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States on implication basic algebras

S. M. Ghasemi Nejad[1], R. A. Borzooei[2] and M. Bakhshi[3]

[1] *Department of Mathematics, Faculty of Shariaty, Tehran Branch, Technical and Vocational University (TVU), Tehran, Iran*
[2] *Department of Mathematics, Shahid Beheshti University, Tehran, Iran*
[3] *Department of Mathematics, University of Bojnord, Bojnord, Iran*

mehrnavazg@gmail.com, borzooei@sbu.ac.ir, bakhshi@ub.ac.ir

Abstract

In this paper, the notions of Bosbach states and state-morphisms on implication basic algebra are introduced, along with their properties and related results. It is proved that Bosbach states coincide with Riečan states on bounded implication basic algebras. Accordingly, the relations between Bosbach states and state-morphisms are discussed. It is proved that sm is a state-morphism on $\mathcal{IB}$ if and only if sm is a Bosbach state and $sm(x \vee y) = sm(x) \vee sm(y)$. In addition, the concept of internal states on implication basic algebras is defined, and accordingly, the notions of IS-prefilters, IS-filters, and IS-congruence relations on implication basic algebras are introduced. Then, it is proved that one-to-one correspondence is available between the set of all IS-filters and IS-congruence relations on implication basic algebras. Finally, the new notion of generalized state maps from an implication basic algebra $\mathcal{IB}_1$ to an arbitrary implication basic algebra $\mathcal{IB}_2$ is defined and generalized state-morphisms and generalized internal states as two types of individual generalized state maps are introduced. This confirmed that the generalized internal states are a generalization of internal states, and the generalized state-morphisms are a generalization of state-morphisms on implication basic algebras. Finally, it is shown that a generalized internal state gs is an internal state on implication basic algebra $\mathcal{IB}$ if $gs^2 = gs$.

Keywords: Basic algebra, implication basic algebra, Bosbach state, Riečan state, state-morphism, internal state, generalized state.

1 Introduction

Logical algebras are the corresponding algebraic foundations for reasoning mechanisms of many fields such as quantum logics, information sciences, computer sciences, artificial intelligence, and so on. Chang [11] suggested MV-algebras as the algebraic counterpart of Łukasiewicz logic [26]. They are the same as Boolean algebras for classical logic (Boolean algebras coincide with idempotent MV-algebras). Łukasiewicz logic and MV-algebras were often used to deal with uncertain information. Chajda et al. [8] introduced basic algebras, as a common generalization of MV-algebras and orthomodular lattices. In addition, Chajda et al. [9] proposed implication basic algebras, as a generalization of orthoimplication algebras and implication reducts of MV-algebras. Chajda indicated that any basic algebra is an implication basic algebra, and any bounded implication basic algebra is considered as a basic algebra. In another study, Mundici in [23] introduced states on MV-algebras as averaging processes for formulas in Łukasiewicz logic. Further, states on MV-algebras generalized the usual probability measures on Boolean algebras, and the states were used as a semantical interpretation of the probability of fuzzy events e. In other words, $s(e)$ is presented as the average of the appearance of the event e, if s is a state, and e is a fuzzy event. A large number of results were obtained, which could connect states with integrals. For instance, Navara [24] indicated that a considerable class of measures on subclasses of MV-algebras is represented by integrals with respect to classical probability measures.

Different approaches to the generalization mainly gave rise to Bosbach and Riečan states as different notions. Riečan [25] introduced the states on BL-algebras as functions defined on these algebras with values in $[0, 1]$. Georgescu [19]

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suggested that Bosbach and Riečan states have a domain as a pseudo-BL algebra and a codomain as the real interval $[0, 1]$. Hence, it is meaningful to extend the notion of states to other algebraic structures and their non-commutative cases. Then, Borzooei and Zahiri, Dvurečenskij, Riečan in [1, 17, 25, 28] introduced the states on other logical algebras such as BL-algebras, hoops, and equality algebras.

In the case of non-commutative fuzzy structures, Dvurečenskij [16], Georgescu [19], and Iorgulescu [22] suggested the states for pseudo-MV-algebras, pseudo-BL algebras, and ℓ-groups with strong units, respectively. In many cases, the evaluation of truth degree on sentences was made in abstract structure, and not in the standard algebra [0, 1]. For this reason, Flaminio and Montagna [18] defined a probability with values in an abstract algebra and presented a unified approach to states and probabilistic many-valued logic in a logical algebraic setting. In addition, they added a unary operation called "internal state" to the language on MV-algebras, which preserves the usual properties of states. Correspondingly, the pair (M, σ) was called a state MV-algebra. Consequently, the internal states were extended and intensively studied in other algebraic structures such as BCK-algebras, BL-algebras, pseudo-hoops and, equality algebras [2, 3, 13, 14], and the like.

Further, generalized Bosbach states of type I and type II were proposed on commutative residuated lattices [20] and non-commutative cases [15] as a common generalization of states and state operators on residuated lattices. Furthermore, generalized Riečan states were defined in [15]. Some other results on generalized states can be found in [12, 27].

Unlike the aforementioned associative generalizations of MV-algebras (pseudo-MV-algebras, etc.), basic algebras are neither commutative nor associative. However, Botur et al. in [4, 5] similarly defined the states and state operators on basic algebras in the case of MV-algebras, although some expected properties were lost in the general setting. Thus, they dealt with states and state operators on commutative basic algebras and certain basic algebras, which retain some essential features of MV-algebras. Chajda [10] indicated the existence of states on non-commutative basic algebras. In addition, we observe that some interesting fields of states are available on implication basic algebras, including state-morphisms, Bosbach states, Riečan states, internal states, etc.

In many cases, the evaluation of truth degree of sentences was made in an abstract structure (MV-algebra, BL-algebra, etc.), and not in the standard algebra $[0, 1]$. Based on this approach, a probability is defined with values in an abstract algebra (in our case, an implication basic algebra). Then, we introduce a new notion of generalized state on implication basic algebras.

This paper is organized as follows. Section 2 focuses on the definitions and some relevant facts about implication basic algebras. Section 3 explains Bosbach and Riečan states and their relations, along with one of the main results proving that any Bosbach state coincides with Riečan state on a bounded implication basic algebra. In Section 4, state-morphisms on implication basic algebras are defined and studied. It is proved that sm is a state-morphism if and only if sm is a Bosbach state and $sm(x \vee y) = sm(x) \vee sm(y)$. Section 5 gives a brief overview of the notion of internal state σ on implication basic algebra $(\mathcal{IB}, \sigma)$, along with its properties. We prove that the combination of a Bosbach state and an internal state on implication basic algebra $\mathcal{IB}$ is a Bosbach state on $\mathcal{IB}$ and the combination of a Riečan state and an internal state on bounded implication basic algebra $\mathcal{B}$ is a Riečan state on $\mathcal{B}$. Then, we introduce the notions of IS-(pre)filter and IS-congruence of $(\mathcal{IB}, \sigma)$, where σ is an internal state on an implication basic algebra $\mathcal{IB}$, and evaluate their properties. We prove that a one-to-one correspondence is available between IS-filters and IS-congruences of $(\mathcal{IB}, \sigma)$. Finally, by using an IS-filter F of $(\mathcal{IB}, \sigma)$, we construct an internal state on $\mathcal{IB}/\mathcal{F}$. In the last section, we introduce the new notion of generalized state maps from an implication basic algebra $\mathcal{IB}_1$ to an arbitrary implication basic algebra $\mathcal{IB}_2$. In addition, generalized state-morphisms and generalized internal states are explained as two types of special generalized state maps. We verify that the generalized internal states are generalization of internal states, and the generalized state-morphisms are generalization of state-morphisms on implication basic algebras and find some conditions for a generalized internal state so that it can be considered as an internal state on an implication basic algebra.

2 Preliminaries

In this section, we recollect some definitions and results about implication basic algebras, which will be used in this paper.

Definition 2.1. [6] *An algebra $(\mathcal{IB}; \to)$ of type (2) is called an implication basic algebra, if for all $x, y, z \in \mathcal{IB}$, it satisfies the following axioms:*
(IB1) $(x \to x) \to x = x$,
(IB2) $(x \to y) \to y = (y \to x) \to x$,
(IB3) $(((x \to y) \to y) \to z) \to (x \to z) = x \to x$.

In any implication basic algebra $(\mathcal{IB};\to)$, for any $x,y\in\mathcal{IB}$, $x\to x=y\to y$ i.e., there exists a constant $1\in\mathcal{IB}$ such that $x\to x=1$. The binary operation $\leq$ defined by $x\leq y$ if and only if $x\to y=1$ is a partial order on $\mathcal{IB}$ and $(\mathcal{IB},\leq)$ is a join semilattice with the greatest element 1, where $x\vee y=(x\to y)\to y=(y\to x)\to x$.

Proposition 2.2. [6] *Let $(\mathcal{IB};\to)$ be an implication basic algebra. Then for any $x,y\in\mathcal{IB}$, the following properties hold:*
(IBA$_1$) $1\to x=x$, $x\to 1=1$.
(IBA$_2$) $y\to(x\to y)=1$.
(IBA$_3$) $(x\vee y)\to y=x\to y$.
(IBA$_4$) *If $x\leq y$ then for all $z\in\mathcal{IB}$, $y\to z\leq x\to z$.*

Definition 2.3. [6] *A nonempty subset F of $\mathcal{IB}$ containing 1 is called a prefilter of $\mathcal{IB}$ if for all $x,y\in\mathcal{IB}$,*
(F1) if $x\in F$ and $x\to y\in F$, then $y\in F$.
A prefilter F of $\mathcal{IB}$ is called a weak filter of $\mathcal{IB}$ if for all $x,y\in\mathcal{IB}$,
(F2) if $x\to y\in F$ and $y\leq x$, then $(y\to z)\to(x\to z)\in F$, for all $z\in\mathcal{IB}$.
A weak filter F of $\mathcal{IB}$ is called a filter of $\mathcal{IB}$ if for all $x,y\in\mathcal{IB}$,
(F3) if $x\to y\in F$ and $y\to x\in F$, then $(z\to x)\to(z\to y)\in F$, for all $z\in\mathcal{IB}$.

Definition 2.4. [6] *An equivalence relation θ on $\mathcal{IB}$ is called a weak congruence if for all $x,y\in\mathcal{IB}$, we have*

$$\text{if } (x,y)\in\theta,\text{ then for all } z\in\mathcal{IB},\ (x\to z,y\to z)\in\theta.$$

A weak congruence θ on $\mathcal{IB}$ is called a congruence if for all $x,y\in\mathcal{IB}$, we have

$$\text{if } (x,y)\in\theta,\text{ then for all } z\in\mathcal{IB},\ (z\to x,z\to y)\in\theta.$$

Theorem 2.5. [6] *Let F be a weak filter of $\mathcal{IB}$. Then the relation θ_F, which is defined by*

$$(x,y)\in\theta_F\Leftrightarrow(x\to y\in F\text{ and } y\to x\in F)$$

is a weak congruence on $\mathcal{IB}$ such that $[1]_{\theta_F}=F$. Moreover, if F is a filter of $\mathcal{IB}$, then θ_F is a congruence on $\mathcal{IB}$.

Theorem 2.6. [6] *Let Φ be a (weak) congruence of $\mathcal{IB}$. Then $[1]_\Phi$ is a (weak) filter of $\mathcal{IB}$ and $\theta_{[1]_\Phi}=\Phi$.*

For a filter F of $\mathcal{IB}$, let θ_F be the congruence, which is described in Theorem 2.5. We denote the set of all equivalence classes of θ_F by $\mathcal{IB}/F=\{x/F\mid x\in\mathcal{IB}\}$. Then $\mathcal{IB}/F$ together with the operations induced from $\mathcal{IB}$ forms an implication basic algebra in which the partial ordering is defined as $x/F\leq y/F$ if and only if $x\to y\in F$. The operation $\to$ is defined by $x/F\to y/F=(x\to y)/F$ and so $x/F\vee y/F=(x\vee y)/F$.

Definition 2.7. [6] *A basic algebra is an algebra $(\mathcal{B};\oplus,\sim,0)$ of type $(2,1,0)$ such that for any $x,y,z\in\mathcal{B}$, it satisfies the following identities:*
(B1) $x\oplus 0=x$,
(B2) $\sim(\sim x)=x$,
(B3) $\sim((\sim x)\oplus y)\oplus y=\sim((\sim y)\oplus x)\oplus x$,
(B4) $\sim(\sim(\sim(x\oplus y)\oplus y)\oplus z)\oplus(x\oplus z)=1$, where $1=\sim 0$.

Theorem 2.8. [6] *Let $(\mathcal{IB};\to)$ be an implication basic algebra with the least element 0. If for any $x,y,z\in\mathcal{IB}$, we define two operations $\sim$ and $\oplus$ by $\sim x=x\to 0$ and $x\oplus y=(x\to 0)\to y$, then $(\mathcal{IB};\oplus,\sim,0)$ is a basic algebra. Conversely, every basic algebra $(\mathcal{B};\oplus,\sim,0)$ is an implication basic algebra with the operation $\to$, which is defined by $x\to y=\sim x\oplus y$, for all $x,y,z\in\mathcal{B}$.*

It is evident that every basic algebra can be redefined in the above signatures [7]. In particular, for all $x,y,z\in\mathcal{B}$, the operation $\to$ allows to reduce the number of defining identities (B1)-(B4) to
(BI1) $(x\to 0)\to 0=x$,
(BI2) $(x\to y)\to y=(y\to x)\to x=x\vee y$,
(BI3) $(((x\to y)\to y)\to z)\to(x\to z)=1$, where $1=0\to 0$.
A basic algebra $\mathcal{B}$ is called *commutative* if for all $x,y\in\mathcal{B}$, it satisfies the identity $x\oplus y=y\oplus x$. Let us note that the binary operation $\oplus$ is not necessarily commutative nor associative. In any basic algebra $(\mathcal{B};\oplus,\sim,0)$, the binary relation $\leq$ defined by $x\leq y$ if and only if $\sim x\oplus y=x\to y=1$ is a partial ordering on $\mathcal{B}$ and $(\mathcal{B};\leq)$ is a lattice with the greatest element 1 and least element 0.

Proposition 2.9. [6] *Let $(\mathcal{B};\oplus,\sim,0)$ be a basic algebra. Then for any $x,y\in\mathcal{B}$, the following properties hold:*
(BA_1) $x\oplus 0=0\oplus x=x, \sim x\oplus x=x\oplus\sim x=1, x\oplus 1=1\oplus x=1$.
(BA_2) $1\to x=x, x\to 1=1, x\to x=1, 0\to x=1$.
(BA_3) $x\leq y$ *if and only if* $\sim y\leq\ \sim x$.
(BA_4) *If* $x\leq y$*, then for all* $z\in\mathcal{B}$*,* $y\to z\leq x\to z$ *and* $x*z\leq y*z$*, where* $*\in\{\oplus,\vee,\wedge\}$.
(BA_5) $x\leq y\oplus x$ *and* $x\leq y\to x$.
(BA_6) $\sim(x\vee y)=\sim x\wedge\sim y$.
(BA_7) $x\odot y=\sim(x\to\ \sim y)$.

Definition 2.10. [4, 10] *Let $\mathcal{B}$ be a basic algebra. A state on $\mathcal{B}$ is a function $s:\mathcal{B}\to[0,1]$ such that, for all $x,y\in\mathcal{B}$,*
(RS1) $s(1)=1$,
(RS2) $s(x\oplus y)=s(x)+s(y)$*, where* $x\leq\ \sim y$.
In this paper, we call this state a Riečan state on $\mathcal{B}$.

Proposition 2.11. [10] *Let $\mathcal{B}$ be a basic algebra. Then, for any Riečan state $s:\mathcal{B}\to[0,1]$ and $x,y\in\mathcal{B}$, we have the following statements*
(i) $s(0)=0$.
(ii) $s(\sim x)=1-s(x)$.
(iii) $x\leq y$ *implies* $s(x)\leq s(y)$.
(iv) $s(x\oplus y)=s(x\wedge\sim y)+s(y)$.

Remark 2.12. *From now on, in this paper, an implication basic algebra is denoted by $\mathcal{IB}=(\mathcal{IB};\to)$ and an implication basic algebra with the least element 0 is denoted by $\mathcal{B}=(\mathcal{B};\to)$, which is a basic algebra (by Theorem 2.8) and so we can define the operations $\oplus$ and $\sim$ and Riečan states on $\mathcal{B}$.*

3 Bosbach state on implication basic algebras

In this section, we introduce the concept of Bosbach state on implication basic algebras, and we get some related results. We prove that any Bosbach state on $\mathcal{B}$ coincides with a Riečan state on $\mathcal{B}$.

Definition 3.1. *A Bosbach state on $\mathcal{IB}$ is a function $s:\mathcal{IB}\to[0,1]$ such that, for all $x,y\in\mathcal{IB}$,*
(BS1) $s(1)=1$,
(BS2) $s(x)+s(x\to y)=s(y)+s(y\to x)$,
(BS3) *there exists an element* $a\in\mathcal{IB}$ *such that* $s(a)=0$.

Example 3.2. *Let $\mathcal{IB}=\{a,b,c,1\}$ be the semilattice whose Hasse diagram is shown in Figure 1 and the operation $\to$ on $\mathcal{IB}$ is given by Table 1. Routine calculations show that $(\mathcal{IB};\to)$ is an implication basic algebra. It is easy to check that the mapping $s:\mathcal{IB}\to[0,1]$ defined by $s(a)=0, s(b)=2/3, s(c)=1/3$ and $s(1)=1$ is a Bosbach state.*

$\to$	a	b	c	1
a	1	b	c	1
b	a	1	b	1
c	a	1	1	1
1	a	b	c	1

Table 1

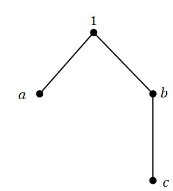

Figure 1

Note. In a Bosbach state s, we can see that the conditions $(BS1)$, $(BS2)$ and $(BS3)$ are independent. Let $(\mathcal{IB};\to)$ be the implication basic algebra from Example 3.2. One can easily check that $s:\mathcal{IB}\to[0,1]$ defined by $s(a)=0$, $s(b)=2/3$, $s(c)=5/6$ and $s(1)=1/2$ satisfies (BS2) and (BS3), but not (BS1). Moreover, $s:\mathcal{IB}\to[0,1]$ defined by $s(a)=1/4$, $s(b)=3/4$, $s(c)=1/2$ and $s(1)=1$ satisfies (BS1) and (BS2), but not (BS3). Finally, $s:\mathcal{IB}\to[0,1]$ defined by $s(a)=s(b)=0$ and $s(c)=s(1)=1$ satisfies (BS1) and (BS3), but not (BS2).

Proposition 3.3. *Let s be a Bosbach state on $\mathcal{IB}$. Then, for all $x,y\in\mathcal{IB}$, the following statements hold:*
(i) $x\leq y$ *implies* $s(x)\leq s(y)$ *and* $s(y\to x)=1-s(y)+s(x)$.
(ii) $s(x\to y)=1-s(x\vee y)+s(y)$.
(iii) $ker(s)=\{x\in\mathcal{IB}\mid s(x)=1\}$ *is a prefilter of* $\mathcal{IB}$.

Proof. (i) Let $x \leq y$. Then $x \to y = 1$. By (BS1) and (BS2), $s(x) - s(y) = s(y \to x) - 1 \leq 0$. Hence $s(x) \leq s(y)$.
(ii) Since $y \leq x \vee y$, by using (i), we get $s((x \vee y) \to y) = 1 - s(x \vee y) + s(y)$. Hence by (IBA_3), we have $s(x \to y) = 1 - s(x \vee y) + s(y)$.
(iii) Let $x, x \to y \in ker(s)$, for $x, y \in \mathcal{IB}$. Then $s(x) + s(x \to y) = 2 = s(y) + s(y \to x)$ and so $s(y) = s(y \to x) = 1$ and $y \in ker(s)$. Hence $ker(s)$ is a prefilter. □

In the following, for an implication basic algebra with least element 0, $(\mathcal{B}; \to)$, we will show that any Bosbach state $s : \mathcal{B} \to [0,1]$ is a Riečan state on $\mathcal{B}$.

Proposition 3.4. *Let s be a Bosbach state on $\mathcal{B}$, then for any $x, y \in \mathcal{B}$, we have*
(i) $s(0) = 0$.
(ii) $s(\sim x) = 1 - s(x)$.
(iii) $s(x \to (x \wedge y)) = 1 - s(x) + s(x \wedge y)$.

Proof. (i) Since 0 is the least element of $\mathcal{B}$, by Proposition 3.3(i), for all $x \in \mathcal{IB}$, $s(0) \leq s(x)$. Then by (BS3), $s(0) = 0$.
(ii) Let $x \in \mathcal{IB}$. Then $s(x) + s(x \to 0) = s(0) + s(0 \to x)$ and by considering Remark 2.12, $s(x) + s(\sim x) = 0 + 1$.
(iii) Since $x \wedge y \leq x$, by using Proposition 3.3(i), we get $s(x \to (x \wedge y)) = 1 - s(x) + s(x \wedge y)$. □

Proposition 3.5. *Let $s : \mathcal{B} \to [0,1]$ be a function such that, for any $x, y \in \mathcal{B}$, $s(0) = 0$ and $s(x \to (x \wedge y)) = s(x \to y)$. Then the following conditions are equivalent:*
(i) *s is a Bosbach state,*
(ii) *$x \leq y$ implies $s(y \to x) = 1 + s(x) - s(y)$,*
(iii) *$s(x \to y) = 1 - s(x) + s(x \wedge y)$.*

Proof. (i) ⇒ (ii) It holds by Proposition 3.3(i).
(ii) ⇒ (iii) By the proof of Proposition 3.4(iii), the proof is clear.
(iii) ⇒ (i) Suppose (iii) holds, for all $x, y \in \mathcal{B}$. Hence, by (IBA_1), we have

$$s(1) = s(x \to 1) = 1 - s(x) + s(x \wedge 1) = 1 - s(x) + s(x) = 1,$$

and by (iii), we have

$$s(x) + s(x \to y) = s(x) + 1 - s(x) + s(x \wedge y) = 1 + s(x \wedge y) = s(y) + 1 - s(y) + s(y \wedge x) = s(y) + s(y \to x).$$

□

Proposition 3.6. *Function $s : \mathcal{B} \to [0,1]$ is a Riečan state on $\mathcal{B}$ if and only if it is a Bosbach state on $\mathcal{B}$.*

Proof. Assume that s is a Riečan state on $\mathcal{B}$. Then by (RS1) and Proposition 2.11(i), $s(1) = 1$ and $s(0) = 0$. By Proposition 2.11(iv),

$$\begin{aligned} s(x) + s(x \to y) &= s(x) + s(\sim x \oplus y) = s(x) + s(\sim x \wedge \sim y) + s(y) = s(y) + s(\sim y \wedge \sim x) + s(x) \\ &= s(y) + s(\sim y \oplus x) = s(y) + s(y \to x). \end{aligned}$$

So s is a Bosbach state. Now, let s be a Bosbach state and $x \leq \sim y$. Then $s(x) + s(x \to \sim y) = s(\sim y) + s((\sim y) \to x)$ and by Proposition 3.4(ii), $s(x) + 1 = 1 - s(y) + s((\sim y) \to x)$. Hence $s(y \oplus x) = s((\sim y) \to x) = s(x) + s(y)$. By (BA_3) and (B2), if $x \leq \sim y$, then $y \leq \sim x$. Hence $s(x \oplus y) = s(y \oplus x)$ and so s is a Riečan state on $\mathcal{B}$. □

The following example shows that in a basic algebra $\mathcal{B}$, the Bosbach state coincides with Riečan state.

Example 3.7. *Let $\mathcal{B} = \{0, a, b, c, d, 1\}$ be the lattice whose Hasse diagram is shown in Figure 2, and the operation $\to$ is given by Table 2(a). Routine calculations show that $(\mathcal{B}; \to)$ is an implication basic algebra with least element 0. Hence, by Remark 2.12, $\mathcal{B}$ with the operations $\sim$ and $\oplus$ which is defined by Table 2(b) is a basic algebra. It is not difficult to show that the mapping $s : \mathcal{B} \to [0,1]$ defined by $s(0) = 0, s(a) = s(b) = 1/3, s(c) = s(d) = 2/3$ and $s(1) = 1$ is a Bosbach and Riečan state.*

The following example shows that there exists an implication basic algebra with least element 0 (a basic algebra) and a Bosbach state s (by Theorem 3.6, a Riečan state s) whose $ker(s)$ is not a filter.

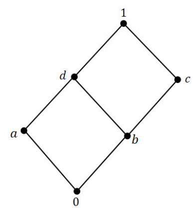

Figure 2

((a)) $(\mathcal{B};\to)$

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	d	c	1	1
b	d	d	1	1	1	1
c	a	a	c	1	d	1
d	b	d	d	c	1	1
1	0	a	b	c	d	1

x	$\sim x$
0	1
a	c
b	d
c	a
d	b
1	0

((b)) $(\mathcal{B};\oplus,\sim)$

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	1	d	1
b	b	d	d	c	1	1
c	c	1	d	c	1	1
d	d	d	1	1	1	1
1	1	1	1	1	1	1

Table 2

Example 3.8. *Let $(\mathcal{B}_i;\oplus_i,\sim_i,0,1)$ (where $i\in I$) be a collection of basic algebras with the same 0 and 1 such that $\mathcal{B}_i\cap\mathcal{B}_j=\{0,1\}$ for all $i,j\in I$. Then the horizontal sum of the B_i's is the algebra $(\bigcup_{i\in I}\mathcal{B}_i,\oplus,\sim,0,1)$ where $\sim x=\sim_i x$ when $x\in\mathcal{B}_i$, $x\oplus y=x\oplus_i y$ when there is $i\in I$ such that $x,y\in\mathcal{B}_i$, and $x\oplus y=y$ otherwise.*
Now, let M be the MV-algebra $\Gamma(\mathbb{Z}\times_{lex}\mathbb{Z})$ where $\mathbb{Z}\times_{lex}\mathbb{Z}$ is the lexicographic product of the linearly ordered groups $\mathbb{Z}$ and $\mathbb{Z}$, i.e., $M=\{(0,x)\mid x=0,1,2,...\}\cup\{(1,x)\mid x=0,-1,-2,...\}$ and $\sim(0,x)=(1,-x)$ and $\sim(1,x)=(0,-x)$, $(0,x)\oplus(0,y)=(0,x+y)$, $(0,x)\oplus(1,y)=(1,\min\{x+y,0\})$ and $(1,x)\oplus(1,y)=(1,0)$. Let $\mathcal{B}_i$ ($2\leq i\in I$) be isomorphic copies of M, say $\mathcal{B}_i=\{0,a_1^{(i)},a_2^{(i)},...,\sim a_2^{(i)},\sim a_1^{(i)},1\}$, and let $\mathcal{B}$ be the horizontal sum of the $\mathcal{B}_i$. Then s defined by $s(0)=s(a_n^{(i)})=0$ and $s(1)=s(\sim a_n^{(i)})=1$ for every $n=1,2,...$ and $i\in I$ is a state on $\mathcal{B}$ (see [5]). The kernel of s is the $ker(s)=\{1,\sim a_1^{(i)},\sim a_2^{(i)},...\mid i\in I\}$. Obviously, $ker(s)$ is not a lattice filter because $\sim(a_m^{(i)})\wedge\sim(a_n^{(j)})=0$ where $i\neq j$, and $ker(s)$ is not a filter.

4 State-morphisms on implication basic algebras

In this section, we define and study state-morphisms on implication basic algebras and investigate their properties. We prove that the kernel of any state-morphism is a filter. Also, we can see that any state-morphism is a Bosbach state, but the converse is not valid in general.
Let $[0,1]_{MV}$ be the standard MV-algebra, where for all $x,y\in[0,1]_{MV}$,

$$x\rightharpoonup y=\min\{1-x+y,1\},\ \ x\oplus y=\min\{x+y,1\},$$

$$x\vee y=\max\{x,y\},\ \ x\wedge y=\min\{x,y\}\text{ and }\sim x=1-x.$$

Definition 4.1. *A state-morphism on $\mathcal{IB}$ is a function $sm:\mathcal{IB}\longrightarrow[0,1]_{MV}$ such that, for all $x,y\in\mathcal{IB}$,*

$$sm(x\to y)=sm(x)\rightharpoonup sm(y).$$

If $\mathcal{IB}$ has a least element 0, then we add the condition $sm(0)=0$.
The state-morphism $1:\mathcal{IB}\longrightarrow[0,1]_{MV}$, which is defined by $1(x)=1$, for any $x\in\mathcal{IB}$, is called a trivial state-morphism.

Proposition 4.2. *Let sm be a state-morphism on $\mathcal{IB}$. Then, for any $x\in\mathcal{IB}$, the following statements hold:*
(i) *sm is a Bosbach state.*
(ii) *$sm(x\vee y)=sm(x)\vee sm(y)$.*

(iii) $sm(\sim x) = 1 - sm(x)$ *and* $sm(x \wedge y) = sm(x) \wedge sm(y)$, *when* $0 \in \mathcal{IB}$.
(iv) $x \leq y$ *implies* $sm(x) \leq sm(y)$.
(v) $ker(sm) = \{x \in \mathcal{IB} \mid sm(x) = 1\}$ *is a filter of* $\mathcal{IB}$.

Proof. (i) Let sm be a state-morphism on $\mathcal{IB}$. Then for any $x, y \in \mathcal{IB}$, we have

$$\begin{aligned} sm(x) + sm(x \to y) &= sm(x) + (sm(x) \rightharpoonup sm(y)) = sm(x) + \min\{1 - sm(x) + sm(y), 1\} \\ &= \min\{1 + sm(x), 1 + sm(y)\} = sm(y) + \min\{1 - sm(y) + sm(x), 1\} \\ &= sm(y) + (sm(y) \rightharpoonup sm(x)) = sm(y) + sm(y \to x). \end{aligned}$$

So sm is a Bosbach state.
(ii) Let $x, y \in \mathcal{IB}$. Since sm is a state-morphism, we get

$$sm(x \vee y) = sm((x \to y) \to y) = (sm(x) \rightharpoonup sm(y)) \rightharpoonup sm(y) = \min\{1 - sm(x) + sm(y), 1\} \rightharpoonup sm(y).$$

If $\min\{1 - sm(x) + sm(y), 1\} = 1 - sm(x) + sm(y)$, then $sm(y) \leq sm(x)$. Hence,

$$sm(x \vee y) = (1 - sm(x) + sm(y)) \rightharpoonup sm(y) = \min\{1 - (1 - sm(x) + sm(y)) + sm(y), 1\} = \min\{sm(x), 1\}.$$

But if $\min\{1 - sm(x) + sm(y), 1\} = 1$, then $sm(x) \leq sm(y)$ and $sm(x \vee y) = sm((x \to y) \to y) = \min\{sm(y), 1\}$. So

$$sm(x \vee y) = sm((x \to y) \to y) = \min\{\max\{sm(x), sm(y)\}, 1\} = \max\{sm(x), sm(y)\} = sm(x) \vee sm(y).$$

(iii) Let $0 \in \mathcal{IB}$ and $x, y \in \mathcal{IB}$. Since sm is a state-morphism, we have

$$sm(\sim x) = sm(x \to 0) = sm(x) \rightharpoonup sm(0) = sm(x) \rightharpoonup 0 = \min\{1 - sm(x) + 0, 1\} = 1 - sm(x) = \sim sm(x).$$

By (BA_6), we have

$$\begin{aligned} sm(x \wedge y) &= sm(\sim(\sim x \vee \sim y)) = 1 - (sm(\sim x) \vee sm(\sim y)) \\ &= 1 - \max\{sm(\sim x), sm(\sim y)\} = 1 - \max\{1 - sm(x), 1 - sm(y)\} \\ &= 1 + \min\{sm(x) - 1, sm(y) - 1\} = \min\{sm(x), sm(y)\} \\ &= sm(x) \wedge sm(y). \end{aligned}$$

(iv) It is obvious that $sm(1) = sm(1 \to 1) = sm(1) \rightharpoonup sm(1) = 1$. Now, let $x \leq y$, for $x, y \in \mathcal{IB}$. Then $x \to y = 1$ and $sm(x \to y) = sm(x) \rightharpoonup sm(y) = \min\{1 - sm(x) + sm(y), 1\} = 1$. So $1 \leq 1 - sm(x) + sm(y)$ and $sm(x) \leq sm(y)$.
(v) We show that $ker(sm)$ satisfies (F1)-(F3).
(F1) Let $x \in ker(sm)$ and $x \to y \in ker(sm)$, for $x, y \in \mathcal{IB}$. Then

$$sm(x) = sm(x \to y) = sm(x) \rightharpoonup sm(y) = 1 \rightharpoonup sm(y) = \min\{sm(y), 1\} = sm(y) = 1.$$

So $y \in ker(sm)$. Hence $ker(sm)$ is a prefilter.
(F2) Suppose that $x \to y \in ker(sm)$ and $y \leq x$, for $x, y \in \mathcal{IB}$. Then by (iv), $sm(y) \leq sm(x)$. Also $sm(x \to y) = sm(x) \rightharpoonup sm(y) = \min\{1 - sm(x) + sm(y), 1\} = 1$ and $sm(x) \leq sm(y)$. So $sm(x) = sm(y)$. For $z \in \mathcal{IB}$, we have $sm((y \to z) \to (x \to z)) = sm(y \to z) \rightharpoonup sm(x \to z) = (\min\{1 - sm(y) + sm(z), 1\}) \rightharpoonup (\min\{1 - sm(x) + sm(z), 1\})$. Since $sm(x) = sm(y)$, we get $sm((y \to z) \to (x \to z)) = 1$ and $(y \to z) \to (x \to z) \in ker(sm)$. Hence $ker(sm)$ is a weak filter. In the same way, (F3) satisfies and $ker(sm)$ is a filter of $\mathcal{IB}$. □

Theorem 4.3. *A Bosbach state* s *on* $\mathcal{IB}$ *is a state-morphism on* $\mathcal{IB}$ *if for all* $x, y \in \mathcal{IB}$,

$$s(x \vee y) = s(x) \vee s(y) = \max\{s(x), s(y)\}.$$

Proof. Let s be a Bosbach state on $\mathcal{IB}$ and $s(x \vee y) = s(x) \vee s(y) = \max\{s(x), s(y)\}$, for all $x, y \in \mathcal{IB}$. According to Proposition 3.3(ii), we have

$$\begin{aligned} s(x \to y) &= 1 + s(y) - s(x \vee y) = 1 + s(y) - (s(x) \vee s(y)) = 1 + s(y) - \max\{s(x), s(y)\} \\ &= \min\{1 + s(y) - s(x), 1\} = s(x) \rightharpoonup s(y). \end{aligned}$$

Then s is a state-morphism. □

Corollary 4.4. *The mapping $sm : \mathcal{IB} \longrightarrow [0,1]_{MV}$ is a state-morphism on $\mathcal{IB}$ if and only if sm is a Bosbach state on $\mathcal{IB}$ and $sm(x \vee y) = sm(x) \vee sm(y)$.*

Proof. It is straightforward from Proposition 4.2(i),(ii) and Theorem 4.3. □

The following examples show that there exists a Bosbach state on implication basic algebra $\mathcal{IB}$ which is not a state-morphism and also there exists a Bosbach state on $\mathcal{IB}$ which is a state-morphism.

Example 4.5. *(i) The Bosbach state s considered in Example 3.2 is not a state-morphism. Since*

$$s(a \to b) = s(b) = 2/3 \neq s(a) \rightharpoonup s(b) = 0 \rightharpoonup 2/3 = \min\{1 - s(a) + s(b), 1\} = 1.$$

Note that $s(b \vee c) = s(b) = 2/3 \neq 1/3 = s(b) \vee s(c)$.
(ii) Let $(\mathcal{IB}; \to)$ be the implication basic algebra from Example 3.2. The mapping $s : \mathcal{IB} \longrightarrow [0,1]$ defined by $s(a) = 0, s(b) = s(c) = s(1) = 1$ is a Bosbach state and $s(x \vee y) = s(x) \vee s(y)$, for all $x, y \in \mathcal{IB}$. Then by Theorem 4.3, s is a state-morphism of $\mathcal{IB}$. It is easy to see that $ker(s) = \{b, c, 1\}$ is a filter of $\mathcal{IB}$.

5 Internal states on implication basic algebras

In this section, we introduce the notion of internal state σ on $\mathcal{IB}$ and investigate some of its properties and discuss on the relations between internal states and Bosbach states on implication basic algebras. Also, we introduce the notions of IS-(pre)filter and IS-congruence of $(\mathcal{IB}, \sigma)$ and study their properties, where σ is an internal state on an implication basic algebra $\mathcal{IB}$. We find some relations between IS-filters and IS-congruences of $(\mathcal{IB}, \sigma)$, and in the sequel, we use an IS-filter F of $(\mathcal{IB}, \sigma)$ to construct an internal state on $\mathcal{IB}/\mathcal{F}$.

Definition 5.1. *An* internal state *on $\mathcal{IB}$ is a function $\sigma : \mathcal{IB} \to \mathcal{IB}$ such that, for all $x, y \in \mathcal{B}$, the following properties hold:*
(IS1) $x \leq y$ implies $\sigma(x) \leq \sigma(y)$,
(IS2) $\sigma(x \to y) = \sigma(x \vee y) \to \sigma(y)$,
(IS3) $\sigma(\sigma(x) \vee \sigma(y)) = \sigma(x) \vee \sigma(y)$,
(IS4) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$.

An internal state σ is said to be faithful if $ker(\sigma) = \{x \in \mathcal{IB} \mid \sigma(x) = 1\} = \{1\}$.

Example 5.2. *The map $1_{\mathcal{IB}}(x) = 1$ and the identity map $Id_{\mathcal{IB}}(x) = x$, for all $x \in \mathcal{IB}$, are internal states on $\mathcal{IB}$ and we call them trivial internal states.*

Proposition 5.3. *Let σ be an internal state on $\mathcal{IB}$. Then, for all $x, y \in \mathcal{IB}$, the following properties hold:*
(i) $\sigma(1) = 1$.
(ii) $\sigma(\sigma(x)) = \sigma(x)$.
(iii) $\sigma(x \to y) \leq \sigma(x) \to \sigma(y)$. *If x and y are comparable, then $\sigma(x \to y) = \sigma(x) \to \sigma(y)$.*
(iv) $\sigma(\mathcal{IB}) = Fix(\sigma) = \{x \in \mathcal{IB} \mid \sigma(x) = x\}$.
(v) $\sigma(\mathcal{IB})$ *is a subalgebra of $\mathcal{IB}$.*
(vi) *If $\sigma(\mathcal{IB}) = \mathcal{IB}$, then σ is the identity map on $\mathcal{IB}$.*

Proof. (i) By (IS2), $\sigma(1) = \sigma(1 \to 1) = \sigma(1 \vee 1) \to \sigma(1) = \sigma(1) \to \sigma(1) = 1$.
(ii) By (IS3), for any $x \in \mathcal{IB}$, $\sigma(\sigma(x)) = \sigma(1 \to \sigma(x)) = \sigma(\sigma(1) \to \sigma(x)) = \sigma(1) \to \sigma(x) = \sigma(x)$.
(iii) Let $x, y \in \mathcal{IB}$. We have $x \leq x \vee y$. Then by (IS1), $\sigma(x) \leq \sigma(x \vee y)$ and by (IBA_4), we have $\sigma(x \vee y) \to \sigma(y) \leq \sigma(x) \to \sigma(y)$. Hence by (IS2), $\sigma(x \to y) = \sigma(x \vee y) \to \sigma(y) \leq \sigma(x) \to \sigma(y)$.
Now, let $x, y \in \mathcal{IB}$ be comparable. If $x \leq y$, then by (IS1), $\sigma(x) \leq \sigma(y)$. So, we have $\sigma(x) \to \sigma(y) = 1 = \sigma(1) = \sigma(x \to y)$. If $y \leq x$, then by (IS2), $\sigma(x) \to \sigma(y) = \sigma(x \vee y) \to \sigma(y) = \sigma(x \to y)$.
(iv) It is obvious that $Fix(\sigma) \subseteq \sigma(\mathcal{IB})$. Let $y \in \sigma(\mathcal{IB})$. Then there exists $x \in \mathcal{IB}$ such that $\sigma(x) = y$. By (iii), $\sigma(y) = \sigma(\sigma(x)) = \sigma(x) = y$ and so $\sigma(\mathcal{IB}) = Fix(\sigma)$.
(v) From (i), (IS1) and (IS3)-(IS4), $\sigma(\mathcal{IB})$ is closed under operators $\to$ and $\vee$. Hence $\sigma(\mathcal{IB})$ is a subalgebra of $\mathcal{IB}$.
(vi) Assume that $\sigma(\mathcal{IB}) = \mathcal{IB}$ and $x \in \mathcal{IB}$. Hence there exists $x_0 \in \mathcal{IB}$ such that $\sigma(x_0) = x$. By (iii), $\sigma(x) = \sigma(\sigma(x_0)) = \sigma(x_0) = x$. So σ is the identity map. □

Proposition 5.4. *Let σ be an internal state on $\mathcal{IB}$. Then, for all $x, y \in \mathcal{IB}$, the following properties hold:*
(i) $ker(\sigma)$ *is a prefilter of $\mathcal{IB}$.*
(ii) *If σ is faithful, then $x < y$ implies $\sigma(x) < \sigma(y)$.*
(iii) *If σ is faithful, then $\sigma(x)$ and x are not comparable or $\sigma(x) = x$.*

Proof. (i) It is obvious that $1 \in ker(\sigma)$. If $x, x \to y \in ker(\sigma)$, then $\sigma(x) = 1 = \sigma(x \to y)$. By Proposition 5.3(iii), we have $1 = \sigma(x \to y) \leq \sigma(x) \to \sigma(y) = 1 \to \sigma(y) = \sigma(y)$. Hence $\sigma(y) = 1$ and $y \in ker(\sigma)$.
(ii) Let σ be faithful. If $x < y$, then $\sigma(x) \leq \sigma(y)$. Suppose $\sigma(x) = \sigma(y)$. By Proposition 5.3(iii),

$$\sigma(y \to x) = \sigma(y) \to \sigma(x) = 1 \quad , \quad y \to x \in ker(\sigma) = \{1\}.$$

So $y \to x = 1$ and $y \leq x$, which is a contradiction.
(iii) Suppose that σ is faithful. Let x and $\sigma(x)$ be comparable and $\sigma(x) \neq x$. Then $x < \sigma(x)$ or $\sigma(x) < x$. By (ii) and Proposition 5.3(ii), we have $\sigma(x) < \sigma(\sigma(x)) = \sigma(x)$, that is a contradiction. Hence $\sigma(x)$ and x are not comparable. □

The following example illustrates Theorem 5.4.

Example 5.5. *Let $\mathcal{IB} = \{a, b, c, d, 1\}$ be the join-semilattice whose Hasse diagram is shown in Figure 3. It is not difficult to show that $\mathcal{IB}$ with the operation $\to$ is defined by Table 3 is an implication basic algebra. Some of the internal states on $\mathcal{IB}$ are given in the table below:*

x	a	b	c	d	1
$\sigma_1(x)$	a	b	a	d	1
$\sigma_2(x)$	a	b	b	d	1
$\sigma_3(x)$	a	b	c	d	1
$\sigma_4(x)$	a	b	d	d	1
$\sigma_5(x)$	1	1	1	1	1

Note that $\sigma_i(x)$, for $i = 1, ..., 4$ are faithful and $x < y$ implies $\sigma_i(x) < \sigma_i(y)$. Also x and $\sigma_i(x)$ are not comparable or $\sigma_i(x) = x$, for $i = 1, ..., 4$.

$\to$	a	b	c	d	1
a	1	d	c	d	1
b	1	1	c	1	1
c	a	b	1	d	1
d	a	a	c	1	1
1	a	b	c	d	1

Table 3: $(\mathcal{IB}; \to)$

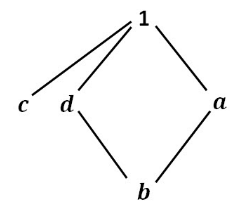

Figure 3

Proposition 5.6. *Let σ be an internal state on $\mathcal{B}$ and $\sigma(0) = 0$. Then, for all $x, y \in \mathcal{B}$, the following statements hold:*
(i) $\sigma({\sim}x) = {\sim}\sigma(x)$.
(ii) $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$.
(iii) $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$.
(iv) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$.
(v) $x \leq {\sim}y$ *implies* $\sigma(x) \leq {\sim}\sigma(y)$ *and* $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$.

Proof. (i) By (IS2), for any $x \in \mathcal{B}$, $\sigma({\sim}x) = \sigma(x \to 0) = \sigma(x \vee 0) \to \sigma(0) = \sigma(x) \to 0 = {\sim}\sigma(x)$.
(ii) By (IS4) and (i), for any $x, y \in \mathcal{B}$,$\sigma(\sigma(x) \oplus \sigma(y)) = \sigma({\sim}\sigma(x) \to \sigma(y)) = \sigma(\sigma({\sim}x) \to \sigma(y)) = \sigma({\sim}x) \to \sigma(y) = {\sim}\sigma(x) \to \sigma(y) = \sigma(x) \oplus \sigma(y)$.
(iii) By (IS4), (BA_7) and (i), for any $x, y \in \mathcal{B}$,
$\sigma(\sigma(x) \odot \sigma(y)) = \sigma({\sim}(\sigma(x) \to {\sim}\sigma(y))) = {\sim}\sigma(\sigma(x) \to \sigma({\sim}y)) = {\sim}(\sigma(x) \to \sigma({\sim}y)) = {\sim}(\sigma(x) \to {\sim}\sigma(y)) = \sigma(x) \odot \sigma(y)$.
(iv) By (IS3), (BA_6) and (i), for any $x, y \in \mathcal{B}$,
$\sigma(\sigma(x) \wedge \sigma(y)) = \sigma({\sim}({\sim}\sigma(x) \vee {\sim}\sigma(y))) = {\sim}(\sigma(\sigma({\sim}x) \vee \sigma({\sim}y))) = {\sim}(\sigma({\sim}x) \vee \sigma({\sim}y)) = {\sim}({\sim}\sigma(x) \vee {\sim}\sigma(y)) = \sigma(x) \wedge \sigma(y)$.
(v) Let $x, y \in \mathcal{B}$ and $x \leq {\sim}y$. Then by (i) and (IS1), $\sigma(x) \leq \sigma({\sim}y) = {\sim}\sigma(y)$ and by (BA_3), $y \leq {\sim}x$. Hence ${\sim}x \vee y = {\sim}x$ and by (IS2), we have:

$$\sigma(x \oplus y) = \sigma({\sim}x \to y) = \sigma({\sim}x \vee y) \to \sigma(y) = \sigma({\sim}x) \to \sigma(y) = {\sim}\sigma(x) \to \sigma(y) = \sigma(x) \oplus \sigma(y).$$

□

In the following example, we show that there is a basic algebra with only trivial internal states.

Example 5.7. *(i) Consider $\mathcal{B} = \{0, a, b, c, d, 1\}$ be the bounded implication basic algebra as Example 3.7. The internal states on $\mathcal{B}$ are given in the table below:*

x	0	a	b	c	d	1
$\sigma_1(x)$	0	b	b	d	d	1
$\sigma_2(x)$	0	a	b	c	d	1
$\sigma_3(x)$	1	1	1	1	1	1

(ii) Let $\mathcal{B} = \{0, a, b, c, 1\}$ be the lattice whose Hasse diagram is shown in Figure 4, and the operation $\rightarrow$ is defined by Table 4. Routine calculations show that $(\mathcal{B}; \rightarrow)$ is an implication basic algebra with least element 0. Hence $\mathcal{B}$ with the operations $\sim$ and $\oplus$ defined by Table 4(b) is a basic algebra. One can easily check that there are just trivial internal states $Id_{\mathcal{B}}, 1_{\mathcal{B}}$ on $\mathcal{B}$.

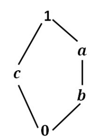

Figure 4

((a)) $(\mathcal{B}; \rightarrow)$

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	a	c	1
b	a	1	1	c	1
c	c	a	b	1	1
1	0	a	b	c	1

((b)) $(\mathcal{B}; \oplus, \sim)$

x	$\sim x$
0	1
a	b
b	a
c	c
1	0

$\oplus$	0	a	b	c	1
0	0	a	b	c	1
a	a	1	1	c	1
b	b	1	a	c	1
c	c	a	b	1	1
1	1	1	1	1	1

Table 4

Theorem 5.8. *Let σ be an internal state on $\mathcal{IB}$. If s is a Bosbach state on $\sigma(\mathcal{IB})$, then the mapping $s_\sigma : \mathcal{IB} \longrightarrow [0,1]$ which for any $x, y \in \mathcal{IB}$ is defined by $s_\sigma(x) = s(\sigma(x))$ is a Bosbach state on $\mathcal{IB}$.*

Proof. It is obvious that $s_\sigma(1) = 1$. Let $x, y \in \mathcal{IB}$. Since s is a Bosbach state, we have

$$s(\sigma(x \vee y)) + s(\sigma(x \vee y) \rightarrow \sigma(x)) = s(\sigma(x)) + s(\sigma(x) \rightarrow \sigma(x \vee y)).$$

Since $\sigma(y) \leq \sigma(x \vee y)$ and $\sigma(y) \rightarrow \sigma(y \vee x) = 1 = \sigma(x) \rightarrow \sigma(x \vee y)$, we have

$$s(\sigma(x \vee y)) + s(\sigma(x \vee y) \rightarrow \sigma(y)) = s(\sigma(y)) + s(\sigma(y) \rightarrow \sigma(x \vee y)) = s(\sigma(y)) + s(\sigma(x) \rightarrow \sigma(x \vee y)).$$

Also, according to (IS2) and (BS2), we have

$$s(\sigma(x \rightarrow y)) = s(\sigma(x \vee y) \rightarrow \sigma(y)) = s(\sigma(y)) + s(\sigma(x) \rightarrow \sigma(x \vee y)) - s(\sigma(x \vee y)),$$

and

$$s(\sigma(y \rightarrow x)) = s(\sigma(x \vee y) \rightarrow \sigma(x)) = s(\sigma(x)) + s(\sigma(x) \rightarrow \sigma(x \vee y)) - s(\sigma(x \vee y)).$$

Therefore,

$$\begin{aligned} s_\sigma(x) + s_\sigma(x \rightarrow y) &= s(\sigma(x)) + s(\sigma(x \rightarrow y)) = s(\sigma(x)) + s(\sigma(x) \rightarrow \sigma(x \vee y)) - s(\sigma(x \vee y)) + s(\sigma(y)) \\ &= s(\sigma(y)) + s(\sigma(y \rightarrow x)) = s_\sigma(y) + s_\sigma(y \rightarrow x). \end{aligned}$$

□

Example 5.9. *Consider the implication basic algebra $\mathcal{IB}$ from Example 5.5 with the internal state σ_1 defined by $\sigma_1(c) = a$ and $\sigma_1(x) = x$, for all $x \in \{a, b, d, 1\}$. It can be calculated that the map $s : \sigma_1(\mathcal{IB}) \longrightarrow [0, 1]$ which is defined by $s(a) = s(d) = 3/4$, $s(b) = 1/2$ and $s(1) = 1$ is a Bosbach state on $\sigma_1(\mathcal{IB})$. Then by Theorem 5.8, the mapping $s_{\sigma_1} : \mathcal{IB} \longrightarrow [0, 1]$ which is defined by*

$$s_{\sigma_1} = s(\sigma_1) = \begin{cases} 3/4, & x \in \{a, c, d\} \\ 1/2, & x = b \\ 1, & x = 1 \end{cases},$$

is a Bosbach state on $\mathcal{IB}$.

Theorem 5.10. *Let σ be an internal state on $\mathcal{B}$ such that $\sigma(0) = 0$. If s is a Riečan state on $\sigma(\mathcal{B})$, then the mapping $s_\sigma : \mathcal{B} \longrightarrow [0, 1]$ which for any $x, y \in \mathcal{IB}$, is defined by $s_\sigma(x) = s(\sigma(x))$, is a Riečan state on $\mathcal{B}$.*

Proof. It is obvious that $s_\sigma(1) = s(\sigma(1)) = 1$. Let $x, y \in \mathcal{B}$ such that $x \leq \sim y$. By Proposition 5.6(iii), $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$. Hence

$$s_\sigma(x \oplus y) = s(\sigma(x \oplus y)) = s(\sigma(x) \oplus \sigma(y)) = s(\sigma(x)) + s(\sigma(y)) = s_\sigma(x) + s_\sigma(y).$$

Therefore, s_σ is an Riečan state on $\mathcal{B}$. □

Example 5.11. *Let $\mathcal{B}$ be the implication basic algebra from Example 5.7 with the internal state σ_1 which is defined by $\sigma_1(a) = \sigma_1(b) = b$, $\sigma_1(c) = \sigma_1(d) = d$ and $\sigma_1(x) = x$, for all $x \in \{0, 1\}$. It is not difficult to show that the mapping $s : \sigma_1(\mathcal{B}) \longrightarrow [0, 1]$ defined by $s(0) = 0$, $s(b) = 1/3$, $s(d) = 2/3$ and $s(1) = 1$ is a Riečan state on $\sigma_1(\mathcal{B})$ and the mapping $s_{\sigma_1} : \mathcal{B} \longrightarrow [0, 1]$ defined by*

$$s_{\sigma_1} = s(\sigma_1) = \begin{cases} x, & x \in \{0, 1\} \\ 1/3, & x \in \{a, b\} \\ 2/3, & x \in \{c, d\} \end{cases},$$

is a Riečan state on $\mathcal{B}$.

In the following, we study the notions of IS-filter and IS-prefilter on implication basic algebras by using internal states.

Definition 5.12. *Let σ be an internal state on $\mathcal{IB}$. A nonempty subset $F \subseteq \mathcal{IB}$ is called an IS-(pre)filter of $(\mathcal{IB}, \sigma)$, if F is a (pre)filter of $\mathcal{IB}$ and $x \in F$ implies $\sigma(x) \in F$. $(\mathcal{IB}, \sigma)$ is called a simple state if it has only two trivial IS-filters $\{1\}$ and $\mathcal{IB}$.*

We denote the set of all IS-filters of $(\mathcal{IB}, \sigma)$ by $\mathcal{ISF}(\mathcal{IB}, \sigma)$.

Example 5.13. *(i) Let $\mathcal{IB}$ be the implication basic algebra from Example 5.5 with internal state σ_1 which is defined by $\sigma_1(a) = \sigma_1(c) = a$ and $\sigma_1(x) = x$, for any $x \in \{b, d, 1\}$. Routine calculations show that*

$$\{\{a, 1\}, \{a, c, 1\}, \{d, 1\}, \{c, d, 1\}, \{a, b, d, 1\}, \{1\}, \mathcal{IB}\},$$

is the set of all filters on $\mathcal{IB}$. One can easily check that

$$\{\{a, 1\}, \{a, c, 1\}, \{d, 1\}, \{a, b, d, 1\}, \{1\}, \mathcal{IB}\},$$

is the set of all IS-filters of $(\mathcal{IB}, \sigma_1)$ and $\{c, d, 1\}$ is not an IS-filter of $(\mathcal{IB}, \sigma_1)$.

(ii) Let $\mathcal{B} = \{0, a, b, c, d, e, 1\}$ be the lattice whose Hasse diagram is shown in Figure 5. Then $\mathcal{B}$ with the operation $\rightarrow$ defined by Table 5 is an implication basic algebra with the least element 0. It is not difficult to check that the set of all prefilters and filters of $\mathcal{B}$ are $\{\{b, d, 1\}, \{e, 1\}\}$ and $\{\{1\}, \mathcal{B}\}$, respectively. Also, the internal states on $\mathcal{B}$ are given by the following table:

x	0	a	b	c	d	e	1
$\sigma_1(x)$	0	a	0	a	a	1	1
$\sigma_2(x)$	0	a	1	0	1	0	1
$\sigma_3(x)$	b	d	b	d	d	1	1
$\sigma_4(x)$	0	a	b	c	d	e	1
$\sigma_5(x)$	1	1	1	1	1	1	1

It is easy to see that IS-prefilters of $(\mathcal{B},\sigma_1),(\mathcal{B},\sigma_2),(\mathcal{B},\sigma_3)$ are $\{\{e,1\}\}$, $\{\{b,d,1\}\}$ and $\{\{e,1\},\{b,d,1\}\}$, respectively.

$\rightarrow$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	a	1	b	c	d	e	1
b	e	a	1	e	1	e	1
c	d	a	d	1	1	1	1
d	c	a	d	e	1	e	1
e	b	a	b	d	d	1	1
1	0	a	b	c	d	e	1

Table 5: $(\mathcal{B};\rightarrow)$

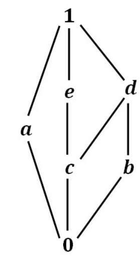

Figure 5

Theorem 5.14. *For $(\mathcal{IB},\sigma)$, the following conditions hold:*
(i) *If F is an IS-filter of $(\mathcal{IB},\sigma)$, then $\sigma(F)$ is a filter of $\sigma(\mathcal{IB})$.*
(ii) *If F is a filter of $\sigma(\mathcal{IB})$, then $\sigma^{-1}(F)$ is an IS-prefilter of $(\mathcal{IB},\sigma)$.*

Proof. (i) Let F be an IS-filter of $(\mathcal{IB},\sigma)$. We claim that $\sigma(F)=F\cap\sigma(\mathcal{IB})$. Let $y\in\sigma(F)$. Then there exists $x\in F$ such that $\sigma(x)=y$ and so $y\in\sigma(\mathcal{IB})$. Since F is an IS-filter of $\mathcal{IB}$ and $x\in F$, we get that $\sigma(x)=y\in F$. Hence $y\in F\cap\sigma(\mathcal{IB})$. Now, suppose that $y\in F\cap\sigma(\mathcal{IB})$. Since $y\in\sigma(\mathcal{IB})$, by Proposition 5.3(iv), we have $\sigma(y)=y$ and so $y\in\sigma(F)$. Hence, $\sigma(F)=F\cap\sigma(\mathcal{IB})$. Now, we show that $\sigma(F)$ is a filter of $\sigma(\mathcal{IB})$.
(F1) It is obvious that $1\in\sigma(F)$. Let $x,x\rightarrow y\in\sigma(F)\subseteq F$, for any $x,y\in\sigma(\mathcal{IB})$. Since F is a filter of $\mathcal{IB}$, $y\in F$ and so

$$y\in F\cap\sigma(\mathcal{IB})=\sigma(F).$$

(F2) Let $x\rightarrow y\in\sigma(F)\subseteq F$ and $y\leq x$, for $x,y\in\sigma(\mathcal{IB})$. Let $z\in\sigma(\mathcal{IB})$. Since F is a filter of $\mathcal{IB}$, we have $(y\rightarrow z)\rightarrow(x\rightarrow z)\in F$. Also, by Proposition 5.3(v), for all $x,y,z\in\sigma(\mathcal{IB})$, we have $(y\rightarrow z)\rightarrow(x\rightarrow z)\in\sigma(\mathcal{IB})$ and so

$$(y\rightarrow z)\rightarrow(x\rightarrow z)\in F\cap\sigma(\mathcal{IB})=\sigma(F).$$

(F3) Let $x\rightarrow y\in\sigma(F)\subseteq F$ and $y\rightarrow x\in\sigma(F)\subseteq F$, for any $x,y\in\sigma(\mathcal{IB})$. Since F is a filter of $\mathcal{IB}$, for $z\in\sigma(\mathcal{IB})$, we have $(z\rightarrow x)\rightarrow(z\rightarrow y)\in F$. Also, by Proposition 5.3(v), we have $(z\rightarrow x)\rightarrow(z\rightarrow y)\in\sigma(\mathcal{IB})$, for all $x,y,z\in\sigma(\mathcal{IB})$, and so

$$(z\rightarrow x)\rightarrow(z\rightarrow y)\in F\cap\sigma(\mathcal{IB})=\sigma(F).$$

(ii) It is obvious that $1\in\sigma^{-1}(F)$. Let $x,x\rightarrow y\in\sigma^{-1}(F)$, for $x,y\in\mathcal{IB}$. Then $\sigma(x),\sigma(x\rightarrow y)\in F$. By Proposition 5.3(iii),(v), $\sigma(x\rightarrow y)\leq\sigma(x)\rightarrow\sigma(y)\in\sigma(\mathcal{IB})$. Since F is a filter of $\sigma(\mathcal{IB})$, we have $\sigma(x)\rightarrow\sigma(y)\in F$. Also $\sigma(x)\in F$ implies $\sigma(y)\in F$, i.e. $y\in\sigma^{-1}(F)$. Then $\sigma^{-1}(F)$ is an IS-prefilter of $(\mathcal{IB},\sigma)$. □

Theorem 5.15. *Let $\sigma(0)=0$ on $(\mathcal{B},\sigma)$. If σ is faithful and $\sigma(\mathcal{B})$ is simple, then $(\mathcal{B},\sigma)$ is a simple state.*

Proof. Let $F\neq 1$ be an IS-filter of $(\mathcal{B},\sigma)$. By Theorem 5.14(i), $\sigma(F)$ is a filter of $\sigma(\mathcal{B})$. Since $\sigma(\mathcal{B})$ is simple, $\sigma(F)=\{1\}$ or $\sigma(F)=\sigma(\mathcal{B})$. By the faithfulness property of σ and $F\neq 1$, we get $\sigma(F)=\sigma(\mathcal{B})$ and $0\in\sigma(F)$. Now, by Theorem 5.14(i), we have $\sigma(F)=F\cap\sigma(\mathcal{B})$. So $0\in F$ and this implies that $\mathcal{B}=F$. Therefore, $(\mathcal{B},\sigma)$ is a simple state. □

The following example shows that the converse of Theorem 5.15 is not valid in general.

Example 5.16. *Let $\mathcal{B}=\{0,a,b,c,d,e,1\}$ be the implication basic algebra with the least element 0 from Example 5.13(ii). It is easy to check that $(\mathcal{B},\sigma_3)$ has two IS-filters, which are $\{1\}$ and $\mathcal{B}$, i.e., $(\mathcal{B},\sigma_3)$ is a simple state. But $\sigma_3(e)=\sigma_3(1)=1$ and σ_3 is not faithful.*

Definition 5.17. *Relation θ on $\mathcal{IB}$ is called an IS-congruence relation on $(\mathcal{IB},\sigma)$, if θ is a congruence relation on $\mathcal{IB}$ and $x\,\theta\,y$ implies $\sigma(x)\,\theta\,\sigma(y)$. We denote the set of all IS-congruences on $(\mathcal{IB},\sigma)$ with $\mathcal{ISC}on(\mathcal{IB},\sigma)$.*

Proposition 5.18. *Let θ and ϕ be two IS-congruences on $(\mathcal{IB},\sigma)$. Then for any $x,y\in\mathcal{IB}$, the following statements hold:*
(i) *$[1]_\theta$ is an IS-filter of $(\mathcal{IB},\sigma)$.*
(ii) *$x\,\theta\,y$ if and only if $(x\rightarrow y)\,\theta\,1$.*
(iii) *$[1]_\theta=\{1\}$ if and only if θ is a trivial congruence.*
(iv) *$[1]_\theta=[1]_\phi$ implies $\theta=\phi$.*

Proof. (i) By Theorem 2.6, $[1]_\theta$ is a filter of $\mathcal{IB}$. Let $x \in [1]_\theta$ and so $x\,\theta 1$. Since $\theta \in \mathcal{ISC}on(\mathcal{IB},\sigma)$, we have $\sigma(x)\,\theta\,\sigma(1) = 1$ and $\sigma(x) \in [1]_\theta$. Hence $[1]_\theta$ is an IS-filter of $\mathcal{IB}$.
(ii) Let $x\,\theta\,y$. Since $y\,\theta\,y$, by congruence property, $x \to y\,\theta\,y \to y = 1$.
(iii) It is obvious.
(iv) This is straightforward by (ii). □

Theorem 5.19. *There exists a one-to-one correspondence between $\mathcal{ISF}(\mathcal{IB},\sigma)$ and $\mathcal{ISC}on(\mathcal{IB},\sigma)$.*

Proof. Let $F \in \mathcal{ISF}(\mathcal{IB},\sigma)$. Then by definition of IS-filter, F is a filter of $\mathcal{IB}$. We define $f : ISF(\mathcal{IB},\sigma) \longrightarrow \mathcal{ISC}on(\mathcal{IB},\sigma)$ by $f(F) = \theta_F$. By Theorem 2.5, θ_F is a congruence on $\mathcal{IB}$. Let $x\,\theta_F y$, for $x, y \in \mathcal{IB}$. Then $x \to y \in F$ and $y \to x \in F$. Since F is an IS-filter of $\mathcal{IB}$, we get $\sigma(x \to y) \in F$ and $\sigma(y \to x) \in F$. By Proposition 5.3(iii), we get $\sigma(x) \to \sigma(y) \in F$ and $\sigma(y) \to \sigma(x) \in F$. So $\sigma(x)\,\theta_F\,\sigma(y)$. Hence $\theta_F \in \mathcal{ISC}on(\mathcal{IB},\sigma)$ and f is well-defined.

Now, we prove that f is an onto map. Let $\phi \in \mathcal{ISC}on(\mathcal{IB},\sigma)$. Then by Proposition 5.18(i), $F = [1]_\phi$ is an IS-filter of $(\mathcal{IB},\sigma)$. By Theorem 2.5 and Proposition 5.18(ii), $x\,\theta_F y$ if and only if $x \to y \in F = [1]_\phi$ and $y \to x \in F = [1]_\phi$ if and only if $x\,\phi\,y$ and $y\,\phi\,x$. Hence $\phi = \theta_{[1]_\phi} = f([1]_\phi)$ and so f is onto.

Finally, we prove that f is a one-to-one map. Consider $F_1, F_2 \in ISF(\mathcal{IB},\sigma)$ such that $f(F_1) = f(F_2)$. So $\theta_{F_1} = \theta_{F_2}$. By Theorem 2.5, $F_1 = [1]_{\theta_{F_1}} = [1]_{\theta_{F_2}} = F_2$. Thus, f is a one-to-one map. Therefore, there exists a one-to-one correspondence between $\mathcal{ISF}(\mathcal{IB},\sigma)$ and $\mathcal{ISC}on(\mathcal{IB},\sigma)$. □

The following example illustrates Theorem 5.19. Also, we show that for an internal state σ, $\sigma(\mathcal{IB})$ is a subalgebra of $\mathcal{IB}$ and if F is an IS-filter of $\mathcal{IB}$, then $\sigma(F) = F \cap \sigma(IB)$.

Example 5.20. *Let $\mathcal{IB}$ be the implication basic algebra from Example 5.5, with internal state σ_2 is defined by $\sigma_2(c) = b$ and $\sigma_2(x) = x$, for all $x \in \{a, b, d, 1\}$. Routine calculations show that $\{F_i \mid i = 1, ..., 7\}$ is the set of all filters on $\mathcal{IB}$, such that $F_1 = \{a, 1\}$, $F_2 = \{d, 1\}$, $F_3 = \{a, b, d, 1\}$, $F_4 = \{1\}$, $F_5 = \mathcal{IB}$, $F_6 = \{a, c, 1\}$ and $F_7 = \{c, d, 1\}$. One can easily check that $\{F_i \mid i = 1, ..., 5\}$ is the set of all IS-filters of $(\mathcal{IB},\sigma_2)$.*
By Proposition 5.3(v), $\sigma_2(\mathcal{IB})$ is a subalgebra of $\mathcal{IB}$ whose Hasse diagram is shown in Figure 6 and the operation $\to$ is defined by Table 6. It is easy to see that the set of filters of $\sigma_2(\mathcal{IB})$ is $\{\{a, 1\}, \{d, 1\}, \{a, b, d, 1\}, \{1\}\}$ (by Theorem 5.14, $\sigma_2(F_i) = F_i \cap \sigma_2(IB)$ for $i = 1, ..., 5$ are filters of $\sigma_2(IB)$).

$\to$	a	b	d	1
a	1	d	d	1
b	1	1	1	1
d	a	a	1	1
1	a	b	d	1

Table 6: $(\sigma_2(\mathcal{IB}); \to)$

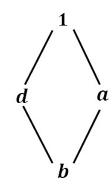

Figure 6

By Theorem 2.6, if ϕ is a congruence of $\mathcal{IB}$, then there exists a filter of $\mathcal{IB}$ such that $\phi = \theta_F$ and $(x, y) \in \theta_F$ if and only if $x \to y \in F$ and $y \to x \in F$. So, we have:

$$\theta_{F_i} = \begin{cases} \{\{a,1\},\{b,d\},\{c\}\}, & i=1 \\ \{\{d,1\},\{a,b,c\}\}, & i=2 \\ \{\{a,b,d,1\},\{c\}\}, & i=3 \\ \{\{1\},\{a\},\{b\},\{c\},\{d\}\}, & i=4 \\ \{\mathcal{IB}\}, & i=5 \\ \{\{a,c,1\},\{b,d\}\}, & i=6 \\ \{\{c,d,1\},\{a,b\}\}, & i=7 \end{cases} \quad , \qquad [1]_{\theta_{F_i}} = \begin{cases} \{a,1\}, & i=1 \\ \{d,1\}, & i=2 \\ \{a,b,d,1\}, & i=3 \\ \{1\}, & i=4 \\ \mathcal{IB}, & i=5 \\ \{a,c,1\}, & i=6 \\ \{c,d,1\}, & i=7 \end{cases}$$

Note that θ_{F_6} and θ_{F_7} are not IS-congruences because $(c, 1) \in \theta_{F_6}$ and $(c, d) \in \theta_{F_7}$ but $(\sigma_2(c), \sigma_2(1)) = (b, 1) \notin \theta_{F_6}$ and $(\sigma_2(c), \sigma_2(d)) = (b, d) \notin \theta_{F_7}$.
It is easy to see that there is a one-to-one correspondence between $\mathcal{ISF}(\mathcal{IB})$ and $\mathcal{ISC}on(\mathcal{IB})$ by the function f : $\mathcal{ISF}(\mathcal{IB},\sigma_2) \longrightarrow \mathcal{ISC}on(\mathcal{IB},\sigma_2)$ which is defined by $f(F_i) = \theta_{F_i}$ for $i = 1, ..., 5$.

Theorem 5.21. *If $F \in ISF(\mathcal{IB},\sigma)$, then $\mu_\sigma : \mathcal{IB}/F \longrightarrow \mathcal{IB}/F$ is an internal state on $\mathcal{IB}/F$ with $\mu_\sigma(x/F) = \sigma(x)/F$.*

Proof. Let $x/F = y/F$. Then $x \to y \in F$ and $y \to x \in F$ and so $\sigma(x \to y) \in F, \sigma(y \to x) \in F$. By Proposition 5.3(iii), we have $\sigma(x) \to \sigma(y) \in F$ and $\sigma(y) \to \sigma(x) \in F$, thus $\sigma(x)/F = \sigma(y)/F$. Hence $\mu_\sigma(x) = \mu_\sigma(y)$ and μ_σ is well-defined. Now, we prove that μ_σ is an internal state on $\mathcal{IB}/F$.

(IS1) Let $x/F \leq y/F$. Same as previous part, we get $\sigma(x)/F \leq \sigma(y)/F$ and $\mu_\sigma(x) \leq \mu_\sigma(y)$.
(IS2) Let $x/F, y/F \in \mathcal{IB}/F$. Then

$$\begin{aligned}\mu_\sigma(x/F \to y/F) = \mu_\sigma((x \to y)/F) &= \sigma(x \to y)/F = (\sigma(x \vee y) \to \sigma(y))/F \\ &= \sigma(x \vee y)/F \to \sigma(y)/F = \mu_\sigma(x/F \vee y/F) \to \mu_\sigma(y/F).\end{aligned}$$

(IS3) Let $x/F, y/F \in \mathcal{IB}/F$. Then

$$\begin{aligned}\mu_\sigma(\mu_\sigma(x/F) \vee \mu_\sigma(y/F)) &= \mu_\sigma((\sigma(x) \vee \sigma(y))/F) = \sigma((\sigma(x) \vee \sigma(y))/F = (\sigma(x) \vee \sigma(y))/F \\ &= \sigma(x)/F \vee \sigma(y)/F = \mu_\sigma(x/F) \vee \mu_\sigma(y/F).\end{aligned}$$

(IS4) Let $x/F, y/F \in \mathcal{IB}/F$. Then

$$\begin{aligned}\mu_\sigma(\mu_\sigma(x/F) \to \mu_\sigma(y/F)) &= \mu_\sigma((\sigma(x) \to \sigma(y))/F) = \sigma((\sigma(x) \to \sigma(y))/F = (\sigma(x) \to \sigma(y))/F \\ &= \sigma(x)/F \to \sigma(y)/F = \mu_\sigma(x/F) \to \mu_\sigma(y/F).\end{aligned}$$

Then μ_σ is an internal state on $\mathcal{IB}$. □

6 Generalized States

In this section, we introduce the new notion of generalized state map by extending the codomain of a state operator to a universal setting X. Moreover, according to the structure of X, we give two special types of generalized state maps, that is, generalized state-morphisms and generalized internal states. Also, we emphasize the relevances between generalized state maps, state-morphisms, and internal states on implication basic algebras and get some important results.

Definition 6.1. *Let $(\mathcal{IB}_1; \to_1)$ and $(\mathcal{IB}_2; \to_2)$ be two implication basic algebras. A map $gs : \mathcal{IB}_1 \longrightarrow \mathcal{IB}_2$ is called a generalized state from $\mathcal{IB}_1$ to $\mathcal{IB}_2$ (or brieflyG-state) if it satisfies the following conditions for all $x, y \in \mathcal{IB}_1$:*
$(GS1)$ If $x_1 \leq_1 x_2$, then $gs(x_1) \leq_2 gs(x_2)$,
$(GS2)$ $gs(x \to_1 y) = gs(x \vee_1 y) \to_2 gs(y)$,
$(GS3)$ $gs(x) \vee_2 gs(y) \in gs(\mathcal{IB}_1)$,
$(GS4)$ $gs(x) \to_2 gs(y) \in gs(\mathcal{IB}_1)$.
Moreover, we give two special types of generalized state maps from $\mathcal{IB}_1$ to $\mathcal{IB}_2$.
(1) If $\mathcal{IB}_2 = ([0,1]_{MV}; \rightharpoonup)$, then gs is called a generalized state-morphism (or brieflyG-state morphism) from $\mathcal{IB}_1$ to $[0,1]_{MV}$,
(2) If $\mathcal{IB}_2 = \mathcal{IB}_1$, then gs is called a generalized internal state (or brieflyGI-state) from $\mathcal{IB}_1$ to $\mathcal{IB}_1$.
Note that the standard MV-algebra $([0,1]_{MV}; \rightharpoonup)$ in Section 4 is an implication basic algebra.

Example 6.2. *Let $(\mathcal{IB}_1; \to_1)$ and $(\mathcal{IB}_2; \to_2)$ be two implication basic algebras. A map $gs : \mathcal{IB}_1 \longrightarrow \mathcal{IB}_2$ such that for all $x \in \mathcal{IB}_1$, $gs(x) = 1_2$ is a generalized state from $\mathcal{IB}_1$ to $\mathcal{IB}_2$, which is called a trivial generalized state.*

Example 6.3. *Let $\mathcal{IB}_1 = \{a_1, b_1, c_1, d_1, 1_1\}$ and $\mathcal{IB}_2 = \{a_2, b_2, c_2, d_2, 1_1\}$ be the join-semilattices whose Hasse diagrams are below (see Figures 7 and 8). We define the operations $\to_1$ and $\to_2$ on $\mathcal{IB}_1$ and $\mathcal{IB}_2$ as shown in Table 7.*

((a)) $(\mathcal{IB}_1; \to_1)$

$\to_1$	a_1	b_1	c_1	d_1	1_1
a_1	1_1	1_1	b_1	b_1	1_1
b_1	b_1	1_1	b_1	a_1	1_1
c_1	b_1	1_1	1_1	a_1	1_1
d_1	1_1	1_1	b_1	1_1	1_1
1_1	a_1	b_1	c_1	d_1	1_1

((b)) $(\mathcal{IB}_2; \to_2)$

$\to_2$	a_2	b_2	c_2	d_2	1_2
a_2	1_2	b_2	c_2	a_2	1_2
b_2	a_2	1_2	b_2	b_2	1_2
c_2	a_2	1_2	1_2	b_2	1_2
d_2	1_2	1_2	b_2	1_2	1_2
1_2	a_2	b_2	c_2	d_2	1_2

Table 7: Implication basic algebras

Routine calculations show that $(\mathcal{IB}_1; \to_1)$ and $(\mathcal{IB}_2; \to_2)$ are implication basic algebras. It is not difficult to see that there exist just trivial generalized states from $\mathcal{IB}_1$ to $\mathcal{IB}_2$.
Also, we can see that there exist five generalized states from $\mathcal{IB}_2$ to $\mathcal{IB}_1$ which are shown in the following table:

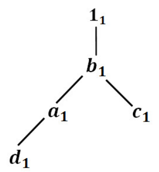

Figure 7: $\mathcal{IB}_1$

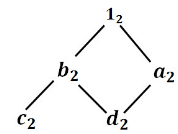

Figure 8: $\mathcal{IB}_2$

x	a_2	b_2	c_2	d_2	1_2
$gs_1(x)$	b_1	b_1	a_1	a_1	1_1
$gs_2(x)$	b_1	b_1	a_1	c_1	1_1
$gs_3(x)$	b_1	b_1	c_1	a_1	1_1
$gs_4(x)$	b_1	b_1	c_1	c_1	1_1
$gs_5(x)$	1_1	1_1	1_1	1_1	1_1

In the following example we show some generalized internal states on $\mathcal{IB}$.

Example 6.4. *Let $\mathcal{IB}$ be the implication basic algebra from Example 5.5. Some of GI-state maps on $\mathcal{IB}$ are given by the following table:*

x	a	b	c	d	1
$gs_1(x)$	a	a	d	1	1
$gs_2(x)$	c	c	a	1	1
$gs_3(x)$	c	c	d	1	1

Example 6.5. *Let $\mathcal{IB} = \{a, b, c, d, e, 1\}$ be the lattice whose Hasse diagram is below (see Figure 9). We define an operation $\to$ on $\mathcal{IB}$, as shown in Table 8. Routine calculations show that $(\mathcal{IB}; \to)$ is an implication basic algebra.*

$\to$	a	b	c	d	e	1
a	1	b	a	d	e	1
b	a	1	e	d	e	1
c	1	1	1	d	1	1
d	a	b	c	1	e	1
e	a	b	b	d	1	1
1	a	b	c	d	e	1

Table 8: $(\mathcal{IB}; \to)$

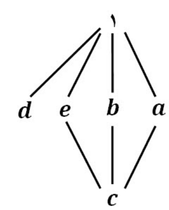

Figure 9

Some of G-state morphisms from $(\mathcal{IB}; \to)$ to $([0,1]_{MV}; \rightharpoonup)$ are given in the following table:

x	a	b	c	d	e	1
$gs_1(x)$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1	1
$gs_2(x)$	$\frac{3}{5}$	1	$\frac{1}{5}$	1	$\frac{1}{5}$	1
$gs_3(x)$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	0	$\frac{1}{5}$	1

Proposition 6.6. *Let $(\mathcal{IB}_1; \to_1)$ and $(\mathcal{IB}_2; \to_2)$ be two implication basic algebras and $gs : \mathcal{IB}_1 \longrightarrow \mathcal{IB}_2$ be a generalized state from $\mathcal{IB}_1$ to $\mathcal{IB}_2$. Then, for all $x, y \in \mathcal{IB}_1$, the following properties hold:*
(i) $gs(1_1) = 1_2$.
(ii) $gs(x \to_1 y) \leq_2 gs(x) \to_2 gs(y)$. *If x and y are comparable, then* $gs(x \to_1 y) = gs(x) \to_2 gs(y)$.
(iii) $gs(\mathcal{IB}_1)$ *is a subalgebra of* $\mathcal{IB}_2$.
(iv) $ker(gs) = \{x \in \mathcal{IB}_1 \mid gs(x) = 1_2\}$ *is a prefilter of* $\mathcal{IB}_1$.
(v) *If gs is faithful, then $x <_1 y$ implies $gs(x) <_2 gs(y)$.*

Proof. (i) By taking $x = y = 1$ in (GS2).
(ii) Let $x, y \in \mathcal{IB}_1$. We have $x \leq_1 x \vee_1 y$. Then by (GS1), $gs(x) \leq_2 gs(x \vee_1 y)$ and by (IBA_4), we have

$$gs(x \vee_1 y) \rightarrow_2 gs(y) \leq_2 gs(x) \rightarrow_2 gs(y).$$

Hence by (GS2), $gs(x \rightarrow_1 y) = gs(x \vee_1 y) \rightarrow_2 gs(y) \leq_2 gs(x) \rightarrow_2 gs(y)$.
Now, let $x, y \in \mathcal{IB}_1$ be comparable. If $x \leq_1 y$, then by (GS1), $gs(x) \leq_2 gs(y)$. So, we have

$$gs(x) \rightarrow_2 gs(y) = 1_2 = gs(1_1) = gs(x \rightarrow_1 y).$$

If $y \leq_1 x$, then by (GS2), $gs(x) \rightarrow_2 gs(y) = gs(x \vee_1 y) \rightarrow_2 gs(y) = gs(x \rightarrow_1 y)$.
(iii) From (i), (GS1) and (GS3)-(GS4), $gs(\mathcal{IB}_1)$ is closed under operators $\rightarrow_2$ and $\vee_2$. Hence $gs(\mathcal{IB}_1)$ is a subalgebra of $\mathcal{IB}_2$.
(iv) It is obvious that $1_1 \in ker(gs)$. If $x, x \rightarrow_1 y \in ker(gs)$, then $gs(x) = 1_2 = gs(x \rightarrow_1 y)$. By (ii), we have

$$1_2 = gs(x \rightarrow_1 y) \leq_2 gs(x) \rightarrow_2 gs(y) = 1_2 \rightarrow_2 gs(y) = gs(y).$$

Hence $gs(y) = 1_2$ and $y \in ker(gs)$.
(v) Let gs be faithful. If $x <_1 y$, then $gs(x) \leq_2 gs(y)$. Suppose $gs(x) = gs(y)$. By (ii), $gs(y \rightarrow_1 x) = gs(y) \rightarrow_2 gs(x) = 1_2$ and so $y \rightarrow_1 x \in ker(gs) = \{1_2\}$. Hence $y \rightarrow_1 x = 1_2$ and $y \leq_1 x$, which is a contradiction. □

In the following, we discuss the relationship between the generalized states and the states on implication basic algebras.

Theorem 6.7. *If $sm : \mathcal{IB} \longrightarrow [0,1]_{MV}$ is a state-morphism on $\mathcal{IB}$, then sm is a G-state morphism from $(\mathcal{IB}; \rightarrow)$ to $([0,1]_{MV}; \rightharpoonup)$.*

Proof. Let sm be a state-morphism on $\mathcal{IB}$ and $x, y \in \mathcal{IB}$. Then $sm(x \rightarrow y) = sm(x) \rightharpoonup sm(y)$ and by (IBA_3), $sm(x \rightarrow y) = sm((x \vee y) \rightarrow y) = sm(x \vee y) \rightharpoonup sm(y)$. Hence (GS4) and (GS2) hold. By Proposition 4.2(i), sm is a Bosbach state and then by Proposition 3.3(i), $(GS1)$ holds. Also, (GS3) follows from Proposition 4.2(ii). □

In the following example, we show that the converse of the above theorem is not valid in general.

Example 6.8. *Let $\mathcal{IB}$ be the implication basic algebra from Example 6.5. It is easy to see that gs_1 is a G-state morphism, but gs_1 is not a state-morphism on $\mathcal{IB}$ because*

$$gs_1(a) \rightharpoonup gs_1(b) = \frac{3}{5} \rightharpoonup \frac{1}{5} = \frac{3}{5} \neq \frac{1}{5} = gs_1(b) = gs_1(a \rightarrow b).$$

Theorem 6.9. *The map $gs : \mathcal{IB} \longrightarrow \mathcal{IB}$ is a GI-state and $gs(gs) = gs^2 = gs$, if and only if gs is an internal state on $\mathcal{IB}$.*

Proof. Let gs be a GI-state from $\mathcal{IB}$ to $\mathcal{IB}$ and $gs^2 = gs$. Then, by (GS4), there exists a $z \in \mathcal{IB}$ such that $gs(z) = gs(x) \rightarrow gs(y)$, for $x, y \in \mathcal{IB}$. We have $gs(gs(x) \rightarrow gs(y)) = gs(gs(z)) = gs(z) = gs(x) \rightarrow gs(y)$ and (IS4) holds. Similarly, we can prove (IS3).

Conversely, let gs be an internal state on $\mathcal{IB}$. Then, by Proposition 5.3(ii), $gs^2 = gs$. Also, by Proposition 5.3(v), $gs(\mathcal{IB})$ is a subalgebra of $\mathcal{IB}$, which implies (GS3) and (GS4). □

Example 6.10. *Let $\mathcal{IB}$ be the implication basic algebra from Example 6.4. It is easy to see that all of the GI-state maps in this example are not internal states on $\mathcal{IB}$ because $gs_i(gs_i(c)) \neq gs_i(c)$ for $i = 1, 2, 3$.*

7 Conclusions

In this paper, we introduced the notions of Bosbach, Riečan states and state-morphisms on implication basic algebras and investigated their properties. We proved that any Bosbach state coincides with a Riečan state on a bounded implication basic algebra. Also, we proved that sm is a state-morphism on $\mathcal{IB}$ if and only if sm is a Bosbach state and $sm(x \vee y) = sm(x) \vee sm(y)$. In the sequel, we applied the notion of internal state σ on an implication basic algebra $\mathcal{IB}$, $(\mathcal{IB}, \sigma)$. We proved that the combination of a Bosbach state and an internal state on an implication basic algebra $\mathcal{IB}$ is a Bosbach state on $\mathcal{IB}$ and a combination of a Riečan state and an internal state on a bounded implication basic algebra $\mathcal{B}$ is a Riečan state on $\mathcal{B}$. Moreover, we defined the notions of IS-(pre)filter and IS-congruence of $(\mathcal{IB}, \sigma)$, where σ is

an internal state on $\mathcal{IB}$, and studied their properties. We obtained there exists a one-to-one correspondence between IS-filters and IS-congruences of $(\mathcal{IB},\sigma)$. In addition, by using an IS-filter F of $(\mathcal{IB},\sigma)$, we constructed an internal state on $\mathcal{IB}/\mathcal{F}$. Finally, we introduced the new notion of generalized state map from an implication basic algebra $\mathcal{IB}_1$ to an arbitrary implication basic algebra $\mathcal{IB}_2$. We gave two types of individual generalized state maps, namely, generalized state-morphisms and generalized internal states. We verified that the generalized internal states are a generalization of internal states, and generalized state-morphisms are a generalization of state-morphisms on implication basic algebras. We showed that a generalized internal state gs is an internal state on implication basic algebra $\mathcal{IB}$ if $gs^2 = gs$.
In the next research, we will extend the types of generalized states (Bosbach states) on implication basic algebras. Also, we will work on some classes of state filters on state implication basic algebras.

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