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C^∞ L-fuzzy manifolds with L-gradation of openness and C^∞ LG-fuzzy mappings of them

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Abstract

In this paper, we generalize all of the fuzzy structures which we have discussed in [14] to L-fuzzy set theory, where $L =< L, \leq, \bigwedge, \bigvee, ' >$ denotes a complete distributive lattice with at least two elements. We define the concept of an LG-fuzzy topological space $(X, \mathfrak{T})$ which X is itself an L-fuzzy subset of a crisp set M and $\mathfrak{T}$ is an L-gradation of openness of L-fuzzy subsets of M which are less than or equal to X. Then we define C^∞ L-fuzzy manifolds with L-gradation of openness and C^∞ LG-fuzzy mappings of them such as LG-fuzzy immersions and LG-fuzzy imbeddings. We fuzzify the concept of the product manifolds with L-gradation of openness and define LG-fuzzy quotient manifolds when we have an equivalence relation on M and investigate the conditions of the existence of the quotient manifolds. We also introduce LG-fuzzy immersed, imbedded and regular submanifolds.

Keywords: C^∞ LG-fuzzy n-manifolds, C^∞ LG -fuzzy mappings, LG-fuzzy quotient manifolds, LG-fuzzy immersion, regular LG-fuzzy submanifolds.

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [21]. Then Chang [1] confined his attention to the more basic concepts of general topology and generalized them to fuzzy topological spaces. In his definition, fuzziness in the concept of openness of a fuzzy subset, is absent. In consequence of the development of fuzzy topology, many authors like Wong [19], Lowen [10] introduced various concepts of fuzzy topology. R. Lowen [11] suggested that the properties should be considered fuzzy, that is, one should be able to measure a degree to which a property holds. E. Lowen and R. Lowen [12] considered compactness degrees, and in [20], investigated measures of separation in [0,1]-topological spaces. In 1985, Shostak [15] gave a new definition of fuzzy topology by introducing a concept of gradation of openness of fuzzy subsets of X. Later, Chattopadhyay [2] et al. attempted to introduce a concept of gradation of openness of a fuzzy set of X by a map $\tau : I^X \to I$ satisfying three weaker conditions than [15] and later in [3] made a slight modification in their definition and rediscovered the Shostak's concept of fuzzy topology. Gregori [7] proved that each gradation of openness δ is the supremum (infimum) of a strictly increasing (decreasing) sequence of gradations of openness which are equivalent to δ. Stadler and Vicente [13] have introduced a new concept of fuzzy topological subspace over each fuzzy subset from the fuzzy topology δ, which coincides with the usual definition in the case that $\mu = X^Y$, $Y \subset X$. In [16], Shostak developed a theory of compactness degrees and connectedness degrees in [0,1]-fuzzy topological spaces, and in [17], brought up a theory of degrees of precompactness and completeness in the so-called Hutton fuzzy uniform spaces. In 2016 Ibedou [9] discussed graded fuzzy topological spaces. While all of the researches about the C^1 or C^∞ fuzzy manifolds, focused on a crisp set, in [14] and in this paper, we demonstrate the possibility of improving current definitions using a new method. In [14], we investigated some properties of a novel fuzzy topological space (X, τ), where X is itself a fuzzy subset of a crisp set M. Perhaps the most important generalization of the aforementioned structures in [14], is the consideration of lattice L beyond the unit interval $I = [0, 1]$. Let $L =< L, \leq, \bigwedge, \bigvee, ' >$ be a complete distributive lattice set with at least 2 elements; 0 is the bottom element and 1 is the top element of L. An L-fuzzy

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subset D of the crisp set M, in Goguen's sense [6], is a function $D:\ M \to L$ and is denoted by $D \in L^M$. In this manuscript, we define the concept of L-fuzzy topological space $(X, \mathfrak{T})$ with the L-gradation of openness, where X is an L-fuzzy subset of a crisp set M. We introduce C^∞ L-fuzzy manifolds $(X, \mathfrak{T})$ with L-gradation of openness, called C^∞ LG-fuzzy manifolds, with a different perception from [5] and [4] and obtain C^∞ n-premanifolds of them. We define C^∞ LG-fuzzy mappings of C^∞ LG-fuzzy manifolds and prove the LG-fuzzy rank theorem. Then we define and discuss LG-fuzzy immersions and LGP-fuzzy imbedding functions. We proceed to define the LG-fuzzy immersed, imbedded submanifolds as well as LG-fuzzy regular submanifolds, and then some theorems about the relations between them are deduced.

2 Preliminaries

Definition 2.1. *Let X be an L-fuzzy subset of M. Then any L-fuzzy subset of M which is less than or equal to X is called an L-fuzzy subset of X. We denote the set of all L-fuzzy subsets of X by L_X^M. If τ as a collection of L-fuzzy subsets of X, satisfies the following conditions, then (X, τ) is called an L-fuzzy topological space (L-fts):*

1) $X,\ \phi \in \tau$,
2) $\{A_i\}_{i \in I} \subseteq \tau \ \Rightarrow\ \bigcup_{i \in I} A_i \in \tau$,
3) $A, B \in \tau \ \Rightarrow\ A \cap B \in \tau$.

Example 2.2. *Let $M = \mathbb{R}^n$ and $X = 1$ be a constant L-fuzzy subset of M. Let $B(a, r, b)$ be an L-fuzzy subset that is equal to zero outside or on the sphere $B(a, r)$ and equal to the function b with values in L, inside $B(a, r)$. We call the L-fuzzy topology induced by*

$$\beta_{Ln} = \{B(a, r, b),\, a \in \mathbb{R}^n,\, r \in \mathbb{R}^+,\ b : B(a, r) \to L,\ \textit{is a function}\},$$

the L-fuzzy Euclidean topology of dimension n and denote it by $\tau_{_{Ln}}$. Therefore we have the L-fuzzy Euclidean topological space $(1_{\mathbb{R}^n},\ \tau_{_{Ln}})$.

Definition 2.3. *Let $\mathfrak{T} : L_X^M \to L$, be a mapping satisfying:*

i) $\mathfrak{T}(X) = \mathfrak{T}(\tilde{0}) = 1$,
ii) $\mathfrak{T}(A \cap B) \geq \mathfrak{T}(A) \wedge \mathfrak{T}(B)$,
iii) $\mathfrak{T}(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathfrak{T}(A_j)$.

Then $\mathfrak{T}$ is called an L-gradation of openness on X and $(X,\ \mathfrak{T})$ is called an LG-fuzzy topological space (L-gfts).
Let $x \in M$ and $A \in L_X^M$. When we write $x \in A$, we mean $x \in suppA$.

Example 2.4. *Let $M = \mathbb{R}^n$ and $X = 1$ be a constant L-fuzzy subset of M. As three useful examples, we define*

$$\mathfrak{T}_{Ln}\ :\ L_X^M \to L, \qquad \mathfrak{T}_{Ln}(B) = \begin{cases} 1 & B \in \tau_{_{Ln}}, \\ 0 & \textit{elsewhere.} \end{cases} \tag{1}$$

and

$$\mathfrak{T}_{Lsup}\ :\ L_X^M \to L, \qquad \mathfrak{T}_{Lsup}(B) = \begin{cases} 1 & B = \tilde{0}, \\ sup\{B(x) : x \in M\} & \tilde{0} \neq B \in \tau_{_{Ln}}, \\ 0 & \textit{elsewhere,} \end{cases} \tag{2}$$

If we set "inf" instead of "sup" in the above definition, then we have L-gradation of openness $\mathfrak{T}_{Linf}$.
Let $\mathfrak{T}_{Ln}$ be any L-gradation of openness on $1_{\mathbb{R}^n}$, such that $supp\mathfrak{T} = \tau_{_{Ln}}$, then we call $(1_{\mathbb{R}^n},\ \mathfrak{T}_{Ln})$ the LG-fuzzy Euclidean topological space.

Definition 2.5. *Let $(X, \mathfrak{T})$ be an L-gfts. Set $supp\mathfrak{T} = \{A \in L_X^M\ :\ \mathfrak{T}(A) > 0\}$, then A is called an LG-open subset of X if $A \in supp\mathfrak{T}$. Furthermore*

1) *Suppose $x \in X$ and $V \in L_X^M$. If there exists an LG-open subset U of X such that $U(x) = V(x)$ and $U \leq V$, then V is called an LG-neighborhood of x in X. We denote the set of all LG-neighborhoods of x in X by $LGN(x)$.*

2) If for all $x, y \in X$, $x \neq y$, there exist two LG-neighborhoods $U_x \in LGN(x)$, $U_y \in LGN(y)$ such that $U_x \cap U_y = 0$. Then $(X, \mathfrak{T})$ is called a Hausdorff L-gfts.

3) For each L-fuzzy subset A of X and any $U \subset suppX$, we define the L-fuzzy subset $\chi_{U,A}$ of X by:

$$\chi_{U,A}(z) = \begin{cases} A(z) & z \in U, \\ 0 & elsewhere. \end{cases}$$

From now on, we write χ_U instead of $\chi_{U,X}$.

4) A is called an LG-closed subset of X if $X - A \in supp\mathfrak{T}$.

5) Let Z be an LG-open subset of X. Define $\mathfrak{T}_Z : L_Z^M \to L$, by $\mathfrak{T}_Z(A) = \mathfrak{T}(A)$. Then $(Z, \mathfrak{T}_Z)$ is called an LG-fuzzy topological subspace of X (L-gtfss).

Definition 2.6. *If $\mathfrak{C} : L_X^M \to L$, satisfies the following conditions:*

i) $\mathfrak{C}(X) = \mathfrak{C}(\tilde{0}) = 1$.

ii) $\mathfrak{C}(A \cup B) \geq \mathfrak{C}(A) \wedge \mathfrak{C}(B)$.

iii) $\mathfrak{C}(\bigcap_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathfrak{C}(A_j)$.

Then $\mathfrak{C}$ is called an L-gradation of closedness on X.

Proposition 2.7. *Let $\mathfrak{C}$ and $\mathfrak{T}$ be L-gradations of closedness and openness respectively on X. Then*

i) The mapping $\mathfrak{T}_{\mathfrak{C}} : L_X^M \to L$, defined by $\mathfrak{T}_{\mathfrak{C}}(A) = \mathfrak{C}(X - A)$, is an L-gradation of openness on X, where $(X - A)$ is an L-fuzzy subset of M defined by $(X - A)(p) = X(p) - A(p)$.

ii) The mapping $\mathfrak{C}_{\mathfrak{T}} : L_X^M \to L$, defined by $\mathfrak{C}_{\mathfrak{T}}(A) = \mathfrak{T}(X - A)$, is an L-gradation of closedness on X.

iii) We have $\mathfrak{C}_{\mathfrak{T}_{\mathfrak{C}}} = \mathfrak{C}$, $\mathfrak{T}_{\mathfrak{C}_{\mathfrak{T}}} = \mathfrak{T}$.

The proof is straightforward.

Proposition 2.8. *Let $\mathfrak{M}_{\mathfrak{T}}(X)$ be the set of all L-gradations of openness on X. We write $\mathfrak{T}_1 \leq \mathfrak{T}_2$, if we have $\mathfrak{T}_1(A) \leq \mathfrak{T}_2(A)$, $\forall A \in L_X^M$. Then $(\mathfrak{M}_{\mathfrak{T}}(X), \leq)$ is a complete lattice.*

Proof. It is clear that the relation $\leq$ between the functions from L_X^M to L, is an equivalence relation. Therefore $(\mathfrak{M}_{\mathfrak{T}}(X), \leq)$ is a partially ordered set. Further we define two mappings $\mathfrak{T}_0, \mathfrak{T}_1 : L_X^M \to L$, by

$$\mathfrak{T}_0(\tilde{0}) = \mathfrak{T}_0(X) = 1, \ \ \mathfrak{T}_0(A) = 0, \ \ \forall A \in L_X^M - \{\tilde{0}, X\}, \ \ \mathfrak{T}_1(A) = 1, \ \ \forall A \in L_X^M.$$

Then $\mathfrak{T}_0, \mathfrak{T}_1$ are two L-gradations of openness on X and we have:

$$\mathfrak{T}_0(A) \leq \mathfrak{T}(A) \leq \mathfrak{T}_1(A), \ \ \forall A \in L_X^M.$$

Hence $\mathfrak{T}_0, \mathfrak{T}_1$ are minimal and maximal elements of $\mathfrak{M}_{\mathfrak{T}}(X)$, respectively.
An arbitrary intersection of gradations of openness on X, is a gradation of openness. Thus any subset of $\mathfrak{M}_{\mathfrak{T}}(X)$, has a lower bound in it. To prove this, let $\{\mathfrak{T}_k, \ k \in K\}$, be an arbitrary family of L-gradations of openness on X. We show that $\mathfrak{T} = \bigwedge_{k \in K} \mathfrak{T}_k$ is an L-gradation of openness on X. Obviously, $\mathfrak{T}(X) = \mathfrak{T}(\tilde{0}) = 1$. Also,

$$\mathfrak{T}(\bigcup_j A_j) = \bigwedge_k \mathfrak{T}_k(\bigcup_j A_j) \geq \bigwedge_k (\bigwedge_j \mathfrak{T}_k(A_j)) = \bigwedge_j (\bigwedge_k \mathfrak{T}_k(A_j) = \bigwedge_j (\mathfrak{T}(A_j),$$

and

$$\mathfrak{T}(A \cap B) = \bigwedge_k \mathfrak{T}_k(A \cap B) \geq \bigwedge_k (\mathfrak{T}_k(A) \wedge \mathfrak{T}_k(B)) \geq \bigwedge_k \mathfrak{T}_k(A) \wedge \bigwedge_k \mathfrak{T}_k(B) \geq \mathfrak{T}(A) \wedge \mathfrak{T}(B).$$

This completes the proof. □

Example 2.9. *Consider $(1_{\mathbb{R}^n}, \mathfrak{T}_{In})$ and $0 \in \mathbb{R}^n$. We show that the fuzzy point 0_1 is an IG-closed subset of $1_{\mathbb{R}^n}$: The fuzzy point $0_1 = \chi_{\{0\}}$ is an I-fuzzy subset of $\mathbb{R}^n$. So,*

$$(1_{\mathbb{R}^n} - 0_1)(x) = 1 - \chi_{\{0\}}(x) = \begin{cases} 1 & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Therefore,

$$(1_{\mathbb{R}^n} - 0_1)(x) = \bigcup_{0 \neq k \in \mathbb{Z}} B(k, 1, 1)(x).$$

Hence, $(1_{\mathbb{R}^n} - 0_1) \in \tau_{I_n}$. Thus $\mathfrak{T}_{In}(1_{\mathbb{R}^n} - 0_1) \geq 0$. So, $1_{\mathbb{R}^n} - 0_1$ is an IG-open set. Hence, 0_1 is an IG-closed subset.

Definition 2.10. *Let $(X, \mathfrak{T})$ be a fuzzy topological space and A, B be any fuzzy subsets of X,*

1) *A fuzzy subset V of X is called an LG-neighborhood of A if there exists an LG-open subset U such that $A \leq U \leq V$. We denote the set of all LG-neighborhoods of A by $LGN(A)$.*
2) *Let $B \leq A$. Then B is called an LG-interior set of A if $A \in LGN(B)$. The union of all LG-interior sets of A is denoted by LGA°.*
3) *The intersection of all LG-closed subsets containing A is called an LG-closure of A and is denoted by $LG\bar{A}$.*
4) *x is called an LG-boundary point of A if for every LG-neighborhood V of x, we have $V \not\leq A$. The set of these points is called an LG-boundary of A and is denoted by $LG\partial A$.*
5) *If x belongs to the LG-closure of $A - \chi_{\{x\}, A}$, then x is called an LG-limited point of A and the set of these points is denoted by LGA'.*
6) *A is said to be an LG-dense subset of X, if $LG\bar{A} = X$.*

From now on, we suppose that M_1, M_2 are two crisp sets, $X \in L^{M_1}, Y \in L^{M_2}$ and $(X, \mathfrak{T}), (Y, \mathfrak{R})$ are two LG-fuzzy topological spaces.

Definition 2.11. *Let $f : M_1 \to M_2$ be a function and $f[X]$ be an L-fuzzy subset of M_2, defined by*

$$f[X](y) = \bigvee \{X(x) \mid x \in f^{-1}(y)\}.$$

If we have $f[X] \leq Y$, then f is called an LG-related function from X to Y and the set of all such functions is denoted by $LGRf(X, Y)$. Furthermore, if we have $\mathfrak{R}(H) \leq \mathfrak{T}(f^{-1}[H])$ for all LG-fuzzy subset H of Y, then f is an L-gradation-preserving LG-related function so it is called an LGP-related function or LGP-fuzzy mapping from X to Y, $f \in LGRf(X, Y)$.

i) *f is called a one-to-one LG-related (LGP-related) function if $f|_{suppX} :\ suppX \to suppY$ is a one-to-one function.*
ii) *f is called an onto LG-related (LGP-related) function if $f[X] = Y$.*

Remark 2.12. *Let $A \in supp\mathfrak{T}$ and $B \in supp\mathfrak{R}$. Let f be an LGP-fuzzy mapping from X to Y such that $f[A] \leq B$. Then we have $\mathfrak{R}(H) \leq \mathfrak{T}(f^{-1}[H])$ for each LG-fuzzy subset of Y and in paticular $H \leq B$. Thus $\mathfrak{R}_B(H) \leq \mathfrak{T}_A(f^{-1}[H])$ for each LG-fuzzy subset H of Y with $H \leq B$. Therefore f can be considered as an LGP-fuzzy mapping of two L-gfts's, $(A, \mathfrak{T}_A)$ and $(B, \mathfrak{R}_B)$. So we can write $f \in LGPRf(A, B)$.*

Definition 2.13. *Let $f \in LGRf(X, Y)$, then*

i) *f is called LG-open if $f[A] \in supp\mathfrak{R} - \{\tilde{0}, Y\}$, $\forall A \in supp\mathfrak{T} - \{\tilde{0}, X\}$ and $f[X] \in supp\mathfrak{R}$.*
ii) *f is called LG-continuous if $f^{-1}[H] \in supp\mathfrak{T} - \{\tilde{0}, X\}$, $\forall H \in supp\mathfrak{R} - \{\tilde{0}, Y\}$ and $f^{-1}[Y] \in supp\mathfrak{T}$.*
iii) *f is called an LG-homeomorphism if it is one -to -one, onto, LG-continuous, LG-open and $f^{-1} \in LGRf(Y, X)$.*
iv) *f is called an LGP-homeomorphism if it is bijective and f, f^{-1} are LGP-fuzzy mapping.*

Proposition 2.14. *Let A, B be LG-open subsets of X, Y respectively. Let $\psi : M_1 \to M_2$ be a function. Then ψ is an LGP-homeomorphism from A to B if and only if ψ satisfies the two following conditions:*

i) $A(p) = B(\psi(p))$ *for all* $p \in A$ *or* $B(q) = A(\psi^{-1}(q))$ *for all* $q \in B$

ii) $\Re(H) = \mathfrak{T}(\psi^{-1}[H])$ *for all LG-fuzzy subset* H *of* B.

Proof. Let ψ satisfies conditions (i) and (ii), then by Definition 2.11 and Remark 2.12 we have $\psi \in LGPRf(A, B)$ and ψ is an LGP-homeomorphism from A to B. Conversely suppose ψ is an LGP-homeomorphism from A to B. We prove that ψ satisfies (i) *and* (ii).

$i)$ Since ψ is bijective, for any $q \in B$, there is exactly one element $p \in A$, such that $\psi^{-1}(q) = \{p\}$. So we have $\psi[A]\,(q) = sup\{A(a)|\ a \in \psi^{-1}(q)\} = A(p)$. On the other hand by Definition 2.11, we have $\psi[A] \leq B$. Hence $A(p) \leq B(q)$. We see $\psi^{-1}[B](p) = B(\psi(p)) = B(q)$. Since by Definition 2.13 (iv), we have $\psi^{-1} \in LGPRf(Y, X)$, then $\psi^{-1}[B] \leq A$. Hence $B(q) \leq A(p)$. Therefore $A(p) = B(q)$. Therefore $A(p) = B(\psi(p))$, for all $p \in A$ and $B(q) = A(\psi^{-1}(q))$, for all $q \in B$. Thus $A = (\psi^{-1}[B])$ and $\psi[A] = B$.

$ii)$ Since $\psi \in LGPRf(A, B)$, we have $\Re(H) \leq \mathfrak{T}(\psi^{-1}[H])$ for all LG-fuzzy subset H of Y, and since $\psi^{-1} \in LGPRf(B, A)$, we have $\mathfrak{T}(D) \leq \Re(\psi[D])$. Set $\psi[D] = H$. Then $D = \psi^{-1}[H]$ by injectivity of ψ. So $\mathfrak{T}(D) \leq \Re(H)$. Hence we have $\mathfrak{T}(\psi^{-1}[H]) = \Re(H)$. □

Proposition 2.15. *Every LGP-fuzzy mapping from* X *to* Y *is an LG-continuous related function, but the converse is not true.*

Proof. Let f be an LGP-fuzzy mapping from X to Y, then $\forall H \in supp\Re - \{\tilde{0}, Y\}$, we have $0 < \Re(H) \leq \mathfrak{T}(f^{-1}[H])$. Hence $f^{-1}[H] \in supp\mathfrak{T} - \{\tilde{0}, X\}$. Therefore f is LG-continuous.
Conversely, we define an LG-continuous function which is not an LGP-fuzzy mapping:
Following Example 2.4, consider $f = id : (1_{\mathbb{R}^n}, \mathfrak{T}_{Ln}) \to (1_{\mathbb{R}^n}, \mathfrak{T}_{Lsup})$. Since $f[1_{\mathbb{R}^n}] = 1_{\mathbb{R}^n}$ and we have

$$f^{-1}[H] = H \in supp\mathfrak{T}_{Ln} - \{\tilde{0}, X\} = \tau_{_{Ln}} - \{\tilde{0}, 1_{\mathbb{R}^n}\}, \qquad \forall H \in supp\mathfrak{T}_{Lsup} - \{\tilde{0}, Y\} = \tau_{_{Ln}} - \{\tilde{0}, 1_{\mathbb{R}^n}\},$$

and $f^{-1}[1_{\mathbb{R}^n}] \in supp\mathfrak{T}_{Ln}$. Therefore f is LG-continuous. Now Let $A = \begin{cases} x^2 & x \in (0, \frac{1}{2}), \\ 0 & \text{elsewhere.} \end{cases}$ Then $A \in \tau_{_{Ln}}$ and $\mathfrak{T}_{Lsup}(f[A]) = \dfrac{1}{4}$. But $\mathfrak{T}_{Ln}(A) = 1$. Hence the condition $\mathfrak{T}_{Ln}(A) \leq \mathfrak{T}_{Lsup}(f[A])$ dose not hold. Hence f is not an LGP-fuzzy mapping. □

3 L-fuzzy topological manifolds with L-gradation of openness

Definition 3.1. *Let* $\mathfrak{T}$ *be an L-gradation of openness on* X. *Then* $(X, \mathfrak{T})$ *is an LG-fuzzy topological space of dimension* n, *if for any* $x \in X$, *there exists an LG-open subset A of* X *containing* x *and an LG-open subset B of* $(1_{\mathbb{R}^n}, \mathfrak{T}_{Ln})$, *together with an LGP-homeomorphism* $\psi \in LGPRf(A, B)$. *The pair* (A, ψ) *is called an LG-local coordinate neighborhood of each* $q \in A$ *and we assign to q the n LG-local coordinates* $x_1(q), x_2(q), ..., x_n(q)$ *of its image* $\psi(q)$ *in* $\mathbb{R}^n$.

Definition 3.2. *Let* $\mathfrak{A} = \{(A_i,\ \psi_i)\ i \in J\}$ *be a collection of LG-local coordinate neighborhoods. Since* ψ_i *is an LGP-homeomorphism for all* $i \in J$, *then for all* $i, j \in J$ *whenever* $A_i \cap A_j \neq \phi$,

$$\psi_j \circ\ \psi_i^{-1} : \psi_i(supp(A_i \cap A_j)) \to \psi_j(supp(A_i \cap A_j)),$$

is an LGP-homeomorphism, that is called an LG-transition function. -

$$\psi_j \circ \psi_i^{-1}(x_1^i,\ x_2^i,\ ...,\ x_n^i) = (x_1^j,\ x_2^j,\ ...,\ x_n^j).$$

If $\psi_i \circ \psi_j^{-1}$ *and* $\psi_j \circ \psi_i^{-1}$ *changing the LG-local coordinates are infinitely differentiable or* C^∞, *we shall say that* $(A_i,\ \psi_i)$ *is* C^∞ *compatible with* $(A_j,\ \psi_j)$ *whenever* $A_i \cap A_j \neq \phi$.

Definition 3.3. *An LG-fuzzy topological space* $(X, \mathfrak{T})$ *is called an LG-fuzzy topological manifold of dimension* n, *if it satisfies the two following conditions:*

i) X *is an LG-fuzzy topological space of dimension* n,

ii) X *is a Hausdorff L-gfts.*

Definition 3.4. *A differentiable or* C^∞ *LG-fuzzy structure on an LG-fuzzy topological manifold* $(X, \mathfrak{T})$, *is a family* $\mathfrak{A} = \{(A_\alpha,\ \psi_\alpha),\ \ \alpha \in J\}$ *of LG-local coordinate neighborhoods such that*

1) $X = \bigcup_{\alpha \in J} A_\alpha$*;*

2) Each pair $(A_\alpha,\ \psi_\alpha)$ *and* $(A_\beta,\ \psi_\beta)$ *are compatible for all* $\alpha,\ \beta \in J$*.*

3) Any LG-local coordinate neighborhood (V, φ) *that is compatible with every* $(A_\alpha,\ \psi_\alpha), \alpha \in J$ *is in* $\mathfrak{A}$ *itself.*

A C^∞ *LG-fuzzy manifold* $(X, \mathfrak{T})$ *is an LG-fuzzy topological manifold with a* C^∞ *LG-fuzzy structure on it. In what follows, for convenience, "LG-fuzzy manifold with LG-fuzzy structure" will mean* C^∞ *LG-fuzzy manifold with* C^∞ *LG-fuzzy structure,*

Example 3.5. *Let* $M = \mathbb{R}^3,\ X : \mathbb{R}^3 \to I,\ X(x) = \begin{cases} 1 & \|x\| = 1, \\ 0 & \|x\| \neq 1. \end{cases}$ *Then* $suppX = S^2$*, the unit sphere. Set*

$$\mathfrak{T}\ :\ I_X^M \to I, \quad \mathfrak{T}(A) = \begin{cases} sup\{A(x) \mid x \in X\} & A \in \tau_{I^n},\ A \leq X, \\ 0 & elsewhere. \end{cases}$$

Then $(X, \mathfrak{T})$ *is an IG-fuzzy manifold of dimension 2.*

Proof. Let $J = \{1, 2, 3\}$. We define six IG-open subsets covering X by:

$$\forall x = (x_1,\ x_2,\ x_3), \quad A_j^\pm(x) = \begin{cases} \pm x_j & \pm\, x_j > 0,\ \|x\| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we show that all $A_j^\pm$ are diffeomorphic to IG-open subset $B : \mathbb{R}^2 \to I$, defined by:

$$\forall y = (y_1, y_2), \quad B(y) = \begin{cases} \sqrt{1 - y_1^2 - y_2^2} & \|y\| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $suppB = B(0,1)$, so $B \in \tau_{I^2}$. We define six bijections $\psi_j^\pm$ from $suppA_j^\pm = \{(x_1,\ x_2,\ x_3) \mid \pm\, x_j > 0,\ \ \|x\| = 1\}$ to $suppB = \{(y_1,\ y_2) | \ \|y\| < 1\}$, for all $j \in J$ by:

$$\psi_1^\pm(x_1, x_2, x_3) = (x_2,\ x_3)\ , \quad (\psi_1^\pm)^{-1}(y_1,\ y_2) = (\pm\sqrt{1 - y_1^2 - y_2^2}\ ,\ y_1,\ y_2).$$

$$\psi_2^\pm(x_1, x_2, x_3) = (x_1,\ x_3)\ , \quad (\psi_2^\pm)^{-1}(y_1,\ y_2) = (y_1, \pm\sqrt{1 - y_1^2 - y_2^2}\ ,\ \ y_2).$$

$$\psi_3^\pm(x_1, x_2, x_3) = (x_1,\ x_2)\ , \quad (\psi_3^\pm)^{-1}(y_1,\ y_2) = (y_1,\ y_2,\ \pm\sqrt{1 - y_1^2 - y_2^2}\ ).$$

Also, it is seen that $\psi_j^\pm \circ (\psi_i^\pm)^{-1}$ is infinitely differentiable for all $i,\ j \in J$. For example:

$$\psi_2^\pm \circ (\psi_1^\pm)^{-1}(y_1,\ y_2) = \psi_2^\pm(\pm\sqrt{1 - y_1^2 - y_2^2}\ ,\ y_1,\ y_2) = (\pm\sqrt{1 - y_1^2 - y_2^2},\ y_2).$$

Therefore, each pair $(A_i^\pm,\ \psi_i^\pm)$ and $(A_j^\pm,\ \psi_j^\pm)$ are compatible, for all $i,\ j \in J$. We see

$$\forall j \in J,\ \ A_j^\pm(x) = \pm x_j = B_j(\psi_j^\pm(x)), \quad \forall x \in A_j^\pm.$$

Let H be an IG-fuzzy subset of $1_{\mathbb{R}^2}$ with $H \leq B$ .We show that $\mathfrak{T}((\psi_j^\pm)^{-1}[H]) = \mathfrak{T}_{Isup}(H)$. Using (2), we have $\mathfrak{T}_{Isup}(H) = sup\{H(a) | \ a \in \mathbb{R}^2\}$. Since $\psi_j^\pm$ is bijective, for each $a \in \mathbb{R}^2$, there exists one and only one element $p \in suppA_j^\pm$ such that $\psi_j^\pm(p) = a\ or (\psi_j^\pm)^{-1}(a) = p$. Hence

$$\mathfrak{T}_{Isup}(H) = sup\{H\big(\psi_j^\pm(p)\big) |\ p \in suppA_j^\pm\} = sup\{\big((\psi_j^\pm)^{-1}[H]\big)(p) |\ p \in suppA_j^\pm\} = \mathfrak{T}\big((\psi_j^\pm)^{-1}[H]\big).$$

Hence $\psi_j^\pm \in IGPRf(A_j^\pm, B)$ is an IGP-homeomorphism for all $j \in J$ and this completes the proof. □

Example 3.6. *The set of natural numbers,* $\mathbb{N}$*, partially ordered by divisibility, is a distributive lattice set, for which the unique supremum is the least common multiple and the unique infimum is the greatest common divisor. Let* $L = \mathbb{N} \cup \{\infty\}$*. Then* L *is a complete lattice. Notice that we denote the top element of any lattice by* 1*, but in this example,* ∞ *is the top element of* $\mathbb{N} \cup \{\infty\}$*. We define the LG-fuzzy Euclidean topological space* $(1_{\mathbb{R}^{mn}},\ \mathfrak{T}_{Lsup})$ *by*

$$1_{\mathbb{R}^{mn}} : \mathbb{R}^{mn} \to L, \qquad 1_{\mathbb{R}^{mn}}\big((a_1, a_2, \ldots, a_{mn})\big) = \infty,$$

$$\mathfrak{T}_{Lmn} : L_{1_{\mathbb{R}^{mn}}}^{\mathbb{R}^{mn}} \to L, \qquad \mathfrak{T}_{Lmn}(D) = \begin{cases} \infty & D \in \tau_{Lmn}, \\ 0 & elsewhere. \end{cases}$$

Let $M = \mathcal{M}_{m\times n}(\mathbb{R})$ and $X \in L^M$ be defined by

$$X\big((a_{ij})\big) = 2 + max\big\{\lfloor\ |a_{ij}|\ \rfloor \mid 1 \le i \le m,\ 1 \le j \le n\big\},$$

where $\lfloor |x| \rfloor$ is equal to the greatest integer less than or equal to $|x|$. There is a bijection ψ from M to $\mathbb{R}^{mn}$:

$$\psi(a_{ij}) = (a_{11},\ \dots,\ a_{1n},\ \dots,\ a_{m1},\ \dots,\ a_{mn}).$$

Hence using ψ and (1), we define

$$\mathfrak{T}\ :\ L_X^M \to L, \qquad \mathfrak{T}(A) = \begin{cases} \infty & \psi[A] \in \tau_{Lmn}, \\ 0 & elsewhere. \end{cases}$$

We show that $(X, \mathfrak{T})$ is an C^{∞} LG-fuzzy mn-manifold. Let B is an L-fuzzy subset of $1_{\mathbb{R}^{mn}}$ defined by

$$B\big((a_1, a_2, \dots, a_{mn})\big) = 2 + max\big\{\lfloor\ |a_k|\ \rfloor \mid 1 \le k \le mn\big\}.$$

We see $B = \psi[X]$. Since supp$B = \mathbb{R}^{mn} = \bigcup_{k=1}^{\infty} B(0,k)$, Hence $B \in \tau_{Lmn}$. Therefore using (1), for each LG-open subset H of $1_{\mathbb{R}^{mn}}$, with $H \le B$, we have $\mathfrak{T}_{Lmn}(H) = \infty = \mathfrak{T}\big(\psi^{-1}[H]\big)$. So by Proposition 2.14, $\psi \in LGPRf(X, B)$ is an LGP-homeomorphism. We can cover $(X, \mathfrak{T})$ by the single LG-coordinate neighborhood (X, ψ). Hence $(X, \mathfrak{T})$ is an C^{∞} LG-fuzzy mn-manifold.

Definition 3.7. *(LG-open submanifolds) Let Z be an LG-open subset of the LG-fuzzy manifold $(X, \mathfrak{T})$. If $\mathfrak{A} = \{(A_\alpha, \psi_\alpha),\ \alpha \in J\}$ is an LG-fuzzy structure on X, then $(Z,\ \mathfrak{T}_Z)$ is an LG-fuzzy topology with LG-fuzzy structure consisting of the LG-coordinate neighborhoods $(A_\alpha \cap Z,\ \psi_\alpha|_{A_\alpha \cap Z})$.*

Example 3.8. *Let $(X, \mathfrak{T})$ be as Example 3.6. We define $Z : \mathcal{M}_{m\times n}(\mathbb{R}) \to L$, $Z(A) = \begin{cases} X(A) & detA \ne 0, \\ 0 & detA = 0. \end{cases}$*
We have $Z \le X$ and $U = suppZ = Gl(n, \mathbb{R})$ is an open subset of $\mathcal{M}_{m\times n}$. Hence we can prove that Z is an LG-open subset of X. Therefore $(Z,\ \mathfrak{T}_Z)$ is an LG-fuzzy submanifold of $(X, \mathfrak{T})$ with the single LG-local coordinate neighborhood $(Z,\ \psi|_Z)$ where ψ is a bijection defined in Example 3.6.

Example 3.9. *Let $L = \mathbb{N} \cup \{\infty\}$ and $M = \mathbb{R}^{n+1}$. Define an L-fuzzy subset*

$$X : M \to L, \quad X(x) = \begin{cases} n+2 & \|x\| = 1, \\ 0 & \|x\| \ne 1, \end{cases}$$

$$\mathfrak{T} : L_X^M \to L, \quad \mathfrak{T}(A) = \begin{cases} \infty & A \in \tau_{L(n+1)},\ A \le X, \\ 0 & elsewhere. \end{cases}$$

Then supp$X = S^n$. Then $(X, \mathfrak{T})$ is an LG-fuzzy manifold of dimension n:

Proof. Let $J = \{1,\ \dots,\ n+1\}$. We define $2(n+1)$ LG-open subsets covering X, $A_j^{\pm} : M \to L$, $j \in J$ by:

$$\forall\, x = (x_1,\ \dots\ x_{n+1}), \quad A_j^{\pm}(x) = \begin{cases} j & \pm\, x_j > 0,\ \|x\| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we show that each $A_j^{\pm}$ is LGP-homeomorphic to the LG-open subset $B_j : \mathbb{R}^n \to L$ defined by:

$$\forall y = (y_1,\ \dots,\ y_n), \quad B_j(y) = \begin{cases} j & \|y\| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

with $2(n+1)$ LG-maps $\psi_i^{\pm} : A_j^{\pm} \to B_j$ defined by:

$$\psi_j^{\pm}(x_1,\ \dots,\ x_{n+1}) = (x_1,\ \dots,\ \widehat{x_j},\ \dots,\ x_{n+1}).$$

$$(\psi_j^{\pm})^{-1}(y_1,\ \dots,\ y_n) = (y_1,\ \dots,\ \pm\sqrt{1 - (y_1^2 +\ \dots\ + y_n^2)}\ ,\ \dots,\ y_n),$$

where $\widehat{x_j}$ means omit x_j. Also it is seen that $\psi_j^{\pm} \circ (\psi_i^{\pm})^{-1}$ is infinitely differentiable for all $i,\ j \in J$ and we have

$$A_j^{\pm}(x) = j = B_j(\psi_j^{\pm}(x)),\ \ \forall x \in A_j^{\pm}\ and\ \forall j \in J.$$

Also using (1), for each LG-open subset H_j of $1_{\mathbb{R}^n}$, with $H_j \leq B_j$, we have

$$\mathfrak{T}_{Ln}(H_j) = \infty = \mathfrak{T}\big(\psi_j^{\pm})^{-1}(H_j)\big)\ \ \forall j \in J.$$

Therefore by Proposition 2.14, $\psi_i^{\pm} \in LGPRf(A_j^{\pm},\ B_j)$, is an LGP-homeomorphism for all $j \in J$. □

Theorem 3.10. *Let (M, τ) be a C^∞ ordinary n-manifold with the C^∞ structure $\mathfrak{W} = \{(U_k,\ \psi_k),\ k \in K\}$. Consider $X = \tilde{1}$, the constant L-fuzzy subset of M. Let δ be the L-fuzzy topology on X generated by $\{\chi_{U_k},\ k \in K\}$. Define*

$$\mathfrak{T}_\tau : L_X^M \to L, \qquad \mathfrak{T}_\tau(A) = \begin{cases} 1 & A \in \delta, \\ 0 & elsewhere. \end{cases}$$

Then $(X, \mathfrak{T}_\tau)$ is an LG-fuzzy manifold of dimension n.

Proof. First we show that $\mathfrak{T}_\tau$ is an L-gradation of openness on X:

1) Since $\tilde{0}, X \in \delta$, therefore $\mathfrak{T}_\tau(\tilde{0}) = \mathfrak{T}_\tau(X) = 1$.

2) For each family of fuzzy subsets $\{A_j\} \subseteq L_X^M$, we have two cases:
i) $\{A_j\}_{j\in J} \subseteq \delta \ \Rightarrow\ \bigcup\limits_{j\in J} A_j \in \delta$ *and* $\mathfrak{T}_\tau(A_j) = 1,\ \forall j \in J \ \Rightarrow\ 1 = \mathfrak{T}_\tau(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathfrak{T}_\tau(A_j) = 1$
ii) $A_j \subseteq (L_X^M - \delta)$, for some $j \in J \ \Rightarrow\ \mathfrak{T}_\tau(A_j) = 0$, for some $j \in J \ \Rightarrow\ \mathfrak{T}_\tau(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathfrak{T}_\tau(A_j) = 0$

3) For every two fuzzy subsets $A, B \in L_X^M$, we have three cases:
i) $A,\ B \in \delta \ \Rightarrow\ A \cap B \in \delta \ \Rightarrow\ 1 = \mathfrak{T}_\tau(A \cap B) \geq \mathfrak{T}_\tau(A) \wedge \mathfrak{T}_\tau(B) = 1$;
ii) $A \in (L_X^M - \delta),\ B \in \delta \ \Rightarrow\ \mathfrak{T}_\tau(A) = 0,\ \mathfrak{T}_\tau(B) = 1 \ \Rightarrow\ \mathfrak{T}_\tau(A \cap B) \geq 0 = \mathfrak{T}_\tau(A) \wedge \mathfrak{T}_\tau(B)$;
iii) $A,\ B \in (L_X^M - \delta) \ \Rightarrow\ \mathfrak{T}_\tau(A) = \mathfrak{T}_\tau(B) = 0 \ \Rightarrow\ \mathfrak{T}_\tau(A \cap B) \geq 0 = \mathfrak{T}_\tau(A) \wedge \mathfrak{T}_\tau(B)$.

Next we prove that $(X, \mathfrak{T}_\tau)$ is an LG-fuzzy manifold:
Let $p \in M$. Then there exists an open subset $U_k,\ k \in K$ s.t. $p \in U_k$ and an open set V_k of $\mathbb{R}^n$ along with a homeomorphism $\psi_k : U_k \to V_k$. Let $\tau^\triangleright$ be the L-fuzzy topology on $1_{\mathbb{R}^n}$ generated by $\{\chi_{V_k},\ k \in K\}$. Then $\tau^\triangleright \subseteq \tau_{Ln}$. Hence we can consider the restriction of $\mathfrak{T}_{Ln}$ on $\tau^\triangleright$. We see

$$\mathfrak{T}_\tau(\chi_{U_k}) = \mathfrak{T}_{Ln}|_{\tau^\triangleright}(\chi_{V_k}).$$

Define LGP-homeomorphisms

$$\psi_k^\triangleright : M \to \mathbb{R}^n, \quad \psi_k^\triangleright(p) = \psi_k(p)\chi_{U_k} = \begin{cases} \psi_k(p) & \text{if } p \in U_k, \\ 0 & elsewhere. \end{cases}$$

Therefore $\psi_k^\triangleright \in LGPRf(\chi_{U_k},\ \chi_{V_k})$. Thus $\mathfrak{W}_\tau = \{(\chi_{U_k},\ \psi_k^\triangleright),\ \ k \in K\}$ is an LG-fuzzy structure on X. □

Theorem 3.11. *Let $(X, \mathfrak{T})$ be an LG-fuzzy n-manifold with LG-fuzzy structure $\mathfrak{A} = \{(A_j,\ \psi_j), j \in J\}$.
If $\mathfrak{T}^\triangleleft = \{suppA \mid \mathfrak{T}(A) > 0\}$, then $(suppX, \mathfrak{T}^\triangleleft)$ is a topological manifold of dimension n called a premanifold with the structure $\mathfrak{A}^\triangleleft = \{(suppA_j,\ \psi_j|_{suppA_j}),\ j \in J\}$ called a prestructure.*

Proof. Since Dom$\mathfrak{T} = L_X^M$, then for all $A \in supp\mathfrak{T}$, we have A is less than X. Hence $suppA \subseteq suppX$.

i) $\mathfrak{T}(\tilde{0}) = \mathfrak{T}(X) = 1 \ \Rightarrow\ \phi = supp\tilde{0} \in supp\mathfrak{T}^\triangleleft$ *and* $suppX \in supp\mathfrak{T}^\triangleleft$.

ii) Let $\{A_j,\ j \in J\} \subseteq \delta$, Then $\mathfrak{T}(A_j) > 0,\ \forall j \in J$. Hence $\mathfrak{T}(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathfrak{T}(A_j) > 0$. Thus $\bigcup\limits_{j\in J} A_j \in supp\mathfrak{T}^\triangleleft$.

iii) Since $\mathfrak{T}(A \cap B) \geq \mathfrak{T}(A) \wedge \mathfrak{T}(B)$. *So* $A, B \in \mathfrak{T}^\triangleleft$ implies that $\mathfrak{T}(A \cap B) > 0$. Thus $A \cap B \in supp\mathfrak{T}^\triangleleft$.

Therefore $\mathfrak{T}^\triangleleft$ is a topology on $suppX$. Let $p \in X$. Then there exists an LG-open subset $A_k,\ k \in K$ such that $p \in A_k$ and there exists an LG-open subset $B_k,\ k \in K$ with an LGP-homeomorphism $\psi_k \in LGPRf(A_k, B_k)$. Hence $p \in suppX,\ A_k(p) = B_k(\psi_k(p))$ and $\psi_k|_{suppA_k} : suppA_k \to suppB_k$ is one-to-one and onto. Therefore $(suppA_k,\ \psi_k|_{suppA_k})$ is a coordinate neighborhood of p. Also $\psi_j \circ \psi_i^{-1}$ is infinitely differentiable for all $i, j \in K$, thus
$(\psi_j|_{supp(A_i\cap A_j)}) \circ (\psi_i^{-1}|_{supp(B_i\cap B_j)})$ is C^∞ for all $i, j \in K$. Hence $(suppX, \mathfrak{T}^\triangleleft)$ is a premanifold. □

Theorem 3.12. *Let (M,τ) be an n-manifold with structure $\mathfrak{W} = \{(U_j,\ \varphi_j),\ \ j \in J\}$. Let $X = \tilde{1}$. Then $(suppX, (\tau^{\triangleright})^{\triangleleft}) = (M,\tau)$.*

Proof. It is clear that $suppX = M$ and for every $U \in \tau$ we have $supp\chi_{_U} = U$. Hence $(\tau^{\triangleright})^{\triangleleft} = \tau$. Since we have $\mathfrak{W}^{\triangleright} = \{(\chi_{_{U_j}},\ \varphi_j^{\triangleright}),\ \ j \in J\}$, then

$$(\mathfrak{W}^{\triangleright})^{\triangleleft} = \{(supp\chi_{_{U_j}},\ \varphi_j^{\triangleright}|_{supp\chi_{_{U_j}}}),\ \ j \in J\} = \{(U_j, \varphi_j^{\triangleright}|_{_{U_j}}),\ \ j \in J\} = \{(U_j,\ \varphi_j),\ \ j \in J\} = \mathfrak{W}.$$

□

Remark 3.13. *Let $(X,\mathfrak{T})$ be an L-fuzzy topological space with L-gradation of openness, then $\mathfrak{T}_{\mathfrak{T}^{\triangleleft}}$ does not necessarily equal $\mathfrak{T}$. We show it by the following example.*

Example 3.14. *Let $M = \mathbb{R}$. We define*

$$X : M \to I, \quad X(x) = \begin{cases} \dfrac{1}{\lfloor x \rfloor} & x \in (2,+\infty), \\ 0 & elsewhere. \end{cases}$$

and $\mathfrak{T} : I_X^M \to I,$ *by* $\mathfrak{T}(A) = \begin{cases} 1 & A \in \tau_{_{I1}},\ A \leq X \\ 0 & elsewhere. \end{cases}$.
Then clearly $(X,\mathfrak{T})$ is an IG-fuzzy manifold of dimension 1. Then by Theorem 3.11, $(suppX,\ \mathfrak{T}^{\triangleleft})$ is a manifold, where $\mathfrak{T}^{\triangleleft} = \{suppA|\ A \in supp\mathfrak{T}\}$. Hence by Theorem 3.10 we have $\mathfrak{T}_{\mathfrak{T}^{\triangleleft}} = \{\chi_{suppA} \mid A \in supp\mathfrak{T}\}$. We have $A(x) \leq X(x) \leq \dfrac{1}{2},\ \forall x \in M$ and $\forall A \in \tau_{_{I1}}$. Hence $1 = \chi_{suppA} \neq A$. Therefore $\mathfrak{T}_{\mathfrak{T}^{\triangleleft}} \neq \mathfrak{T}$.

4 LG-fuzzy quotient manifolds

Definition 4.1. *Let M be a crisp set and $\sim$ be an equivalence relation on it. If A is an L-fuzzy subset of M such that $A(y) = A(x)$ whenever $y \sim x$, then we define the L-fuzzy subset:*

$$\frac{A}{\sim} : \frac{M}{\sim} \to L, \quad \frac{A}{\sim}([x]) = A(x),\ \ \forall x \in M,$$

where $[x] = \{y \mid x \sim y\}$. Since $A \leq X$, thus $\frac{A}{\sim} \leq \frac{X}{\sim}$ and hence $\frac{A}{\sim} \in L^{\frac{M}{\sim}}_{\frac{X}{\sim}}$.

Theorem 4.2. *Let $(X,\mathfrak{T})$ be an LG-fuzzy topological space, such that $X(y) = X(x)$ whenever $y \sim x$, then $(\dfrac{X}{\sim}, \dfrac{\mathfrak{T}}{\sim})$ is an LG-fuzzy topological space, called the LG-fuzzy quotient space, where*

$$\frac{\mathfrak{T}}{\sim} :\ L^{\frac{M}{\sim}}_{\frac{X}{\sim}} \to L, \quad \frac{\mathfrak{T}}{\sim}(\frac{A}{\sim}) = \mathfrak{T}(A).$$

Proof. We show that all elements of $L^{\frac{M}{\sim}}_{\frac{X}{\sim}}$ are in the form $\frac{A}{\sim}$ for some $A \in L_X^M$. Let B be an L-fuzzy subset of $\frac{M}{\sim}$ less than $\frac{X}{\sim}$. We define L-fuzzy subset A of M by $A(x) = B([x]),\ \forall x \in M$. Let $x \sim y$ so $[x] = [y]$, then $A(x) = B([x]) = B([y]) = A(y)$ and thus $\frac{A}{\sim} = B$. Also,

1) $\frac{\mathfrak{T}}{\sim}(\frac{\tilde{0}}{\sim}) = \mathfrak{T}(\tilde{0}) = 1,\ \ \frac{\mathfrak{T}}{\sim}(\frac{X}{\sim}) = \mathfrak{T}(X) = 1.$

2) $\dfrac{\mathfrak{T}}{\sim}(\dfrac{A_1}{\sim} \cap \dfrac{A_2}{\sim}) = \dfrac{\mathfrak{T}}{\sim}(\dfrac{A_1 \cap A_2}{\sim}) = \mathfrak{T}(A_1 \cap A_2) \geq \mathfrak{T}(A_1) \wedge \mathfrak{T}(A_2) = \dfrac{\mathfrak{T}}{\sim}(\dfrac{A_1}{\sim}) \wedge \dfrac{\mathfrak{T}}{\sim}(\dfrac{A_2}{\sim}).$

3) Let $\{A_j\}_{j\in J}$ be a sequence of L-fuzzy subsets of X, such that $\forall j \in J$, $A_j(y) = A_j(x)$, whenever $y \sim x$, then

$$\bigcup_{j\in J} \frac{A_j}{\sim}\ [y] = sup\{\ \frac{A_j}{\sim}[y],\ j \in J\} = sup\{\ A_j(y),\ j \in J\} = sup\{\ A_j(x),\ j \in J\} = \bigcup_{j\in J} \frac{A_j}{\sim}\ [x],$$

$$\frac{\mathfrak{T}}{\sim}(\bigcup_{j\in J} \frac{A_j}{\sim}) = \frac{\mathfrak{T}}{\sim}(\frac{\bigcup_{j\in J} A_j}{\sim}) = \mathfrak{T}(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathfrak{T}(A_j) = \bigwedge_{j\in J} \frac{\mathfrak{T}}{\sim}(\frac{A_j}{\sim}).$$

Hence $\frac{\mathfrak{T}}{\sim}$ is a gradation of openness on $\frac{X}{\sim}$. □

This LG-fuzzy topology is nontrivial when $L = I$, because for each $\alpha \in I,\ \alpha X(x) = \alpha X(y)$ whenever $x \sim y$. Hence $\alpha X \in supp\frac{\mathfrak{T}}{\sim}$.

Definition 4.3. *Consider an LG-fuzzy quotient space* $(\frac{X}{\sim}\ ,\frac{\mathfrak{T}}{\sim})$. *The equivalence relation* $\sim$ *is called an LG-open relation if for each fuzzy subset* $A \in supp\mathfrak{T}$ *we have* $\frac{A}{\sim} \in supp\frac{\mathfrak{T}}{\sim}$.

Theorem 4.4. *Let* $(X, \mathfrak{T})$ *be an LG-fuzzy manifold and* $\sim$ *be an LG-open relation. Then* $(\frac{X}{\sim}\ ,\frac{\mathfrak{T}}{\sim})$ *is an LG-fuzzy topological space of dimension* n *called LG-fuzzy quotient topological space of dimension* n. *If* $supp\mathfrak{T}$ *has a countable basis, then* $\frac{\mathfrak{T}}{\sim}$ *has a countable basis.*

Proof. Let $(A,\ \psi)$ be an LG-locally coordinate neighborhood of $p \in X$ and $\psi \in LGPRf(A, B)$. Since $\sim$ is an LG-open relation, then we have $\frac{\mathfrak{T}}{\sim}(\frac{A}{\sim}) > 0$. We define a corresponding relation $\sim^*$ on $suppB$ as follows:

$$a \sim^* b \iff \psi^{-1}(a) \sim \psi^{-1}(b) \quad for\ all\ \ a, b \in suppB.$$

Clearly $\sim^*$ is a reflexive and symmetric relation. Let $a \sim^* b$ *and* $b \sim^* c$. Then we have $\psi^{-1}(a) \sim \psi^{-1}(b),$ and $\psi^{-1}(b) \sim \psi^{-1}(c)$. Since $\sim$ is transitive, so $\psi_1^{-1}(a) \sim \psi_1^{-1}(c)$. Hence $a \sim^* c$. Therefore $\sim^*$ is transitive and so it is an equivalence relation. Since we have

$$a \sim^* b \implies \psi^{-1}(a) \sim \psi^{-1}(b) \implies A(\psi^{-1}(a)) = A(\psi^{-1}(b)) \implies B(a) = B(b).$$

Therefore, $\frac{B}{\sim^*}$ is well-defined. Now we define

$$\frac{\psi}{\sim} : supp(\frac{A}{\sim}) \to supp(\frac{B}{\sim^*}), \quad \frac{\psi}{\sim}([p]) = [\psi(p)].$$

We see

$$[p] = [q] \iff p \sim q \iff \psi(p) \sim^* \psi(q) \iff [\psi(p)] = [\psi(q)] \iff \frac{\psi}{\sim}([p]) = \frac{\psi}{\sim}([q]).$$

Therefore, $\frac{\psi}{\sim}$ is a well-defined and one to one function. Since ψ is onto, we see

$$\forall a \in suppB, \quad \exists p \in suppA \ \ s.t.\ \ a = \psi(p) \quad \implies \quad \frac{\psi}{\sim}([p]) = [\psi(p)] = [a].$$

Hence, $\frac{\psi}{\sim}$ is onto. We have

$$\frac{B}{\sim}([a]) = B(a) = A(\psi^{-1}(a)) = \frac{A}{\sim}([\psi^{-1}(a)]) = \frac{A}{\sim}\big((\frac{\psi}{\sim})^{-1}([a])\big).$$

On the other hand for each LG-open subset H of $1_{\mathbb{R}^n}$ which $H \leq B$, we have

$$\frac{\mathfrak{T}_{Ln}}{\sim}(\frac{H}{\sim}) = \mathfrak{T}_{Ln}(H) = 1 = \mathfrak{T}\big(\psi^{-1}(H)\big) = \frac{\mathfrak{T}}{\sim}(\frac{\psi^{-1}(H)}{\sim}).$$

Hence, $\frac{\psi}{\sim} \in LGPRf(\frac{A}{\sim},\ \frac{B}{\sim})$. Therefore, if $\mathfrak{A} = \big\{(A_j,\ \psi_j),\ j \in J\big\}$ be a C^∞ LG-structure of X. Then $\frac{\mathfrak{A}}{\sim} = \{(\frac{A_j}{\sim},\ \frac{\psi_j}{\sim}),\ j \in J\}$ is a C^∞ LG-fuzzy structure of $(\frac{X}{\sim}, \frac{\mathfrak{T}}{\sim})$. Now, suppose that $supp\mathfrak{T}$ has a countable basis $\beta = \{A_i,\ i \in \mathbb{N}\}$. Let $\frac{A}{\sim} \in supp\frac{\mathfrak{T}}{\sim}$ and $A = \bigcup_{j \in K} A_j$. Since $K \subseteq \mathbb{N}$. Hence for all $y \in X$ we have:

$$\frac{A}{\sim}([y]) = A(y) = \bigcup_{j \in J} A_j(y) = sup\{\ A_j(y),\ j \in K\} = sup\{\ \frac{A_j}{\sim}([y]),\ j \in K\} = \bigcup_{j \in K} \frac{A_j}{\sim}([y]).$$

Therefore, $\frac{\beta}{\sim} = \{\frac{A_i}{\sim},\ i \in K\}$ is a countable basis for $supp\frac{\mathfrak{T}}{\sim}$. □

Example 4.5. *Consider the IG-fuzzy Euclidean topological space* $(1_{\mathbb{R}},\ \mathfrak{T}_{I1})$. *We define a relation on* $\mathbb{R}$ *as follows:* $\forall x, y \in 1_{\mathbb{R}},\quad x \sim y \quad if \quad x - y \in \mathbb{Z}$. *Since* $1_{\mathbb{R}}(y) = 1_{\mathbb{R}}(x) = 1$ *whenever* $y \sim x$. *Hence* $\frac{1_{\mathbb{R}}}{\sim}$ *is well-defined and hence by Theorem 4.2 we have the IG-fuzzy quotient topological space* $(\frac{1_{\mathbb{R}}}{\sim},\ \frac{\mathfrak{T}_{I1}}{\sim})$. *We show that it is an IG-fuzzy topological manifold:*
Let $A : \mathbb{R} \to I,\quad A(x) = x - \lfloor x \rfloor$. *Since*

$$suppA = \mathbb{R} - \mathbb{Z} = \bigcup_{k \in \mathbb{Z}} (k,\ k+1) = \bigcup_{k \in \mathbb{Z}} B(k + \frac{1}{2},\ \frac{1}{2}).$$

Then we see:

$$A = \bigcup_{k \in \mathbb{Z}} B(k + \frac{1}{2},\ \frac{1}{2},\ b_k), \quad where \quad b_k : B(k + \frac{1}{2},\ \frac{1}{2}) \to I, \quad b_k(x) = x - \lfloor x \rfloor.$$

Therefore by Example 2.2, $A \in \tau_{_{I1}}$ *and hence by Example 2.4,* $\mathfrak{T}_{I1}(A) = 1$. *Since for all* $x, y \in \mathbb{R}$ *we have*

$$x \sim y \quad \Rightarrow \quad y = x + k,\ \ k \in \mathbb{Z} \quad \Rightarrow \quad \lfloor y \rfloor = k + \lfloor x \rfloor.$$

$$y - \lfloor y \rfloor = k + x - (k + \lfloor x \rfloor) = x - \lfloor x \rfloor \quad \Rightarrow \quad A(y) = A(x). \tag{3}$$

Hence $\frac{A}{\sim}$ *is well-defined. So* $\frac{\mathfrak{T}_{I1}}{\sim}(\frac{A}{\sim}) = \mathfrak{T}_{I1}(A) = 1$. *Define*

$$B : \mathbb{R} \to I,\quad B(y) = \begin{cases} y & y \in (0,1), \\ 0 & otherwise. \end{cases}$$

We can write $B = B(\frac{1}{2},\ \frac{1}{2}, b)$, *where* $b : B(\frac{1}{2},\ \frac{1}{2}) \to I,\ b(y) = y$. *So* $B \in \tau_{_{I1}}$. *Hence* $\mathfrak{T}_{I1}(B) = 1 = \frac{\mathfrak{T}_{I1}}{\sim}(\frac{A}{\sim})$. *We define* $\psi : supp\frac{A}{\sim} \to suppB$ *by* $\psi[x] = x - \lfloor x \rfloor$. *If* $[x] = [y]$, *then it means that* $x \sim y$, *so by* (3), *we have* $x - \lfloor x \rfloor = y - \lfloor y \rfloor$. *Hence* $\psi([x]) = \psi([y])$. *Therefore* ψ *is well-defined. We show that* ψ *is injective:*

$$\psi([x]) = \psi([y]) \quad \Rightarrow \quad x - \lfloor x \rfloor = y - \lfloor y \rfloor \quad \Rightarrow \quad y - x = \lfloor y \rfloor - \lfloor x \rfloor \in \mathbb{Z} \quad \Rightarrow \quad x \sim y \quad \Rightarrow \quad [x] = [y].$$

Since $\forall t \in (0,1),\ \lfloor t \rfloor = 0$, *so* $\psi([t]) = t - \lfloor t \rfloor = t$. *Therefore* ψ *is surjective. So by Definition 2.11 we have* $\frac{A}{\sim}([x]) = B(\psi([x]))$. *On the other hand for each LG-open subset* H *of* $1_{\mathbb{R}^n}$ *with* $H \leq B$, *we have*

$$\frac{\mathfrak{T}_{I1}}{\sim}\big(\psi^{-1}(H)\big) = \frac{\mathfrak{T}_{I1}}{\sim}(\frac{H}{\sim}) = \mathfrak{T}_{I1}(H) = 1.$$

Hence $\psi \in IGPRf(\frac{A}{\sim},\ B)$. *Thus we have a single IG-local coordinate neighborhood* $(\frac{A}{\sim},\ \psi)$ *for all points of* $\frac{1_{\mathbb{R}}}{\sim}$.

Proposition 4.6. *Consider all of the hopotheses of Theorem 4.4. Let* $R = \{(x, y) \mid x \sim y\}$. *Then* $(\frac{X}{\sim}\ ,\frac{\mathfrak{T}}{\sim})$ *is a Hausdorff L-gfts if and only if* $\chi_{_{R,X \times X}}$ *is an LG-closed subset of* $X \times X$.

Proof. Let $(\frac{X}{\sim},\ \frac{\mathfrak{T}}{\sim})$ be a Hausdorff L-gfts, then for each $[x],\ [y] \in \frac{X}{\sim}$ where $[x] \neq [y]$, there exist $\frac{U}{\sim},\ \frac{V}{\sim} \in supp\frac{\mathfrak{T}}{\sim}$ such that $[x] \in \frac{U}{\sim},\ [y] \in \frac{V}{\sim}$ and $\frac{U}{\sim} \cap \frac{V}{\sim} = \phi$. So for each $(x, y) \in (X \times X)$ where $(x, y) \notin R$ there exist $U,\ V \in supp\mathfrak{T}$, such that $U \cap V = \phi$ *and* $(x, y) \in (U \times V)$. we show that $supp(U \times V) \cap R = \phi$.

$$(a, b) \in supp(U \times V) \cap R \quad \Rightarrow \quad a \in U,\ b \in V,\ a \sim b \Rightarrow \quad [a] \in \frac{U}{\sim},\ [b] \in \frac{V}{\sim},\ [a] = [b] \Rightarrow \quad \frac{U}{\sim} \cap \frac{V}{\sim} \neq \phi,$$

that is a contradition. It means that for each $(x, y) \in supp\big(X \times X - \chi_{_{R,X \times X}}\big)$ there exists $U \times V \in supp\big(\mathfrak{T} \times \mathfrak{T}\big)$ such that $(x, y) \in U \times V \leq \big(X \times X - \chi_{_{R,X \times X}}\big)$. Therefore, $\big(\mathfrak{T} \times \mathfrak{T}\big)\big(X \times X - \chi_{_{R,X \times X}}\big) > \mathfrak{T}(U) \wedge \mathfrak{T}(V) > 0$. Hence $\chi_{_{R,X \times X}}$ is an LG-closed subset of $X \times X$.

Conversely, suppose that $\chi_{R,X\times X}$ is an LG-closed subset of $X \times X$. Then $\big(X \times X - \chi_{R,X\times X}\big)$ is an LG-open subset. By Theorem 3.11,

$$supp\big(X \times X - \chi_{R,X\times X}\big) \in (\mathfrak{T} \times \mathfrak{T})^{\triangleleft}.$$

Hence $supp(X \times X) - R$ is an ordinary open subset. So for each $(x, y) \in \big(supp(X \times X) - R\big)$, there exists an open subsets $U \times V \in (\mathfrak{T} \times \mathfrak{T})^{\triangleleft}$ such that $(x, y) \in U \times V \subseteq \big(supp(X \times X) - R\big)$. We show that $U \cap V = \phi$

$$a \in U \cap V \quad \Rightarrow \quad a \sim a \ \ and \ \ (a, a) \in U \times V \quad \Rightarrow \quad (a, a) \in (U \times V) \cap R,$$

that is a contradition. It means that for each $(x, y) \in supp(X \times X)$ where $(x, y) \notin R$, there exist $U,\ V \in supp\mathfrak{T}$, such that $U \cap V = \phi$ and $(x, y) \in U \times V$. Since $\sim$ is an LG-open relation, then we have $\mathfrak{T}(\frac{U}{\sim}) \geq 0$ and $\mathfrak{T}(\frac{V}{\sim}) \geq 0$. Therefore for each $[x],\ [y] \in \frac{X}{\sim}$ where $[x] \neq [y]$, there exist $\frac{U}{\sim},\ \frac{V}{\sim} \in supp\frac{\mathfrak{T}}{\sim}$ such that $[x] \in \frac{U}{\sim},\ [y] \in \frac{V}{\sim}$ and $\frac{U}{\sim} \cap \frac{V}{\sim} = \phi$. So $(\frac{X}{\sim},\ \frac{\mathfrak{T}}{\sim})$ is a Hausdorff L-gfts. □

5 LG-fuzzy product manifolds

The concept of the product of fuzzy topological spaces was introduced by C. K. Wong [19] and later by Hutton [8]. We define and investigate LG-fuzzy product manifolds by the following theorem:

Theorem 5.1. *Let $X \in L^{M_1},\ X_2 \in L^{M_2}$ and $(X_1, \mathfrak{T}_1),\ (X_2, \mathfrak{T}_2)$ be two LG-fuzzy manifolds of dimensions $m,\ n$ and with the LG-fuzzy structures $\mathfrak{A}_i = \{(A_{\alpha_i}, \psi_{\alpha_i})|\ \alpha_i \in K_i\},\ i = 1, 2$ respectively. Then $(X_1 \times X_2,\ \mathfrak{T}_1 \times \mathfrak{T}_2)$ is an LG-fuzzy manifold of dimension $m + n$.*

Proof. We define for all $A_1 \in L^{M_1}_{X_1},\ \ A_2 \in L^{M_2}_{X_2}$:

$$A_1 \times A_2 \in L^{M_1\times M_2}_{X_1\times X_2}, \quad (A_1 \times A_2)(x, y) = A_1(x) \wedge A_2(y),$$

and

$$(\mathfrak{T}_1 \times \mathfrak{T}_2)(A_1 \times A_2) = \mathfrak{T}_1(A_1) \wedge \mathfrak{T}_2(A_2).$$

It can be verified that $\mathfrak{T}_1 \times \mathfrak{T}_2$ is an L-gradation of openness on $X_1 \times X_2$. Now let $p_1 \in X_1,\ p_2 \in X_2$. Then there exist two LG-open subsets A_i of X_i containing $p_i,\ for\ i = 1, 2$ and two LG-open subsets B_i of LG-fuzzy Euclidean spaces of dimension m, n, respectively together with two LGP-homeomorphisms $\psi_i \in (A_i, B_i)$. Therefore for any $(p_1, p_2) \in X_1 \times X_2$, there exists an LG-open subset $A_1 \times A_2$ of $X_1 \times X_2$ containing $(p_1,\ p_2)$ and an LG-open subset $B_1 \times B_2$ of LG-fuzzy Euclidean space of dimension $m+n$, together with an LGP-homeomorphism $\psi_1 \times \psi_2 \in (A_1 \times A_2,\ B_1 \times B_2)$ such that for each LG-open subsets $H_1 \leq B_1,\ \ H_2 \leq B_2$ we have

$$\mathfrak{T}_{L(m+n)}\ (H_1 \times H_2) = (\mathfrak{T}_1 \times \mathfrak{T}_2)\ \big(\psi_1^{-1}(H_1) \times \psi_2^{-1}(H_2)\big),$$

where $(\psi_1 \times \psi_2)\ (x_1,\ x_2) = (\psi_1(x_1),\ \psi_2(x_2)) \in \mathbb{R}^{m+n}$. One can prove easily that

$$\mathfrak{A}_1 \times \mathfrak{A}_2 = \big\{\big((A_{\alpha_1} \times A_{\alpha_2}), (\psi_{\alpha_1} \times \psi_{\alpha_2})\big) \mid \alpha_1 \in K_1,\ \alpha_2 \in K_2\big\},$$

is an LG-fuzzy structure on $X_1 \times X_2$. □

Example 5.2. *Let $M = \mathbb{R}^2,\ \ X = \chi_{S^1}$. One can easily prove that $(X, \mathfrak{T}_{I3})$ is an LG-fuzzy manifold of dimension 1, similarly to Example 3.6. Then $(X \times X,\ \mathfrak{T}_{I3} \times \mathfrak{T}_{I3})$ is an LG-fuzzy manifold of dimensions 2 and $supp(X \times X) = S^1 \times S^1$.*

6 C^∞ LG-fuzzy mappings of LG-fuzzy manifolds

The concept of the fuzzy vector space (V, η) over a field F was defined in [18]. We extend this definition by L-fuzzificaion:

Definition 6.1. *An L-fuzzy vector space (V, η) or ηV over a field F is an ordinary vector space V over the field F, with a map $\eta : V \to L$ satisfying the following conditions for all $a, b \in V$ and $r \in F$.*

1) $\eta(a + b) \geq min\{\eta(a),\ \eta(b)\}$,

2) $\eta(-a) = \eta(a)$,

3) $\eta(0) = 1$,

4) $\eta(ra) \geq \eta(a)$,

Definition 6.2. *Let* $(X, \mathfrak{T})$ *be an LG-fuzzy manifold of dimension* n, $U \in supp\mathfrak{T}$ *and* $V \in supp\mathfrak{T}_{L1}$. *The LG-related function* f *from* U *to* V, *is called a* C^∞ *LG-fuzzy mapping, if for every* $p \in U$,

$$\hat{f} = f \circ \psi^{-1} : \psi(supp(A \cap U)) \rightarrow suppV,$$

is C^∞ *where* (A, ψ) *is an LG-local coordinate neighborhood of* p.

We denote the set of all C^∞ LG-fuzzy mappings from an LG-open subset U of X, containing p to $1_{\mathbb{R}}$, by $C_L^\infty(p)$. If we define $\eta : C_L^\infty(p) \rightarrow L$, $\eta(f) = A(p)$, where (A, ψ) is an LG-coordinate neighborhood of p, then $C_L^\infty(p)$ may be considered as an L-fuzzy vector space $(C_L^\infty(p), \eta)$. Let $\psi(q) = (x_1, \ldots, x_n)$, $\forall q \in supp(A \cap U)$. Then $\hat{f}(x_1, \ldots, x_n) = y(q)$, and since $\hat{f}$ is C^∞, there exist all partial derivatives of any order of y.

Example 6.3. *In Example 3.8, if we define* $f : \mathcal{M}_{m\times n} \rightarrow \mathbb{R}$, $f((a_{ij})) = det((a_{ij}))$, *then using the single IG-local coordinate neighborhood* $(Z, \psi|_Z)$, *we have* $\hat{f} = f \circ \psi$ *is* C^∞. *Hence* $f \in C_I^\infty(p)$ *for all* $p \in Z$.

From now on, we suppose that M_1, M_2 are two crisp sets, $X \in L^{M_1}$, $Y \in L^{M_2}$ such that $(X, \mathfrak{T})$, $(Y, \mathfrak{R})$ are two LG-fuzzy manifolds of dimension n, m and LG-fuzzy structures $\mathfrak{A} = \{(A_i, \psi_i),\ i \in K\}$ *and* $\mathfrak{D} = \{(D_j, \varphi_j),\ j \in J\}$ respectively and $U \in supp\mathfrak{T}$, $V \in supp\mathfrak{D}$.

Definition 6.4. *An LG-fuzzy function* $F \in LGRf(U, V)$ *is a* C^∞ *LG-fuzzy mapping if for every* $p \in U$,

$$\hat{F} = \varphi \circ F \circ \psi^{-1} : \psi(supp(A \cap U)) \rightarrow \varphi(supp(B \cap V)),$$

is C^∞ *where* (A, ψ), (B, φ) *are LG-local coordinate neighborhoods of* p *and* $F(p)$ *respectively.* $F \in LGRf(U, V)$ *is called a LG-diffeomorphism if it is an LG-homeomorphism and* F, F^{-1} *are* C^∞.

More precisely, if $\psi(q) = (x_1, \ldots, x_n)$, $\forall q \in supp(A \cap U)$ and $\varphi(w) = (y_1, \ldots, y_m)$, $\forall w \in B$, then

$$\hat{F}(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)),$$

and each $y_i = f_i(x_1, \ldots, x_n)$ is C^∞ on $\psi(A)$.

Definition 6.5. *The rank of* $F \in LGRf(X, Y)$ *at* p *is equal to the rank at* $x = \psi(q)$ *of the Jacobian matrix:*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}_x .$$

Example 6.6. *Let* $M_1 = M_2 = \mathbb{R}^2$, $L = I$ *and* $X : M_1 \rightarrow I$, $Y : M_2 \rightarrow I$ *be defined by:*

$$X(x_1, x_2) = \begin{cases} 1 & \|x\| = 1, \\ 0 & \|x\| \neq 1, \end{cases} \quad and \quad Y(y_1, y_2) = \begin{cases} 1 & \|y\| = 1, \\ 0 & \|y\| \neq 1. \end{cases}$$

If we define $\mathfrak{T} : I_X^{M_1} \rightarrow I$, *and* $\mathfrak{R} : I_Y^{M_2} \rightarrow I$, *by*

$$\mathfrak{T}(A) = \begin{cases} 1 & A \in supp\mathfrak{T}_{I2},\ A \leq X, \\ 0 & elsewhere. \end{cases} \quad and \quad \mathfrak{R}(D) = \begin{cases} 1 & D \in supp\mathfrak{T}_{I2},\ D \leq Y, \\ 0 & elsewhere. \end{cases}$$

In a similar manner to the Example 3.5, we can prove that $(X, \mathfrak{T})$, $(Y, \mathfrak{R})$ *are IG-fuzzy manifolds. Let*

$$F : M_1 \rightarrow M_2, \quad F(x_1,\ x_2) = (x_1 - x_2,\ \sqrt{2x_1x_2}).$$

We prove that $F \in IGRf(X, Y)$ *is a* C^∞ *IG-fuzzy mapping. First we show* $F|suppX : suppX \rightarrow suppY$ *is well-defined and* $F[X] = Y$:

$$(x_1,\ x_2) \in S^1 \Rightarrow\ x_1^2 + x_2^2 = 1 \Rightarrow\ (x_1 - x_2)^2 + (\sqrt{2x_1x_2})^2 = 1 \Rightarrow\ F(x_1,\ x_2) \in S^1,$$

$$F[X](y_1,\ y_2) = \bigvee\{X(x_1,\ x_2) :\ (x_1, x_2) \in F^{-1}(y_1,\ y_2)\} = 1 = Y(y_1,\ y_2).$$

Let (A_1^+, ψ_1^+), $(D_2^+,\ \varphi_2^+)$ *be IG-local coordinate neighborhoods on* X, Y *respectively, then we see:*

$$\varphi_2^+ \circ F \circ (\psi_1^+)^{-1}(y) = \varphi_2^+ \circ F(\sqrt{1 - y^2}\ ,\ y) = \varphi_2^+(\sqrt{1 - y^2}\ - y,\ \sqrt{2y\sqrt{1 - y^2}}), = \sqrt{1 - y^2}\ - y,$$

is C^∞. *Similarly, one can show that* $\varphi_i^\pm \circ F \circ (\psi_j^\pm)^{-1}$ *is* C^∞ *for all* $i, j = 1, 2$.

Example 6.7. *Let* $F:\mathbb{R}\to\mathbb{R}^2,\ F(t)=(cos(t-\frac{\pi}{2}),\ sin(t-\frac{\pi}{2}))$. *Then* $F\in IGPRf(1_{\mathbb{R}},\ 1_{\mathbb{R}^2})$ *and* $\operatorname{rank}F=1$ *at every point of* X.

Theorem 6.8. *(LG-fuzzy rank theorem) Let* $F\in LGRf(U,Y)$ *be a* C^∞ *fuzzy mapping and* $\operatorname{rank}F=k$ *at every point of* X. *If* $p\in X$, *then there exist LG-local coordinate neighborhoods* (A,ψ), (B,φ) *such that*

$$\psi(p)=(0,\ldots,0)\in\mathbb{R}^n,\ \ \varphi(F(p))=(0,\ldots,0)\in\mathbb{R}^m,$$

and $\hat{F}=\varphi\circ F\circ\psi^{-1}$ *is given by:*

$$\hat{F}(x_1,\ \ldots,\ x_n)=(x_1,\ \ldots,\ x_k,0,\ldots,0). \tag{4}$$

Proof. Using Theorem 3.11, we see that $(suppX,\mathfrak{T}^{\triangleleft})$ and $(suppY,\mathfrak{R}^{\triangleleft})$ are two topological manifolds of dimension n,m with the structures $\mathfrak{A}^{\triangleleft}=\{(suppA_i,\ \psi_i|_{suppA_i}),\ i\in K\}$ *and* $\mathfrak{D}^{\triangleleft}=\{(suppD_j,\ \varphi_j|_{suppD_j}),\ j\in J\}$ respectively. Also $F|_{suppX}:suppX\to suppY$ is a C^∞ mapping and rank$F=k$ at every point of X. Fix $p\in suppX$, then by the rank theorem, there exist coordinate neighborhoods $(suppA,\ \psi|_{suppA})$, $(suppD,\ \varphi|_{suppD})$ of p *and* $F(p)$ respectively such that

$$\psi|_{suppA}(p)=(0,\ldots,0)\in\mathbb{R}^n,\quad \varphi|_{suppD}(F|_{suppX}(p))=(0,\ldots,0)\in\mathbb{R}^m,$$

and $\hat{F}|_{\psi(suppA)}=\varphi|_{suppD}\circ F|_{suppA}\circ\psi^{-1}|_{\psi(suppA)}$ is given by:

$$\hat{F}|_{\psi(suppA)}(x_1,\ \ldots,\ x_n)=(x_1,\ \ldots,\ x_k,0,\ldots,0).$$

Therefore the LG-fuzzy rank theorem holds for LG-fuzzy manifolds. □

Remark 6.9. *We can cover* X *and* $\tilde{X}=F[X]$ *by these LG-local coordinate neighborhoods* $\mathfrak{A}=\{(A_s,\psi_s)|\ s\in S\}$, *and* $\mathfrak{D}=\{(D_s,\varphi_s)|\ s\in S\}$ *respectively where* $S\subseteq K$. *Since* $\mathfrak{A}$ *is an LG-structure of* X, *one can show that* $\mathfrak{D}$ *is an LG-structure of* $F[X]$. *If* F *is an LG-diffeomorphism, then we have* $\operatorname{rank}F=\dim X=\dim Y$.

Definition 6.10. *The* C^∞ *L-related function* $F\in LGRf(X,Y)$ *is an LG-fuzzy immersion (submersion) if* $\operatorname{rank}F=\dim X(=\dim Y)$. *at every point of* X.

Theorem 6.11. *Let* $F\in LGRf(X,Y)$ *be a* C^∞ *LG-fuzzy mapping. If* F *is an injective LG-fuzzy immersion, then* $(\tilde{X},\tilde{\mathfrak{T}})$ *is an LG-fuzzy submanifold of dimension* n, *called an LG-fuzzy immersed submanifold and* $F\in LGRf(X,\tilde{X})$ *is an LG-diffeomorphism.*

Proof. F establishes a one-to-one correspondence between $suppX$ and $F(suppX)$. Thus, $F\in LGRf(X,\tilde{X})$ is one-to-one and onto. Since for each $q\in F(suppX)$, there exists only one $p\in suppX$ such that $F^{-1}(q)=\{p\}$, hence

$$\tilde{X}(q)=F[X](q)=sup\{X(a)|\ F(a)=q\}=X(p).$$

Since $F\in LGRf(X,Y)$, we have $F[X]\leq Y$; therefore $\tilde{X}\leq Y$. Hence $\tilde{X}$ is an LG-fuzzy subset of Y. We use F to endow $\tilde{X}$ with an LG-structure

$$\tilde{\mathfrak{D}}=\{(D_s|_{\tilde{X}},\ \pi\circ\varphi_s|_{\tilde{X}})\ |\ (D_s,\ \varphi_s)\in\mathfrak{D},\ \ \forall s\in S\},$$

where $\pi(y_1,\ \ldots,\ y_m)=(y_1,\ \ldots,y_n)$ is the projection and an LG-fuzzy topology

$$\tilde{\mathfrak{T}}:L_{\tilde{X}}^{M_2}\to L,\quad \tilde{\mathfrak{T}}(H)=\mathfrak{T}(F^{-1}[H]).$$

Then $(\tilde{X},\tilde{\mathfrak{T}})$ is an LG-fuzzy manifold of dimension n, called an LG-fuzzy immersed submanifold and $F\in LGRf(X,\tilde{X})$ is an L-gradation-preserving. Therefore F is an LGP-diffeomorphism. □

In general, gradation of openness $\tilde{\mathfrak{T}}$ and the LG-fuzzy structure on $\tilde{X}$ depend on F as well as X, i.e. $(\tilde{X},\ \tilde{\mathfrak{T}})$ is not a submanifold of $(Y,\mathfrak{R})$. So we add the condition of LG-continuity of $F,\ F^{-1}$ in the following definition:

Definition 6.12. *An LG-fuzzy imbedding is a one-to-one LG-fuzzy immersion* $F\in LGRf(X,Y)$ *with* $F\in LGRf(X,\tilde{X})$ *is an LG-homeomorphism from* X *to* $\tilde{X}=F[X]$ *as an LG-fuzzy subspace of* Y. *The image of an LG-fuzzy imbedding is called an LG-fuzzy imbedded submanifold.*

Theorem 6.13. *Let* $F\in LGRf(X,Y)$ *be an LG-fuzzy immersion. Then for each* $p\in X$, *exists an LG-neighborhood* A *of* p *such that* $F|_{suppA}$ *is an LG-fuzzy imbedding.*

Proof. According to Theorem 6.8, we may choose (A,ψ) and (D,φ), the LG-local coordinate neighborhoods of p and $F(p)$, respectively, such that (4) holds. Since $F[A]=D$ and D is an LG-open subset of Y, hence L-gradation of openness $\tilde{\mathfrak{T}}$ of $F[A]$, is the same as its L-gradation of openness $\mathfrak{D}|_D$ as an L-gftss of Y, i.e. $\tilde{\mathfrak{T}}(H)=\mathfrak{T}(F^{-1}[H])=\mathfrak{D}(H)$, for all LG-open subset H of D. On the other hand, ψ *and* φ are LGP-homeomorphisms, hence $\hat{F}$ is an LGP-homeomorphism of $\psi[A]$ and $\varphi[D]$. Therefore $F|_{suppA}$ is a homeomorphism, and thus the theorem holds. □

Example 6.14. *Let $Z=\chi_{(1,+\infty)}$. Then Z is an LG-open subset of $1_{\mathbb{R}}$. If $\mathfrak{T}_Z=\mathfrak{T}_{I1}|_z$, then $(Z,\ \mathfrak{T}_Z)$ is an LG-fuzzy submanifold of $(1_{\mathbb{R}},\ \mathfrak{T}_{I1})$. Let $W=B((0,0),1,1)$, then W is an LG-open subset of $1_{\mathbb{R}^2}$. Consider $F:\mathbb{R}\to\mathbb{R}^2$, $F(t)=(\frac{1}{t}\ cos2\pi t,\ \frac{1}{t}\ sin2\pi t)$. Then $F\in IGRf(Z,\ W)$ and* $\operatorname{rank} F=1$ *at every point of Z. We see $F^{-1}(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$, so F is a one-to-one LG-fuzzy immersion. Since $F\in IGRf(Z,\tilde{Z})$ is an LG-homeomorphism, F is an LG-fuzzy imbedding.*

7 LG-fuzzy submanifolds of LG-fuzzy manifolds

Definition 7.1. *An LG-fuzzy subset N of an LG-fuzzy manifold $(X,\mathfrak{T})$, is said to have the LG-fuzzy k-submanifold property if each $p\in N$ has an LG-local coordinate neighborhood (A,ψ) on X with LG-local coordinates $x_1,x_2,\ldots,x_n$ such that $\psi(p)=(0,\ldots,0)\in\mathbb{R}^n$, and*

$$\psi(suppA\cap suppN)=\big\{(x_1,x_2,\ldots,x_n)\in\psi(A)\mid x_{k+1}=\ \ldots\ =x_n=0\big\}.$$

If N has this property, LG-coordinate neighborhoods of this type are called preferred LG-local coordinates.

Denote by $\pi:\mathbb{R}^n\to\mathbb{R}^k$, $k\le n$, the projection to the first k coordinates. Using the notation above, we may state the following proposition:

Theorem 7.2. *Let $N\le X$ have the LG-fuzzy k-submanifold property. Then each preferred LG-local coordinate system (A,ψ) of X defines an LG-local coordinate neighborhood (A',ψ') on N where $A'=A\cap N$, $\psi'=\pi\circ\psi|_{A'}$. Therefore the inclusion $i\in LGRf(N,X)$ is an LG-fuzzy imbedding.*

Proof. Since N is an LG-open subset of X, thus $(N,\mathfrak{T}_N)$ is an LG-fuzzy topological subspace of X. Then (A',ψ') are LG-coordinate neighborhoods covering N, where $A'=A\cap N$ is an LG-open subset of N and $\psi'=\pi\circ\psi|_{A'}$ is an LGP-homeomorphism. Suppose that for two preferred neighborhoods (A'_1,ψ'_1) and (A'_2,ψ'_2), A'_1,A'_2 have a nonempty intersection. We know that the change of LG-local coordinates is given by LGP-homeomorphisms $\psi'_1\circ\psi_2^{'-1}$ and $\psi'_2\circ\psi_1^{'-1}$ which we must show to be C^∞. Let

$$\gamma(x_1,\ \ldots,\ x_k)=(x_1,\ \ldots,\ x_k,0,\ldots,0)\in\mathbb{R}^n,$$

so that $\pi\circ\gamma$ is the identity on $\mathbb{R}^k$.This map γ is C^∞. Hence its restriction to $\psi'(A')$, an LG-open subset of $\mathbb{R}^k$, is C^∞; thus $\psi'^{-1}=\psi\circ\gamma$ is C^∞, since it is a composition of C^∞ maps. On the other hand, $\psi'=\pi\circ\psi$ so ψ' is a C^∞ map on A'. Hence $\psi'_1\circ\psi_2^{'-1}$ is C^∞. If $y_i=f_i(x_1,\ \ldots,\ x_k), i=1,...,k$, are the functions giving $\psi'_1\circ\psi_2^{'-1}$, which we know to be C^∞, then it can easily be checked that $\psi_1\circ\psi_2^{-1}$ is given by $y_i=f_i(x_1,\ \ldots,\ x_k,0,\ldots,0), i=1,...,k$. Therefore $\psi'_1\circ\psi_2^{'-1}$ is C^∞ by Definition 3.2. Thus the totality of these LG-neighborhoods define a unique differentiable structure on N. In preferred LG-local coordinates (A',ψ'), $\in LGRf(N,X)$ is given on V by

$$\psi\circ i\circ\psi^{'-1}(x_1,\ \ldots,\ x_k)=(x_1,\ \ldots,\ x_k,0,\ldots,0).$$

So the map i is clearly an LG-fuzzy immersion. Because we have taken the relative LG-fuzzy topology on N, the fuzzy map i is by Definition 2.13 (iii) an LG-homeomorphism to its image $i(N)$, with the LG-fuzzy subspace topology, that is, i is an LG-fuzzy imbedding. □

Definition 7.3. *A regular LG-fuzzy submanifold of an LG-fuzzy manifold $(X,\mathfrak{T})$ is any LG-fuzzy topological subspace N with the LG-fuzzy submanifold property and with the structure that the corresponding preferred LG-local coordinate neighborhoods determine on it.*

Example 7.4. *Let $M=\mathbb{R}^3$, $X:\mathbb{R}^3\to I$, $X(x)=\left\{\begin{array}{ll}1 & \|x\|=1,\\ 0 & \|x\|\neq 1.\end{array}\right.$. Then $suppX=S^2$, the unit sphere. Let $\mathfrak{T}:I_X^M\to I$, $\mathfrak{T}(A)=\left\{\begin{array}{ll}1 & A\in\tau_{I^3},\ A\le X\\ 0 & elsewhere.\end{array}\right.$. We shall see that X is an LG-fuzzy submanifold of $(1_{\mathbb{R}^3},\ \mathfrak{T}_{I3})$. If*

$q = (x_1, x_2, x_3)$ is an arbitrary point in supp$X$, it cannot lie on more than one coordinate axis. For convenience, we assume that it does not lie on the x_3-axis. We introduce the spherical LG-local coordinates (ρ, θ, φ); they are defined on $1_{\mathbb{R}^3-\{x_3-axis\}}$ and if $(1, \theta_0, \varphi_0)$ are the LG-coordinates of q, we may change them a little so that it is replaced by $\tilde{\rho} = \rho - 1$, $\tilde{\theta} = \theta - \theta_0$, and $\tilde{\varphi} = \varphi - \varphi_0$. Then it defines an LG-coordinate neighborhood of q, with q having LG-coordinates $(0, 0, 0)$ and with the LG-open subset V of X.

Remark 7.5. *So far, we have defined three classes of LG-fuzzy submanifolds of an LG-fuzzy n-manifold $(X, \mathfrak{T})$. The first of these, which we usually simply call an LG-fuzzy submanifold, was defined (in 6.11) as the image $N = F[N']$ of an LG-fuzzy immersion F of N' into X. Since $F : N' \to N \le X$ is one-to-one and onto, we coduct (as part of the definition) carry over to N the LG-fuzzy topology and LG-fuzzy structure of N'; LG-open subsets of N are the images of LG-open sets of N' and LG-coordinate neighborhoods (A, ψ) of N are of the form $A = F[A']$, $\psi = \psi' \circ F^{-1}$, where (A', ψ') is an LG-local coordinate neighborhood of N'. The fact that F is LG-continuous shows that the LG-fuzzy topology of N gained in this way is in general finer than its relative LG-fuzzy topology as an LG-fuzzy subspace of X , that is, if D is LG-open subset of X, then $D \cap N$ is LG-open subset of N, but there may be LG-open subsets of N which are not of this form.*
An LG-fuzzy imbedding is a particular type of LG-fuzzy immersion, one in which A is LG-open subset of N if and only if $A = F[U'] = D \cap N$ for some LG-open subset D of X so that the LG-fuzzy topology of the submanifold $N = F[N']$ is exactly its relative LG-fuzzy topology as an LG-fuzzy topological subspace of X. An LG-fuzzy imbedded submanifold is so a special type of (immersed) LG-fuzzy submanifold.
Ultimately, if $N \le X$ is an LG-fuzzy regular submanifold, then it is also an LG-fuzzy imbedded submanifold since the inclusion $i : N \to M$ is an LG-fuzzy imbedding as we proved in 7.2.

Theorem 7.6. *Let $F \in LGRf(N', X)$ be an LG-fuzzy imbedding of an LG-fuzzy manifold N' of dimension k in an LG-fuzzy manifold of dimension n. Then $N = F[N']$ has the LG-fuzzy k-submanifold property and thus N is an LG-fuzzy regular submanifold. As such, it is LG-diffeomorphic to N' with respect to the LG-fuzzy mapping $F \in LGRf(N', N)$.*

Proof. Let $q = F(p)$ be any point of N. According to Theorem 7.2 (and its proof), there are (A, ψ) and (B, φ), LG-local coordinate neighborhoods of p and $F(p)$, respectively, such that (4) holds. If $F[A] = V \le N$, then the LG-neighborhood V would be a preferred LG-local coordinate neighborhood relative to N. To deduce this result, we should use the fact that F is an LG-imbedding. This denotes at least that $F[A]$ is a relatively LG-open subset of N, that is, $F[A] = W \cap N$, where W is LG-open subset of X. Since $F[A] \le V$, we can suppose $W \le V$. Thus $\varphi[W]$ is an LG-open subset of $\varphi[B]$ containing the origin in $\mathbb{R}^n$ and $\varphi[F[A]] \le \varphi[W]$, which is a slice S of $\varphi[V]$, $S = \{x \in \varphi[V] \mid x_{k+l} = \ldots = x_m = 0\}$. Hence we may select an (smaller) LG-open subset $\varphi[V'] \le \varphi[W]$ *and* $\varphi' = \varphi|_{suppV'}$. This is an LG-local coordinate neighborhood of q for which $F[A] \cap V' = V' \cap N$; furthermore, taking $A' = F^{-1}[V']$, we see that (A', ψ'), with $\psi' = \psi|_{suppA'}$, is an LG-local coordinate neighborhood of p and the pair (A', ψ') and (V', φ') have exactly the properties needed in 7.1 and $F[A'] = V' \le N$. This proves at the same time, that N has the LG-fuzzy k-submanifold property. This is true since the inverse of $F \in LGRf(N', N)$ is given in the preferred LG-local coordinates $(V',\ \pi \circ \varphi')$ and (A', ψ') by $\hat{F}^{-l}(x_l, ,\ \ldots,\ x_k) = (x_1,\ \ldots,\ x_k)$, which is C^∞. □

Remark 7.7. *Suppose that $N \le X$ is an LG-fuzzy immersed submanifold and that $q \in N$. Then there is an LG-neighborhood (V, ψ) of q, with $\psi(p) = (0, \ldots, 0)$ such that the slice $S' \subseteq suppV$, consisting of all points of V whose last $n - k$ coordinates vanish, is an LG-open set and an LG-local coordinate neighborhood of the LG- fuzzy submanifold structure of N is given by LG-local coordinate map*

$$\psi'(q) = \pi \circ \psi(q) = (x_1(q),\ \ldots,\ x_k(q)).$$

Theorem 7.8. *If $F \in LGRf(N, X)$ is a one-to-one LG-fuzzy immersion and N is a compact L-gfts, then F is an LG-fuzzy imbedding and $\tilde{N} = F[N]$ an LG-fuzzy regular submanifold.*

Proof. Since F is LG-continuous and both N and $\tilde{N}$ are Hausdorff L-gfts's, we have an LG-continuous (one-to-one) mapping from a compact L-gfts to a Hausdorff L-gfts. Since an LG-closed subset K of N is compact, so $F(K)$ is compact and therefore LG-closed. Thus F takes LG-closed subsets of N to LG-closed subsets of X, and since F is one-to-one and onto, it takes LG-open subsets to LG-open subsets as well. It follows that F^{-1} is LG-continuous, so $F \in LGRf(N, \tilde{N})$ is an LG-homeomorphism and therefore an LG-imbedding. □

Theorem 7.9. *Let $F \in LGRf(X, Y)$ be a C^∞ LG-fuzzy mapping. Suppose that F has constant* rank *k on X and that $q \in F(X)$. Let D denotes $F^{-1}(q)$; then χ_D is an LG-closed, LG-fuzzy regular submanifold of X of dimension $n - k$.*

Proof. Let $p \in D$; since F has constant rank k on an LG-neighborhood of p, we may find LG-local coordinate neighborhoods (A, ψ), (B, φ) such that (4) holds. By Example 2.9, the fuzzy point 0_1 is an IG-closed subset of $\mathbb{R}^n$, then $\chi_{\{q\}}$, is an LG-closed subset of Y. Hence χ_D is an LG-closed subset since the inverse image of $\chi_{\{q\}}$, under a continuous map, is LG-closed. We shall show that χ_D has the LG-fuzzy $n-k$ submanifold property. This means that the only points of D mapped onto q are those whose first k coordinates are zero, that is,

$$suppA \cap D = \psi^{-1}(\psi \circ F^{-1} \circ \varphi^{-1}(0)) = \psi^{-1}(\hat{F}^{-1}(0)) = \psi^{-1}\{x \in \psi(A) \mid x_1 = ... = x_k = 0\}.$$

Hence χ_D is a regular LG-fuzzy $(n-k)$-submanifold since it has the LG-fuzzy submanifold property. □

Corollary 7.10. *If $F \in LGRf(X, Y)$ is a C^∞ LG-fuzzy mapping of LG-fuzzy manifolds,* $\dim X = n \leq m = \dim Y$, *and* $\operatorname{rank} F = n$ *at every point of $D = F^{-1}(q)$, then χ_D is an LG-closed, regular LG-fuzzy submanifold of X. The corollary holds because at $p \in A$, F has the maximum* rank *possible, namely m. It follows from the independence of* rank *on LG-local coordinates that, in some LG-neighborhood of p in N, F also has this* rank*; thus the* rank *of F is m on an LG-open subset of N containing A. But such an LG-fuzzy subset is itself an LG-fuzzy n-manifold (an LG-open submanifold) to which we may apply the theorem.*

8 Conclusion

In this paper, we generalize all of the fuzzy structures which we have discussed in [14] to L-fuzzy set theory, where $L =< L, \leq, \bigwedge, \bigvee,' >$ denotes a complete distributive lattice with at least two elements. We define the concept of an LG-fuzzy topological space $(X, \mathfrak{T})$ which X is itself an L-fuzzy subset of a crisp set M and $\mathfrak{T}$ is an L-gradation of openness of L-fuzzy subsets of M which are less than or equal to X. Then we define C^∞ L-fuzzy manifolds with L-gradation of openness and C^∞ LG-fuzzy mappings of them such as LG-fuzzy immersions and LG-fuzzy imbeddings. We fuzzify the concept of the product manifolds with L-gradation of openness and define LG-fuzzy quotient manifolds when we have an equivalence relation on M and investigate the conditions of the existence of the quotient manifolds. We also introduce LG-fuzzy immersed, imbedded and regular submanifolds.

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