \) when \( x < x_0 \), we can obtain that \( G(x, F(\alpha, y)) = G(x, \alpha) \). Therefore, it holds that \( F(\alpha, x, y) = G(\alpha, x) = G(x, F(\alpha, y)) \).

\[\square\]

**Remark 5.6.** (i) For Case (i) in Theorem 5.5, when \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( G \) must be a semi-nullnorm and the cross-migrativity is completely determined by their underlying semi-t-conorms \( T_F \) and \( T_G \).

(ii) For Case (ii) in Theorem 5.5, when \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( G \) must be a semi-nullnorm and the cross-migrativity is completely determined by the cross-migrativity of \( F \) on the domain \( x \geq x_0, y \geq b \) or \( x > x_0, y \geq b \), and by the values of \( G(x, b) \). These results are different from the above obtained results.

5.3 \( \min(b, d) < \alpha < \max(a, c) \)

For this case, there are four different subcases to be considered: (1) \( a \leq c \leq b \leq d \), (2) \( a \leq c, b \geq d \), (3) \( a \geq c, b \leq d \), (4) \( a \geq c, b \geq d \). Now we start Subcase (1).

**Lemma 5.7.** Let \( a = \min(a, c) < \alpha < \min(b, d) = d \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( a < \alpha < b = c = d \).
Proof. Assume that \( c < \alpha < d \), taking \( x = 0, y = 1 \), then we get that \( F(G(\alpha, 0), 1) = F(\alpha, 1) = \alpha = G(0, F(\alpha, 1)) = G(0, \alpha) = c \), and then \( \alpha = c \) contradicts \( c < \alpha < d \), thus \( c < \alpha \leq c \).

Taking \( x = 1, y = 1 \), then we can obtain that \( F(G(\alpha, 1), 1) = F(c, 1) = c = G(1, F(\alpha, 1)) = G(1, \alpha) = d \), then \( a < \alpha < c = d \) and \( a \leq b \leq d = c \).

Taking \( x = 1, y = 0 \), then we have that \( F(G(\alpha, 1), 0) = F(c, 0) = b = G(1, F(\alpha, 0)) = d \), and then \( a < \alpha < b = c = d \).

\[ \square \]

**Theorem 5.8.** Let \( a = \min(a, c) < \alpha < \max(b, d) = d \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). Then \( F \) is \( \alpha \)-cross-migrative over \( G \) if and only if the following two results are true.

(i) \( a < \alpha < b = c = d \).

(ii) \( G(x, \alpha) = G(\alpha, x) \), for all \( x \in [0, c] \).

Proof. (\( \Rightarrow \)) (i) is clearly established from Lemma 5.7. To prove (ii), consider \( x, y \in [0, c] \) in Eq. (8), since \( a < \alpha \leq cG\left(\frac{\alpha}{x}, \frac{x}{\alpha}\right) \leq b \) and \( F \) is \( \alpha \)-cross-migrative over \( G \), we have that \( G(\alpha, x) = F(G(\alpha, x), y) = G(x, F(\alpha, y)) = G(x, \alpha) \) for all \( x \in [0, c] \). Thus the result is proved.

(\( \Leftarrow \)) Suppose that \( x, y \in [0, 1] \), we consider the following cases:

If \( x \geq c \), then, from \( a < \alpha < b = c = d \) and the structures of \( F \) and \( G \), we have that \( F(G(\alpha, x), y) = F(c, y) = c = G(\alpha, x) = G(x, F(\alpha, y)) \) for all \( y \in [0, 1] \).

If \( x \leq c \), then, since \( a < \alpha < b = c = d \), \( G(x, \alpha) = G(\alpha, x) \) for all \( x \in [0, c] \) and the structures of \( F \) and \( G \), we obtain that \( F(G(\alpha, x), y) = F(cS_G\left(\frac{\alpha}{x}, \frac{x}{\alpha}\right), y) = cS_G\left(\frac{\alpha}{x}, \frac{x}{\alpha}\right) = G(\alpha, x) = G(x, \alpha) = G(x, F(\alpha, y)) \) for all \( y \in [0, 1] \). \( \square \)

**Remark 5.9.** For Subcase \( a = \min(a, c) < \alpha < \max(b, d) = d \), the \((\alpha, G)\)-cross-migrativity of \( F \) is completely determined by a portion of \( \alpha \)-section of \( G \) and has nothing to do with the remaining of \([0, 1]^2\).

For Subcase (2), there is the following lemma.

**Lemma 5.10.** Let \( a = \min(a, c) < \alpha < \max(b, d) = d \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( a < \alpha \leq c = d = b \) or \( a \leq c = d \leq a < b \).

Proof. Assume that \( c < \alpha < d \), then, taking \( x = 1, y = 0 \), we can get that \( \alpha = F(\alpha, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) \) and \( G(1, F(\alpha, 0)) = G(1, \alpha) = d \), which contradicts \( \alpha < d \). Taking Thus it follows that \( a < \alpha \leq c = d \) and \( \alpha < d \).

If \( a < \alpha \leq c \), then, taking \( x = 1, y = 0 \) in Eq. (8), we can get that \( c = F(\alpha, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, \alpha) = d \). So \( a < \alpha \leq c = d \leq b \).

If \( d \leq a < b \), then, taking \( x = 0, y = 1 \) in Eq. (8), we can get that \( d = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, \alpha) = c \). So \( a \leq c = d \leq a < b \).

In a word, we have always that \( a < \alpha \leq c = d \leq b, \) or \( a \leq c = d \leq a < b \). \( \square \)

**Theorem 5.11.** Let \( a = \min(a, c) < \alpha < \max(b, d) = d \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). Then \( F \) is \( \alpha \)-cross-migrative over \( G \) if and only if one of the following conditions is true

(i) \( a < \alpha \leq c = d \leq b \) and \( G(x, \alpha) = G(\alpha, x) \), for all \( x \in [0, c] \).

(ii) \( a \leq c = d \leq a < b \), and \( G(x, \alpha) = G(\alpha, x) \), for all \( x \in [c, 1] \).

Proof. (\( \Rightarrow \)) We can directly get from Lemma 5.10 that \( a < \alpha \leq c = d \leq b \) or \( a \leq c = d \leq a < b \).

On the one hand, for the case \( a < \alpha \leq c = d \leq b \), consider \( x, y \in [0, c] \) in Eq. (8), from the structures of \( F \) and \( G \) and \( a < \alpha \leq cS_G\left(\frac{\alpha}{x}, \frac{x}{\alpha}\right) \leq c \leq b \), we have that \( G(\alpha, x) = F(cS_G\left(\frac{\alpha}{x}, \frac{x}{\alpha}\right), y) = G(x, F(\alpha, y)) = G(x, \alpha) \) for all \( x \in [0, c] \). Thus (i) is proved.

On the other hand, for the case \( a \leq c = d \leq a < b \), consider \( x, y \in [c, 1] \) in Eq. (8), the same procedure may be easily adapted to obtaining \( G(x, \alpha) = G(\alpha, x) \) for all \( x \in [c, 1] \). Thus (ii) is proved.

(\( \Leftarrow \)) For Case (i), we need to consider the following cases:

If \( x \geq c, y \in [0, 1] \), from (i) and structures of \( F \) and \( G \), then it follows that \( F(G(\alpha, x), y) = F(c, y) = c = G(x, \alpha) = G(x, F(\alpha, y)) \).

If \( x \leq c, y \in [0, 1] \), from (i) and structures of \( F \) and \( G \), then it holds that \( F(G(\alpha, x), y) = F(cS_G\left(\frac{\alpha}{x}, \frac{x}{\alpha}\right), y) = cS_G\left(\frac{\alpha}{x}, \frac{x}{\alpha}\right) = G(x, \alpha) = G(x, F(\alpha, y)) \).

For Case (ii), we can prove that the result is true by using the same methods. \( \square \)
Remark 5.12. For the case $a = \min(a, c) < \alpha < \max(b, d) = b$, the $(\alpha, G)$-cross-migrativity of $F$ is completely determined by a portion of $\alpha$-section of $G$, and has nothing to do with the remaining of $[0, 1]^2$.

For Subcase (3), there is the following lemma.

Lemma 5.13. Let $c = \min(a, c) < \alpha < \max(b, d) = d$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$, then $F$ is not $\alpha$-cross-migrative over $G$.

Proof. Assume that $a \leq \alpha \leq b$, then, taking $x = 0$, $y = 1$ in Eq. (8), we have that $\alpha = F(\alpha, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = c$, which contradicts $\alpha > c$. Assume that $c < \alpha < a$, then, taking $x = 0$, $y = 1$ in Eq. (8), we have that $a = F(\alpha, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = c$, which contradicts $\alpha > c$. Assume that $b < \alpha < d$, then, taking $x = 0$, $y = 1$ in Eq. (8), we have that $\alpha = F(\alpha, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = c$, which contradicts $\alpha > c$.

Therefore, $F$ is not $\alpha$-cross-migrative over $G$.

For Subcase (4), there is the following lemma.

Lemma 5.14. Let $c = \min(a, c) < \alpha < \max(b, d) = b$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. If $F$ is $\alpha$-cross-migrative over $G$, then $a = c = d < \alpha < b$.

Proof. Assume that $c < \alpha < d$, then, taking $x = 1$, $y = 0$ in Eq. (8), we can get that $\alpha = F(\alpha, 0) = F(G(\alpha, 1), 0) = G(0, F(\alpha, 0)) = G(0, 0) = c$. On the one hand, taking $x = 0$, $y = 0$ in Eq. (8), we can get that $d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = G(0, 0) = c$. On the other hand, taking $x = 0$, $y = 1$ in Eq. (8), we can get that $a = F(c, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = c$.

Sum up, it follows that $a = c = d < \alpha < b$.

Theorem 5.15. Let $c = \min(a, c) < \alpha < \max(b, d) = b$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. Then $F$ is $\alpha$-cross-migrative over $G$ if and only if

(i) $a = c = d < \alpha < b$.

(ii) $G(x, \alpha) = G(\alpha, x)$, for all $x \in [a, 1]$.

Proof. ($\Rightarrow$) We can directly get from Lemma 5.14 that (i) is true. To prove (ii), from (i) and the structures of $F$ and $G$ and $a \leq a + (1 - a)T_{G}(\frac{\alpha - a}{1 - a}, \frac{\alpha - a}{1 - a}) \leq \alpha < b$, we have that $G(\alpha, x) = F(G(\alpha, x), y) = G(x, F(\alpha, y)) = G(x, \alpha)$ for all $x \in [a, 1]$. Thus (ii) is proved.

($\Leftarrow$) Suppose that $x, y \in [0, 1]$. If $x \leq a, y \in [0, 1]$, from (i) and the structures of $F$ and $G$, then $F(G(\alpha, x), y) = F(a, y) = a = G(x, \alpha) = G(x, F(\alpha, y))$. If $x \geq a, y \in [0, 1]$, from (i), (ii), $a \leq a + (1 - a)T_{G}(\frac{\alpha - a}{1 - a}, \frac{\alpha - a}{1 - a}) \leq \alpha < b$ and the structures of $F$ and $G$, then $F(G(\alpha, x), y) = F(a + (1 - a)T_{G}(\frac{\alpha - a}{1 - a}, \frac{\alpha - a}{1 - a}), y) = a + (1 - a)T_{G}(\frac{\alpha - a}{1 - a}, \frac{\alpha - a}{1 - a}) = G(\alpha, x) = G(x, F(\alpha, y))$.

\[\begin{array}{c|c|c}
1 & b & T_{F} \\
\hline
a & x & S_{F} \\
\hline
0 & a & b
\end{array}\]

\[\begin{array}{c|c|c}
1 & c & T_{G} \\
\hline
\hline
S_{G} & c & 0 \\
\hline
\end{array}\]

Fig. 3. Structures of $F$ (left) and $G$ (right) in Section 5

Remark 5.16. For the case $c = \min(a, c) < \alpha < \max(b, d) = b$, the $(\alpha, G)$-cross-migrativity of $F$ is completely determined by a portion of $\alpha$-section of $G$, and has nothing to do with the remaining of $[0, 1]^2$. 
6 Cross-migrativity of $F \in \mathcal{F}_{a,b}$ with $a \leq b$ over $G \in \mathcal{F}_{c,d}$ with $c \geq d$

In this section, depending on the order relationship among $\alpha, a, b, c, d$, we will consider three different cases: (1) $\alpha \leq \min(a, d)$, (2) $\alpha \geq \max(b, c)$, (3) $\min(a, d) < \alpha < \max(b, c)$. It is very interesting that the three cases have the same results with Section 5. Next, let us consider the first case.

6.1 $\alpha \leq \min(a, d)$

Lemma 6.1. Let $\alpha \leq \min(a, d)$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If $F$ is $\alpha$-cross-migrative over $G$, then $\alpha \leq a \leq c = d \leq b$.

Proof. Assume that $d < a$, then, taking $x = 0, y = 1$ in Eq. 8, we have that $a = F(\alpha, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = c$, which contradicts $a > c$. Therefore $a \leq c$.

Assume that $d < a$, then, taking firstly $x = 1, y = 1$ in Eq. 8, we obtain that $c = F(c, 1) = F(G(\alpha, 1), 1) = G(1, F(\alpha, 1)) = G(1, a) = a$. So it follows that $d < a = c$. Further, taking $x = 1, y = 0$ in Eq. 8, then we get that $a = c = F(c, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, \alpha) = d$. Therefore it holds that $a = d$, but it contradicts $d < a$.

Thus we have $\alpha \leq a \leq c = d \leq b$.

Theorem 6.2. Let $\alpha \leq \min(a, d)$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. Then $F$ is $\alpha$-cross-migrative over $G$ if and only if one of the following two cases holds:

(i) If $\alpha \leq a = c = d \leq b$, then $S_F$ is $\frac{\alpha}{a}$-cross-migrative over $S_G$.

(ii) If $\alpha \leq a < c = d \leq b$, and let $x_0 = \sup\{x \in [0, 1]|G(\alpha, x) \leq a\}$, then one of the following two subcases holds.

(a) If $G(\alpha, x_0) = a$, then $G(x, a) = \begin{cases} G(\alpha, x) = G(x, \alpha) & \text{if } x > x_0, \\ a & \text{if } x \leq x_0, \end{cases}$ and $F(G(\alpha, x), y) = G(x, F(\alpha, y))$ when $x \leq x_0, y \leq a$.

(b) If $G(\alpha, x_0) < a$, then $G(x, a) = \begin{cases} G(\alpha, x) = G(x, \alpha) & \text{if } x \geq x_0, \\ a & \text{if } x < x_0, \end{cases}$ and $F(G(\alpha, x), y) = G(x, F(\alpha, y))$ when $x < x_0, y \leq a$.

Proof. The proof is omitted because it is similar to that of Theorem 5.2.

6.2 $\alpha \geq \max(b, c)$

Lemma 6.3. Let $\alpha \geq \max(b, c)$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If $F$ is $\alpha$-cross-migrative over $G$, then $a \leq c = d \leq b \leq a$.

Proof. Assume that $d < a$, then, taking $x = 0, y = 1$ in Eq. 8, we can get that $a = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 0)) = G(0, a) = c$. Thus it follows that $d < a = c \leq b$. Further, taking $x = 0, y = 0$ in Eq. 8, then we can get that $d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = c$, then $d = c = a$, which contradicts the assumption $d < a$. Therefore it follows that $d \geq a$.

Next, taking $x = 0, y = 1$ in Eq. 8, then we can get that $d = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 0)) = G(0, a) = c$.

Finally, assume that $b < c$, then taking $x = 1, y = 0$, we can get that $b = F(\alpha, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = c$, which contradicts the assumption $b < c$. Therefore it follows that $b \geq c$. Thus we prove that $b \geq d = c \geq a$.

Theorem 6.4. Let $\alpha \geq \max(b, c)$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. Then $F$ is $\alpha$-cross-migrative over $G$ if and only if one of the following results is true

(i) If $\alpha \leq c = d = b \leq a$, then $T_F$ is $\frac{\alpha-b}{1-b}$-cross-migrative over $T_G$.

(ii) If $\alpha \leq c = d < b \leq a$, and write $x_0 = \inf\{x \in [0, 1]|G(\alpha, x) \geq b\}$, then one of the following two subcases holds.

(a) If $G(\alpha, x_0) = b$, then $G(x, b) = \begin{cases} G(\alpha, x) = G(x, \alpha) & \text{if } x < x_0, \\ b & \text{if } x \geq x_0, \end{cases}$ and $F(G(\alpha, x), y) = G(x, F(\alpha, y))$ when $x \geq x_0, y \geq b$.
Proof. The proof is omitted because it is similar to that of Theorem 5.5.

6.3 \( \min(a, d) < \alpha < \max(b, c) \)

For this case, there are four different subcases to be considered: (1) \( a \leq d, b \leq c \), (2) \( a \leq d, b \geq c \), (3) \( a \geq d, b \leq c \), (4) \( a \geq d, b \geq c \). Now we start Subcase (1).

Lemma 6.5. Let \( a = \min(a, d) < \alpha < \max(b, c) = c \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \), if \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( a < \alpha < b = c = d \).

**Proof.** Assume that \( d \leq \alpha < c \), then, taking \( x = 1, y = 1 \) in Eq. [5], we get that \( c = F(c, 1) = F(G(\alpha, 1), 1) = G(1, F(\alpha, 1)) = G(1, \alpha) = \alpha \), which contradicts \( d \leq \alpha < c \), thus \( a < \alpha < d \).

Taking \( x = 1, y = 1 \) in Eq. [5], then we get that \( c = F(c, 1) = F(G(\alpha, 1), 1) = G(1, F(\alpha, 1)) = G(1, \alpha) = d \), so \( d = c \).

Further, taking \( x = 1, y = 0 \) in Eq. [5], then we have that \( b = F(c, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = d \).

Sum up, we have proven that \( a < \alpha < b = c = d \).

Theorem 6.6. Let \( a = \min(a, d) < \alpha < \max(b, c) = c \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \). Then \( F \) is \( \alpha \)-cross-migrative over \( G \) if and only if

(i) \( a < \alpha < b = c = d \).

(ii) \( G(x, \alpha) = G(\alpha, x) \), for all \( x \in [0, c] \).

**Proof.** The proof is omitted because it is similar to that of Theorem 5.8.

For Subcase (2), we have the following lemma.

Lemma 6.7. Let \( a = \min(a, d) < \alpha < \max(b, c) = b \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( a < \alpha \leq d = c \leq b \) or \( a \leq d = c < \alpha < b \).

**Proof.** Assume that \( d < \alpha < c \), then, taking \( x = 0, y = 1 \) in Eq. [5], we can get that \( d = F(d, 0) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, \alpha) = \alpha \), which contradicts the assumption \( d < \alpha \). Thus it follows that \( a < \alpha \leq d \) or \( c \leq \alpha < b \).

If \( a < \alpha \leq d \), then, taking \( x = 1, y = 0 \) in Eq. [5], we can get that \( c = F(c, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, \alpha) = d \). Therefore \( a < \alpha \leq d = c \leq b \).

If \( c \leq \alpha < b \), then, taking \( x = 0, y = 0 \) in Eq. [5], we can get that \( d = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, \alpha) = \alpha \). Hence \( a \leq c = d \leq \alpha < b \).

Theorem 6.8. Let \( a = \min(a, c) < \alpha < \max(b, d) = b \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \leq d \). Then \( F \) is \( \alpha \)-cross-migrative over \( G \) if and only if one of the following conditions is true

(i) \( a < \alpha \leq c = d \leq b \), and \( G(x, \alpha) = G(\alpha, x) \), for all \( x \in [0, c] \).

(ii) \( a \leq c = d \leq \alpha < b \), and \( G(x, \alpha) = G(\alpha, x) \), for all \( x \in [c, 1] \).

**Proof.** The proof is omitted because it is similar to that of Theorem 5.11.

For Subcase (3), we have the following lemma.

Lemma 6.9. Let \( d = \min(a, d) < \alpha < \max(b, c) = c \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \), then \( F \) is not \( \alpha \)-cross-migrative over \( G \).

**Proof.** Assume that \( \alpha \leq b \), then taking \( x = 0, y = 0 \) in Eq. [5], we can get that \( d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = G(0, \alpha) = \alpha \), which contradicts \( \alpha > b \). Again taking \( x = 0, y = 0 \) in Eq. [5], we can get that \( d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = G(0, b) = b \). Note that \( d = \min(a, d) \leq a \leq b \) and \( b = d \), then it gets that \( \alpha > a = d = b \). Finally, taking \( x = 0, y = 1 \) in Eq. [5], we can get that \( d = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, \alpha) = \alpha \), which contradicts \( \alpha > a = d = b \).

Therefore \( F \) is not \( \alpha \)-cross-migrative over \( G \).
For Subcase (4), we have the following lemma.

**Lemma 6.10.** Let \( d = \min(a, d) < \alpha < \max(b, c) = b \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \). If \( F \) is \( \alpha \)-cross-migrative over \( G \), then \( a = c = d < \alpha < b \).

**Proof.** Assume that \( \alpha \leq c \), then taking \( x = 0, y = 0 \) in Eq.(8), we can get that \( d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = G(0, \alpha) = \alpha \), which contradicts \( d < \alpha \). Thus it follows that \( \alpha > c \). Again taking \( x = 0, y = 0 \) in Eq.(8), then we can get that \( d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = G(0, \alpha) = c \). Note that \( d = \min(a, d) \leq a \), then it gets that \( \alpha > d = c \). Finally, taking \( x = 0, y = 1 \) in Eq.(8), then we can get that \( a = F(d, a) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, \alpha) = d \).

Hence it holds that \( a = c = d < \alpha < b \).

**Theorem 6.11.** Let \( \min(a, c) < \alpha < \max(b, d) \), \( F \in \mathcal{F}_{a,b} \) with \( a \leq b \) and \( G \in \mathcal{F}_{c,d} \) with \( c \geq d \). Then \( F \) is \( \alpha \)-cross-migrative over \( G \) if and only if

(i) \( a = c = d < \alpha < b \).

(ii) \( G(x, \alpha) = G(\alpha, x) \), for all \( x \in [a, 1] \).

**Proof.** The proof is omitted because it is similar to that of Theorem 5.15.

**Remark 6.12.** By comparing results of Section 5, note that the cross-migrativity of a semi-t-operator \( F \in \mathcal{F}_{a,b} \) over another semi-t-operator \( G \in \mathcal{F}_{c,d} \) with \( a \leq b \) and \( c \geq d \) is the same with the case \( a \leq b, c \leq d \).

Fig.4. Structures of \( F \) (left) and \( G \) (right) in Section 6

### 7 Conclusions

In the paper, depending on the order relationship of \( \alpha, a, b, c, d \), we studied the cross-migrative property between semi-t-operators and gave all solutions of the cross-migrativity equation of a semi-t-operator \( F \in \mathcal{F}_{a,b} \) over another semi-t-operator \( G \in \mathcal{F}_{c,d} \). We have found that \( G \) degrades into a semi-nullnorm except the case \( a \geq b, c \geq d \) if \( F \) is \( \alpha \)-cross-migrative over \( G \). But for the case \( a \geq b, c \geq d \), there must be \( a = c, b = d \), which doesn’t mean \( F = G \). Specially, the cross-migrative property between two semi-t-operators is always determined by their underlying operators. In the future, we will intensively study the cross-migrative property between semi-t-conorms and (or) semi-t-norms. Meanwhile, according to summarizing all of results in this paper, it is easy to find that the semi-nullnorms play the important role.

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