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Applications of generalized fixed points theorems to the existence of uncertain differential equations with finite delay

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Abstract

In this paper, we study the existence and uniqueness of solutions to initial value problems associated to ordinary fuzzy differential equations with finite delay in the setting of a generalized Hukuhara derivative. Our approach is based on the results of fixed points of weakly contractive mappings on partially ordered metric spaces.

Keywords: Fuzzy differential equations, generalized Hukuhara differentiability, contractive mappings, finite delay.

1 Introduction

Differential equations with delay describe phenomena in which the unknown function of the system depends, not only on the state of the system at a given instant, but also, on the history of the trajectory until that instant [16, 25]. The class of differential equations with delay covers a large variety of differential equations with applications in biology, engineering, physics, forecasting and other experimental sciences. See for instance [13] for a mathematical delay model related to the dynamics of a proliferating cell population, and [21, 31, 37] for other applications including models of cell cycle considered in chemotherapy. When we integrate fuzzy phenomena into delay differential equations we get fuzzy differential equations with delay. In particular, fuzzy differential equations with finite delay are used to study the dynamical system with uncertain or vague data; thus, if the underlying structure of the model depends upon subjective choices, one way to incorporate these characteristics into the model is using the aspects of fuzziness through the theory of fuzzy differential equations (FDE), and in particular, those with finite delay.

In order to study FDE, depending on the notion of fuzzy diferentiability, several settings have been considered. The first and the most popular approach is using the Hukuhara differentiability (H-differentiability) for fuzzy valued functions (see, for instance [18, 29, 34]). However, this approach has the drawback that the solution of a fuzzy differential equation needs to have increasing length of the support, so, in this case, the qualitative theory is poor in comparison to ordinary differential equations [3, 5, 4, 10, 11]. In order to overcome this weakness, some alternatives have been proposed; in fact, in [2], the concept of generalized differentiability (GH) was introduced, which allows us to obtain new solutions to fuzzy differential equations [3, 4]. This concept of differentiability is based on four forms of lateral derivatives. The differentiability in the first form (i) is coincident with the H-differentiability and then the generalized differentiability is more general than H-differentiability. Contrary to the previous case, if we consider a fuzzy initial value problem (FIVP) with the differentiability in the second form (ii), the solution needs to have decreasing length of the support. Thus, if we consider a FIVP with differentiability in the first form (i) or in the second form (ii), then we do not have periodic solutions. However, since the GH-differentiability in the third form (iii) and fourth form (iv) are linked with the so-called switching points, combining the obtained solutions using GH-differentiability in the first form (i) and in the second form (ii), we can obtain periodic solutions and many other solutions for a FIVP having better

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qualitative properties. So, the interest of the existence and uniqueness of solutions to FIVP considering the concept of GH-differentiability in the first form (i) and in the second form (ii) is very important. In this direction, several results of existence and uniqueness of solutions have been obtained, see for instance [2, 3, 8, 18, 23, 26, 29, 34, 41, 42].

In the setting of ordinary FDE with finite delay, some results of the existence of fuzzy solutions were obtained in [20, 24, 29] by using the classical Banach contraction principle. Also, in [25], the authors studied the local existence and uniqueness of solution to fuzzy delay differential equations driven by Liu process. Recently, in [38], the authors analyze some fuzzy delay differential equations by employing the concepts of granular differentiability. Motivated by the results of [20, 24, 29], in this paper, we study the following Cauchy problem of ordinary FDE with finite delay

$$\begin{cases} u'(t) = f(t, u_t), & t \in [0,T], \\ u(t) = \eta(t), & t \in [-\tau, 0], \ \tau > 0, \end{cases} \tag{1}$$

where u is the unknown, $f\colon [0,T] \times C([-\tau,0],\mathcal{F}^1) \longrightarrow \mathcal{F}^1$ is continuous, $\eta \in C([-\tau,0],\mathcal{F}^1)$, and the derivative u' is considered in the sense of the GH-derivative. In order to obtain the existence and uniqueness of solution for the FIVP (1), in place of using the classical Banach fixed point theorem, we apply some fixed point theorems, established in [17, 28], on weakly contractive functions defined on partially ordered sets. Precisely, under a generalized contractive-like property over comparable elements, which is weaker than the classical Lipschitz condition, we prove the existence and uniqueness of two types of solutions for FIVP (1), corresponding to the two different types of the GH-differentiability (see Theorems 4.5 and 4.6 below). Uniqueness is understood in the sense that the solutions considered do not have switching points. Indeed, using the concept of switching point we can construct other solutions for FIVP (1) combining the two types of GH-differentiability (see Example 4.10 below, and Example 4.2 in [42]). It should be noted that the space of fuzzy numbers is not a Banach space, but it is a semilinear and partially ordered metric space.

This paper is organized as follows: In Section 2, we give some preliminaries about the GH-derivative, which will be necessary to study the existence and uniqueness of solutions for FIVP associated to FDE with finite delay. In Section 3, we present some fixed point results of weakly contractive mappings on partially ordered sets. In Section 4, we prove a result on the existence and uniqueness of a (i)-solution, as well as a (ii)-solution for Problem (1).

2 Preliminaries

We start by recalling some preliminaries about the fuzzy sets defined on $\mathbb{R}^n$. A fuzzy set on $\mathbb{R}^n$ is a mapping $u\colon \mathbb{R}^n \to [0,1]$, where the value $u(x)$ denotes the degree of membership of the element x to the fuzzy set u. For $0 < \alpha \leq 1$, the α-level of u is defined by the set $[u]^\alpha = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}$. For $\alpha = 0$, the support of u is defined as the set $[u]^0 = \text{supp}(u) = \overline{\{x \in \mathbb{R}^n \mid u(x) > 0\}}$. We denote

$$\mathcal{F}^n = \{u : \mathbb{R}^n \to [0,1] | u \text{ satisfies } (i) - (iv) \text{ below}\},$$

where

- (i) u is normal, that is, there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$.
- (ii) u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$, for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$.
- (iii) u is upper semicontinuous.
- (iv) $[u]^0$ is compact.

In particular, if $u \in \mathcal{F}^1$ we say that u is a fuzzy number.

According to Zadeh's Extension Principle [12], operations of addition and scalar multiplication on $\mathcal{F}^n$ are defined as:

$$(u+v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\}, \qquad \text{and} \qquad (\lambda u)(x) = \begin{cases} u(\frac{x}{\lambda}) & \lambda \neq 0, \\ \chi_{\{0\}}(x) & \lambda = 0, \end{cases}$$

where $\chi_{\{0\}}$ is the characteristic function of $\{0\}$. Moreover, the following relations hold:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad \text{and} \qquad [\lambda u]^\alpha = \lambda [u]^\alpha, \quad \forall u, v \in \mathcal{F}^n, \quad \forall \alpha \in [0,1].$$

Definition 2.1. *Let $u, v, w \in \mathcal{F}^n$. An element w is called the Hukuhara difference (H-difference, for short) of u and v, if it verifies the equation $u = v + w$. If the H-difference exists, it will be denoted by $u \ominus_H v$. Clearly, $u \ominus_H u = \{0\}$, and if $u \ominus_H v$ exists, it is unique.*

The space $\mathcal{F}^n$ is a complete metric space with the distance $d_\infty(u,v)$ given by

$$d_\infty(u,v) = \sup_{\alpha\in[0,1]} d([u]^\alpha,[v]^\alpha), \quad \forall u,v \in \mathcal{F}^n,$$

where $d(\cdot,\cdot)$ is the well known Pompeiu-Hausdorff distance on the space $\mathcal{K}_c^n$ of all nonempty, compact and convex subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. We recall that the couple $(\mathcal{K}_c^n, d)$ is also a complete metric space (cf. [35]).

Theorem 2.2. [15] *Let $u \in \mathcal{F}^1$ be a fuzzy number with α-levels given by $[u]^\alpha = [u_l^\alpha, u_r^\alpha]$. Then u is completely determined by any pair $u = (u_l, u_r)$ of functions $u_l, u_r : [0,1] \to \mathbb{R}$, defining the endpoints of the α-level sets, satisfying the following conditions:*

(i) *$u_l : \alpha \to u_l^\alpha \in \mathbb{R}$ is a bounded nondecreasing left-continuous function on $(0,1]$ and it is right-continuous at 0.*

(ii) *$u_r : \alpha \to u_r^\alpha \in \mathbb{R}$ is a bounded nonincreasing left-continuous function on $(0,1]$ and it is right-continuous at 0.*

(iii) *$u_l^\alpha \leq u_r^\alpha$, for any $\alpha \in [0,1]$.*

In $\mathcal{F}^1$, we consider the partial order $\lesssim$ (also considered in [29, 28]) defined as follows: if $u,v \in \mathcal{F}^1$ with $[u]^\alpha = [u_l^\alpha, u_r^\alpha]$ and $[v]^\alpha = [v_l^\alpha, v_r^\alpha]$, then

$$u \lesssim v \;\Leftrightarrow\; u_l^\alpha \leq v_l^\alpha \text{ and } u_r^\alpha \leq v_r^\alpha,\ \forall \alpha \in [0,1].$$

If $u \lesssim v$, we may use the notation $v \gtrsim u$. It is easy to see that for $u,v,w,z \in \mathcal{F}^1$,

$$\text{if } u \lesssim v, \text{ then } u + w \lesssim v + w,$$

and

$$\text{if } u \lesssim v \text{ and } w \lesssim z, \text{ then } u \ominus_H (-1)w \lesssim v \ominus_H (-1)z,$$

provided $u \ominus_H (-1)w$ and $v \ominus_H (-1)z$ exist. On the other hand, we say that $(u_k)_{k\in\mathbb{N}} \subset \mathcal{F}^1$ is a nondecreasing sequence if $u_k \lesssim u_{k+1}$ for all $k \in \mathbb{N}$; analogously, $(v_k)_{k\in\mathbb{N}} \subset \mathcal{F}^1$ is a nonincreasing sequence if $v_{k+1} \lesssim v_k$ for all $k \in \mathbb{N}$. The following result holds.

Lemma 2.3. [29] *On $\mathcal{F}^1$ the following properties hold:*

(i) *Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{F}^1$ be a nondecreasing sequence. If $u_k \to u$ in $\mathcal{F}^1$, then $u_k \lesssim u$ for all $k \in \mathbb{N}$.*

(ii) *Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{F}^1$ be a nonincreasing sequence. If $u_k \to u$ in $\mathcal{F}^1$, then $u_k \gtrsim u$ for all $k \in \mathbb{N}$.*

(iii) *Every pair of elements of $\mathcal{F}^1$ has an upper bound and a lower bound in $\mathcal{F}^1$.*

Let J be a compact interval and denote by $C(J,\mathcal{F}^1)$ the set of all continuous functions on the interval J with values in $\mathcal{F}^1$. We consider the following partial order on $C(J,\mathcal{F}^1)$:

$$f,g \in C(J,\mathcal{F}^1),\ f \lesssim g \iff f(t) \lesssim g(t), \quad \forall t \in J. \tag{2}$$

For $C(J \times J', \mathcal{F}^1)$, with J and J' closed intervals, we say that $f \lesssim g$ if and only if $f(s,t) \lesssim g(s,t)$ for all $(s,t) \in J \times J'$. On $C(J,\mathcal{F}^1)$, we consider the metric $D(\cdot,\cdot)$ defined by

$$D(f,g) = \sup_{t\in J} d_\infty(f(t),g(t)), \quad f,g \in C(J,\mathcal{F}^1). \tag{3}$$

Then, $(C(J,\mathcal{F}^1), D)$ is a complete metric space (cf. [12]). For our aims, we will use the weighted metric $D_\rho(\cdot,\cdot)$ on $C(J,\mathcal{F}^1)$ defined by (cf. [29])

$$D_\rho(f,g) = \sup_{t\in J} e^{-\rho t} d_\infty(f(t),g(t)), \quad f,g \in C(J,\mathcal{F}^1),$$

for $\rho > 0$ fixed, which is equivalent to $D(\cdot,\cdot)$ defined in (3) since J is a compact interval. On the space $C(J \times J', \mathcal{F}^1)$, for J and J' closed intervals, we will consider the metric

$$D_\rho(f,g) = \sup_{(s,t)\in J\times J'} e^{-\rho(s+t)} d_\infty(f(s,t),g(s,t)), \quad f,g \in C(J \times J, \mathcal{F}^1).$$

Then, the following result holds:

Lemma 2.4. [29] *On $C(J,\mathcal{F}^1)$ the following properties hold:*

(i) If $(f_k)_{k\in\mathbb{N}} \subset C(J,\mathcal{F}^1)$ is a nondecreasing sequence, with the order $\lesssim$ defined in (2), such that $f_k \to f$ in $C(J,\mathcal{F}^1)$, then $f_k \lesssim f$ for all $k\in\mathbb{N}$.

(ii) If $(f_k)_{k\in\mathbb{N}} \subset C(J,\mathcal{F}^1)$ is a nonincreasing sequence, with the order $\lesssim$ defined in (2), such that $f_k \to f$ in $C(J,\mathcal{F}^1)$, then $f_k \gtrsim f$ for all $k\in\mathbb{N}$.

(iii) Every pair of elements of $C(J,\mathcal{F}^1)$ has an upper bound and a lower bound in $C(J,\mathcal{F}^1)$.

In recent years, several authors have established different concepts of fuzzy differentiability due to the necessity of the enlargement of the class of differentiable fuzzy functions and treat, in a best way, the analysis of FDE (cf. [1, 2, 3, 7, 18, 27, 35, 40, 41, 42]). A first generalization of the concept of H-differentiability [34] is given by the following definition.

Definition 2.5. [2] *Let $f\colon (a,b) \to \mathcal{F}^n$ and $t_0 \in (a,b)$. f is said to be strongly generalized differentiable (or GH-differentiable for short) at t_0, if there exists an element $f'(t_0)\in\mathcal{F}^n$ such that*

(i) there exist the differences $f(t_0+h)\ominus_H f(t_0)$, $f(t_0)\ominus_H f(t_0-h)$ and

$$\lim_{h\to 0^+}\frac{f(t_0+h)\ominus_H f(t_0)}{h} = \lim_{h\to 0^+}\frac{f(t_0)\ominus_H f(t_0-h)}{h} = f'(t_0),$$

or

(ii) there exist the differences $f(t_0)\ominus_H f(t_0+h)$, $f(t_0-h)\ominus_H f(t_0)$ and

$$\lim_{h\to 0^+}\frac{f(t_0)\ominus_H f(t_0+h)}{(-h)} = \lim_{h\to 0^+}\frac{f(t_0-h)\ominus_H f(t_0)}{(-h)} = f'(t_0),$$

or

(iii) there exist the differences $f(t_0+h)\ominus_H f(t_0)$, $f(t_0-h)\ominus_H f(t_0)$ and

$$\lim_{h\to 0^+}\frac{f(t_0+h)\ominus_H f(t_0)}{h} = \lim_{h\to 0^+}\frac{f(t_0-h)\ominus_H f(t_0)}{(-h)} = f'(t_0),$$

or

(iv) there exist the differences $f(t_0)\ominus_H f(t_0+h)$, $f(t_0)\ominus_H f(t_0-h)$, and

$$\lim_{h\to 0^+}\frac{f(t_0)\ominus_H f(t_0+h)}{(-h)} = \lim_{h\to 0^+}\frac{f(t_0)\ominus_H f(t_0-h)}{h} = f'(t_0).$$

Here, the limit is taken in the metric space $(\mathcal{F}^n, d_\infty)$.

We say that f is (i)-GH-differentiable (respectively, (ii)-GH-differentiable, (iii)-GH-differentiable and (iv)-GH-differentiable) at t_0 if f is GH-differentiable in the first form (i) (respectively, second form (ii), third form (iii) and fourth form (iv)) of Definition 2.5.

Notice that the (i)-GH-differentiability coincides with the H-differentiability. Thus, the GH-differentiability generalizes the H-differentiability and it is more adequate to solve fuzzy differential equations [2].

Given a fuzzy function $f\colon (a,b)\longrightarrow \mathcal{F}^1$ and $t\in(a,b)$, the α-levels of $f(t)$ are compact intervals, that is, $[f(t)]^\alpha = [f_l^\alpha(t), f_r^\alpha(t)]$, where f_l^α and f_r^α are real functions on (a,b) which are the endpoint functions of f. The following result establishes a connection between the GH-differentiability of f and its endpoint functions f_l^α and f_r^α.

Theorem 2.6. [6] *Let $f:(a,b)\to\mathcal{F}^1$ be a fuzzy function. If f is GH-differentiable at $t_0\in(a,b)$, then we have the following cases:*

(a) If f is (i)-GH-differentiable at $t_0\in(a,b)$ then, for each $\alpha\in[0,1]$, f_l^α and f_r^α are differentiable functions at t_0 and

$$[f'(t_0)]^\alpha = \left[(f_l^\alpha)'(t_0)\ ,\ (f_r^\alpha)'(t_0)\right].$$

(b) *If f is (ii)-GH-differentiable at $t_0 \in (a,b)$ then, for each $\alpha \in [0,1]$, f_l^α and f_r^α are differentiable functions at t_0 and*

$$[f'(t_0)]^\alpha = \left[(f_r^\alpha)'(t_0)\ ,\ (f_l^\alpha)'(t_0)\right].$$

(c) *If f is (iii)-GH-differentiable at $t_0 \in (a,b)$ then, there exist the lateral derivatives $(f_l^\alpha)'_{+/-}$ and $(f_r^\alpha)'_{+/-}$ at t_0 and*

$$[f'(t_0)]^\alpha = \left[(f_l^\alpha)'_+(t_0)\ ,\ (f_r^\alpha)'_+(t_0)\right] = \left[(f_r^\alpha)'_-(t_0)\ ,\ (f_l^\alpha)'_-(t_0)\right].$$

(d) *If f is (iv)-GH-differentiable at $t_0 \in (a,b)$ then, there exist the lateral derivatives $(f_l^\alpha)'_{+/-}$ and $(f_r^\alpha)'_{+/-}$ at t_0 and*

$$[f'(t_0)]^\alpha = \left[(f_r^\alpha)'_+(t_0)\ ,\ (f_l^\alpha)'_+(t_0)\right] = \left[(f_l^\alpha)'_-(t_0)\ ,\ (f_r^\alpha)'_-(t_0)\right].$$

Given a fuzzy function f, for each $\alpha \in [0,1]$, we define the length function $len(f)_\alpha$ given by $len(f)_\alpha(t) = f_r^\alpha - f_l^\alpha$. Then, from Theorem 2.6, we can distinguish two kind of GH-differentiable fuzzy functions in relation to the differentiability of its endpoint functions. The (i) and (ii)-GH-differentiability are linked with the monotonicity of the length function while (iii) and (iv)-GH- differentiability are linked with the change of monotonicity of the length function. More precisely, the following result holds:

Proposition 2.7. [39] *Let $f : (a,b) \to \mathcal{F}^1$ be a fuzzy function.*

(i) *If f is (i)-GH-differentiable, then for each $\alpha \in [0,1]$, the function $len(f)_\alpha$ is nondecreasing.*

(ii) *If f is (ii)-GH-differentiable, then for each $\alpha \in [0,1]$, the function $len(f)_\alpha$ is nonincreasing.*

Proof. The proof is a consequence of Proposition 3 in [7]. □

Example 2.8. [42] *Let $f : \mathbb{R} \to \mathcal{F}^1$ defined by $f(t) = C \cdot t$, where C is a fuzzy number such that $[C]^\alpha = [\alpha - 1, 1 - \alpha]$ for all $\alpha \in [0,1]$. Then*

$$[f(t)]^\alpha = [(\alpha - 1)\,|t|\,, (1-\alpha)\,|t|] \quad \text{and} \quad len(f)_\alpha = 2(1-\alpha)\,|t|\,,$$

for all $\alpha \in [0,1]$. In this case f is GH-differentiable and $f'(t) = C$ for all $t \in \mathbb{R}$. Now, on the interval $(-\infty, 0)$ f is (ii)-GH-differentiable and $len(f)_\alpha$ is nonincreasing while on the interval $(0, +\infty)$ f is (i)-GH-differentiable and $len(f)_\alpha$ is nondecreasing. On the other hand, f is (iii)-GH-differentiable at $t = 0$ and in this point there is a change of monoticity of the function $len(f)_\alpha$.

It is interesting to see how the switch between the two cases (i) and (ii) can occur.

Definition 2.9. [40] *Let $f\colon (a,b) \longrightarrow \mathcal{F}^1$ be a GH-differentiable fuzzy function. An element $t_0 \in (a,b)$ is called a switching point for the GH-differentiability of f, if for every neighborhood of t_0, there exist points $t_1 < t_0 < t_2$ such that*

(I) *(type I) f is (i)-GH-differentiable at t_1, while f is not (ii)-GH-differentiable at t_1, and f is (ii)-GH-differentiable at t_2, while f is not (i)-GH-differentiable at t_2, or*

(II) *(type II) f is (ii)-GH-differentiable at t_1, while f is not (i)-GH-differentiable at t_1, and f is (i)-GH-differentiable at t_2, while f is not (ii)-GH-differentiable at t_2.*

Proposition 2.10. *Let $f\colon (a,b) \longrightarrow \mathcal{F}^1$ be a GH-differentiable fuzzy function and $t_0 \in (a,b)$.*

(i) *If t_0 is a switching point for the GH-differentiability of f of type I, then f is (iv)-GH-differentiable at t_0.*

(ii) *If t_0 is a switching point for the GH-differentiability of f of type II, then f is (iii)-GH-differentiable at t_0.*

Example 2.11. *Let $f : (0, 3\pi) \to \mathcal{F}^1$ defined by*

$$f(t)(x) = \begin{cases} x + 2 + t + 2\sin(t) & \text{if } x \in [-2 - t - 2\sin(t), -1 - t - 2\sin(t)], \\ 1 & \text{if } x \in [-1 - t - 2\sin(t), t + 2\sin(t)], \\ -x + t + 1 + 2\sin(t) & \text{if } x \in [t + 2\sin(t), 1 + t + 2\sin(t)], \\ 0 & \text{otherwise.} \end{cases}$$

Then, for each $\alpha \in [0,1]$ it holds

$$[f(t)]^\alpha = [f_l^\alpha(t), f_r^\alpha(t)] = [\alpha - 2 - t - 2\sin(t), -\alpha + 1 + t + 2\sin(t)]\,.$$

Since the endpoint functions f_l^α and f_r^α are differentiable on $(0, 3\pi)$, then f is GH-differentiable on $(0, 3\pi)$. It holds

- *f is (i)-GH-differentiable on $\left(0, \frac{2}{3}\pi\right)$ and $\left(\frac{4}{3}\pi, \frac{8}{3}\pi\right)$.*
- *f is (ii)-GH-differentiable on $\left(\frac{2}{3}\pi, \frac{4}{3}\pi\right)$.*
- *The points $t_1 = \frac{2}{3}\pi$, $t_2 = \frac{4}{3}\pi$ and $t_3 = \frac{8}{3}\pi$ are three switching points for GH-differentiability of f on $(0, 3\pi)$ and, from Definition 2.9, we have that t_1 and t_3 are switching points of type I, while t_2 is a switching point of type II.*
- *From Proposition 2.10, we have that f is (iv)-GH-differentiable at t_1 and t_3, and f is (iii)-GH-differentiable at t_2.*

Moreover, the length function is given by $len(f)_\alpha(t) = -2\alpha + 3 + 2t + 4\sin(t)$, and $\left(len(f)_\alpha\right)'(t) = 2 + 4\cos(t)$, for each $\alpha \in [0,1]$. The length function is nondecreasing on $\left(0, \frac{2}{3}\pi\right)$ and $\left(\frac{4}{3}\pi, \frac{8}{3}\pi\right)$ and it is nonincreasing on $\left(\frac{2}{3}\pi, \frac{4}{3}\pi\right)$, in accordance with Proposition 2.7 (see Figure 3.).

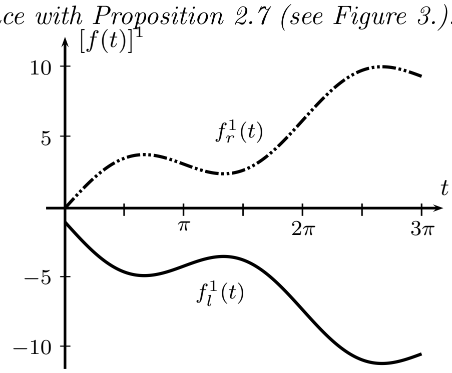

Figure 1: Set-valued $[f(t)]^1$.

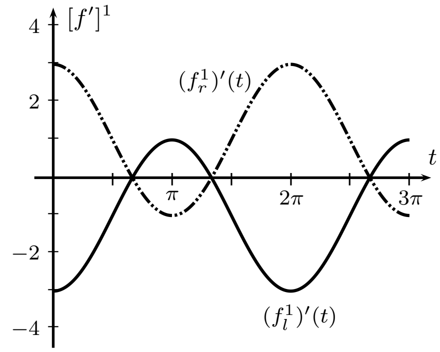

Figure 2: Lateral derivatives.

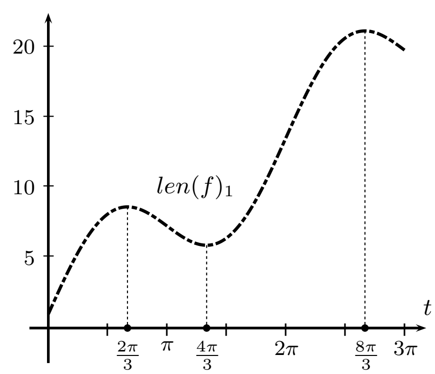

Figure 3: Length function $len(f)_1$.

Figure 1 shows the set-valued function $[f(t)]^1$ where we can see three points where the length of $[f(t)]^1$ changes monotonicity which corresponds to the switching points for GH-differentiablity of f. Figure 2 shows the derivative of the endpoint functions of $[f(t)]^1$ and Figure 3 shows the length function $len(f)_1$ where we can see the switching points for GH-differentiablity of f.

Now, we present some definitions and properties of the fuzzy integral which will be used in Section 4. Given $f : [a,b] \longrightarrow \mathcal{F}^1$, we say that f is integrably bounded, if there exists an integrable function $h : [a,b] \longrightarrow \mathbb{R}$, such that $||f(t)||_{\mathcal{F}} \leq h(t)$ for all $t \in [a,b]$, where $||u||_{\mathcal{F}} = d_\infty(u, 0)$, $\forall u \in \mathcal{F}^1$. Moreover, f is called strongly measurable if the set-valued mapping $f_\alpha : [a,b] \longrightarrow \mathcal{K}_c^1$ defined as $f_\alpha(t) = [f(t)]^\alpha$, is measurable for all $\alpha \in [0,1]$ (see [18]).

Definition 2.12. [18] *A fuzzy function $f : [a,b] \longrightarrow \mathcal{F}^1$ is integrable if f is integrably bounded and strongly measurable.*

Definition 2.13. [18] *Let $f : [a,b] \longrightarrow \mathcal{F}^1$ and $\alpha \in [0,1]$. The integral of f on $[a,b]$, denoted by $\int_a^b f(t)\, dt$, is defined levelwise by the equation*

$$\left[\int_a^b f(t)\, dt\right]^\alpha = \int_a^b [f(t)]^\alpha\, dt = \left\{\int_a^b v(t)\, dt \,|\, v : [a,b] \longrightarrow \mathbb{R} \text{ is a measurable selection for } [f(t)]^\alpha\right\}.$$

The following theorem summarizes some basic properties of the integral of fuzzy functions.

Theorem 2.14. [18] *Let $f, g\colon [a,b] \longrightarrow \mathcal{F}^1$ be two continuous fuzzy functions and $c \in \mathbb{R}$. Then,*

(i) $\int_a^b (f+g)(t)dt = \int_a^b f(t)dt + \int_a^b g(t)dt.$

(ii) $\int_a^b cf(t)dt = c\int_a^b f(t)dt.$

(iii) $d_\infty(f(t), g(t))$ *is integrable.*

(iv) $d_\infty(\int_a^b f(t)dt, \int_a^b g(t)dt) \leq \int_a^b d_\infty(f(t), g(t))dt.$

(v) *If* $f \lesssim g$, *then* $\int_a^b f(t)\ dt \lesssim \int_a^b g(t)dt.$

Theorem 2.15. [3] *Let $f\colon [a,b] \longrightarrow \mathcal{F}^1$ be continuous. Then*

(i) *The fuzzy function $F(x) = \int_a^x f(t)\ dt$ is (i)-GH-differentiable and $F'(x) = f(x)$.*

(ii) *The fuzzy function $G(x) = \int_x^b f(t)\ dt$ is (ii)-GH-differentiable and $G'(x) = -f(x)$.*

Lemma 2.16. *Let $x \in \mathcal{F}^1$ be such that the functions x_l and x_r defined as in Theorem 2.2 are differentiable, with x_l increasing and x_r decreasing on $[0,1]$, such that there exist the constants $c_1 > 0$, $c_2 < 0$ satisfying $(x_l^\alpha)' \geq c_1$ and $(x_r^\alpha)' \leq c_2$ for all $\alpha \in [0,1]$. Let $f\colon [a,b] \longrightarrow \mathcal{F}^1$ be continuous with respect to t and let M, M_1, M_2 be constants such that $len([f(t)]^1) \leq M$ for all $t \in [a,b]$, $\left|\frac{\partial f_l^\alpha(t)}{\partial \alpha}\right| \leq M_1$ and $\left|\frac{\partial f_r^\alpha(t)}{\partial \alpha}\right| \leq M_2$. Moreover, suppose that b is such that $b \leq \frac{c_1}{M_1}$, $b \leq \frac{|c_2|}{M_2}$ and $b \leq \frac{len([x]^1)}{M}$. If*

(a) $x_l(1) < x_r(1)$

or if

(b) $x_l(1) = x_r(1)$ *and the set $[f(s)]^1$ consists of exactly one element for any $s \in [a,b]$,*

then the H-difference $x \ominus_H \int_a^t f(s)\ ds$ exists for any $t \in [a,b]$.

Proof. The proof follows from Lemma 2.2 in [3]. □

3 Fixed points results of weakly contractive mappings

Some fixed point results of weakly contractive mappings on complete metric spaces have been obtained in [14, 17, 22, 23, 29, 30, 28, 36] and some references there in. Briefly, we recall some fixed point results obtained in [17] in a complete metric space endowed with a partial order by using altering distance functions.

Definition 3.1. [19] *An altering distance function is a function $\psi\colon [0,\infty) \to [0,\infty)$ such that*

(i) *ψ is continuous and nondecreasing;*

(ii) *$\psi(t) = 0$ if and only if $t = 0$.*

Definition 3.2. [14] *Let (X,d) be a metric space and $f\colon X \to X$ be a function. It is said that f is a weakly contractive mapping if*

$$\psi(d(f(x), f(y))) \leq \psi(d(x,y)) - \phi(d(x,y)), \quad \forall x, y \in X, \tag{4}$$

where ψ and ϕ are altering distance functions.

Considering the class of weakly contractive mappings, in [14], Theorem 2.1, the authors proved the following fixed point theorem is a generalization of the Banach contraction mapping principle, which in turns generalizes the fixed points results in [19] and [36].

Theorem 3.3. ([14], Theorem 2.1). *Let (X,d) be a complete metric space and let $f : X \to X$ be a weakly contractive mapping satisfying the inequality (4) where ψ and ϕ are both continuous and monotone nondecreasing functions. Then f has a unique fixed point.*

The following example shows the existence of a unique fixed point, as an application of Theorem 3.3, which cannot derived from the classical Banach contraction mapping (cf. [14]).

Example 3.4. *Let* $X = [0,1] \cup \{2,3,4,...\}$ *and*

$$d(x,y) = \begin{cases} |x-y| & \text{if } x, y \in [0,1], x \neq y, \\ x+y & \text{if at least one of } x \text{ or } y \notin [0,1], \text{ and } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then, (X,d) *is a complete metric space. Now, let* $\psi, \phi : [0,\infty) \to [0,\infty)$ *defined as*

$$\psi(t) = \begin{cases} t & \text{if } t \in [0,1], \\ t^2 & \text{if } t \in (1,\infty), \end{cases} \qquad \text{and} \qquad \phi(t) = \begin{cases} \frac{t^2}{2} & \text{if } t \in [0,1], \\ \frac{1}{2} & \text{if } t \in (1,\infty). \end{cases}$$

and consider $f : X \to X$ *defined as*

$$f(x) = \begin{cases} x - \frac{1}{2}x^2 & \text{if } x \in [0,1], \\ x-1 & \text{if } x \in \{2,3,...\}. \end{cases}$$

Then, f *is weakly contractive and, by Theorem 3.3,* f *has a unique fixed point. It is seen that the unique fixed point is* $x = 0$.

Now we recall some fixed point theorems involving altering distance functions in the context of ordered metric spaces, including the case when f is not necessarily continuous (cf.[17]). Let $(X, \leq)$ be a partially ordered set. A function $f: X \longrightarrow X$ is monotone nondecreasing if every all $x, y \in X$ with $x \leq y$, implies that $f(x) \leq f(y)$. The function f is monotone nonincreasing if for all $x, y \in X$ with $x \leq y$, implies that $f(x) \geq f(y)$.

Theorem 3.5. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X,d) *is a complete metric space. Let* $f: X \longrightarrow X$ *be a monotone nondecreasing function such that*

$$\psi(d(f(x), f(y))) \leq \psi(d(x,y)) - \phi(d(x,y)), \text{ for } x \geq y, \tag{5}$$

for some altering distance functions ψ *and* ϕ*. Suppose also that either* f *is continuous or* X *is such that*

$$\text{if a nondecreasing sequence } (x_k)_{k\in\mathbb{N}} \text{ is convergent to } x \in X, \text{ then } x_k \leq x, \text{ for all } k \in \mathbb{N}. \tag{6}$$

If there exists $x_0 \in X$ *such that* $x_0 \leq f(x_0)$*, then* f *has a fixed point.*

Proof. The proof of Theorem 3.5 in the case $f: X \longrightarrow X$ being a continuous and nondecreasing mapping is given in [17], Theorem 2.1. The proof of Theorem 3.5 in the case when X verifies (6) is given in [17], Theorem 2.2. □

Remark 3.6. *Theorem 3.5 generalizes the Banach contraction mapping principle for weakly contractive mappings on complete metric spaces (Theorem 3.3) (cf. Theorem 2.1 in [14] and Theorem 2.2 in [28]).*

Next result is analogous to Theorem 3.5 for a monotone nondecreasing function $f: X \longrightarrow X$ such that (5) holds, and verifying the counterpart of the condition (6) for nonincreasing sequences.

Theorem 3.7. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X,d) *is a complete metric space. Let* $f: X \longrightarrow X$ *be a monotone nondecreasing function such that (5) holds. Suppose also that either* f *is continuous or* X *is such that if a nonincreasing sequence* $(x_k)_{k\in\mathbb{N}}$ *is convergent to* $x \in X$*, then* $x \leq x_k$ *for all* $k \in \mathbb{N}$*. If there exists* $x_0 \in X$ *such that* $x_0 \geq f(x_0)$*, then* f *has a fixed point.*

Proof. The proof of Theorem 3.7 is analogous to the proof of Theorem 3.5 (cf. [17]). For convenience for the reader, we sketch it. First assume that $f: X \longrightarrow X$ is continuous. If $f(x_0) = x_0$, then the proof is finished. Then suppose that $x_0 > f(x_0)$. In this case, since f is nondecreasing it holds that

$$x_0 > f(x_0) \geq f^2(x_0) \geq f^3(x_0) \geq ... \geq f^n(x_0) \geq f^{n+1}(x_0) \geq ...$$

Making $x_{n+1} = f(x_n)$, from (5) we get

$$\psi(d(x_n, x_{n+1})) = \psi(d(f(x_{n-1}), f(x_n))) \leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)) \leq \psi(d(x_{n-1}, x_n)). \tag{7}$$

Since ψ is nondecreasing we get $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0-1}) = 0$, then $x_{n_0} = f(x_{n_0-1}) = x_{n_0-1}$ and thus x_{n_0-1} is a fixed point. Otherwise, if $d(x_{n+1}, x_n) \neq 0$ for all $n \in \mathbb{N}$, taking into account that $\{d(x_n, x_{n+1})\}$ is decreasing, there exists $M \geq 0$ such that $d(x_{n+1}, x_n) \to M$, as n goes to ∞. Passing to the limit as $n \to \infty$ in (7) we get

$$\psi(M) \leq \psi(M) - \phi(M) \leq \psi(M),$$

which gives that $\phi(M) = 0$, and since ϕ is an altering distance function it holds that $M = 0$. In conclusion, $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Following the proof of Theorem 2.2 in [17], $\{x_n\}$ is a Cauchy sequence and, by the continuity of f, its limit is a fixed point. The proof in the case when X is such that if a nonincreasing sequence $(x_k)_{k\in\mathbb{N}}$ is convergent to $x \in X$, then $x \leq x_k$ for all $k \in \mathbb{N}$, is very similar to the second part of the proof of Theorem 3.5 (cf. [17], Theorem 2.2.). □

Next theorem gives a sufficient condition for the uniqueness of the fixed point provided by Theorems 3.5 and 3.7.

Theorem 3.8. [17] *Under the assumption of Theorem 3.5 (respectively, Theorem 3.7), if every pair of elements of X has an upper bound or a lower bound, then f has a unique fixed point. Moreover, if $\overline{x}$ is the fixed point of f, then for all $x \in X$, $\lim_{k\to\infty} f^k(x) = \overline{x}$.*

Remark 3.9. *Some examples to appreciate that the hyphoteses in Theorems 3.3, 3.5, 3.7 and 3.8 are necessary can be found in [28].*

4 Fuzzy differential equations with finite delay

There is an extensive literature on differential equations of functions with finite delay and applications; in particular, FDE with finite delay using Hukuhara derivative have been studied in [29]. The main purpose of this section is to apply the results of Section 3 to study the FIVP with finite delay under the concept of GH-derivative, using some generalizations of fixed point theorems in partially ordered metric spaces.

For $\tau > 0$, we consider on $C([-\tau, 0], \mathcal{F}^1)$ the metric $D(\cdot, \cdot)$ defined in (3) over the interval $[-\tau, 0]$. If $u \in C(J_0, \mathcal{F}^1)$, where $J_0 := [-\tau, T]$, $T > 0$, and $t \in J_0$ is nonnegative, then the translation $u_t \in C([-\tau, 0], \mathcal{F}^1)$ of u on $[-\tau, 0]$ is defined as the restriction of u to the interval $[t - \tau, t]$, that is, $u_t(s) = u(t + s)$ with $-\tau \leq s \leq 0$. We consider the following ordinary FDE with finite delay:

$$\begin{cases} u'(t) = f(t, u_t), & t \in J = [0, T], \\ u(t) = \eta(t), & t \in [-\tau, 0], \end{cases} \tag{8}$$

where $f\colon J \times C([-\tau, 0], \mathcal{F}^1) \longrightarrow \mathcal{F}^1$ is continuous, $\eta \in C([-\tau, 0], \mathcal{F}^1)$, and the derivative u' is considered in the sense of the GH-derivative.

Definition 4.1. *A solution for (8) is a fuzzy function $u \in C(J_0, \mathcal{F}^1) \cap C^1(J, \mathcal{F}^1)$ satisfying (8). A solution for (8) is called (i)-solution if it is (i)-GH-differentiable. A solution for (8) is called (ii)-solution if it is (ii)-GH-differentiable.*

Lemma 4.2. *A fuzzy function $u \in C(J_0, \mathcal{F}^1) \cap C^1(J, \mathcal{F}^1)$ is a (i)-solution of (8) if and only if it satisfies the following integral equation*

$$\begin{cases} u(t) = \eta(t), & \text{if } t \in [-\tau, 0], \\ u(t) = \eta(0) + \displaystyle\int_0^t f(s, u_s)\, ds, & \text{if } t \in J. \end{cases}$$

Proof. Suppose that $u \in C(J_0, \mathcal{F}^1) \cap C^1(J, \mathcal{F}^1)$ is a (i)-solution of (8). Then

$$\eta(t) = u_0(t) = u(t + 0) = u(t), \quad \forall t \in [-\tau, 0].$$

The fuzzy differential equation $u'(t) = f(t, u_t)$ with $t \in J$ is equivalent to the integral equation

$$u(t) = \eta(0) + \int_0^t f(s, u_s)\, ds,$$

where $u \in C^1(J, \mathcal{F}^1)$ is a (i)-solution of (8). As $u \in C([-\tau, 0], \mathcal{F}^1) \cap C(J, \mathcal{F}^1)$, we will see that $u \in C(J_0, \mathcal{F}^1)$. For this, we prove that u is continuous at $t = 0$. If $h > 0$, then there exist $u(0 + h) \ominus_H u(0)$ and $u(0 - h) \ominus_H u(0)$ and

$$u(0 + h) \ominus_H u(0) = \int_0^h f(s, u_s) ds \overset{h\to 0}{\longrightarrow} \chi_{\{0\}},$$

$$u(0 - h) \ominus_H u(0) = \eta(-h) \ominus_H u(0) \overset{h\to 0}{\longrightarrow} \chi_{\{0\}},$$

where the limit is taken in the metric space $(\mathcal{F}^1, d_\infty)$. Thus, $u \in C(J_0, \mathcal{F}^1)$, and therefore, the proof is finished. □

Lemma 4.3. *A fuzzy function $u \in C(J_0, \mathcal{F}^1) \cap C^1(J, \mathcal{F}^1)$ is a (ii)-solution of (8) if, and only if, it satisfies the integral equation*

$$\begin{cases} u(t) = \eta(t), & \text{if } t \in [-\tau, 0], \\ u(t) = \eta(0) \ominus_H \left(- \int_0^t f(s, u_s)\, ds \right), & \text{if } t \in J, \end{cases}$$

provided the corresponding H-difference exists.

Proof. The proof of this result is analogous to that of Lemma 4.2. □

We use the existence of upper and lower solutions in order to ensure the existence and uniqueness of solutions for (8). We recall that the existence of lower and upper solutions for FDE is obtained through a monotone iterative approach (cf. [9, 32, 33]).

Definition 4.4. *A fuzzy function $\mu \in C(J_0, \mathcal{F}^1) \cap C^1(J, \mathcal{F}^1)$ is said a lower solution to Problem (8), if*

$$\mu'(t) \precsim f(t, \mu_t), \quad t \in J, \quad \mu_0 \precsim \eta.$$

If μ is (i)-GH-differentiable (respectively, (ii)-GH-differentiable), then μ is said a lower (i)-solution (respectively, a lower (ii)-solution).
A fuzzy function $\mu \in C(J_0, \mathcal{F}^1) \cap C^1(J, \mathcal{F}^1)$ is said an upper solution to Problem (8), if

$$\mu'(t) \succsim f(t, \mu_t), \quad t \in J, \quad \mu_0 \succsim \eta.$$

If μ is (i)-GH-differentiable (respectively, (ii)-GH-differentiable), then μ is said an upper (i)-solution (respectively, an upper (ii)-solution).

Our main results on the existence and uniqueness of solution to FIVP with finite delay are given by the following theorems.

Theorem 4.5. *Suppose that there exists a lower (i)-solution $\mu \in C(J_0, \mathcal{F}^1) \cap C^1(J, \mathcal{F}^1)$ to Problem (8). Let $f\colon J \times C([-\tau, 0], \mathcal{F}^1) \longrightarrow \mathcal{F}^1$ be continuous such that:*

$(H1)$ f is nondecreasing in the second variable, that is, if $\eta \succsim \lambda$ then $f(t, \eta) \succsim f(t, \lambda)$, $\forall t \in J$,

$(H2)$ f verifies

$$d_\infty(f(t, u_s), f(t, v_s)) \leq D(u_s, v_s) \text{ for all } u \succsim v, \quad t \in J.$$

Then, Problem (8) has a unique (i)-solution defined on J_0.

Proof. We will apply Theorem 3.5. For this, we define the operator $\mathcal{A}_1 \colon C(J_0, \mathcal{F}^1) \longrightarrow C(J_0, \mathcal{F}^1)$ by

$$[\mathcal{A}_1 u](t) = \begin{cases} \eta(t), & \text{if } t \in [-\tau, 0], \\ \eta(0) + \int_0^t f(s, u_s)\, ds, & \text{if } t \in J. \end{cases}$$

If $u \in C(J_0, \mathcal{F}^1)$ is a fixed point of $\mathcal{A}_1$, then $u \in C(J_0, \mathcal{F}^1) \cap C^1(J, \mathcal{F}^1)$ is (i)-solution of Problem (8) and conversely. Now, let $u \succsim v$ on J_0 (so, $u_s \succsim v_s, \forall s \in J$). If $t \in [-\tau, 0]$, $[\mathcal{A}_1 u](t) = \eta(t) = [\mathcal{A}_1 v](t)$, and if $t \in J$, from the assumption $(H1)$, we have that

$$[\mathcal{A}_1 u](t) = \eta(0) + \int_0^t f(s, u_s)\, ds \succsim \eta(0) + \int_0^t f(s, v_s)\, ds = [\mathcal{A}_1 v](t), \text{ for all } u \succsim v.$$

Thus, $\mathcal{A}_1 u \succsim \mathcal{A}_1 v$ provided $u \succsim v$, and therefore, the operator $\mathcal{A}_1$ is nondecreasing. For $u \succsim v$, if $t \in [-\tau, 0]$, $d_\infty([\mathcal{A}_1 u](t), [\mathcal{A}_1 v](t)) = d_\infty(\eta(t), \eta(t)) = 0$ and, if $t \in J$, from the assumption $(H2)$, it holds

$$\begin{aligned} d_\infty([\mathcal{A}_1 u](t), [\mathcal{A}_1 v](t)) &= d_\infty\left(\eta(0) + \int_0^t f(s, u_s) ds, \eta(0) + \int_0^t f(s, v_s) ds\right) \leq \int_0^t d_\infty(f(s, u_s), f(s, v_s))\, ds \\ &\leq \int_0^t D(u_s, v_s)\, ds = \int_0^t \max_{-\tau \leq r \leq 0} d_\infty(u_s(r), v_s(r)) ds. \end{aligned}$$

Then,

$$\begin{aligned}
D_\rho(\mathcal{A}_1 u, \mathcal{A}_1 v) &= \sup_{t\in[-\tau,T]} \left\{ d_\infty\left([\mathcal{A}_1 u](t), [\mathcal{A}_1 v](t)\right) e^{-\rho t} \right\} \leq \sup_{t\in J} \left\{ \int_0^t \max_{-\tau\leq r\leq 0} d_\infty(u_s(r), v_s(r)) ds\ e^{-\rho t} \right\} \\
&= \sup_{t\in J} \left\{ \int_0^t \max_{-\tau\leq r\leq 0} d_\infty(u(s+r), v(s+r)) ds\ e^{-\rho t} \right\} \leq \sup_{t\in J} \left\{ D_\rho(u,v) \int_0^t \max_{-\tau\leq r\leq 0} e^{\rho(s+r)} ds\ e^{-\rho t} \right\} \\
&\leq D_\rho(u,v) \sup_{t\in J} \left\{ \int_0^t e^{\rho s} ds\ e^{-\rho t} \right\} = D_\rho(u,v) \sup_{t\in J} \left\{ \frac{1-e^{-\rho t}}{\rho} \right\} = \frac{1-e^{-\rho T}}{\rho}\ D_\rho(u,v).
\end{aligned}$$

Thus, $D_\rho(\mathcal{A}_1 u, \mathcal{A}_1 v) \leq \frac{1-e^{-\rho T}}{\rho}\ D_\rho(u,v)$. Let γ be an increasing altering distance function. Then, it holds that

$$\gamma\left(D_\rho(\mathcal{A}_1 u, \mathcal{A}_1 v)\right) \leq \gamma\left(D_\rho(u,v)\ \frac{1-e^{-\rho T}}{\rho} \right) = \gamma(D_\rho(u,v)) - \left[\gamma(D_\rho(u,v)) - \gamma\left(D_\rho(u,v)\ \frac{1-e^{-\rho T}}{\rho} \right) \right].$$

Then, taking $\Phi(t) = \gamma(t) - \gamma\left(\frac{1-e^{-\rho T}}{\rho} t\right)$, it follows that Φ is an altering distance function and

$$\gamma(D_\rho(\mathcal{A}_1 u, \mathcal{A}_1 v)) \leq \gamma(D_\rho(u,v)) - \Phi(D_\rho(u,v)).$$

Finally, notice that if $t \in [-\tau, 0]$ and $\mu(0) \lesssim \eta(0)$, $\mu(t) \lesssim \eta(t) = [\mathcal{A}_1\mu](t)$, and if $t \in J$,

$$\mu(t) = \mu(0) + \int_0^t \mu'(s)\ ds \lesssim \eta(0) + \int_0^t f(s,\mu_s)\ ds = [\mathcal{A}_1\mu](t).$$

Thus, $\mu \leq \mathcal{A}_1\mu$. In this manner the operator $\mathcal{A}_1$ verifies the hypotheses of the Theorem 3.5, and therefore, $\mathcal{A}_1$ has a fixed point in $C(J_0, \mathcal{F}^1)$. Since, by Lemma 2.4, every pair of elements of $C(J_0, \mathcal{F}^1)$ has an upper bound, then by Theorem 3.8, it follows that the operator $\mathcal{A}_1$ has a unique fixed point. □

Theorem 4.6. *Suppose that there exists a lower (ii)-solution $\mu \in C(J_0, \mathcal{F}^1) \cap C^1(J, \mathcal{F}^1)$ for Problem (8). Let $f\colon J \times C([-\tau, 0], \mathcal{F}^1) \longrightarrow \mathcal{F}^1$ be a continuous fuzzy function satisfying:*

($H1$) *f is nondecreasing in the second variable, that is, if $\eta \gtrsim \lambda$ then $f(t,\eta) \gtrsim f(t,\lambda)$, $\forall t \in J$,*

($H2$) *f verifies*

$$d_\infty(f(t,u_s), f(t,v_s)) \leq D(u_s, v_s) \text{ for all } u \gtrsim v, \quad t \in J.$$

($H3$) *$len([u_0(t)]^\alpha) \geq len\left(\left[\int_0^t f(s,u_s)\ ds\right]^\alpha\right)$ for all $\alpha \in [0,1]$, $(u_0(t))_l^\alpha - \int_0^t f_l^\alpha(s)\ ds$ is increasing with respect to α, and $(u_0(t))_r^\alpha - \int_0^t f_r^\alpha(s)\ ds$ is decreasing with respect to α, where $[u_0(t)]^\alpha = [(u_0(t))_l^\alpha, (u_0(t))_r^\alpha]$ and $\left[\int_0^t f(s,u_s)\ ds\right]^\alpha = \left[\int_0^t f_l^\alpha(s)\ ds, \int_0^t f_r^\alpha(s)\ ds\right]$.*

Then, Problem (8) has a unique (ii)-solution defined on J_0.

Proof. We define the operator $\mathcal{A}_2\colon C(J_0, \mathcal{F}^1) \longrightarrow C(J_0, \mathcal{F}^1)$ by:

$$[\mathcal{A}_2 u](t) = \begin{cases} \eta(t), & \text{if } t \in [-\tau, 0], \\ \eta(0) \ominus_H \left(-\int_0^t f(s,u_s)\ ds \right), & \text{if } t \in J. \end{cases}$$

If $u \in C(J_0, \mathcal{F}^1)$ is a fixed point of $\mathcal{A}_2$, then $u \in C(J_0, \mathcal{F}^1) \cap C^1(J, \mathcal{F}^1)$ is (ii)-solution of Problem (8) and conversely. The hypothesis ($H3$) guarantees the existence of the H-difference in the definition of the operator $\mathcal{A}_2$. On the other hand, similar to the proof of Theorem 4.5, the operator $\mathcal{A}_2$ is nondecreasing and it verifies

$$\gamma(D_\rho(\mathcal{A}_2 u, \mathcal{A}_2 v)) \leq \gamma(D_\rho(u,v)) - \Phi(D_\rho(u,v)),$$

where Φ is defined as $\Phi(t) = \gamma(t) - \gamma\left(\frac{1-e^{-\rho T}}{\rho} t\right)$, with γ increasing. Moreover, $\mu \gtrsim \mathcal{A}_2\mu$. Therefore, the operator $\mathcal{A}_2$ verifies the hypothesis of the Theorem 3.7, and hence, $\mathcal{A}_2$ has a fixed point in $C(J_0, \mathcal{F}^1)$. Since every pair of elements of $C(J_0, \mathcal{F}^1)$ has an upper bound, it follows that the operator $\mathcal{A}_2$ has a unique fixed point. □

Remark 4.7. *The condition $(H3)$ of Theorem 4.6 can be safisyed if u_0 and f verifies the hypotheses of Lemma 2.16.*

Remark 4.8. *Replacing the existence of a lower (i)-solution (lower (ii)-solution, respectively) by an upper (i)-solution (upper (ii)-solution, respectively) for Problem (8), the conclusion of the Theorem 4.5 (Theorem 4.6, respectively) is still valid. Indeed, if μ is an upper (i)-solution to Problem (8), we have*

$$\mu(t) \gtrsim \eta(t) = [\mathcal{A}_1\mu](t), \quad t \in [-\tau, 0],$$
$$\mu(t) = \mu(0) + \int_0^t \mu'(s)\ ds \gtrsim \eta(0) + \int_0^t f(s, \mu_s)\ ds = [\mathcal{A}_1\mu](t), \quad t \in J.$$

Therefore, $\mu \gtrsim \mathcal{A}_1\mu$, which corresponds to the case considered in the Theorem 4.5. Respectively, if μ is a lower (ii)-solution to the Problem (8), using (2), we have

$$\mu(t) \lesssim \eta(t) = [\mathcal{A}_2\mu](t), \quad t \in [-\tau, 0],$$
$$\mu(t) = \mu(0) \ominus_H \left(-\int_0^t \mu'(s)\ ds\right) \lesssim \eta(0) \ominus_H \left(\int_0^t f(s, \mu_s)\ ds\right) = [\mathcal{A}_2\mu](t), \quad t \in J.$$

Thus, $\mu \lesssim \mathcal{A}_2\mu$, which corresponds to the case considered in Theorem 4.6. The existence of a solution for Problem (8) follows from Theorem 3.7, because $C(J_0, \mathcal{F}^1)$ verifies that if a nonincreasing sequence $(u_k)_{k\in\mathbb{N}}$ is convergent to $u \in C(J_0, \mathcal{F}^1)$, then $u \lesssim u_k$ for all $k \in \mathbb{N}$. Given that every pair of elements of $C(J_0, \mathcal{F}^1)$ has an upper bound, the operator $\mathcal{A}_i$, $i = 1, 2$, has a unique fixed point.

Remark 4.9. *Using Theorems 4.5, 4.6 and Remark 4.8, the solutions to Problem (8) can be obtained as $\lim_{k\to\infty} \mathcal{A}_i^k u$, $i = 1, 2$, for any $u \in C(J_0, \mathcal{F}^1)$. In particular, for upper solutions, $(\mathcal{A}_i^k(\mu))_{k\in\mathbb{N}}$, $i = 1, 2$, are nondecreasing sequences and convergent in $C(J_0, \mathcal{F}^1)$ to the corresponding solution of Problem (8).*

Next we present an example to illustrate our main results.

Example 4.10. *A classical model of fuzzy differential equation with finite delay is given by the fuzzy Malthusian growth model. This is a natural example because the birth rate does not changes immediately as soon as new individuals born; indeed, new individuals need a time for the development of their bodies and their reproduction. An illustrative example of this population dynamics model is given by the following FIVP:*

$$\begin{cases} P'(t) = \lambda P(t-1), & t \in [0, 10], \\ P(t) = P_0, & t \in [-1, 0], \end{cases}$$

where λ denotes the growth rate and the initial data P_0 is a fuzzy number. For simplicity we consider P_0 as being $(1, 2, 3)$ which has α-levels $[\alpha + 1, 3 - \alpha]$, $\alpha \in [0, 1]$. Observe that the assumptions of Theorems 4.5, 4.6 are verified. If we consider the (i)-GH-derivative of P, then we get the following delay differential system

$$\begin{cases} (P_l^\alpha)'(t) = \lambda P_l^\alpha(t-1), & t \in [0, 10], \\ (P_r^\alpha)'(t) = \lambda P_r^\alpha(t-1), & t \in [0, 10], \\ (P_l^\alpha)(t) = \alpha + 1, & t \in [-1, 0], \\ (P_r^\alpha)(t) = 3 - \alpha, & t \in [-1, 0]. \end{cases} \tag{9}$$

In order to solve (9) we can argue as in [20, 24] (method of steps). If $0 \le t \le 1$, then $-1 \le t - 1 \le 0$. Therefore, for $0 \le t \le 1$ we have $P_l^\alpha(t-1) = \alpha + 1$ and $P_r^\alpha(t-1) = 3 - \alpha$. Consequently, for $0 \le t \le 1$ we get

$$P_l^\alpha(t) = (\alpha + 1) + \int_0^t \lambda(\alpha + 1)ds = (\alpha + 1)(1 + \lambda t),$$
$$P_r^\alpha(t) = (3 - \alpha) + \int_0^t \lambda(3 - \alpha)ds = (3 - \alpha)(1 + \lambda t).$$

Now, if $1 \le t \le 2$, then $0 \le t - 1 \le 1$. Therefore, for $0 \le t \le 2$ we have $P_l^\alpha(t-1) = (\alpha + 1)(1 + \lambda(t-1))$ and $P_r^\alpha(t-1) = (3 - \alpha)(1 + \lambda(t-1))$. Consequently, for $1 \le t \le 2$ we get

$$P_l^\alpha(t) = (\alpha + 1)(1 + \lambda) + \int_1^t \lambda(\alpha + 1)(1 + \lambda(s-1))ds = (\alpha + 1)\left(1 + \lambda t + \frac{\lambda^2 t^2}{2} - \lambda^2 t + \frac{\lambda^2}{2}\right),$$
$$P_r^\alpha(t) = (3 - \alpha)(1 + \lambda) + \int_1^t \lambda(3 - \alpha)(1 + \lambda(s-1))ds = (3 - \alpha)\left(1 + \lambda t + \frac{\lambda^2 t^2}{2} - \lambda^2 t + \frac{\lambda^2}{2}\right).$$

Since Theorem 4.5 guarantee the uniqueness of (i)-solution, we can continue with the previous procedure in order to construct the (i)-solution defined on the whole interval $[0, 10]$*. On the other hand, if we consider the (ii)-GH-derivative of P, then we get the following delay differential system*

$$\begin{cases} (P_l^\alpha)'(t) = \lambda P_r^\alpha(t-1), & t \in [0,10], \\ (P_r^\alpha)'(t) = \lambda P_l^\alpha(t-1), & t \in [0,10], \\ (P_l^\alpha)(t) = \alpha + 1, & t \in [-1,0], \\ (P_r^\alpha)(t) = 3 - \alpha, & t \in [-1,0]. \end{cases} \tag{10}$$

In order to solve (10) we also use the method of steps. Indeed, if $0 \le t \le 1$*, then* $-1 \le t-1 \le 0$*. Therefore, for* $0 \le t \le 1$ *we have* $P_r^\alpha(t-1) = 3-\alpha$ *and* $P_l^\alpha(t-1) = \alpha+1$*. Consequently, for* $0 \le t \le 1$ *we get*

$$P_l^\alpha(t) = (\alpha+1) + \int_0^t \lambda(3-\alpha)ds = (\alpha+1) + \lambda(3-\alpha)t,$$

$$P_r^\alpha(t) = (3-\alpha) + \int_0^t \lambda(\alpha+1)ds = (3-\alpha) + \lambda(\alpha+1)t.$$

Now, if $1 \le t \le 2$*, then* $0 \le t-1 \le 1$*. Therefore, for* $1 \le t \le 2$ *we have* $P_l^\alpha(t-1) = (\alpha+1) + \lambda(3-\alpha)(t-1)$ *and* $P_r^\alpha(t-1) = (3-\alpha) + \lambda(\alpha+1)(t-1)$*. Consequently, for* $1 \le t \le 2$ *we get*

$$P_l^\alpha(t) = (\alpha+1) + \lambda(3-\alpha) + \lambda(3-\alpha)(t-1) + \lambda(\alpha+1)\left(\frac{t^2}{2} - t + \frac{1}{2}\right),$$

$$P_r^\alpha(t) = (3-\alpha) + \lambda(\alpha+1) + \lambda(\alpha+1)(t-1) + \lambda(3-\alpha)\left(\frac{t^2}{2} - t + \frac{1}{2}\right).$$

Since Theorem 4.6 guarantee the uniqueness of (ii)-solution, we can continue with the previous procedure in order to construct the (ii)-solution defined on the whole interval $[0, 10]$.

Notice that we can combine the (i) and (ii)-derivatives in order to get other solutions. For instance, we can consider the solution P_2 *obtained previously (in terms of the* α*-levels) by using the (ii)-GH-derivative on the interval* $[0, 1]$*, and then consider the following FIVP*

$$\begin{cases} P'(t) = \lambda P(t-1), & t \in [1,10], \\ P(t) = P_2(1), & t \in [0,1]. \end{cases} \tag{11}$$

Thus, taking the (i)-solution of (11) we can construct a third solution $P_3(t)$.

5 Conclusions

Using some results of fixed point of weakly contractive mappings on partially ordered metric spaces, we analyzed the existence and uniqueness of solutions for fuzzy initial value problems with finite delay FIVP in the setting of the generalized Hukuhara derivative (GH-derivative). In particular, since the definition of GH-differentiability is based on four types of lateral derivatives, we proved the existence and uniqueness of an (i)-solution, as well as an (ii)-solution for a FIVP.

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