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# Generalized convex combination of implications on bounded lattices

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**Abstract**

In this paper, the notions of the linear and g-convex combination for implications which extend the notion of convex combination of fuzzy implications on the unit interval to bounded lattices are introduced. A necessary and sufficient condition for the g-convex combination to be an implication is determined. Some basic properties of the g-convex combinations are discussed. Also, some sets which are defined by the linear (g-convex) combination of two implications on a bounded lattice are studied and the relationships between them are discussed. Moreover, the lattice theoretical structure of the mentioned sets is investigated.

*Keywords:* Fuzzy implication, linear combination, convex combination, bounded lattice.

## 1 Introduction

Fuzzy implication functions generalize the classical implication to fuzzy logic. They have a significant role in many applications, viz., approximate reasoning, fuzzy control, fuzzy image processing, etc. (see [1, 5, 6, 13, 14, 25]), fuzzy mathematical morphology [14], fuzzy DI-subsethood measures [4, 3], data mining [28]. The great quantity of applications has lead systematically to implications from the theoretical point of view.

As an extension of the unit interval $[0, 1]$, like in the case of other logical operators, the problem of introducing and investigating implications on bounded lattices has been attractive for many researchers. In this sense, Ma and Wu [12] have introduced them at first.

In the literature, there are some methods in order to generate new logical operators from given ones [1, 15, 16, 26, 27]. Especially, the convex combination of logical operators, which is one of the remarkable generating method, provides new logical operators from two given ones [7, 17, 18, 19, 22, 21, 23, 24]. In this sense, the convex combination of fuzzy implications $I$ and $J$ on the unit interval $[0, 1]$ is given as follows (see [1, 18, 20]): for $\lambda \in [0, 1]$,

$$K(x, y) = \lambda I(x, y) + (1 - \lambda)J(x, y).$$

According to our best knowledge so far, the notion of convex combination for two implications on a bounded lattice is not encountered in the literature. In this paper, we introduce the generalized convex combination notion for two implications on a bounded lattice. We show that the generalized convex combination notion coincides with the notion for the convex combination of two fuzzy implications on the unit interval $[0, 1]$. Main aim of this paper is to extend the notion of convex combinations for fuzzy implications defined on the unit interval $[0, 1]$ to bounded lattices and investigate some properties of the new operator. The paper is organized as follows. In Section 2, we shortly recall some basic notions. In Section 3, we introduce the notions of the linear and g-convex combination of implications which extend the notions of the convex combination of implications on the unit interval to bounded lattices. We present a necessary and sufficient condition for the linear combination of implications on bounded lattices to be an implication. We show that the notion of the convex combination of two fuzzy implications is a special g-convex combination. We determine the greatest one in the class of the g-convex combinations of two implications on bounded lattices. We investigate that the

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g-convex combination of implications on bounded lattices preserves some basic properties such as (NP), (IP), (OP) and (EP) under which conditions. We present the relationships between the implications and their g-convex combinations. We determine the conjugacy of the g-convex combination for two implications on bounded lattices. We show that the $N'$-reciprocal of the g-convex combination of two implications coincides with the g-convex combination of the $N'$-reciprocals of the implications. We determine the g-convex combination of the least and the greatest implications. We present the set of the elements satisfying (IP) w.r.t. the g-convex combination of two implications is a complete lattice. In Section 4, we study the sets denoted by $(X)^{\perp}$, $(X)_{\perp}$, ${}^{\perp}(X)$ and ${}_{\perp}(X)$ which are defined by the linear (g-convex) combination of two implications on a bounded lattice and discuss the relationships between them. We prove that the sets $(X)^{\perp}$, $(X)_{\perp}$, ${}^{\perp}(X)$ and ${}_{\perp}(X)$ are in shape of intervals under some conditions. Also, we investigate the lattice theoretical structure of the mentioned sets.

## 2 Preliminaries

In this section, we recall some basic notions and results.

**Definition 2.1.** [2] *A bounded lattice $(L,\leq)$ is a lattice which has the top and bottom elements, which are written as $1$ and $0$, respectively, that is, there exist two elements $1,0\in L$ such that $0\leq x\leq 1$, for all $x\in L$.*

**Definition 2.2.** [11, 12] *A binary operation $T$ $(S)$ on a bounded lattice $L$ is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element $1$ $(0)$.*

*The followings are the four basic t-norms $T_M, T_P, T_{LK}$ and $T_D$ on the unit interval $[0,1]$ given by respectively:*

$$T_M(x,y)=\min(x,y),\qquad T_P(x,y)=xy,$$

$$T_{LK}(x,y)=\max(x+y-1,0),\qquad T_D(x,y)=\begin{cases}0 & (x,y)\in[0,1)^2,\\ \min(x,y) & \text{otherwise.}\end{cases}$$

*The t-norms $T_M$ and $T_D$ are generalized to a bounded lattice $L$, respectively, as follows.*

$$T_\wedge(x,y)=x\wedge y \quad \text{and} \quad T_D(x,y)=\begin{cases}y & x=1,\\ x & y=1,\\ x\wedge y & \text{otherwise.}\end{cases}$$

*The followings are the four basic t-conorms $S_M, S_P, S_{LK}$ and $S_D$ on the unit interval $[0,1]$ given by respectively:*

$$S_M(x,y)=\max(x,y),\qquad S_P(x,y)=x+y-xy,$$

$$S_{LK}(x,y)=\min(x+y,1),\qquad S_D(x,y)=\begin{cases}1 & (x,y)\in(0,1]^2,\\ \max(x,y) & \text{otherwise.}\end{cases}$$

*The t-conorms $S_M$ and $S_D$ are generalized to a bounded lattice $L$, respectively, as follows.*

$$S_\vee(x,y)=x\vee y \quad \text{and} \quad S_D(x,y)=\begin{cases}y & x=0,\\ x & y=0,\\ x\vee y & \text{otherwise.}\end{cases}$$

**Definition 2.3.** [11, 12] *A t-norm $T$ is distributive over a t-conorm $S$ if for all $x,y,z\in L$,*

$$T(x,S(y,z))=S(T(x,y),T(x,z)).$$

**Definition 2.4.** [1, 10, 9] *A function $I:L^2\to L$ on a bounded lattice $(L,\leq,0,1)$ is called an implication if it satisfies the following conditions:*

***(I1)*** *$I$ is a decreasing operation on the first variable, that is, for every $a,b\in L$ with $a\leq b$, $I(b,y)\leq I(a,y)$ for all $y\in L$.*

***(I2)*** *$I$ is an increasing operation on the second variable, that is, for every $a,b\in L$ with $a\leq b$, $I(x,a)\leq I(x,b)$ for all $x\in L$.*

***(I3)*** *$I(0,0)=1$.*

***(I4)*** *$I(1,1)=1$.*

***(I5)*** *$I(1,0)=0$.*

Denote by $\mathcal{F}$ the set of all implications on a bounded lattice $L$.

**Example 2.5.** [1] *The followings are well-known implications on the unit interval* $[0,1]$.

$$I_{LK}(x,y)=\min(1,1-x+y), \qquad I_{RC}(x,y)=1-x+xy,$$

$$I_{KD}(x,y)=\max(1-x,y), \qquad I_{GD}(x,y)=\begin{cases}1 & x\le y,\\ y & x>y,\end{cases}$$

$$I_{GG}(x,y)=\begin{cases}1 & x\le y,\\ \frac{y}{x} & x>y,\end{cases} \qquad I_{RS}(x,y)=\begin{cases}1 & x\le y,\\ 0 & x>y,\end{cases}$$

$$I_{YG}(x,y)=\begin{cases}1 & x=0 \ \ and \ \ y=0,\\ y^x & x>0 \ \ or \ \ y>0,\end{cases} \qquad I_{WB}(x,y)=\begin{cases}1 & x<1,\\ y & x=1,\end{cases}$$

$$I_{FD}(x,y)=\begin{cases}1 & x\le y,\\ \max(1-x,y) & x>y.\end{cases}$$

*The least and the greatest implications are respectively given by:*

$$I_0(x,y)=\begin{cases}1 & x=0 \ \ or \ \ y=1,\\ 0 & x>0 \ \ and \ \ y<1,\end{cases} \qquad I_1(x,y)=\begin{cases}1 & x<1 \ \ or \ \ y>0,\\ 0 & x=1 \ \ and \ \ y=0.\end{cases}$$

**Definition 2.6.** [1, 10, 12] *Let* $(L,\le,0,1)$ *be a bounded lattice. A decreasing function* $N:L\to L$ *is called a negation if* $N(0)=1$ *and* $N(1)=0$. *A negation* $N$ *on* $L$ *is called strong if it is an involution, i.e.,* $N(N(x))=x$, *for all* $x\in L$.

*The weakest and strongest negations on* $L$ *are given by respectively*

$$N_{D_1}(x)=\begin{cases}0 & x\ne 0,\\ 1 & x=0,\end{cases} \qquad N_{D_2}(x)=\begin{cases}1 & x\ne 1,\\ 0 & x=1.\end{cases}$$

*The natural negation of an implication* $I$ *on a bounded lattice is the function* $N_I:L\to L$ *defined by* $N_I(x)=I(x,0)$, *for all* $x\in L$.

*Let* $N$ *be a fuzzy negation and* $I$ *be an implication on a bounded lattice* $L$. *The implication* $I_N:L^2\to L$ *defined by* $I_N(x,y)=I(N(y),N(x))$, *for any* $x,y\in L$, *is called the* $N$*-reciprocal of* $I$.

**Example 2.7.** *Here, one can easily check that the following functions are implications on a bounded lattice* $L$ *(see [10]): for a t-norm* $T$, *a strong negation* $N$ *and the natural negation* $N_I$,

$$I(x,y)=\begin{cases}y & x=1,\\ N_I(x) & y=0,\\ 1 & otherwise,\end{cases} \qquad J(x,y)=N(x)\vee y, \qquad K(x,y)=\begin{cases}N(x)\vee y & x>y,\\ 1 & otherwise.\end{cases}$$

**Definition 2.8.** [1, 26] *An implication* $I$ *on a bounded lattice* $L$ *is said to satisfy*

*(i) the left neutrality property if for all* $y\in L$

$$I(1,y)=y. \tag{NP}$$

*(ii) the left ordering property if for all* $x,y\in L$

$$x\le y\Rightarrow I(x,y)=1. \tag{LOP}$$

*(iii) the right ordering property if for all* $x,y\in L$

$$I(x,y)=1\Rightarrow x\le y. \tag{ROP}$$

*(iv) the ordering property if for all* $x,y\in L$

$$x\le y\Leftrightarrow I(x,y)=1. \tag{OP}$$

*(v) the identity principle if for all* $x\in L$

$$I(x,x)=1. \tag{IP}$$

*(vi) the exchange principle if for all* $x,y,z\in L$

$$I(x,I(y,z))=I(y,I(x,z)). \tag{EP}$$

*(vii) the weak exchange ability principle if for all* $x,y,z\in L$

$$x\le I(y,z)\Leftrightarrow y\le I(x,z). \tag{WE}$$

**Definition 2.9.** [1] *Let $S$ be a t-conorm and $N$ a negation on a bounded lattice $L$. We say that the pair $(S, N)$ satisfies the law of excluded middle if for all $x \in L$*

$$S(N(x), x) = 1. \tag{LEM}$$

**Definition 2.10.** [1] *If $I$ is an implication on a bounded lattice $L$ and $\phi : L \to L$ is an order-preserving bijection, then the operation $I_\phi : L^2 \to L$ given by $I_\phi(x, y) = \phi^{-1}(I(\phi(x), \phi(y)))$ is also an implication. This implication is called $\phi$-conjugate of $I$.*

*An order-preserving bijection does not always exist for a bounded lattice. For this, the lattice must be autodual.*

**Definition 2.11.** [1, 18, 20] *Let $I$ and $J$ be two fuzzy implications on $[0, 1]$. The operation $K$ defined as follows is called a convex combination of $I$ and $J$ for any $x, y \in [0, 1]$ and $\lambda \in [0, 1]$, $K(x, y) = \lambda I(x, y) + (1 - \lambda)J(x, y)$.*

**Definition 2.12.** [8] *(i) An operation $M$ on a lattice is called $\vee$-distributive in the second place if for any $a, b_1, b_2 \in L$ $M(a, b_1 \vee b_2) = M(a, b_1) \vee M(a, b_2)$.*

*(ii) Let $A$ be an infinite indexing set. An operation $M$ on a complete lattice is called infinitely $\vee$-distributive in the second place if for any $a \in L$ and $\{b_\tau \mid \tau \in A\} \subseteq L$, $M(a, \vee_{\tau \in A} b_\tau) = \vee_{\tau \in A} M(a, b_\tau)$.*

*The (infinitely) $\wedge$-distributivity in the first place of any operation can be defined as similar to Definition 2.12.*

*(iii) An operation $M$ on a lattice is called (infinitely) $\vee$-distributive if it is (infinitely) $\vee$-distributive in both the first and second places.*

*(iv) The (infinitely) $\wedge$-distributivity of any operation can be defined similarly.*

## 3 G-convex combination

In this section, the notions of the linear and g-convex combination of implications, which extend the notion of the convex combination of implications on the unit interval to bounded lattices, are introduced and a necessary and sufficient condition for the linear combination of implications on bounded lattices to be an implication is discussed. The basic properties such as (NP), (IP), (OP) and (EP) of the convex combination are studied. The conjugacy of the g-convex combination for two implications on bounded lattices are presented.

**Definition 3.1.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$. For $a \in L$, define the operation $K^{I,J}_{a,T,S,N}$ as follows,*

$$K^{I,J}_{a,T,S,N}(x, y) = S(T(a, I(x, y)), T(N(a), J(x, y))) \quad \text{for all} \quad x, y \in L \tag{1}$$

*where $T, S$ and $N$ are respectively a t-norm, a t-conorm and a negation on $L$.*

*The operation $K^{I,J}_{a,T,S,N}$ defined by formula (1) is called the linear combination of the implications $I$ and $J$ by means of $T, S$ and $N$ for $a \in L$.*

The following Theorem 3.2 gives us a necessary and sufficient condition making the linear combination $K^{I,J}_{a,T,S,N}$ an implication on $L$.

**Theorem 3.2.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. Then, $K^{I,J}_{a,T,S,N}$ is an implication on $L$ if and only if $S(a, N(a)) = 1$.*

*Proof.* Let $K^{I,J}_{a,T,S,N}$ be an implication on $L$. Then, we have that

$$1 = K^{I,J}_{a,T,S,N}(1, 1) = S(T(a, I(1, 1)), T(N(a), J(1, 1))) = S(T(a, 1), T(N(a), 1)) = S(a, N(a)).$$

Conversely, let $S(a, N(a)) = 1$ for $a \in L$. We shall show that $K^{I,J}_{a,T,S,N}$ is an implication.

(I1) Let $x \le z$ for $x, z \in L$. Then, for any $y \in L$, $I(x, y) \ge I(z, y)$ and $J(x, y) \ge J(z, y)$.

By the monotonicity of $T$, $T(a, I(x, y)) \ge T(a, I(z, y))$ and $T(N(a), J(x, y)) \ge T(N(a), J(z, y))$.

By the monotonicity of $S$, it is clear that

$$K^{I,J}_{a,T,S,N}(x, y) = S(T(a, I(x, y)), T(N(a), J(x, y))) \ge S(T(a, I(z, y)), T(N(a), J(z, y))) = K^{I,J}_{a,T,S,N}(z, y).$$

Thus, $K^{I,J}_{a,T,S,N}$ is a decreasing operation in the first place.

(I2) Similarly, it can be shown that $K^{I,J}_{a,T,S,N}$ is an increasing operation in the second place.

(I3) $K^{I,J}_{a,T,S,N}(1, 1) = S(T(a, I(1, 1)), T(N(a), J(1, 1))) = S(T(a, 1), T(N(a), 1)) = S(a, N(a)) = 1.$

(I4) $K^{I,J}_{a,T,S,N}(0,0) = S(T(a,I(0,0)),T(N(a),J(0,0))) = S(T(a,1),T(N(a),1)) = S(a,N(a)) = 1.$

(I5) $K^{I,J}_{a,T,S,N}(1,0) = S(T(a,I(1,0)),T(N(a),J(1,0))) = S(T(a,0),T(N(a),0)) = S(0,0) = 0.$

Thus, $K^{I,J}_{a,T,S,N}$ is an implication on $L$. □

**Definition 3.3.** *The implication $K^{I,J}_{a,T,S,N}$ is called the generalized convex combination of the implications $I$ and $J$ by means of $T,S$ and $N$ for any $a \in L$ with $S(a,N(a)) = 1$.*

For reader convenience, the operation $K^{I,J}_{a,T,S,N}$ given in Definition 3.3 will be called as the g-convex combination of $I$ and $J$.

**Remark 3.4.** *Take $L = [0,1]$, $S = S_{LK}$, $T = T_P$ and $N(x) = N_C(x) = 1 - x$ in (1).*
*Since $aI(x,y) + (1-a)J(x,y) \leq a + (1-a) = 1$, we have that*

$$\begin{aligned} K^{I,J}_{a,T,S,N}(x,y) &= S_{LK}(T_P(a,I(x,y)),T_P(N_C(a),J(x,y))) = S_{LK}(aI(x,y),(1-a)J(x,y)) \\ &= \min(aI(x,y) + (1-a)J(x,y),1) = aI(x,y) + (1-a)J(x,y). \end{aligned}$$

*Then, the operation $K^{I,J}_{a,T,S,N}$ coincides with the notion for the convex combination of two fuzzy implications on the unit interval $[0,1]$. This is why the operation $K^{I,J}_{a,T,S,N}$ is named as the generalized convex combination.*

*In this paper, we will denote by $K^{I,J}_a$ the g-convex combination of $I$ and $J$, when $S = S_{LK}$, $T = T_P$ and $N(x) = N_C(x) = 1 - x$.*

**Remark 3.5.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$.*

*(i) If $(S,N)$ is a pair satisfying (LEM), then the convex combination operator $K^{I,J}_{a,T,S,N}$ is an implication.*

*(ii) Since any t-conorm satisfies (LEM) with the greatest fuzzy negation $N_{D_2}$ by Remark 2.3.10 (i) [1], the convex combination operator $K^{I,J}_{a,T,S,N_{D_2}}$ is always an implication on $L$.*

*(iii) If $a \vee N(a) = 1$, then it is clear that $K^{I,J}_{a,T,S,N}$ is an implication because $a \vee N(a) \leq S(a,N(a))$.*

*(iv) If $S = S_D$ and $N$ is a strong negation, since $S_D(N(a),a) = 1$ for all $a \in L$, the linear combination $K^{I,J}_{a,T,S,N}$ is the g-convex combination of $I$ and $J$.*

*(v) If $L$ is a Boolean algebra and $N(x) = x'$, where $x'$ denotes the complement of $x$, then $K^{I,J}_{a,T,S,N}$ is always an implication since $1 = a \vee N(a) \leq S(a,N(a))$.*

Let us give an example for the g-convex combination of two implications on the unit interval $[0,1]$.

**Example 3.6.** *Consider the t-norm $T : [0,1]^2 \to [0,1]$ defined by* (see [11])

$$T(x,y) = \begin{cases} 0 & (x,y) \in [0,0.5]^2, \\ 2(x-0.5)(y-0.5)+0.5 & (x,y) \in (0.5,1]^2 \\ \min(x,y) & \textit{otherwise}, \end{cases}$$

*the t-conorm $S_{LK}$ and the classical negation $N_C$ on $[0,1]$. Let us obtain the g-convex combination of the Weber implication $I_{WB}$ and the Fodor implication $I_{FD}$ for $a = \frac{1}{3}$. By the definition of the g-convex combination, it is clear that for any $x,y \in [0,1]$,*

$$K^{I_{WB},I_{FD}}_{\frac{1}{3},T,S_{LK},N_C}(x,y) = S_{LK}(T(\frac{1}{3},I_{WB}(x,y)),T(\frac{2}{3},I_{FD}(x,y))).$$

*1. Let $x < 1$.*

*1.1. If $x \leq y$, then*

$$K^{I_{WB},I_{FD}}_{\frac{1}{3},T,S_{LK},N_C}(x,y) = S_{LK}(T(\frac{1}{3},1),T(\frac{2}{3},1)) = S_{LK}(\frac{1}{3},\frac{2}{3}) = 1.$$

*1.2. Let $x > y$.*

*1.2.1. If $(x,y) \in [0,0.5]^2$, we obtain that*

$$\begin{aligned} K^{I_{WB},I_{FD}}_{\frac{1}{3},T,S_{LK},N_C}(x,y) &= S_{LK}(\frac{1}{3},T(\frac{2}{3},\max(1-x,y))) = S_{LK}(\frac{1}{3},T(\frac{2}{3},1-x)) \\ &= S_{LK}(\frac{1}{3},2(\frac{2}{3}-\frac{1}{2})(1-x-\frac{1}{2})+\frac{1}{2}) = S_{LK}(\frac{1}{3},\frac{2-x}{3}) \\ &= \min(1-\frac{x}{3},1) = 1-\frac{x}{3}. \end{aligned}$$

*1.2.2. Let* $(x,y)\in(0.5,1]^2$*. Then,*

$$\begin{aligned}K^{I_{WB},I_{FD}}_{\frac{1}{3},T,S_{LK},N_C}(x,y) &= S_{LK}(\frac{1}{3},T(\frac{2}{3},\max(1-x,y)))=S_{LK}(\frac{1}{3},T(\frac{2}{3},y))\\ &= S_{LK}(\frac{1}{3},2(\frac{2}{3}-\frac{1}{2})(y-\frac{1}{2})+\frac{1}{2})=\min(\frac{y+2}{3},1)=\frac{y+2}{3}.\end{aligned}$$

*1.2.3.* $(x,y)\notin[0,\frac{1}{2}]^2\cup(\frac{1}{2},1]^2$*. Suppose that* $\max(1-x,y)>\frac{1}{2}$*. If* $\max(1-x,y)=1-x$*, then* $x<\frac{1}{2}$*. Since* $x>y$*, we have that* $y<\frac{1}{2}$*. This contradicts to* $(x,y)\notin[0,\frac{1}{2}]^2$*. If* $\max(1-x,y)=y>\frac{1}{2}$*, then it would be* $x>\frac{1}{2}$ *since* $x>y$*. This is a contradiction again since* $(x,y)\notin(\frac{1}{2},1]^2$*. Thus,* $\max(1-x,y)\leq\frac{1}{2}$*. In this case,*

$$\begin{aligned}K^{I_{WB},I_{FD}}_{\frac{1}{3},T,S_{LK},N_C}(x,y) &= S_{LK}(\frac{1}{3},T(\frac{2}{3},\max(1-x,y)))=S_{LK}(\frac{1}{3},\min(\frac{2}{3},\max(1-x,y)))\\ &= S_{LK}(\frac{1}{3},\max(1-x,y))=\frac{1}{3}+\max(1-x,y).\end{aligned}$$

*2. Let* $x=1$*. If* $y\leq\frac{1}{2}$*, then*

$$K^{I_{WB},I_{FD}}_{\frac{1}{3},T,S_{LK},N_C}(x,y)=S_{LK}(T(\frac{1}{3},y),T(\frac{2}{3},y))=S_{LK}(0,T(\frac{2}{3},y))=T(\frac{2}{3},y)=\min(\frac{2}{3},y)=y.$$

*Let* $y>\frac{1}{2}$*. Then,*

$$K^{I_{WB},I_{FD}}_{\frac{1}{3},T,S_{LK},N_C}(1,y)=S_{LK}(T(\frac{1}{3},y),T(\frac{2}{3},y))=S_{LK}(\frac{1}{3},2(\frac{2}{3}-\frac{1}{2})(y-\frac{1}{2})+\frac{1}{2})=\min(\frac{2+y}{3},1)=\frac{2+y}{3}.$$

*Thus, we obtain that*

$$K^{I_{WB},I_{FD}}_{\frac{1}{3},T,S_{LK},N_C}(x,y)=\begin{cases}1 & x<1,\ x\leq y,\\ 1-\frac{x}{3} & x<1,\ x>y,\ (x,y)\in[0,0.5]^2,\\ \frac{1}{3}+\max(1-x,y) & x<1,\ x>y,\ (x,y)\notin[0,0.5]^2\cup(0.5,1]^2,\\ y & x=1,\ y\leq\frac{1}{2},\\ \frac{y+2}{3} & x=1,\ y>\frac{1}{2}\ \text{or}\ x<1,\ x>y,\ (x,y)\in[0,0.5]^2.\end{cases}$$

The following is an illustrating example for the g-convex combination of two implications on a bounded lattice.

**Example 3.7.** *Consider the lattice* $(L=\{0,a,b,c,1\},\leq)$ *whose lattice diagram is displayed in Figure 1.*

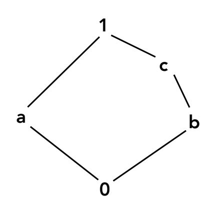


Figure 1: $(L,\leq,0,1)$

*Then, it is clear that the function* $N$ *defined by,*

$$N(x)=\begin{cases}a & x=a,\\ c & x=b,\\ b & x=c,\\ 1 & x=0,\\ 0 & x=1,\end{cases}$$

*is a negation on* $L$*. Take the implications defined by* $I_1(x,y)=\begin{cases}1 & x\leq y,\\ 0 & \text{otherwise},\end{cases}$ *and* $I_2(x,y)=N(x)\vee y$*.*

*Let* $S=S_D$ *and* $T=T_\wedge$*. Notice that* $S_D(a,N(a))=1$ *although* $a\vee N(a)\neq 1$*. Then, for all* $x,y\in L$

$$\begin{aligned}K^{I_1,I_2}_{a,T,S,N}(x,y) &= S_D(T_\wedge(a,I_1(x,y)),T_\wedge(N(a),I_2(x,y)))=S_D(T_\wedge(a,I_1(x,y)),T_\wedge(a,I_2(x,y)))\\ &= S_D(a\wedge I_1(x,y),a\wedge I_2(x,y))=S_D(a\wedge I_1(x,y),a\wedge(N(x)\vee y)).\end{aligned}$$

*If $x \leq y$, it can be easily seen that $K^{I_1,I_2}_{a,T,S,N}(x,y) = S_D(a \wedge 1, a \wedge (N(x) \vee y)) = 1$. Otherwise, it is clear that*

$$K^{I_1,I_2}_{a,T,S,N}(x,y) = S_D(a \wedge 0, a \wedge (N(x) \vee y)) = S_D(0, a \wedge (N(x) \vee y)) = a \wedge (N(x) \vee y).$$

*Thus,* $K^{I_1,I_2}_{a,T,S,N}(x,y) = \begin{cases} 1 & x \leq y, \\ a \wedge (N(x) \vee y) & \text{otherwise.} \end{cases}$

In Example 3.8, we investigate the linear combination of two implications $I$ and $J$, when the implications $I$ and $J$ are the greatest or the least ones.

**Example 3.8.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$.*

*(i) If $I = I_0$ and $J = I_1$, then* $K^{I,J}_{a,T,S,N}(x,y) = \begin{cases} S(a, N(a)) & x = 0 \text{ or } y = 1, \\ 0 & x = 1 \text{ and } y = 0, \\ N(a) & \text{otherwise.} \end{cases}$

*(ii) If $I = J = I_0$, then* $K^{I,J}_{a,T,S,N}(x,y) = \begin{cases} S(a, N(a)) & x = 0 \text{ or } y = 1, \\ 0 & x > 0 \text{ and } y < 1. \end{cases}$

*(iii) If $I = J = I_1$, then* $K^{I,J}_{a,T,S,N}(x,y) = \begin{cases} S(a, N(a)) & x < 1 \text{ or } y > 0, \\ 0 & x = 1 \text{ and } y = 0. \end{cases}$

*(iv) If $J = I_0$, then* $K^{I,J}_{a,T,S,N}(x,y) = \begin{cases} S(a, N(a)) & x = 0 \text{ or } y = 1, \\ T(a, I(x,y)) & \text{otherwise.} \end{cases}$

*Now, let us verify (i) since the proofs (ii) and (iii) are straightforward. Let $x = 0$. Then,*

$$K^{I,J}_{a,T,S,N}(0,y) = S(T(a, I_0(0,y)), T(N(a), I_1(0,y))) = S(T(a,1), T(N(a),1)) = S(a, N(a)).$$

*Let $x = 1$. If $y = 0$, then $K^{I,J}_{a,T,S,N}(x,y) = 0$ is clear. Let $y \neq 0$. If $y = 1$, then $K^{I,J}_{a,T,S,N}(x,y) = S(a, N(a))$. If $0 < y < 1$, then*

$$K^{I,J}_{a,T,S,N}(x,y) = S(T(a, I_0(x,y)), T(N(a), I_1(x,y))) = S(0, T(N(a),1)) = S(0, N(a)) = N(a).$$

*Let $0 < x < 1$. If $y = 1$, it is obvious that $K^{I,J}_{a,T,S,N}(x,y) = S(a, N(a))$. Let $y < 1$. In this case, we have that*

$$K^{I,J}_{a,T,S,N}(x,y) = S(T(a, I_0(x,y)), T(N(a), I_1(x,y))) = S(0, T(N(a),1)) = S(0, N(a)) = N(a).$$

The following gives us the greatest one in the class of the g-convex combinations of two implications on a bounded lattice.

**Proposition 3.9.** *Let $(L, \leq, 0, 1)$ be a bounded lattice. For any $a \in L$, $K^{I_1,I_1}_{a,T_M,S_D,N_{D_2}}$ is the greatest in the class of the g-convex combinations of two implications on $L$. Moreover, $K^{I_1,I_1}_{a,T_M,S_D,N_{D_2}} = I_1$.*

*Proof.* Let $a \in L$ be arbitrary. For any t-norm $T$, t-conorm $S$, negation $N$ and two implications $I$ and $J$, it is clear that

$$K^{I,J}_{a,T,S,N}(x,y) \leq K^{I_1,I_1}_{a,T_M,S_D,N_{D_2}}(x,y), \text{ for any } x, y \in L.$$

Also, for every $x, y \in L$,

$$K^{I_1,I_1}_{a,T_M,S_D,N_{D_2}}(x,y) = \begin{cases} S_D(a, N_{D_2}(a)) & x = 0 \text{ or } y = 1, \\ 0 & x = 1 \text{ and } y = 0. \end{cases} = \begin{cases} 1 & x = 0 \text{ or } y = 1, \\ 0 & x = 1 \text{ and } y = 0. \end{cases} = I_1(x,y).$$

□

Now, let us investigate under which conditions the properties (NP), (IP) and (LOP) are preserved.

**Proposition 3.10.** *Let $L$ be a bounded lattice, $I$ and $J$ two implications on $L$ and $K^{I,J}_{a,T,S,N}$ the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$.*

*(i) If $I$ and $J$ satisfy the identity principle (IP), then $K^{I,J}_{a,T,S,N}$ satisfies (IP).*

*(ii) If $T$ is distributive over $S$ and both $I$ and $J$ satisfy (NP), then $K^{I,J}_{a,T,S,N}$ satisfies (NP).*

*(iii) If both $I$ and $J$ satisfy (LOP), then $K^{I,J}_{a,T,S,N}$ satisfies (LOP).*

*Proof.* (i) Let $I$ and $J$ satisfy (IP). For any $x \in L$, $I(x,x) = J(x,x) = 1$. Then,

$$K^{I,J}_{a,T,S,N}(x,x) = S(T(a,I(x,x)),T(N(a),J(x,x))) = S(T(a,1),T(N(a),1)) = S(a,N(a)) = 1.$$

(ii) Let $I$ and $J$ satisfy (NP). Then, for any $y \in L$, $I(1,y) = J(1,y) = y$. Thus,

$$K^{I,J}_{a,T,S,N}(1,y) = S(T(a,I(1,y)),T(N(a),J(1,y))) = S(T(a,y),T(N(a),y)) = T(S(a,N(a)),y) = T(1,y) = y.$$

(iii) Let $x, y \in L$ with $x \leq y$. If $I$ and $J$ satisfy (LOP), then

$$K^{I,J}_{a,T,S,N}(x,y) = S(T(a,I(x,y)),T(N(a),J(x,y))) = S(T(a,1),T(N(a),1)) = S(a,N(a)) = 1,$$

which shows that $K^{I,J}_{a,T,S,N}$ satisfies (LOP). □

**Corollary 3.11.** *Let $L$ be a distributive lattice, $I$ and $J$ be two implications on $L$. Let $K^{I,J}_{a,\wedge,\vee,N}$ be the linear combination of $I$ and $J$, and $a \in L$ with $a \vee N(a) = 1$. If $I$ and $J$ satisfy (NP), then $K^{I,J}_{a,\wedge,\vee,N}$ satisfies (NP).*

**Corollary 3.12.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$, and $K^{I,J}_{a,T,\vee,N}$ be the linear combination of $I$ and $J$ with $a \vee N(a) = 1$. If $T$ is a $\vee$-distributive t-norm and both $I$ and $J$ satisfy (NP), then $K^{I,J}_{a,T,\vee,N}$ satisfies (NP).*

**Remark 3.13.** *Even if $I$ and $J$ are two implications satisfying (OP), their g-convex combination need not satisfy (OP). Let us investigate the following example.*

**Example 3.14.** *Consider two implications $I_{GG}$ and $I_{GD}$. Let $S = S_D$, $T = T_\wedge$ and $N$ be any negation. For any $a \in [0,1]$,*

$$K^{I_{GG},I_{GD}}_{a,T_\wedge,S_D,N}(\frac{2}{3},\frac{1}{3}) = S_D(T_\wedge(\frac{1}{3},I_{GD}(\frac{2}{3},\frac{1}{3})),T_\wedge(\frac{2}{3},I_{GG}(\frac{2}{3},\frac{1}{3})) = S_D(T_\wedge(\frac{1}{3},\frac{1}{3}),T_\wedge(\frac{2}{3},\frac{1}{2})) = S_D(\frac{1}{3},\frac{1}{2}) = 1.$$

*Since $K^{I_{GG},I_{GD}}_{a,T_\wedge,S_D,N}(\frac{2}{3},\frac{1}{3}) = 1$ but $\frac{2}{3} \not\leq \frac{1}{3}$, $K^{I_{GG},I_{GD}}_{a,T_\wedge,S_D,N}$ doesn't satisfy (OP).*

**Proposition 3.15.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a,N(a)) = 1$. If $T$ is distributive over $S$ and $I \leq J$, then $I \leq K^{I,J}_{a,T,S,N} \leq J$.*

*Proof.* Let $T$ be distributive over $S$ and $I \leq J$. For any $x, y \in L$,

$$\begin{aligned} I(x,y) &= T(1,I(x,y)) = T(S(a,N(a)),I(x,y)) = S(T(a,I(x,y)),T(N(a),I(x,y))) \\ &\leq S(T(a,I(x,y)),T(N(a),J(x,y))) = K^{I,J}_{a,T,S,N}(x,y) \leq S(T(a,J(x,y)),T(N(a),J(x,y))) \\ &= T(S(a,N(a)),J(x,y)) = T(1,J(x,y)) = J(x,y). \end{aligned}$$

Thus, $I \leq K^{I,J}_{a,T,S,N} \leq J$. □

**Remark 3.16.** *(i) If $L$ is distributive, $T = \wedge$ and $S = \vee$, then Proposition 3.15 is also true.*
*(ii) Let $T$ be $\vee$-distributive, $I \leq J$ and $a \vee N(a) = 1$. Then,*

$$\begin{aligned} I(x,y) &= T(1,I(x,y)) = T(a \vee N(a),I(x,y)) = T(a,I(x,y)) \vee T(N(a),I(x,y)) \\ &\leq S(T(a,I(x,y)),T(N(a),J(x,y))) = K^{I,J}_{a,T,S,N}(x,y) \end{aligned}$$

*holds. Although, $K^{I,J}_{a,T,S,N} \leq J$ need not be true. The following example proves that this situation may not be satisfied.*

**Example 3.17.** *Consider the lattice $L$ whose Hasse diagram is as shown in Figure 2.*

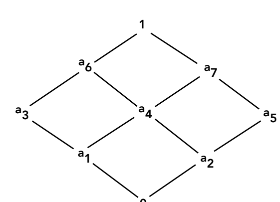


Figure 2: Lattice diagram of $L$

*One can easily check that $L$ is a distributive lattice. Take $T = T_\wedge$, let the t-conorm $S$ and the strong negation $N$ on $L$ be respectively as follows;*

Table 1: T-conorm $S$ on $L$

| $S$ | $0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $1$ |
| $a_1$ | $a_1$ | $a_1$ | $a_4$ | $1$ | $a_4$ | $1$ | $1$ | $1$ | $1$ |
| $a_2$ | $a_2$ | $a_4$ | $a_2$ | $1$ | $a_4$ | $1$ | $1$ | $1$ | $1$ |
| $a_3$ | $a_3$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a_4$ | $a_4$ | $a_4$ | $a_4$ | $1$ | $a_4$ | $1$ | $1$ | $1$ | $1$ |
| $a_5$ | $a_5$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a_6$ | $a_6$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a_7$ | $a_7$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

and

$$N(x) = \begin{cases} 1 & \text{if } x = 0, \\ a_6 & \text{if } x = a_1, \\ a_7 & \text{if } x = a_2, \\ a_5 & \text{if } x = a_3, \\ a_4 & \text{if } x = a_4, \\ a_3 & \text{if } x = a_5, \\ a_1 & \text{if } x = a_6, \\ a_2 & \text{if } x = a_7, \\ 0 & \text{if } x = 1. \end{cases}$$

*Let $I = J = I_{GD}$, where $I_{GD}(x,y) = \begin{cases} 1 & x \le y, \\ y & \text{otherwise}, \end{cases}$ and $a = a_5$. Since $L$ is a distributive lattice, it is obvious that $T_\wedge$ is $\vee$-distributive. Also, it is clear that $a_5 \vee N(a_5) = 1$. By Remark 3.16, we have that $I = I_{GD} \le K^{I,J}_{a_5,T_\wedge,S,N}$. On the other hand, it is obtained that $K^{I,J}_{a_5,T_\wedge,S,N} \neq I_{GD}$ since $K^{I,J}_{a_5,T_\wedge,S,N}(1,a_6) = 1$ and $I_{GD}(1,a_6) = a_6$.*

**Proposition 3.18.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a,N(a)) = 1$.*
*(i) If $a = 0$, then $K^{I,J}_{0,T,S,N} = J$.*
*(ii) If $a = 1$, then $K^{I,J}_{1,T,S,N} = I$.*
*(iii) If $T$ is distributive over $S$ and $I = J$, then $K^{I,J}_{a,T,S,N} = I = J$.*
*(iv) If $N(N(a)) = a$, then $K^{I,J}_{a,T,S,N} = K^{J,I}_{N(a),T,S,N}$.*
*(v) $N_{K^{I,J}_{a,T,S,N}}(x) = S(T(a,N_I(x)),T(N(a),N_J(x)))$*

*Proof.* (i) Let $a = 0$. Then, for all $x, y \in L$,

$$K^{I,J}_{0,T,S,N}(x,y) = S(T(0,I(x,y)),T(N(0),J(x,y))) = S(0,J(x,y)) = J(x,y).$$

(ii) Let $a = 1$. Then, for all $x, y \in L$,

$$K^{I,J}_{1,T,S,N}(x,y) = S(T(1,I(x,y)),T(N(1),J(x,y))) = S(I(x,y),0) = I(x,y).$$

(iii) Let $T$ be distributive over $S$ and $I = J$. Then,

$$K^{I,J}_{a,T,S,N}(x,y) = S(T(a,I(x,y)),T(N(a),I(x,y))) = T(S(a,N(a)),I(x,y)) = T(1,I(x,y)) = I(x,y).$$

(iv) Let $N(N(a)) = a$.

$$K^{J,I}_{N(a),T,S,N}(x,y) = S(T(N(a),J(x,y)),T(N(N(a)),I(x,y))) = S(T(a,I(x,y)),T(N(a),J(x,y))) = K^{I,J}_{a,T,S,N}(x,y).$$

(v) The proof is straightforward. □

**Proposition 3.19.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a,N(a)) = 1$ and $N(N(a)) = a$. If $S = \vee$ and $T$ is $\vee$-distributive, then*

$$N_{K^{I,J}_{a,T,S,N}} \vee N_{K^{I,J}_{N(a),T,S,N}} = N_{K^{I\vee J,I\vee J}_{a,T,S,N}}.$$

*Proof.* Let $S = \vee$ and $T$ be $\vee$-distributive. Then, for any $x \in L$,

$$\begin{aligned}(N_{K^{I,J}_{a,T,S,N}} \vee N_{K^{I,J}_{N(a),T,S,N}})(x) &= N_{K^{I,J}_{a,T,S,N}}(x) \vee N_{K^{I,J}_{N(a),T,S,N}}(x) = K^{I,J}_{a,T,S,N}(x,0) \vee K^{I,J}_{N(a),T,S,N}(x,0)\\ &= T(a, I(x,0)) \vee T(N(a), J(x,0)) \vee T(N(a), I(x,0)) \vee T(N(N(a)), J(x,0))\\ &= T(a, I(x,0) \vee J(x,0)) \vee T(N(a), I(x,0) \vee J(x,0))\\ &= T(a, (I \vee J)(x,0)) \vee T(N(a), (I \vee J)(x,0)) = K^{I\vee J, I\vee J}_{a,T,S,N}(x,0) = N_{K^{I\vee J,I\vee J}_{a,T,S,N}}(x).\end{aligned}$$

□

Let us determine the conjugacy of the g-convex combinations for two implications on a bounded lattice.

**Proposition 3.20.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$, and $\phi : L \to L$ be an order-preserving bijection. Let $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$. Then,*

$$(K^{I,J}_{a,T,S,N})_\phi = K^{I_\phi,J_\phi}_{\phi^{-1}(a),T_\phi,S_\phi,N_\phi}.$$

*Proof.* For any $x, y \in L$,

$$\begin{aligned}(K^{I,J}_{a,T,S,N})_\phi(x,y) &= \phi^{-1}(K^{I,J}_{a,T,S,N}(\phi(x),\phi(y))) = \phi^{-1}(S(T(a, I(\phi(x),\phi(y))), T(N(a), J(\phi(x),\phi(y)))))\\ &= \phi^{-1}(S(T(a, \phi(\phi^{-1}(I(\phi(x),\phi(y))))), T(N(a), \phi(\phi^{-1}(J(\phi(x),\phi(y)))))))\\ &= \phi^{-1}(S(T(a, \phi(I_\phi(x,y))), T(N(a), \phi(J_\phi(x,y)))))\\ &= \phi^{-1}(S(T(\phi(\phi^{-1}(a)), \phi(I_\phi(x,y))), T(\phi(\phi^{-1}(N(a))), \phi(J_\phi(x,y)))))\\ &= \phi^{-1}(S(\phi(\phi^{-1}(T(\phi(\phi^{-1}(a)), \phi(I_\phi(x,y))))), \phi(\phi^{-1}(T(\phi(\phi^{-1}(N(a))), \phi(J_\phi(x,y)))))))\\ &= \phi^{-1}(S(\phi(T_\phi(\phi^{-1}(a), I_\phi(x,y))), \phi(T_\phi(\phi^{-1}(N(a)), J_\phi(x,y)))))\\ &= S_\phi(T_\phi(\phi^{-1}(a), I_\phi(x,y)), T_\phi(\phi^{-1}(N(a)), J_\phi(x,y)))\\ &= S_\phi(T_\phi(\phi^{-1}(a), I_\phi(x,y)), T_\phi(\phi^{-1}(N(\phi(\phi^{-1}(a)))), J_\phi(x,y)))\\ &= S_\phi(T_\phi(\phi^{-1}(a), I_\phi(x,y)), T_\phi(N_\phi(\phi^{-1}(a)), J_\phi(x,y))) = K^{I_\phi,J_\phi}_{\phi^{-1}(a),T_\phi,S_\phi,N_\phi}(x,y).\end{aligned}$$

□

Now, we will investigate the $N'$-reciprocal of the g-convex combination of two implications on a bounded lattice.

**Proposition 3.21.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$, and $N' : L \to L$ be a negation. Let $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$. Then, $N'$-reciprocal of $K^{I,J}_{a,T,S,N}$ is the g-convex combination of $I_{N'}$ and $J_{N'}$.*

*Proof.*

$$\begin{aligned}(K^{I,J}_{a,T,S,N})_{N'}(x,y) &= K^{I,J}_{a,T,S,N}(N'(y), N'(x)) = S(T(a, I(N'(y), N'(x))), T(N(a), J(N'(y), N'(x))))\\ &= S(T(a, I_{N'}(x,y)), T(N(a), J_{N'}(x,y))) = K^{I_{N'},J_{N'}}_{a,T,S,N}(x,y).\end{aligned}$$

□

**Proposition 3.22.** *Let $L$ be a bounded lattice, $I$ be an implication on $L$ and $K^{I,I}_{a,T,S,N}$ be the linear combination of $I$ for $a \in L$. Suppose that $a \vee N(a) = 1$, $a \wedge N(a) = 0$, $S$ is an $\wedge$-distributive t-conorm, and $T$ is $\vee$-distributive t-norm on $L$. If $I$ satisfies (WE), then $K^{I,I}_{a,T,S,N}$ satisfies (WE).*

*Proof.* Let $x \le K^{I,I}_{a,T,S,N}(y,z)$. Then,

$$\begin{aligned}x \le K^{I,I}_{a,T,S,N}(y,z) &= S(T(a, I(y,z)), T(N(a), I(y,z))) \le S(a \wedge I(y,z), N(a) \wedge I(y,z))\\ &= S(a, N(a)) \wedge S(a, I(y,z)) \wedge S(I(y,z), N(a)) \wedge S(I(y,z), I(y,z))\\ &= S(a, I(y,z)) \wedge S(I(y,z), N(a)) \wedge S(I(y,z), I(y,z))\\ &= S(a \wedge N(a), I(y,z)) \wedge S(I(y,z), I(y,z)) = I(y,z) \wedge S(I(y,z), I(y,z)) = I(y,z),\end{aligned}$$

whence $x \le I(y,z)$. Since $I$ satisfies (WE), we have that $y \le I(x,z)$. Then,

$$K^{I,I}_{a,T,S,N}(x,z) = S(T(a, I(x,z)), T(N(a), I(x,z))) \ge S(T(a,y), T(N(a),y)) \ge T(a,y) \vee T(N(a),y) = T(a \vee N(a), y) = y.$$

Similarly, it can be easily shown that $x \le K^{I,I}_{a,T,S,N}(y,z)$ if $y \le K^{I,I}_{a,T,S,N}(x,z)$. Thus, $K^{I,I}_{a,T,S,N}$ satisfies (WE). □

The following results give us some information about the algebraic structures of some sets defined by the g-convex combination of any two implications on a bounded lattice.

**Proposition 3.23.** *Let $L$ be a complete lattice, $I$ and $J$ be two implications on $L$. Let $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$. Then, the set defined by*

$$M^{\leq}_{K^{I,J}_{a,T,S,N}} = \{y \in L |\ K^{I,J}_{a,T,S,N}(1, y) \leq y\},$$

*is a complete lattice.*

*Proof.* Since $K^{I,J}_{a,T,S,N}(1, 1) = 1$, $1 \in M^{\leq}_{K^{I,J}_{a,T,S,N}}$ is clear. Let $\{x_i |\ i \in A\} \subseteq M^{\leq}_{K^{I,J}_{a,T,S,N}}$ for all $i \in A$. Then, for any $i \in A$, $K^{I,J}_{a,T,S,N}(1, x_i) \leq x_i$. Since for any $i \in A$,

$$K^{I,J}_{a,T,S,N}(1, \wedge_i x_i) = S(T(a, I(1, \wedge_i x_i)), T(N(a), J(1, \wedge_i x_i))) \leq S(T(a, I(1, x_i)), T(N(a), J(1, x_i))) = K^{I,J}_{a,T,S,N}(1, x_i) \leq x_i,$$

we have that $K^{I,J}_{a,T,S,N}(1, \wedge_{i\in A} x_i) \leq \wedge_{i\in A} x_i$. Then, $\wedge_{i\in A} x_i \in M^{\leq}_{K^{I,J}_{a,T,S,N}}$. By Theorem 6 [2], $M^{\leq}_{K^{I,J}_{a,T,S,N}}$ is a complete lattice. □

**Proposition 3.24.** *Let $L$ be a complete lattice, $I$ and $J$ be two implications on $L$. Let $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$. Then, the set defined by*

$$M^{\geq}_{K^{I,J}_{a,T,S,N}} = \{y \in L |\ K^{I,J}_{a,T,S,N}(1, y) \geq y\},$$

*is a complete lattice.*

*Proof.* Since $K^{I,J}_{a,T,S,N}(1, 0) = 0 \geq 0$, it is clear that $0 \in M^{\geq}_{K^{I,J}_{a,T,S,N}}$. Let $\{x_i |\ i \in A\} \subseteq M^{\geq}_{K^{I,J}_{a,T,S,N}}$, for all $i \in A$. Then, for all $i \in A$,

$$K^{I,J}_{a,T,S,N}(1, \vee_i x_i) = S(T(a, I(1, \vee_i x_i)), T(N(a), J(1, \vee_i x_i))) \geq S(T(a, I(1, x_i)), T(N(a), J(1, x_i))) = K^{I,J}_{a,T,S,N}(1, x_i) \geq x_i,$$

whence $K^{I,J}_{a,T,S,N}(1, \vee_i x_i) \geq \vee_i x_i$. Then, $\vee_i x_i \in M^{\geq}_{K^{I,J}_{a,T,S,N}}$. This shows that $M^{\geq}_{K^{I,J}_{a,T,S,N}}$ is a complete lattice by Theorem 6 [2]. □

Denote by $K^{I,J}_{a,T,S,N} \downarrow (\{1\} \times L)$ the restriction of $K^{I,J}_{a,T,S,N}$ to the set $\{1\} \times L$. The following proposition gives us a relationship between $K^{I,J}_{a,T,S,N} \downarrow (\{1\} \times L)$ and the algebraic structure of the set $M^{=}_{K^{I,J}_{a,T,S,N}}$.

**Proposition 3.25.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$. Let $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$. $K^{I,J}_{a,T,S,N} \downarrow (\{1\} \times L)$ is $\vee$ $(\wedge)$-distributive iff the set defined by $M^{=}_{K^{I,J}_{a,T,S,N}} = \{y \in L |\ K^{I,J}_{a,T,S,N}(1, y) = y\}$ is a join (meet) sublattice.*

*Proof.* Let $K^{I,J}_{a,T,S,N} \downarrow (\{1\} \times L)$ be $\vee$-distributive and $x, y \in M^{=}_{K^{I,J}_{a,T,S,N}}$. Then, we have

$$K^{I,J}_{a,T,S,N}(1, x \vee y) = K^{I,J}_{a,T,S,N}(1, x) \vee K^{I,J}_{a,T,S,N}(1, y) = x \vee y,$$

whence $x \vee y \in M^{=}_{K^{I,J}_{a,T,S,N}}$.

Conversely, let $M^{=}_{K^{I,J}_{a,T,S,N}}$ be a join sublattice. Then, $x \vee y \in M^{=}_{K^{I,J}_{a,T,S,N}}$ for any $x, y \in M^{=}_{K^{I,J}_{a,T,S,N}}$. Thus,

$$K^{I,J}_{a,T,S,N}(1, x \vee y) = x \vee y = K^{I,J}_{a,T,S,N}(1, x) \vee K^{I,J}_{a,T,S,N}(1, y).$$

□

**Remark 3.26.** *Note that the following relationship between the sets $M^{=}_{K^{I,J}_{a,T,S,N}}$, $M^{\leq}_{K^{I,J}_{a,T,S,N}}$ and $M^{\geq}_{K^{I,J}_{a,T,S,N}}$*

$$M^{=}_{K^{I,J}_{a,T,S,N}} = M^{\leq}_{K^{I,J}_{a,T,S,N}} \cap M^{\geq}_{K^{I,J}_{a,T,S,N}},$$

*holds.*

**Example 3.27.** *Take the lattice $L$ in Example 3.17, the strong negation $N$ and the t-conorm $S$ defined in Example 3.17. Let $T = T_\wedge$ and $I = J = I_{GD}$, where $I_{GD}(x,y) = \begin{cases} 1 & x \le y, \\ y & otherwise. \end{cases}$ For $a = a_5$, it can be easily observed that*
$K^{I,J}_{a_5,T_\wedge,S,N}(1,a_1) = a_1$, $K^{I,J}_{a_5,T_\wedge,S,N}(1,a_2) = a_2$, $K^{I,J}_{a_5,T_\wedge,S,N}(1,a_3) = a_3$,
$K^{I,J}_{a_5,T_\wedge,S,N}(1,a_4) = a_4$, $K^{I,J}_{a_5,T_\wedge,S,N}(1,a_5) = a_5$, $K^{I,J}_{a_5,T_\wedge,S,N}(1,a_6) = K^{I,J}_{a_5,T_\wedge,S,N}(1,a_7) = 1$, *whence* $a_6, a_7 \notin M^{=}_{K^{I,J}_{a_5,T_\wedge,S,N}}$. *Thus,* $M^{=}_{K^{I,J}_{a_5,T_\wedge,S,N}}$ *is not a sublattice of $L$.*

**Proposition 3.28.** *Let $I_1$ and $I_2$ be two right continuous implications on the unit interval $[0,1]$ and $K^{I_1,I_2}_a$ be the g-convex combination of $I_1$ and $I_2$. Then, the set $M^{=}_{K^{I_1,I_2}_a}$ is a complete lattice on $[0,1]$.*

*Proof.* Since $K^{I_1,I_2}_a(1,0) = 0$ and $K^{I_1,I_2}_a(1,1) = 1$, we obtain that $0, 1 \in M^{=}_{K^{I_1,I_2}_a}$. Let $\{y_i |\ i \in A\} \subseteq M^{=}_{K^{I_1,I_2}_a}$. It is clear that

$$y_i = K^{I_1,I_2}_a(1,y_i) = aI_1(1,y_i) + (1-a)I_2(1,y_i).$$

Since $I_1$ and $I_2$ are two right continuous implications, we have that

$$K^{I_1,I_2}_a(1,\wedge_i y_i) = aI_1(1,\wedge_i y_i) + (1-a)I_2(1,\wedge_i y_i) = \wedge_i(aI_1(1,y_i) + (1-a)I_2(1,y_i)) = \wedge_i y_i,$$

whence $\wedge_i y_i \in M^{=}_{K^{I_1,I_2}_a}$. Thus, $M^{=}_{K^{I_1,I_2}_a}$ is a complete lattice on $[0,1]$ by Theorem 6 [2]. □

Let us determine the set $M^{=}_{K^{I,J}_a}$ for two given implications in Example 3.29.

**Example 3.29.** *Take the following implication on the unit interval $[0,1]$ (see [1]):*

$$I(x,y) = \begin{cases} 0 & (x,y) \in [\frac{7}{10},1] \times [0,\frac{6}{10}], \\ \frac{1}{2} & (x,y) \in [\frac{4}{10},\frac{7}{10}] \times [0,\frac{6}{10}], \\ 1 & otherwise. \end{cases}$$

*Let $K^{I,J}_a(1,y) = aI_{RS}(1,y) + (1-a)I(x,y)$ for $a \in (0,1)$ and $y \in M^{=}_{K^{I,J}_a}$. Since $K^{I,J}_a(1,1) = 1$, we have that $1 \in M^{=}_{K^{I,J}_a}$. Let $0 \le y \le \frac{6}{10}$. Then, $y = K^{I,J}_a(1,y) = aI_{RS}(1,y) + (1-a)I(1,y) = a0 + (1-a)0 = 0$.*
*If $\frac{6}{10} < y < 1$, then $y = K^{I,J}_a(1,y) = aI_{RS}(1,y) + (1-a)I(1,y) = a0 + (1-a)1 = 1-a$. Thus, $M^{=}_{K^{I,J}_a} = \{0, 1-a, 1\}$.*

**Proposition 3.30.** *Let $L$ be a complete lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a,N(a)) = 1$. Let the function $\alpha : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ be defined by $\alpha(I,J) = K^{I,J}_{a,T,S,N}$. If $T$ and $S$ are continuous t-norm and t-conorm, respectively, then for any fixed implication $I \in \mathcal{F}$, the set defined by*

$$\mu = \{\alpha(I,J) = K^{I,J}_{a,T,S,N} |\ J \in \mathcal{F}\},$$

*is a complete sublattice.*

*Proof.* Let $\{K^{I,J_i}_{a,T,S,N} |\ i \in A\} \subseteq \mu$ be arbitrary. Then, we have that for any $x, y \in L$

$$\begin{aligned} (\wedge_{i\in M} K^{I,J_i}_{a,T,S,N})(x,y) &= \wedge_i S(T(a,I(x,y)), T(N(a),J_i(x,y))) = S(T(a,I(x,y)), \wedge_i T(N(a),J_i(x,y))) \\ &= S(T(a,I(x,y)), T(N(a),\wedge_i J_i(x,y))) = K^{I,\wedge_i J_i}_{a,T,S,N} \in \mathcal{F}. \end{aligned}$$

Similarly, it can be shown that $\alpha(I,\vee_i J_i) = K^{I,\vee_i J_i}_{a,T,S,N} \in \mathcal{F}$. Thus, $\mu$ is a complete sublattice by Theorem 6 [2]. □

# 4 The sets $(X_I)^\perp$, $(X_I)_\perp$, ${}^\perp(X_I)$ and ${}_\perp(X_I)$

In this section, we will study the sets denoted by $(X)^\perp$, $(X)_\perp$, ${}^\perp(X)$ and ${}_\perp(X)$ which are defined for the linear (g-convex) combination of two implications on a bounded lattice and discuss the relationships between them. We will prove that the sets $(X)^\perp$, $(X)_\perp$, ${}^\perp(X)$ and ${}_\perp(X)$ are in shape of intervals. Also, we will investigate the lattice theoretical structure of the mentioned sets.

**Definition 4.1.** *Let $L$ be a bounded lattice and $I : L \times L \to L$ be a function such that $I(.,y)$ is decreasing and $I(x,.)$ is increasing. For any subset $X \subseteq L$, define the following sets,*

$$(X_I)^\perp = \{y \in L |\ I(x,y) = 1 \text{ for all } x \in X\}, \qquad (X_I)_\perp = \{y \in L |\ I(x,y) = 0 \text{ for all } x \in X\},$$
$${}^\perp(X_I) = \{y \in L |\ I(y,x) = 1 \text{ for all } x \in X\}, \qquad {}_\perp(X_I) = \{y \in L |\ I(y,x) = 0 \text{ for all } x \in X\}.$$

**Remark 4.2.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear (g-convex) combination of $I$ and $J$ for $a \in L$. We will write $(X)^{\perp}$, $(X)_{\perp}$, ${}^{\perp}(X)$ and ${}_{\perp}(X)$ instead of $(X_{K^{I,J}_{a,T,S,N}})^{\perp}$, $(X_{K^{I,J}_{a,T,S,N}})_{\perp}$, ${}^{\perp}(X_{K^{I,J}_{a,T,S,N}})$ and ${}_{\perp}(X_{K^{I,J}_{a,T,S,N}})$, respectively.*

**Proposition 4.3.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. Then,*
*(i) $\emptyset^{\perp} = \emptyset_{\perp} = {}^{\perp}\emptyset = {}_{\perp}\emptyset = L$.*
*(ii) $K^{I,J}_{a,T,S,N}$ is the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$ iff $\{\emptyset\}^{\perp} = L$ and ${}^{\perp}\{1\} = L$.*
*(iii) If $X_1 \subseteq X_2$, then $X_2^{\perp} \subseteq X_1^{\perp}$.*
*(iv) $X \subseteq {}^{\perp}(X^{\perp})$ and $X \subseteq {}_{\perp}(X_{\perp})$ for $X \subseteq L$.*

*Proof.* (i) Let $X = \emptyset$. Then, it is clear that $\emptyset^{\perp} \subseteq L$. Let $\ell \in L$. If $\ell \notin \emptyset^{\perp}$, there would exist an element $x \in \emptyset$ such that $K^{I,J}_{a,T,S,N}(x, \ell) \neq 1$. This is a contradiction. Thus, $\ell \in \emptyset^{\perp}$. Hence, $\emptyset^{\perp} = L$. Similarly, it can be shown that $\emptyset_{\perp} = {}^{\perp}\emptyset = {}_{\perp}\emptyset = L$.

(ii) :⇒ Let $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$. Let $X = \{0\}$. Then, for any $y \in L$, since

$$K^{I,J}_{a,T,S,N}(0, y) = S(T(a, I(0, y)), T(N(a), J(0, y))) = S(T(a, 1), T(N(a), 1)) = S(a, N(a)) = 1,$$

it is clear that $y \in \{0\}^{\perp}$. Thus, $\{0\}^{\perp} = L$. Now, let $X = \{1\}$. For any $y \in L$, since

$$K^{I,J}_{a,T,S,N}(y, 1) = S(T(a, I(y, 1)), T(N(a), J(y, 1))) = S(T(a, 1), T(N(a), 1)) = S(a, N(a)) = 1,$$

we have that $\{1\}^{\perp} = L$.

⇐: The converse of the proof is trivial.

(iii) Let $X_1 = \emptyset$. Since $X_1^{\perp} = L$, it is clear that $X_2^{\perp} \subseteq X_1^{\perp}$. Let $X_1 \neq \emptyset$. In this case, $X_2 \neq \emptyset$. If $X_2^{\perp} = \emptyset$, then it is clear that $X_2^{\perp} \subseteq X_1^{\perp}$. Let $X_2^{\perp} \neq \emptyset$ and $y \in X_2^{\perp}$ be arbitrary. For any $x \in X_2$, $K^{I,J}_{a,T,S,N}(x, y) = 1$. Then, for any $x \in X_1$, we have that $K^{I,J}_{a,T,S,N}(x, y) = 1$, whence $X_2^{\perp} \subseteq X_1^{\perp}$.

(iv) Let $X^{\perp} = K^{I,J}_{a,T,S,N}$ for any $X \subseteq L$ and let $x \in X$ be arbitrary. For any $k \in K = X^{\perp}$, $K^{I,J}_{a,T,S,N}(t, k) = 1$ for all $t \in X$. Since $x \in X$, it must be $K^{I,J}_{a,T,S,N}(x, k) = 1$, whence $x \in {}^{\perp}K$. Thus, $X \subseteq {}^{\perp}K = {}^{\perp}(X^{\perp})$. Similarly, it can be shown that $X \subseteq {}_{\perp}(X_{\perp})$.

□

**Remark 4.4.** *For any set $\emptyset \neq X \subseteq [0, 1]$, it is possible that $X^{\perp} = \emptyset$. If we take $S = \vee$, $N(x) = 1 - x$ and $a = \frac{1}{2}$, we have that*

$$K^{I,J}_{a,T,S,N}(x, y) = T(a, I(x, y)) \vee T(N(a), J(x, y)) \leq a \vee N(a) = a.$$

*Then, $K^{I,J}_{a,T,S,N}(x, y) \neq 1$ for any $x, y \in [0, 1]$. Thus, $X^{\perp} = \emptyset$ for every $\emptyset \neq X \subseteq [0, 1]$.*

In Proposition 4.5, we will determine the algebraic structure of the set $(X^{\perp}, \leq)$ under some conditions.

**Proposition 4.5.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$. If $K^{I,J}_{a,T,S,N}$ is $\wedge$-distributive in the second place, for any $X \subseteq L$, then $(X^{\perp}, \leq)$ is a sublattice.*

*Proof.* Let $X = \emptyset$. Then, it is clear that $\emptyset^{\perp} = L$ by Proposition 4.3 (i). Since $1 \in X^{\perp}$, it is obvious that $X^{\perp} \neq \emptyset$. Let $y \in X^{\perp}$ and $y \leq y'$. For any $x \in X$, since $1 = K^{I,J}_{a,T,S,N}(x, y) \leq K^{I,J}_{a,T,S,N}(x, y')$, we have that $K^{I,J}_{a,T,S,N}(x, y') = 1$. then, $y' \in X^{\perp}$. Thus, $(X^{\perp}, \leq)$ is a join sublattice.

For any $y_1, y_2 \in X^{\perp}$ and $x \in X$ since $K^{I,J}_{a,T,S,N}(x, y_1 \wedge y_2) = K^{I,J}_{a,T,S,N}(x, y_1) \wedge K^{I,J}_{a,T,S,N}(x, y_2) = 1 \wedge 1 = 1$, we have that $y_1 \wedge y_2 \in X^{\perp}$. Thus, $(X^{\perp}, \leq)$ is a meet sublattice. □

**Remark 4.6.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$.*
*(i) If $K^{I,J}_{a,T,S,N}$ is the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$, then $1 \in X^{\perp}$ for any $X \subseteq L$. Thus, $X^{\perp} \neq \emptyset$ for any $X \subseteq L$.*
*(ii) If $K^{I,J}_{a,T,S,N}$ is the linear combination of $I$ and $J$ for $a \in L$, then $X^{\perp} \neq \emptyset$ iff $1 \in X^{\perp}$.*

**Corollary 4.7.** *Let $L$ be a complete lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$. If $K^{I,J}_{a,T,S,N}$ is infinitely $\wedge$-distributive in the second place, $(X^{\perp}, \leq)$ is a sublattice.*

**Proposition 4.8.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$. Then, $X^{\perp}$ is in shape of an interval. Moreover, if $L$ is a lattice of finite length, then there exists an element $x_0 \in X^{\perp}$ such that $X^{\perp} = [x_0, 1]$.*

*Proof.* Let $y_1, y_2 \in X^{\perp}$ with $y_1 \leq y \leq y_2$. For any $x \in X$,

$$1 = K^{I,J}_{a,T,S,N}(x, y_1) \leq K^{I,J}_{a,T,S,N}(x, y) \leq K^{I,J}_{a,T,S,N}(x, y_2) = 1.$$

Then, $K^{I,J}_{a,T,S,N}(x, y) = 1$. Thus, $y \in X^{\perp}$. This shows that $X^{\perp}$ is in shape of an interval. If $L$ is a lattice of finite length and $X^{\perp} \subseteq L$, then the length of $X^{\perp}$ is also finite. There must exist the bottom element of this interval, that is , there exists an element $x_0 \in X^{\perp}$ such that $x_0 \leq y$ for any $y \in X^{\perp}$. Since $x_0 \leq 1$, $x_0, 1 \in X^{\perp}$. Thus, we have that $y \in X^{\perp}$ for all $y \in L$ with $x_0 \leq y \leq 1$. That is, $X^{\perp} = [x_0, 1]$. □

**Corollary 4.9.** *Let $L$ be a bounded lattice and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. If $X^{\perp}$ is a non-empty set, then $X^{\perp}$ is in shape of an interval. Moreover, if $L$ is a lattice of finite length, then there exists an element $x_0 \in X^{\perp}$ such that $X^{\perp} = [x_0, 1]$.*

The following Proposition 4.10 and Proposition 4.12 give us the lattice theoretical structure of $(X_{\perp}, \leq)$ under some special conditions.

**Proposition 4.10.** *Let $L$ be a complete lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. Let $K^{I,J}_{a,T,S,N}$ be $\vee$-distributive in the second place. If $X_{\perp} \neq \emptyset$ for any $X \subseteq L$, then $(X_{\perp}, \leq)$ is a complete sublattice.*

*Proof.* Let $\{y_i | \ i \in A\} \subseteq X_{\perp}$. For any $x \in X$, since $K^{I,J}_{a,T,S,N}(x, \vee_i y_i) = \vee_i K^{I,J}_{a,T,S,N}(x, y_i) = 0$, we have that $\vee_i y_i \in X_{\perp}$.

Since $K^{I,J}_{a,T,S,N}(x, \wedge_i y_i) \leq \wedge_i K^{I,J}_{a,T,S,N}(x, y_i) = 0$, we have that $K^{I,J}_{a,T,S,N}(x, \wedge_i y_i) = 0$. Then, $\wedge_i y_i \in X_{\perp}$. By Theorem 6 [2], $(X_{\perp}, \leq)$ is a complete sublattice. □

**Remark 4.11.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. If $y_1 \in X_{\perp}$ and $y_2 \leq y_1$, then $y_2 \in X_{\perp}$. Moreover, $X_{\perp} \neq \emptyset$ if and only if $0 \in X_{\perp}$.*

**Proposition 4.12.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. If $L$ is a lattice of finite length and $X_{\perp} \neq \emptyset$, there exists an element $x_0 \in L$ such that $X_{\perp} = [0, x_0]$.*

*Proof.* Let $y_1, y_2 \in X_{\perp}$ with $y_1 \leq y \leq y_2$. For any $x \in X$, since $0 = K^{I,J}_{a,T,S,N}(x, y_1) \leq K^{I,J}_{a,T,S,N}(x, y) \leq K^{I,J}_{a,T,S,N}(x, y_2) = 0$, we have that $K^{I,J}_{a,T,S,N}(x, y) = 0$, whence $y \in X_{\perp}$. Thus, $X_{\perp}$ is in shape of an interval. If $L$ is a lattice of finite length and $X_{\perp} \subseteq L$, the length of $X_{\perp}$ is also finite. There must exist the top element of this interval, that is, there exists an element $x_0 \in X_{\perp}$ such that $y \leq x_0$ for any $y \in X_{\perp}$. Since $0 \leq y \leq x_0$ and $y \in X_{\perp}$, $0 \in X_{\perp}$. Thus, we have that $0 \leq y \leq x_0$ for any $y \in X_{\perp}$. That is, $X_{\perp} = [0, x_0]$. □

The following Proposition 4.13, Proposition 4.14 and Proposition 4.15 give us the lattice theoretical structure of ${}_{\perp}X$ under some special conditions.

**Proposition 4.13.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. Then, ${}_{\perp}X$ is a join sublattice.*

*Proof.* Let $y_1 \in {}_{\perp}X$ and $y_1 \leq y_2$. Since $K^{I,J}_{a,T,S,N}(y_2, x) \leq K^{I,J}_{a,T,S,N}(y_1, x) = 0$, $K^{I,J}_{a,T,S,N}(y_2, x) = 0$. Then, $y_2 \in {}_{\perp}X$. Thus, $({}_{\perp}X, \leq)$ is a join sublattice. □

**Proposition 4.14.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. If ${}_{\perp}X \neq \emptyset$, then it is in shape of an interval. If $L$ is a lattice of finite length, then there exists $y_1 \in {}_{\perp}X$ such that ${}_{\perp}X = [y_1, 1]$.*

*Proof.* The proof is straightforward. □

**Proposition 4.15.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. If ${}^{\perp}X \neq \emptyset$, then it is in shape of an interval. If $L$ is a lattice of finite length, then there exists $y \in {}^{\perp}X$ such that ${}^{\perp}X = [0, y]$.*

**Remark 4.16.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. If $K^{I,J}_{a,T,S,N}$ is a g-convex combination, then ${}^{\perp}X \neq \emptyset$.*

**Proposition 4.17.** *Let $L$ be a bounded lattice, $I$ be an implication on $L$ satisfying (EP). For any $X \subseteq L$,*

$$I({}_{\perp}({}^{\perp}X_I), X) \subseteq ({}^{\perp}X_I)^{\perp}.$$

*Proof.* Let $m \in {}_{\perp}({}^{\perp}X_I)$ and $x \in X$. Then, for any $x' \in {}^{\perp}X_I$, $I(m, x') = 0$. Since $x' \in {}^{\perp}X_I$, for any $x^* \in X$,

$$I(x', x^*) = 1.$$

Especially, for $x^* = x$, $I(x', x) = 1$ holds. Since $I$ satisfies (EP), $I(x', I(m, x)) = I(m, I(x', x)) = I(m, 1) = 1$. Thus, $I(m, x) \in ({}^{\perp}X_I)^{\perp}$. □

**Proposition 4.18.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $X \subseteq L$.*

*(i) If $K^{I,J}_{a,T,S,N}$ is the g-convex combination of $I$ and $J$ for $a \in L$ with $S(a, N(a)) = 1$, then $(X_I)^{\perp} \cap (X_J)^{\perp} \subseteq X^{\perp}$.*

*(ii) If $K^{I,J}_{a,T,S,N}$ is the linear combination of $I$ and $J$ for $a \in L$, then $(X_I)_{\perp} \cap (X_J)_{\perp} \subseteq X_{\perp}$.*

*Proof.* (i) Let $m \in (X_I)^{\perp} \cap (X_J)^{\perp}$. Then, $m \in (X_I)^{\perp}$ and $m \in (X_J)^{\perp}$. For all $x \in X$, $I(x, m) = 1$ and $J(x, m) = 1$. Then,

$$K^{I,J}_{a,T,S,N}(x, m) = S(T(a, I(x, m)), T(N(a), J(x, m))) = S(T(a, 1), T(N(a), 1)) = S(a, N(a)) = 1.$$

Thus, $m \in X^{\perp}$.

(ii) Let $m \in (X_I)_{\perp} \cap (X_J)_{\perp}$. Then, $m \in (X_I)_{\perp}$ and $m \in (X_J)_{\perp}$. For all $x \in X$, $I(x, m) = 0$ and $J(x, m) = 0$. Since

$$K^{I,J}_{a,T,S,N}(x, m) = S(T(a, I(x, m)), T(N(a), J(x, m))) = S(T(a, 0), T(N(a), 0)) = 0,$$

we have that $x \in X_{\perp}$. □

**Proposition 4.19.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. For all $x \in X$ and for any $y \in (X_I)_{\perp} \cup (X_J)_{\perp}$,*

$$K^{I,J}_{a,T,S,N}(x, y) \leq (a \wedge I(x, y)) \vee (N(a) \wedge J(x, y)).$$

*Proof.* Let $y \in (X_I)_{\perp} \cup (X_J)_{\perp}$. Then, $y \in (X_I)_{\perp}$ or $y \in (X_J)_{\perp}$. Suppose that $y \in (X_I)_{\perp}$. Then, for all $x \in X$,

$$\begin{aligned} K^{I,J}_{a,T,S,N}(x, y) &= S(T(a, I(x, y)), T(N(a), J(x, y))) = S(T(a, 0), T(N(a), J(x, y))) \\ &= S(0, T(N(a), J(x, y))) = T(N(a), J(x, y)) \leq N(a) \wedge J(x, y). \end{aligned}$$

Let $y \in (X_J)_{\perp}$. Then, for all $x \in X$,

$$\begin{aligned} K^{I,J}_{a,T,S,N}(x, y) &= S(T(a, I(x, y)), T(N(a), J(x, y))) = S(T(a, I(x, y)), T(N(a), 0)) \\ &= S(T(a, I(x, y)), 0) = T(a, I(x, y)) \leq a \wedge I(x, y). \end{aligned}$$

Thus, we have that $K^{I,J}_{a,T,S,N}(x, y) \leq (a \wedge I(x, y)) \vee (N(a) \wedge J(x, y))$. □

**Proposition 4.20.** *Let $L$ be a bounded lattice, $I$ and $J$ be two implications on $L$ and $K^{I,J}_{a,T,S,N}$ be the linear combination of $I$ and $J$ for $a \in L$. For all $x \in X$ and for any $y \in (X_I)^{\perp} \cup (X_J)^{\perp}$,*

$$K^{I,J}_{a,T,S,N}(x, y) \geq a \wedge N(a).$$

*Proof.* Let $y \in (X_I)^{\perp} \cup (X_J)^{\perp}$. Then, $y \in (X_I)^{\perp}$ or $y \in (X_J)^{\perp}$. Suppose that $y \in (X_I)^{\perp}$. Then, for all $x \in X$,

$$K^{I,J}_{a,T,S,N}(x, y) = S(T(a, I(x, y)), T(N(a), J(x, y))) = S(T(a, 1), T(N(a), J(x, y))) = S(a, T(N(a), J(x, y))) \geq a.$$

Similarly, if $y \in (X_J)^{\perp}$, we have that $K^{I,J}_{a,T,S,N}(x, y) \geq N(a)$. Thus, $K^{I,J}_{a,T,S,N}(x, y) \geq a \wedge N(a)$.

□

## 5 Conclusions

In this paper, the notion of convex combinations for fuzzy implications defined on the unit interval $[0,1]$ was extended to bounded lattices. In this sense, the notions of the linear combination and the g-convex combination for implications on bounded lattices were introduced. Some properties of the new operator such as (NP), (IP), (OP) and (EP) were discussed. It was shown that the greatest one in the class of the g-convex combination of two implications was the greatest implication on a bounded lattice. The classical notion of convex combination for two fuzzy implications was proven to be a special case of the notion of g-convex combination of two implications on a bounded lattice. The fact that the $N'$-reciprocal of the g-convex combination of two implications coincides with the g-convex combination of the $N'$-reciprocals of the implications was presented. The sets denoted by $(X)^{\perp}$, $(X)_{\perp}$, ${}^{\perp}(X)$ and ${}_{\perp}(X)$ which are defined for the linear (g-convex) combination of two implications on a bounded lattice were discussed and the relationships between them were studied. The sets $(X)^{\perp}$, $(X)_{\perp}$, ${}^{\perp}(X)$ and ${}_{\perp}(X)$ were proven to be in shape of intervals. The lattice theoretical structure of these sets were investigated.

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