C∞ L-fuzzy manifolds with L-gradation of openness and C∞ LG-fuzzy mappings of them

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Abstract
In this paper, we generalize all of the fuzzy structures which we have discussed in [14] to L-fuzzy set theory, where \(L = \langle L, \leq, \wedge, \vee', > \) denotes a complete distributive lattice with at least two elements. We define the concept of an LG-fuzzy topological space \((X, \mathfrak{T})\) which \(X\) is itself an L-fuzzy subset of a crisp set \(M\) and \(\mathfrak{T}\) is an \(L\)-gradation of openness of \(L\)-fuzzy subsets of \(M\) which are less than or equal to \(X\). Then we define \(C^\infty\) L-fuzzy manifolds with \(L\)-gradation of openness and \(C^\infty\) LG-fuzzy mappings of them such as \(LG\)-fuzzy immersions and \(LG\)-fuzzy imbeddings. We fuzzify the concept of the product manifolds with \(L\)-gradation of openness and define \(LG\)-fuzzy quotient manifolds when we have an equivalence relation on \(M\) and investigate the conditions of the existence of the quotient manifolds. We also introduce \(LG\)-fuzzy immersed, imbedded and regular submanifolds.

Keywords: \(C^\infty\) LG-fuzzy n-manifolds, \(C^\infty\) LG -fuzzy mappings, LG-fuzzy quotient manifolds, LG-fuzzy immersion, regular LG-fuzzy submanifolds.

1 Introduction
The concept of fuzzy sets was introduced by Zadeh [2]. Then Chang [1] confined his attention to the more basic concepts of general topology and generalized them to fuzzy topological spaces. In his definition, fuzziness in the concept of openness of a fuzzy subset, is absent. In consequence of the development of fuzzy topology, many authors like Wong [19], Lowen [10] introduced various concepts of fuzzy topology. R. Lowen [11] suggested that the properties should be considered fuzzy, that is, one should be able to measure a degree to which a property holds. E. Lowen and R. Lowen [12] considered compactness degrees, and in [20], investigated measures of separation in \([0,1]\)-topological spaces. In 1985, Shostak [15] gave a new definition of fuzzy topology by introducing a concept of gradation of openness of fuzzy subsets of \(X\). Later, Chattopadhyay [2] et al. attempted to introduce a concept of gradation of openness of a fuzzy set of \(X\) by a map \(\tau : I^X \to I\) satisfying three weaker conditions than [15] and later in [9] made a slight modification in their definition and rediscovered the Shostak’s concept of fuzzy topology. Gregori [7] proved that each gradation of openness \(\delta\) is the supremum (infimum) of a strictly increasing (decreasing) sequence of gradations of openness which are equivalent to \(\delta\). Stadler and Vicente [13] have introduced a new concept of fuzzy topological subspace over each fuzzy subset from the fuzzy topology \(\delta\), which coincides with the usual definition in the case that \(\mu = X^Y, Y \subset X\). In [16], Shostak developed a theory of compactness degrees and connectedness degrees in \([0,1]\)-fuzzy topological spaces, and in [17], brought up a theory of degrees of precompactness and completeness in the so-called Hutton fuzzy uniform spaces. In 2016 Ibedou [9] discussed graded fuzzy topological spaces. While all of the researches about the \(C^1\) or \(C^\infty\) fuzzy manifolds, focused on a crisp set, in [14] and in this paper, we demonstrate the possibility of improving current definitions using a new method. In [14], we investigated some properties of a novel fuzzy topological space \((X, \tau)\), where \(X\) is itself a fuzzy subset of a crisp set \(M\). Perhaps the most important generalization of the aforementioned structures in [14], is the consideration of lattice \(L\) beyond the unit interval \(I = [0, 1]\). Let \(L = \langle L, \leq, \wedge, \vee', > \) be a complete distributive lattice set with at least 2 elements; 0 is the bottom element and 1 is the top element of \(L\). An \(L\)-fuzzy
subset \( D \) of the crisp set \( M \), in Goguen’s sense \[6\], is a function \( D : M \rightarrow L \) and is denoted by \( D \in L^M \). In this manuscript, we define the concept of \( L \)-fuzzy topological space \((X, \mathcal{T})\) with the \( L \)-gradation of openness, where \( X \) is an \( L \)-fuzzy subset of a crisp set \( M \). We introduce \( C^\infty \) \( L \)-fuzzy manifolds \((X, \mathcal{T})\) with \( L \)-gradation of openness, called \( C^\infty \) \( LG \)-fuzzy manifolds, with a different perception from \([5]\) and \([4]\) and obtain \( C^\infty \) \( n \)-premanifolds of them. We define \( C^\infty \) \( LG \)-fuzzy mappings of \( C^\infty \) \( LG \)-fuzzy manifolds and prove the \( LG \)-fuzzy rank theorem. Then we define and discuss \( LG \)-fuzzy immersions and \( LGP \)-fuzzy imbedding functions. We proceed to define the \( LG \)-fuzzy immersed, imbedded submanifolds as well as \( LG \)-fuzzy regular submanifolds, and then some theorems about the relations between them are deduced.

2 Preliminaries

**Definition 2.1.** Let \( X \) be an \( L \)-fuzzy subset of \( M \). Then any \( L \)-fuzzy subset of \( M \) which is less than or equal to \( X \) is called an \( L \)-fuzzy subset of \( X \). We denote the set of all \( L \)-fuzzy subsets of \( X \) by \( L^M_X \). If \( \tau \) as a collection of \( L \)-fuzzy subsets of \( X \), satisfies the following conditions, then \((X, \tau)\) is called an \( L \)-fuzzy topological space (\( L \)-fts):

1) \( X, \phi \in \tau \),
2) \( \{A_i\}_{i \in I} \subseteq \tau \Rightarrow \bigcup_{i \in I} A_i \in \tau \),
3) \( A, B \in \tau \Rightarrow A \cap B \in \tau \).

**Example 2.2.** Let \( M = \mathbb{R}^n \) and \( X = 1 \) be a constant \( L \)-fuzzy subset of \( M \). Let \( B(a, r, b) \) be an \( L \)-fuzzy subset that is equal to zero outside or on the sphere \( B(a, r) \) and equal to the function \( b \) with values in \( L \), inside \( B(a, r) \). We call the \( L \)-fuzzy topology induced by

\[
\beta_{L_n} = \{B(a, r, b), a \in \mathbb{R}^n, r \in \mathbb{R}^+, b : B(a, r) \rightarrow L, \text{ is a function}\},
\]

the \( L \)-fuzzy Euclidean topology of dimension \( n \) and denote it by \( \tau_{L_n} \). Therefore we have the \( L \)-fuzzy Euclidean topological space \((1_{\mathbb{R}^n}, \tau_{L_n})\).

**Definition 2.3.** Let \( \mathcal{T} : L^M_X \rightarrow L \), be a mapping satisfying:

i) \( \mathcal{T}(X) = \mathcal{T}(\emptyset) = 1 \),
ii) \( \mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B) \),
iii) \( \mathcal{T}(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathcal{T}(A_j) \).

Then \( \mathcal{T} \) is called an \( L \)-gradation of openness on \( X \) and \( (X, \mathcal{T}) \) is called an \( LG \)-fuzzy topological space (\( L \)-gfts). Let \( x \in M \) and \( A \in L^M_X \). When we write \( x \in A \), we mean \( x \in \text{supp}A \).

**Example 2.4.** Let \( M = \mathbb{R}^n \) and \( X = 1 \) be a constant \( L \)-fuzzy subset of \( M \). As three useful examples, we define

\[
\mathcal{T}_{L_n} : L^M_X \rightarrow L, \quad \mathcal{T}_{L_n}(B) = \begin{cases} 1 & B \in \tau_{L_n}, \\ 0 & \text{elsewhere}, \end{cases}
\]

and

\[
\mathcal{T}_{Lsup} : L^M_X \rightarrow L, \quad \mathcal{T}_{Lsup}(B) = \begin{cases} 1 & \sup\{B(x) : x \in M\} B = \emptyset, \\ 0 & B \neq \emptyset \in \tau_{L_n}, \text{ elsewhere}, \end{cases}
\]

If we set “inf” instead of “sup” in the above definition, then we have \( L \)-gradation of openness \( \mathcal{T}_{Linf} \).

Let \( \mathcal{T}_{L_n} \) be any \( L \)-gradation of openness on \( 1_{\mathbb{R}^n} \), such that \( \text{supp}\mathcal{T} = \tau_{L_n} \), then we call \((1_{\mathbb{R}^n}, \mathcal{T}_{L_n})\) the \( LG \)-fuzzy Euclidean topological space.

**Definition 2.5.** Let \((X, \mathcal{T})\) be an \( L \)-gfts. Set \( \text{supp}\mathcal{T} = \{A \in L^M_X : \mathcal{T}(A) > 0\} \), then \( A \) is called an \( LG \)-open subset of \( X \) if \( A \in \text{supp}\mathcal{T} \). Furthermore

1) Suppose \( x \in X \) and \( V \in L^M_X \). If there exists an \( LG \)-open subset \( U \) of \( X \) such that \( U(x) = V(x) \) and \( U \leq V \), then \( V \) is called an \( LG \)-neighborhood of \( x \) in \( X \). We denote the set of all \( LG \)-neighborhoods of \( x \) in \( X \) by \( LGN(x) \).
2) If for all \(x, y \in X\), \(x \neq y\), there exist two \(L\)-neighborhoods \(U_x \in \text{LGN}(x)\), \(U_y \in \text{LGN}(y)\) such that \(U_x \cap U_y = 0\). Then \((X, \mathfrak{T})\) is called a Hausdorff \(L\)-gtfs.

3) For each \(L\)-fuzzy subset \(A\) of \(X\) and any \(U \subset \supp X\), we define the \(L\)-fuzzy subset \(\chi_{U, A}\) of \(X\) by:

\[
\chi_{U, A}(z) = \begin{cases} A(z) & z \in U, \\ 0 & \text{elsewhere.} \end{cases}
\]

From now on, we write \(\chi_u\) instead of \(\chi_{U, X}\).

4) \(A\) is called an \(L\)-closed subset of \(X\) if \(X - A \in \supp \mathfrak{T}\).

5) Let \(Z\) be an \(L\)-open subset of \(X\). Define \(\mathfrak{T}_Z : L^M_X \rightarrow L\), by \(\mathfrak{T}_Z(A) = \mathfrak{T}(A)\). Then \((Z, \mathfrak{T}_Z)\) is called an \(L\)-fuzzy topological subspace of \(X\) (\(L\)-gtfss).

**Definition 2.6.** If \(\mathfrak{C} : L^M_X \rightarrow L\), satisfies the following conditions:

1) \(\mathfrak{C}(X) = \mathfrak{C}(\tilde{0}) = 1\).
2) \(\mathfrak{C}(A \cup B) \geq \mathfrak{C}(A) \wedge \mathfrak{C}(B)\).
3) \(\mathfrak{C}\left(\bigcap_{j \in J} A_j\right) \geq \bigwedge_{j \in J} \mathfrak{C}(A_j)\).

Then \(\mathfrak{C}\) is called an \(L\)-gradation of closedness on \(X\).

**Proposition 2.7.** Let \(\mathfrak{C}\) and \(\mathfrak{T}\) be \(L\)-gradations of closedness and openness respectively on \(X\). Then

1) The mapping \(\mathfrak{T}_{\mathfrak{C}} : L^M_X \rightarrow L\), defined by \(\mathfrak{T}_{\mathfrak{C}}(A) = \mathfrak{C}(X - A)\), is an \(L\)-gradation of openness on \(X\), where \((X - A)\) is an \(L\)-fuzzy subset of \(M\) defined by \((X - A)(p) = X(p) - A(p)\).
2) The mapping \(\mathfrak{C}_{\mathfrak{T}} : L^M_X \rightarrow L\), defined by \(\mathfrak{C}_{\mathfrak{T}}(A) = \mathfrak{T}(X - A)\), is an \(L\)-gradation of closedness on \(X\).
3) We have \(\mathfrak{T}_{\mathfrak{C}_\mathfrak{T}} = \mathfrak{C}\), \(\mathfrak{C}_{\mathfrak{T}_{\mathfrak{C}}} = \mathfrak{T}\).

The proof is straightforward.

**Proposition 2.8.** Let \(\mathfrak{M}_{\mathfrak{T}}(X)\) be the set of all \(L\)-gradations of openness on \(X\). We write \(\mathfrak{T}_1 \leq \mathfrak{T}_2\), if we have \(\mathfrak{T}_1(A) \leq \mathfrak{T}_2(A)\), \(\forall A \in L^M_X\). Then \((\mathfrak{M}_{\mathfrak{T}}(X), \leq)\) is a complete lattice.

**Proof.** It is clear that the relation \(\leq\) between the functions from \(L^M_X\) to \(L\), is an equivalence relation. Therefore \((\mathfrak{M}_{\mathfrak{T}}(X), \leq)\) is a partially ordered set. Further we define two mappings \(\mathfrak{T}_0, \mathfrak{T}_1 : L^M_X \rightarrow L\), by

\[
\mathfrak{T}_0(\tilde{0}) = \mathfrak{T}_0(X) = 1, \quad \mathfrak{T}_0(A) = 0, \quad \forall A \in L^M_X - \{\tilde{0}, X\}, \quad \mathfrak{T}_1(A) = 1, \quad \forall A \in L^M_X.
\]

Then \(\mathfrak{T}_0, \mathfrak{T}_1\) are two \(L\)-gradations of openness on \(X\) and we have:

\[
\mathfrak{T}_0(A) \leq \mathfrak{T}(A) \leq \mathfrak{T}_1(A), \quad \forall A \in L^M_X.
\]

Hence \(\mathfrak{T}_0, \mathfrak{T}_1\) are minimal and maximal elements of \(\mathfrak{M}_{\mathfrak{T}}(X)\), respectively.

An arbitrary intersection of gradations of openness on \(X\), is a gradation of openness. Thus any subset of \(\mathfrak{M}_{\mathfrak{T}}(X)\), has a lower bound in it. To prove this, let \(\{\mathfrak{T}_k, k \in K\}\), be an arbitrary family of \(L\)-gradations of openness on \(X\). We show that \(\mathfrak{T} = \bigwedge_{k \in K} \mathfrak{T}_k\) is an \(L\)-gradation of openness on \(X\). Obviously, \(\mathfrak{T}(X) = \mathfrak{T}(\tilde{0}) = 1\). Also,

\[
\mathfrak{T}\left(\bigcup_j A_j\right) = \bigwedge_k \mathfrak{T}_k\left(\bigcup_j A_j\right) \geq \bigwedge\left(\bigwedge_k \mathfrak{T}_k(A_j)\right) = \bigwedge\left(\bigwedge_k \mathfrak{T}_k(A_j)\right) = \bigwedge_j (\mathfrak{T}(A_j),
\]

and

\[
\mathfrak{T}(A \cap B) = \bigwedge_k \mathfrak{T}_k(A \cap B) \geq \bigwedge_k (\mathfrak{T}_k(A) \wedge \mathfrak{T}_k(B)) \geq \bigwedge_k \mathfrak{T}_k(A) \wedge \bigwedge_k \mathfrak{T}_k(B) \geq \mathfrak{T}(A) \wedge \mathfrak{T}(B).
\]

This completes the proof. \(\square\)
Example 2.9. Consider \((1_{\mathbb{R}^n}, \mathcal{I}_{1n})\) and \(0 \in \mathbb{R}^n\). We show that the fuzzy point \(0_1\) is an IG-closed subset of \(1_{\mathbb{R}^n}\): The fuzzy point \(0_1 = \chi_{(0)}\) is an I-fuzzy subset of \(\mathbb{R}^n\). So,

\[
(1_{\mathbb{R}^n} - 0_1)(x) = 1 - \chi_{(0)}(x) = \begin{cases} 1 & x \neq 0, \\ 0 & x = 0. \end{cases}
\]

Therefore,

\[
(1_{\mathbb{R}^n} - 0_1)(x) = \bigcup_{0 \leq k \in \mathbb{Z}} B(k, 1, 1)(x).
\]

Hence, \((1_{\mathbb{R}^n} - 0_1) \in \tau_{1n}\). Thus \(\mathcal{I}_{1n}(1_{\mathbb{R}^n} - 0_1) \geq 0\). So, \(1_{\mathbb{R}^n} - 0_1\) is an IG-open set. Hence, \(0_1\) is an IG-closed subset.

Definition 2.10. Let \((X, \mathfrak{S})\) be a fuzzy topological space and \(A, B\) be any fuzzy subsets of \(X\),

1) A fuzzy subset \(V\) of \(X\) is called an LG-neighborhood of \(A\) if there exists an LG-open subset \(U\) such that \(A \subseteq U \subseteq V\). We denote the set of all LG-neighborhoods of \(A\) by \(\text{LGN}(A)\).

2) Let \(B \subseteq A\). Then \(B\) is called an LG-interior set of \(A\) if \(A \in \text{LGN}(B)\). The union of all LG-interior sets of \(A\) is denoted by \(\text{LGA}\).

3) The intersection of all LG-closed subsets containing \(A\) is called an LG-closure of \(A\) and is denoted by \(\text{LGC}(A)\).

4) \(x\) is called an LG-boundary point of \(A\) if for every LG-neighborhood \(V\) of \(x\), we have \(V \nsubseteq A\). The set of these points is called an LG-boundary of \(A\) and is denoted by \(\text{LG∂}(A)\).

5) If \(x\) belongs to the LG-closure of \(A - \chi(x), A\), then \(x\) is called an LG-limited point of \(A\) and the set of these points is denoted by \(\text{LGL}(A)\).

6) \(A\) is said to be an LG-dense subset of \(X\), if \(\text{LG∂}(X) = X\).

From now on, we suppose that \(M_1, M_2\) are two crisp sets, \(X \in L^{M_1}, Y \in L^{M_2}\) and \((X, \mathfrak{S}), (Y, \mathcal{R})\) are two LG-fuzzy topological spaces.

Definition 2.11. Let \(f : M_1 \rightarrow M_2\) be a function and \(f\{X\}\) be an L-fuzzy subset of \(M_2\), defined by

\[
f\{X\}(y) = \bigvee \{X(x) \mid x \in f^{-1}(y)\}.
\]

If we have \(f\{X\} \leq Y\), then \(f\) is called an LG-related function from \(X\) to \(Y\) and the set of all such functions is denoted by \(\text{LGRf}(X, Y)\). Furthermore, if we have \(\mathcal{R}(H) \leq \mathfrak{S}(f^{-1}[H])\) for all LG-fuzzy subset \(H\) of \(Y\), then \(f\) is an L-gradation-preserving LG-related function so it is called an LGP-related function or LGP-fuzzy mapping from \(X\) to \(Y\), \(f \in \text{LGRf}(X, Y)\).

i) \(f\) is called a one-to-one LG-related (LGP-related) function if \(f|\supp X : \supp X \rightarrow \supp Y\) is a one-to-one function.

ii) \(f\) is called an onto LG-related (LGP-related) function if \(f\{X\} = Y\).

Remark 2.12. Let \(A \in \supp \mathfrak{S}\) and \(B \in \supp \mathcal{R}\). Let \(f\) be an LGP-fuzzy mapping from \(X\) to \(Y\) such that \(f[A] \leq B\). Then we have \(\mathcal{R}(H) \leq \mathfrak{S}(f^{-1}[H])\) for each LG-fuzzy subset \(H\) of \(Y\) and in particular \(H \leq B\). Thus \(\mathcal{R}(H) \leq \mathfrak{S}(f^{-1}[H])\) for each LG-fuzzy subset \(H\) of \(Y\) with \(H \leq B\). Therefore \(f\) can be considered as an LGP-fuzzy mapping of two LGft's, \((A, \mathfrak{S}_A)\) and \((B, \mathcal{R}_B)\). So we can write \(f \in \text{LGRf}(A, B)\).

Definition 2.13. Let \(f \in \text{LGRf}(X, Y)\), then

i) \(f\) is called LG-open if \(f[A] \subseteq \supp \mathcal{R} - \{0, Y\}, \forall A \in \supp \mathfrak{S} - \{0, X\}\) and \(f[X] \subseteq \supp \mathcal{R}\).

ii) \(f\) is called LG-continuous if \(f^{-1}[H] \subseteq \supp \mathfrak{S} - \{0, X\}, \forall H \in \supp \mathcal{R} - \{0, Y\}\) and \(f^{-1}[Y] \subseteq \supp \mathfrak{S}\).

iii) \(f\) is called an LG-homeomorphism if it is one-to-one, onto, LG-continuous, LG-open and \(f^{-1} \in \text{LGRf}(Y, X)\).

iv) \(f\) is called an LGP-homeomorphism if it is bijective and \(f, f^{-1}\) are LGP-fuzzy mapping.

Proposition 2.14. Let \(A, B\) be LG-open subsets of \(X, Y\) respectively. Let \(\psi : M_1 \rightarrow M_2\) be a function. Then \(\psi\) is an LGP-homeomorphism from \(A\) to \(B\) if and only if \(\psi\) satisfies the two following conditions:
\[ (LGPRf) \]

\[ (\forall q \in B, \exists p \in A : \psi^{-1}(q) = \{ p \}) \]  

So we have \( \psi[A] = \sup \{ A(\alpha) \mid \alpha \in \psi^{-1}(q) \} = A(p) \). On the other hand by Definition 2.11 we have \( \psi[A] \leq B \). Hence \( A(p) \leq B(q) \). We see \( \psi^{-1}[B](p) = B(\psi(p)) = B(q) \). Since by Definition 2.13 (iv), we have \( \psi^{-1} \in LGPRf(Y, X) \), then \( \psi^{-1}[B] \leq A \). Hence \( B(q) \leq A(p) \). Therefore \( A(p) = B(q) \). Therefore \( A(p) = B(\psi(p)) \), for all \( p \in A \) and \( B(q) = A\psi^{-1}(q) \), for all \( q \in B \). Thus \( A = \psi^{-1}[B] \) and \( \psi[A] = B \).

ii) Since \( \psi \in LGPRf(A, B) \), we have \( \mathcal{R}(H) \leq \mathcal{T}(\psi^{-1}[H]) \) for all \( LG-fuzzy \) subset \( H \) of \( Y \), and since \( \psi^{-1} \in LGPRf(B, A) \), we have \( \mathcal{T}(D) \leq \mathcal{R}(\psi[D]) \). Set \( \psi[D] = H \). Then \( D = \psi^{-1}[H] \) by injectivity of \( \psi \). So \( \mathcal{T}(D) \leq \mathcal{R}(H) \).

Hence we have \( \mathcal{T}(\psi^{-1}[H]) = \mathcal{R}(H) \).

\[ \Box \]

**Proposition 2.15.** Every \( LG-fuzzy \) mapping from \( X \) to \( Y \) is an \( LG-continuous \) related function, but the converse is not true.

**Proof.** Let \( f \) be an \( LG-fuzzy \) mapping from \( X \) to \( Y \), then \( \forall H \in \supp \mathcal{R} \setminus \{ 0, Y \} \), we have \( 0 < \mathcal{R}(H) \leq \mathcal{T}(f^{-1}[H]) \).

Hence \( f^{-1}[H] \in \supp \mathcal{T} \setminus \{ 0, X \} \). Therefore \( f \) is \( LG-continuous \).

Conversely, we define an \( LG-continuous \) function which is not an \( LG-fuzzy \) mapping: Following Example 2.4 consider \( f = id : (\mathbb{R}^n, \mathcal{T}_L) \rightarrow (\mathbb{R}^n, \mathcal{T}_{L,S}). \) Since \( f[\mathbb{R}^n] = \mathbb{R}^n \) and we have \( f^{-1}[H] = H \in \supp \mathcal{T}_L \setminus \{ 0, X \} = \tau_L \setminus \{ 0, 1_{\mathbb{R}^n} \} \), \( \forall H \in \supp \mathcal{T}_{L,S} \setminus \{ 0, Y \} = \tau_{L,S} \setminus \{ 0, 1_{\mathbb{R}^n} \} \), and \( f^{-1}[1_{\mathbb{R}^n}] \in \supp \mathcal{T}_L \).

Therefore \( f \) is \( LG-continuous \). Now let \( A = \left\{ \begin{array}{ll} x^2 & \text{if } x \in (0, \frac{1}{2}) \\ 0 & \text{elsewhere.} \end{array} \right. \) Then \( A \in \tau_L \) and \( \mathcal{T}_{L,S}(f[A]) = \frac{1}{2} \). But \( \mathcal{T}_L(A) = 1 \). Hence the condition \( \mathcal{T}_L(A) \leq \mathcal{T}_{L,S}(f[A]) \) does not hold. Hence \( f \) is not an \( LG-fuzzy \) mapping.

\[ \Box \]

### 3 **L-fuzzy topological manifolds with L-gradation of openness**

**Definition 3.1.** Let \( \mathcal{T} \) be an \( L-gradation \) of openness on \( X \). Then \( (X, \mathcal{T}) \) is an \( LG-fuzzy \) topological space of dimension \( n \), if for any \( x \in X \), there exists an \( LG-open \) subset \( A \) of \( X \) containing \( x \) and an \( LG-open \) subset \( B \) of \( \{ \mathbb{R}_n, \mathcal{T}_L \} \), together with an \( LG-fuzzy \) topological \( \psi \in LGPRf(A, B) \). The pair \( (A, \psi) \) is called an \( LG-local \) coordinate neighborhood of each \( q \in A \) and we assign to \( q \) the \( n \) \( LG-local \) coordinates \( x_1(q), x_2(q), ..., x_n(q) \) of its image \( \psi(q) \) in \( \mathbb{R}^n \).

**Definition 3.2.** Let \( \mathfrak{A} = \{(A_i, \psi_i) \in J \) be a collection of \( LG-local \) coordinate neighborhoods. Since \( \psi_i \) is an \( LG-homeomorphism \) for all \( i \in J \), then for all \( i, j \in J \) whenever \( A_i \cap A_j \neq \emptyset \),

\[ \psi_j \circ \psi_i^{-1} : \psi_i(\supp(A_i \cap A_j)) \rightarrow \psi_j(\supp(A_i \cap A_j)) \]

is an \( LG-homeomorphism \), that is called an \( LG-transition \) function.

\[ \psi_j \circ \psi_i^{-1}(x_1^i, x_2^i, ..., x_n^i) = (x_1^j, x_2^j, ..., x_n^j). \]

If \( \psi_i \circ \psi_i^{-1} \) and \( \psi_j \circ \psi_j^{-1} \) changing the \( LG-local \) coordinates are infinitely differentiable or \( C^\infty \), we shall say that \( (A_i, \psi_i) \) is \( C^\infty \) compatible with \( (A_j, \psi_j) \) whenever \( A_i \cap A_j \neq \emptyset \).

**Definition 3.3.** An \( LG-fuzzy \) topological space \( (X, \mathcal{T}) \) is called an \( LG-fuzzy \) topological manifold of dimension \( n \), if it satisfies the two following conditions:

i) \( X \) is an \( LG-fuzzy \) topological space of dimension \( n \),

ii) \( X \) is a \( Hausdorff \) \( L-fuzzy \).

**Definition 3.4.** A differentiable or \( C^\infty \) \( LG-fuzzy \) structure on an \( LG-fuzzy \) topological manifold \( (X, \mathcal{T}) \), is a family \( \mathfrak{A} = \{(A_\alpha, \psi_\alpha), \alpha \in J \} \) of \( LG-local \) coordinate neighborhoods such that
1) \( X = \bigcup_{\alpha \in J} A_\alpha; \)

2) Each pair \((A_\alpha, \psi_\alpha)\) and \((A_\beta, \psi_\beta)\) are compatible for all \(\alpha, \beta \in J.\)

3) Any LG-local coordinate neighborhood \((V, \varphi)\) that is compatible with every \((A_\alpha, \psi_\alpha), \alpha \in J\) is in \(\mathfrak{X}\) itself.

A \(C^\infty\) LG-fuzzy manifold \((X, \mathfrak{X})\) is an LG-fuzzy topological manifold with a \(C^\infty\) LG-fuzzy structure on it. In what follows, for convenience, “LG-fuzzy manifold with LG-fuzzy structure” will mean \(C^\infty\) LG-fuzzy manifold with \(C^\infty\) LG-fuzzy structure.

Example 3.5. Let \(M = \mathbb{R}^3, X : \mathbb{R}^3 \rightarrow I, X(x) = \begin{cases} 1 & \|x\| = 1, \\ 0 & \|x\| \neq 1. \end{cases}\) Then \(\text{supp}X = S^2, \) the unit sphere. Set

\[ \mathfrak{T} : I_X^M \rightarrow I, \quad \mathfrak{T}(A) = \begin{cases} \sup\{A(x) \mid x \in X\} & A \in \tau_I, A \leq X, \\ 0 & \text{elsewhere.} \end{cases} \]

Then \((X, \mathfrak{T})\) is an IG-fuzzy manifold of dimension 2.

Proof. Let \(J = \{1, 2, 3\}\). We define six IG-open subsets covering \(X\) by:

\[ \forall x = (x_1, x_2, x_3), \quad A_j^\pm(x) = \begin{cases} \pm x_j & \pm x_j > 0, \|x\| = 1, \\ 0 & \text{otherwise.} \end{cases} \]

Then we show that all \(A_j^\pm\) are diffeomorphic to IG-open subset \(B : \mathbb{R}^2 \rightarrow I\), defined by:

\[ \forall y = (y_1, y_2), \quad B(y) = \begin{cases} \sqrt{1 - y_1^2 - y_2^2} & \|y\| < 1, \\ 0 & \text{otherwise.} \end{cases} \]

Since \(\text{supp}B = B(0, 1)\), so \(B \in \tau_I\). We define six bijections \(\psi_j^\pm\) from \(\text{supp}A_j^\pm = \{(x_1, x_2, x_3) \mid \pm x_j > 0, \|x\| = 1\}\) to \(\text{supp}B = \{(y_1, y_2) \| y \| < 1\}\), for all \(j \in J\) by:

\[ \psi_j^\pm(x_1, x_2, x_3) = (x_2, x_3), \quad (\psi_j^\pm)^{-1}(y_1, y_2) = (\pm \sqrt{1 - y_1^2 - y_2^2} , y_1, y_2). \]

\[ \psi_j^\pm(x_1, x_2, x_3) = (x_1, x_3), \quad (\psi_j^\pm)^{-1}(y_1, y_2) = (y_1, \pm \sqrt{1 - y_1^2 - y_2^2} , y_2). \]

Also, it is seen that \(\psi_j^\pm \circ (\psi_i^\pm)^{-1}\) is infinitely differentiable for all \(i, j \in J\). For example:

\[ \psi_j^\pm \circ (\psi_i^\pm)^{-1}(y_1, y_2) = \psi_j^\mp(\pm \sqrt{1 - y_1^2 - y_2^2} , y_1, y_2) = (\pm \sqrt{1 - y_1^2 - y_2^2} , y_1, y_2). \]

Therefore, each pair \((A_i^\pm, \psi_i^\pm)\) and \((A_j^\pm, \psi_j^\pm)\) are compatible, for all \(i, j \in J\). We see

\[ \forall j \in J, \quad A_j^\pm(x) = \pm x_j = B_j(\psi_j^\pm(x)), \quad \forall x \in A_j^\pm. \]

Let \(H\) be an IG-fuzzy subset of \(1_{\mathbb{R}^2}\) with \(H \leq B\). We show that \(\mathfrak{T}((\psi_j^\pm)^{-1}[H]) = \mathfrak{T}_{\text{supp}}(H)\). Using [2], we have \(\mathfrak{T}_{\text{supp}}(H) = \sup\{H(a) \mid a \in \mathbb{R}^2\}\). Since \(\psi_j^\pm\) is bijective, for each \(a \in \mathbb{R}^2\), there exists one and only one element \(p \in \text{supp}A_j^\pm\) such that \(\psi_j^\pm(p) = a \text{ or } (\psi_j^\pm)^{-1}(a) = p\). Hence

\[ \mathfrak{T}_{\text{supp}}(H) = \sup\{H(\psi_j^\pm(p)) \mid p \in \text{supp}A_j^\pm\} = \sup\{( (\psi_j^\pm)^{-1}[H](p) \mid p \in \text{supp}A_j^\pm\} = \mathfrak{T}((\psi_j^\pm)^{-1}[H]). \]

Hence \(\psi_j^\pm \in \text{IGPR}f(A_j^\pm, B)\) is an IG-homeomorphism for all \(j \in J\) and this completes the proof.

Example 3.6. The set of natural numbers, \(\mathbb{N}\), partially ordered by divisibility, is a distributive lattice set, for which the unique supremum is the least common multiple and the unique infimum is the greatest common divisor. Let \(L = \mathbb{N} \cup \{\infty\}\). Then \(L\) is a complete lattice. Notice that we denote the top element of any lattice by \(1\), but in this example, \(\infty\) is the top element of \(\mathbb{N} \cup \{\infty\}\). We define the LG-fuzzy Euclidean topological space \((1_{\mathbb{R}^{mn}}, \mathfrak{T}_{\text{supp}})\) by

\[ 1_{\mathbb{R}^{mn}} : \mathbb{R}^{mn} \rightarrow L, \quad 1_{\mathbb{R}^{mn}}((a_1, a_2, \ldots, a_{mn})) = \infty, \]
\[ \mathfrak{T}_{L,mn} : L^{R_{mn}}_{1R_{mn}} \to L, \quad \mathfrak{T}_{L,mn}(D) = \begin{cases} 0 & D \in \tau_{L,mn}, \\ \infty & \text{elsewhere.} \end{cases} \]

Let \( M = M_{m \times n}(\mathbb{R}) \) and \( X \in L^M \) be defined by

\[ X((a_{ij})) = 2 + \max\{|a_{ij}| \mid 1 \leq i \leq m, \ 1 \leq j \leq n\}, \]

where \(|x|\) is equal to the greatest integer less than or equal to \(|x|\). There is a bijection \( \psi \) from \( M \) to \( \mathbb{R}^{mn} \):

\[ \psi(a_{ij}) = (a_{i1}, \ldots, a_{1n}, \ldots, a_{m1}, \ldots, a_{mn}). \]

Hence using \( \psi \) and \([\mathbb{I}]\), we define

\[ \mathfrak{T} : L^{M}_X \to L, \quad \mathfrak{T}(A) = \begin{cases} \infty & \psi[A] \in \tau_{L,mn}, \\ 0 & \text{elsewhere.} \end{cases} \]

We show that \((X, \mathfrak{T})\) is an \( \mathcal{C}^\infty \) \( LG \)-fuzzy \( mn \)-manifold. Let \( B \) be an \( L \)-fuzzy subset of \( 1_{R^{mn}} \) defined by

\[ B((a_{1}, a_{2}, \ldots, a_{mn})) = 2 + \max\{|a_{k}| \mid 1 \leq k \leq mn\}. \]

We see \( B = \psi[X] \). Since \( \text{ supp}\, B = \mathbb{R}^{mn} \cup \bigcup_{k=1}^{n} \mathcal{B}(0,k) \), Hence \( B \in \tau_{L,mn} \). Therefore using \([\mathbb{I}]\), for each \( LG \)-open subset \( H \) of \( 1_{R^{mn}} \), with \( H \subseteq B \), we have \( \mathfrak{T}_{L,mn}(H) = \infty = \mathfrak{T}(\psi^{-1}[H]) \). So by Proposition \([2.14] \), \( \psi \) is a \( LG \)-homeomorphism. We can cover \((X, \mathfrak{T})\) by the single \( LG \)-coordinate neighborhood \((X, \psi)\). Hence \((X, \mathfrak{T})\) is an \( \mathcal{C}^\infty \) \( LG \)-fuzzy \( mn \)-manifold.

**Definition 3.7.** (LG-open submanifolds) Let \( Z \) be an LG-open subset of the LG-fuzzy manifold \((X, \mathfrak{T})\). If \( \mathfrak{A} = \{(A_{\alpha}, \psi_{\alpha}) , \ \alpha \in J\} \) is an LG-fuzzy structure on \( X \), then \((Z, \mathfrak{T}_z)\) is an LG-fuzzy topology with LG-fuzzy structure consisting of the LG-coordinate neighborhoods \((A_{\alpha} \cap Z, \psi_{\alpha}|_{A_{\alpha} \cap Z})\).

**Example 3.8.** Let \((X, \mathfrak{T})\) be as Example \([3.6] \). We define \( Z : M_{m \times n}(\mathbb{R}) \to L, \quad Z(A) = \begin{cases} X(A) & \det A \neq 0, \\ 0 & \det A = 0. \end{cases} \)

We have \( Z \subseteq X \) and \( U = \text{ supp}\, Z = \text{GL}(n,\mathbb{R}) \) is an open subset of \( M_{m \times n} \). Hence we can prove that \( Z \) is an LG-open subset of \( X \). Therefore \((Z, \mathfrak{T}_z)\) is an LG-fuzzy submanifold of \((X, \mathfrak{T})\) with the single LG-local coordinate neighborhood \((Z, \psi|_Z)\) where \( \psi \) is a bijection defined in Example \([3.6]\).

**Example 3.9.** Let \( L = \mathbb{N} \cup \{\infty\} \) and \( M = \mathbb{R}^{n+1} \). Define an \( L \)-fuzzy subset

\[ X : M \to L, \quad X(x) = \begin{cases} n+2 \quad & \|x\| = 1, \\ 0 \quad & \|x\| \neq 1, \end{cases} \]

\[ \mathfrak{T} : L^{M}_X \to L, \quad \mathfrak{T}(A) = \begin{cases} \infty \quad & A \in \tau_{L(n+1)}, A \subseteq X, \\ 0 \quad & \text{elsewhere.} \end{cases} \]

Then \( \text{ supp}\, X = S^n \). Then \((X, \mathfrak{T})\) is an LG-fuzzy manifold of dimension \( n \):

**Proof.** Let \( J = \{1, \ldots, n+1\} \). We define \( 2(n+1) \) LG-open subsets covering \( X \), \( A_j^\pm : M \to L \), \( j \in J \) by:

\[ \forall x = (x_1, \ldots, x_{n+1}), \quad A_j^\pm(x) = \begin{cases} j \quad & \pm x_j > 0, \|x\| = 1, \\ 0 \quad & \text{otherwise.} \end{cases} \]

Then we show that each \( A_j^\pm \) is LGP-homeomorphic to the LG-open subset \( B_j : \mathbb{R}^{n} \to L \) defined by:

\[ \forall y = (y_1, \ldots, y_n), \quad B_j(y) = \begin{cases} j \quad & \|y\| < 1, \\ 0 \quad & \text{otherwise.} \end{cases} \]

with \( 2(n+1) \) LG-maps \( \psi_j^\pm : A_j^\pm \to B_j \) defined by:

\[ \psi_j^\pm(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x_j}, \ldots, x_{n+1}). \]

\[ (\psi_j^\pm)^{-1}(y_1, \ldots, y_n) = (y_1, \ldots, \pm \sqrt{1 - (y_1^2 + \ldots + y_n^2)}, \ldots, y_n), \]
where $\widehat{x}_j$ means omit $x_j$. Also it is seen that $\psi_j^+ \circ (\psi_i^+)^{-1}$ is infinitely differentiable for all $i, j \in J$ and we have

$$A_j^+(x) = j = B_j(\psi_j^+(x)), \quad \forall x \in A_j^+ \quad \text{and} \quad \forall j \in J.$$ 

Also using $[1]$, for each $L^G$-open subset $H_j$ of $1_{\mathbb{R}^n}$ with $H_j \subseteq B_j$, we have

$$\mathfrak{T}_{Ln}(H_j) = \infty = \mathfrak{T}(\psi_j^+)^{-1}(H_j) \quad \forall j \in J.$$ 

Therefore by Proposition $2.14$, $\psi_j^+ \in LGPRf(A_j^+, B_j)$, is an $LGP$-homeomorphism for all $j \in J$. \hfill $\square$

**Theorem 3.10.** Let $(M, \tau)$ be a $C^\infty$ ordinary n-manifold with the $C^\infty$ structure $\mathfrak{M} = \{(U_k, \psi_k), k \in K\}$. Consider $X = \widetilde{1}$, the constant $L$-fuzzy subset of $M$. Let $\delta$ be the $L$-fuzzy topology on $X$ generated by $\{\chi_{U_k}, k \in K\}$. Define

$$\mathfrak{T}_\tau : L^M_X \to L, \quad \mathfrak{T}_\tau(A) = \begin{cases} 1 & A \in \delta, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $(X, \mathfrak{T}_\tau)$ is an $L$-fuzzy manifold of dimension $n$.

**Proof.** First we show that $\mathfrak{T}_\tau$ is an $L$-gradation of openness on $X$:

1) Since $\widetilde{0}, X \in \delta$, therefore $\mathfrak{T}_\tau(\widetilde{0}) = \mathfrak{T}_\tau(X) = 1$.

2) For each family of fuzzy subsets $\{A_j\} \subseteq L^M_X$, we have two cases:
   
   i) $\{A_j\}_{j \in J} \subseteq \delta \Rightarrow \bigcup_{j \in J} A_j \in \delta$ and $\mathfrak{T}_\tau(A_j) = 1, \forall j \in J \Rightarrow 1 = \mathfrak{T}_\tau(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathfrak{T}_\tau(A_j) = 1$

   ii) $A_j \subseteq (L^M_X - \delta)$, for some $j \in J \Rightarrow \mathfrak{T}_\tau(A_j) = 0$, for some $j \in J \Rightarrow \mathfrak{T}_\tau(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathfrak{T}_\tau(A_j) = 0$

3) For every two fuzzy subsets $A, B \in L^M_X$, we have three cases:

   i) $A, B \in \delta \Rightarrow A \cap B \in \delta \Rightarrow 1 = \mathfrak{T}_\tau(A \cap B) \geq \mathfrak{T}_\tau(A) \land \mathfrak{T}_\tau(B) = 1$

   ii) $A \subseteq (L^M_X - \delta), B \in \delta \Rightarrow \mathfrak{T}_\tau(A) = 0, \mathfrak{T}_\tau(B) = 1 \Rightarrow \mathfrak{T}_\tau(A \cap B) \geq 0 = \mathfrak{T}_\tau(A) \lor \mathfrak{T}_\tau(B)$

   iii) $A, B \subseteq (L^M_X - \delta) \Rightarrow \mathfrak{T}_\tau(A) = \mathfrak{T}_\tau(B) = 0 \Rightarrow \mathfrak{T}_\tau(A \cap B) \geq 0 = \mathfrak{T}_\tau(A) \lor \mathfrak{T}_\tau(B)$.

Next we prove that $(X, \mathfrak{T}_\tau)$ is an $L$-fuzzy manifold:

Let $p \in M$. Then there exists an open subset $U_k, k \in K$ s.t. $p \in U_k$ and an open set $V_k$ of $\mathbb{R}^n$ along with a homeomorphism $\psi_k : U_k \to V_k$. Let $\tau^\phi$ be the $L$-fuzzy topology on $1_{\mathbb{R}^n}$ generated by $\{\chi_{V_k}, k \in K\}$. Then $\tau^\phi \subseteq \tau_{Ln}$. Hence we can consider the restriction of $\mathfrak{T}_{Ln}$ on $\tau^\phi$. We see

$$\mathfrak{T}_\tau(\chi_{U_k}) = \mathfrak{T}_{Ln}|_{\tau^\phi}(\chi_{V_k}).$$

Define $LGP$-homeomorphisms

$$\psi_k^\phi : M \to \mathbb{R}^n, \quad \psi_k^\phi(p) = \psi_k(p)\chi_{U_k} = \begin{cases} \psi_k(p) & \text{if } p \in U_k, \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore $\psi_k^\phi \in LGPRf(\chi_{U_k}, \chi_{V_k})$. Thus $\mathfrak{M}_\tau = \{(\chi_{U_k}, \psi_k^\phi), k \in K\}$ is an $L$-fuzzy structure on $X$. \hfill $\square$

**Theorem 3.11.** Let $(X, \mathfrak{T})$ be an $L$-fuzzy $n$-manifold with $L$-fuzzy structure $\mathfrak{A} = \{(A_j, \psi_j), j \in J\}$. If $\mathfrak{T}_A = \{\text{supp}A \mid \mathfrak{T}(A) > 0\}$, then $(\text{supp}X, \mathfrak{T}_A)$ is a topological manifold of dimension $n$ called a premanifold with the structure $\mathfrak{T}_A = \{((\text{supp}A_j, \psi_j|_{\text{supp}A_j}), j \in J\}$ called a prestructure.

**Proof.** Since $\operatorname{Dom}\mathfrak{T} = L^M_X$, then for all $A \in \text{supp}\mathfrak{T}$, we have $A$ is less than $X$. Hence $\text{supp}A \subseteq \text{supp}X$.

i) $\mathfrak{T}(\widetilde{0}) = \mathfrak{T}(X) = 1 \Rightarrow \phi = \text{supp}\widetilde{0} \in \text{supp}\mathfrak{T}_A$ and $\text{supp}X \in \text{supp}\mathfrak{T}_A$.

ii) Let $\{A_j, j \in J\} \subseteq \delta$, Then $\mathfrak{T}(A_j) > 0, \forall j \in J$. Hence $\mathfrak{T}(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathfrak{T}(A_j) > 0$. Thus $\bigcup_{j \in J} A_j \in \text{supp}\mathfrak{T}_A$.

iii) Since $\mathfrak{T}(A \cap B) \geq \mathfrak{T}(A) \lor \mathfrak{T}(B)$. So $A, B \in \mathfrak{T}_A$ implies that $\mathfrak{T}(A \cap B) > 0$. Thus $A \cap B \in \text{supp}\mathfrak{T}_A$.

Therefore $\mathfrak{T}_A$ is a topology on $\text{supp}X$. Let $p \in X$. Then there exists an $L^G$-open subset $A_k, k \in K$ such that $p \in A_k$ and there exists an $L^G$-open subset $B_k, k \in K$ with an $L^G$-homeomorphism $\psi_k \in LGPRf(A_k, B_k)$. Hence $p \in \text{supp}X, A_k(p) = B_k(\psi_k(p))$ and $\psi_k|_{\text{supp}A_k} : \text{supp}A_k \to \text{supp}B_k$ is one-to-one and onto. Therefore $(\text{supp}A_k, \psi_k|_{\text{supp}A_k})$ is a coordinate neighborhood of $p$. Also $\psi_j \circ \psi_i^{-1}$ is infinitely differentiable for all $i, j \in K$, thus $(\psi_j|_{\text{supp}(A_i \cap A_j)}) \circ (\psi_i^{-1}|_{\text{supp}(B_i \cap B_j)})$ is $C^\infty$ for all $i, j \in K$. Hence $(\text{supp}X, \mathfrak{T}_A)$ is a premanifold. \hfill $\square$
Theorem 3.12. Let \((M, \tau)\) be an n-manifold with structure \(\mathfrak{M} = \{(U_j, \varphi_j)\}, j \in J\). Let \(X = \tilde{1}\). Then 
\((\text{supp}X, (\tau^p)^\circ) = (M, \tau)\).

Proof. It is clear that \(\text{supp}X = M\) and for every \(U \in \tau\) we have \(\text{supp}X_u = U\). Hence \((\tau^p)^\circ = \tau\). Since we have 
\[\mathfrak{M}^p = \{(x_{U_j}, \varphi_j^p), j \in J\}, \]
then 
\[\mathfrak{M}^p = \{(\text{supp}X_{U_j}, \varphi_j^p|_{\text{supp}X_{U_j}}), j \in J\} = \{(U_j, \varphi_j)|_{U_j}, j \in J\} = \mathfrak{M}^p = \mathfrak{M}.\]

\[\blacksquare\]

Remark 3.13. Let \((X, \mathfrak{T})\) be an L-fuzzy topological space with L-gradation of openness, then \(\mathfrak{T}_{\mathfrak{T}_{\mathfrak{L}}}\) does not necessarily equal \(\mathfrak{T}\). We show it by the following example.

Example 3.14. Let \(M = \mathbb{R}\). We define 
\[X : M \to I, \quad X(x) = \begin{cases} \frac{1}{x} & x \in (2, +\infty), \\ 0 & \text{elsewhere}. \end{cases} \]
and \(\mathfrak{T} : I^M_X \to I\), by \(\mathfrak{T}(A) = \begin{cases} 1 & A \in \tau_{11}, A \leq X \\ 0 & \text{elsewhere}. \end{cases} \)
Then clearly \((X, \mathfrak{T})\) is an IG-fuzzy manifold of dimension 1. Then by Theorem 3.11 \((\text{supp}X, \mathfrak{T}^\circ)\) is a manifold, where \(\mathfrak{T}^\circ = \{\text{supp}A | A \in \text{supp}\mathfrak{T}\}\). Hence by Theorem 3.10 we have \(\mathfrak{T}_{\mathfrak{T}_{\mathfrak{L}}} = \{\chi_{\text{supp}A} | A \in \text{supp}\mathfrak{T}\}\). We have 
\[A(x) \leq X(x) \leq \frac{1}{2}, \forall x \in M \text{ and } \forall A \in \tau_{11}. \text{ Hence } 1 = \chi_{\text{supp}A} \neq A. \text{ Therefore } \mathfrak{T}_{\mathfrak{T}_{\mathfrak{L}}} \neq \mathfrak{T}.\]

4 \ IG-fuzzy quotient manifolds

Definition 4.1. Let \(M\) be a crisp set and \(\sim\) be an equivalence relation on it. If \(A\) is an L-fuzzy subset of \(M\) such that \(A(y) = A(x)\) whenever \(y \sim x\), then we define the L-fuzzy subset:
\[A \sim M \to L, \quad A \sim (\left[x\right]) = A(x), \quad \forall x \in M,\]
where \([x] = \{y | x \sim y\}\). Since \(A \leq X\), thus \(A \sim X \sim\) and hence \(A \sim L^M_{X \sim}\).

Theorem 4.2. Let \((X, \mathfrak{T})\) be an LG-fuzzy topological space, such that \(X(y) = X(x)\) whenever \(y \sim x\), then \(\frac{X}{\sim} \sim \frac{\mathfrak{T}}{\sim} : L^M_{X \sim} \to L\), \(\frac{X}{\sim} \sim \frac{A}{\sim} = \mathfrak{T}(A)\).

Proof. We show that all elements of \(L^M_{X \sim}\) are in the form \(\frac{A}{\sim}\) for some \(A \in L^M_X\). Let \(B\) be an L-fuzzy subset of \(\frac{M}{\sim}\) less than \(\frac{X}{\sim}\). We define L-fuzzy subset \(A \sim M\) of \(M\) by \(A(x) = B([x]), \forall x \in M\). Let \(x \sim y\) so \([x] = [y]\), then \(A(x) = B([x]) = B([y]) = A(y)\) and thus \(A \sim B\). Also,

1) \(\frac{X}{\sim} \sim \frac{0}{\sim} = \mathfrak{T}(0) = 1, \quad \frac{X}{\sim} \sim \frac{X}{\sim} = \mathfrak{T}(X) = 1.\)

2) \(\frac{X}{\sim} \sim \frac{A_1 \sim \cap A_2 \sim}{\sim} = \mathfrak{T} \frac{A_1 \cap A_2}{\sim} = \mathfrak{T}(A_1 \cap A_2) \geq \mathfrak{T}(A_1) \wedge \mathfrak{T}(A_2) = \frac{X}{\sim} \sim \frac{A_1}{\sim} \wedge \frac{X}{\sim} \sim \frac{A_2}{\sim}.\)

3) Let \(\{A_j\}_{j \in J}\) be a sequence of L-fuzzy subsets of \(X\), such that \(\forall j \in J, A_j(y) = A_j(x)\), whenever \(y \sim x\), then 
\[\bigcup_{j \in J} \frac{A_j}{\sim} [y] = \sup \{\frac{A_j}{\sim} [y], j \in J\} = \sup \{A_j(y), j \in J\} = \sup \{A_j(x), j \in J\} = \bigcup_{j \in J} \frac{A_j}{\sim} [x],\]
\[\frac{X}{\sim} \sim \frac{\bigcup_{j \in J} A_j}{\sim} = \mathfrak{T} \frac{\bigcup_{j \in J} A_j}{\sim} = \mathfrak{T}(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathfrak{T}(A_j) = \bigwedge_{j \in J} \frac{X}{\sim} \sim \frac{A_j}{\sim}.\]
Hence $\frac{\mathbb{T}}{\sim}$ is a gradation of openness on $X$. □

This $LG$-fuzzy topology is nontrivial when $L = I$, because for each $a \in I$, $aX(x) = aX(y)$ whenever $x \sim y$. Hence $aX \in \text{supp}\frac{\mathbb{T}}{\sim}$.

**Definition 4.3.** Consider an $LG$-fuzzy quotient space $(X, \mathbb{T})$. The equivalence relation $\sim$ is called an $LG$-open relation if for each fuzzy subset $A \in \text{supp}\mathbb{T}$ we have $A \sim \mathbb{T} \sim A \sim \mathbb{T}$.

**Theorem 4.4.** Let $(X, \mathbb{T})$ be an $LG$-fuzzy manifold and $\sim$ be an $LG$-open relation. Then $(X, \mathbb{T})$ is an $LG$-fuzzy topological space of dimension $n$ called $LG$-fuzzy quotient topological space of dimension $n$. If $\text{supp}\mathbb{T}$ has a countable basis, then $\frac{\mathbb{T}}{\sim}$ has a countable basis.

**Proof.** Let $(A, \psi)$ be an $LG$-locally coordinate neighborhood of $p \in X$ and $\psi \in LGPRf(A, B)$. Since $\sim$ is an $LG$-open relation, then we have $\frac{\mathbb{T}}{\sim}(A) > 0$. We define a corresponding relation $\sim^*$ on $\text{supp}B$ as follows:

$$a \sim^* b \iff \psi^{-1}(a) \sim \psi^{-1}(b) \text{ for all } a, b \in \text{supp}B.$$

Clearly $\sim^*$ is a reflexive and symmetric relation. Let $a \sim^* b$ and $b \sim^* c$. Then we have $\psi^{-1}(a) \sim \psi^{-1}(b)$, and $\psi^{-1}(b) \sim \psi^{-1}(c)$. Since $\sim$ is transitive, so $\psi^{-1}(a) \sim \psi^{-1}(c)$. Hence $a \sim^* c$. Therefore $\sim^*$ is transitive and so it is an equivalence relation. Since we have

$$a \sim^* b \implies \psi^{-1}(a) \sim \psi^{-1}(b) \implies A(\psi^{-1}(a)) = A(\psi^{-1}(b)) \implies B(a) = B(b).$$

Therefore, $B \sim^*$ is well-defined. Now we define

$$\psi: \text{supp}\frac{A}{\sim} \to \text{supp}\frac{B}{\sim^*}, \quad \psi([p]) = [\psi(p)].$$

We see

$$[p] = [q] \iff p \sim q \iff \psi(p) \sim^* \psi(q) \iff [\psi(p)] = [\psi(q)] \iff \psi([p]) = \psi([q]).$$

Therefore, $\psi$ is a well-defined and one to one function. Since $\psi$ is onto, we see

$$\forall a \in \text{supp}B, \exists p \in \text{supp}A \text{ s.t. } a = \psi(p) \implies \psi([p]) = [\psi(p)] = [a].$$

Hence, $\psi$ is onto. We have

$$\frac{B}{\sim}([a]) = B(a) = A(\psi^{-1}(a)) = \frac{A}{\sim}([\psi^{-1}(a)]) = \frac{A}{\sim}((\psi^{-1})([a])).$$

On the other hand for each $LG$-open subset $H$ of $1_{\mathbb{R}^n}$ which $H \leq B$, we have

$$\frac{\mathbb{T}}{\sim} \mathbb{A} \sim H \mathbb{T} \sim = \mathbb{A} \sim \mathbb{T} \sim 1 \sim = \mathbb{T} \sim (\psi^{-1}(H)) = \frac{\mathbb{T}}{\sim} \mathbb{A} \sim (\psi^{-1}(H)).$$

Hence, $\psi \in LGPRf\left(\frac{A}{\sim}, \frac{B}{\sim}\right)$. Therefore, if $\mathbb{A} = \{(A_j, \psi_j), j \in J\}$ be a $C^\infty$ $LG$-structure of $X$. Then $\frac{\mathbb{A}}{\sim} = \left\{\left(\frac{A_j}{\sim}, \frac{\psi_j}{\sim}\right), j \in J\right\}$ is a $C^\infty$ $LG$-fuzzy structure of $(X, \mathbb{T})$. Now, suppose that $\text{supp}\mathbb{T}$ has a countable basis $\beta = \{A_i, i \in \mathbb{N}\}$. Let $\frac{A}{\sim} \in \text{supp}\frac{\mathbb{T}}{\sim}$ and $A = \bigcup_{j \in K} A_j$. Since $K \subseteq \mathbb{N}$. Hence for all $y \in X$ we have:

$$\frac{A}{\sim}([y]) = A(y) = \bigcup_{j \in A} A_j(y) = sup\{A_j(y), j \in K\} = sup\left\{\frac{A_j}{\sim}([y]), j \in K\right\} = \bigcup_{j \in K} \frac{A_j}{\sim}([y]).$$

Therefore, $\beta \sim = \left\{\frac{A_i}{\sim}, i \in K\right\}$ is a countable basis for $\text{supp}\frac{\mathbb{T}}{\sim}$. □
Example 4.5. Consider the IG-fuzzy Euclidean topological space \((1_R, \Sigma_{11})\). We define a relation on \(R\) as follows: 
\[ \forall x, y \in 1_R, \ x \sim y \text{ if } x - y \in \mathbb{Z}. \] 
Since \(1_R(y) = 1_R(x) = 1\) whenever \(y \sim x\). Hence \(1_R \sim \) is well-defined and hence by Theorem 4.2 we have the IG-fuzzy quotient topological space \((1_R \sim, \Sigma_{11} \sim)\). We show that it is an IG-fuzzy topological manifold: 
Let \(A : R \rightarrow I, \ A(x) = x - [x]. \) Since 
\[ \text{supp} A = R - \mathbb{Z} = \bigcup_{k \in \mathbb{Z}} (k, k + 1) = \bigcup_{k \in \mathbb{Z}} B(k + \frac{1}{2}, \frac{1}{2}). \] 
Then we see: 
\[ A = \bigcup_{k \in \mathbb{Z}} B(k + \frac{1}{2}, \frac{1}{2}, b_k), \text{ where } b_k : B(k + \frac{1}{2}, \frac{1}{2}) \rightarrow I, \ b_k(x) = x - [x]. \] 
Therefore by Example 2.2 \(A \in \tau_i, \) and hence by Example 2.4 \(\Sigma_{11}(A) = 1. \) Since for all \(x, y \in R\) we have 
\[ x \sim y \Rightarrow y = x + k, \ k \in \mathbb{Z} \Rightarrow [y] = k + [x]. \] 
\[ y - [y] = k + x - (k + [x]) = x - [x] \Rightarrow A(y) = A(x). \] (3) 
Hence \(A \sim \) is well-defined. So \(\Sigma_{11}(A \sim) = \Sigma_{11}(A) = 1. \) Define 
\[ B : R \rightarrow I, \ B(y) = \begin{cases} y & y \in (0, 1), \\ 0 & \text{otherwise}. \end{cases} \] 
We can write \(B = B(\frac{1}{2}, \frac{1}{2}, b), \) where \(b : B(\frac{1}{2}, \frac{1}{2}) \rightarrow I, \) \(b(y) = y. \) So \(B \in \tau_i. \) Hence \(\Sigma_{11}(B) = 1 = \Sigma_{11}(A \sim). \) We define \(\psi : \text{supp} A \sim \rightarrow \text{supp} B \) by \(\psi([x]) = x - [x]. \) If \([x] = [y], \) then it means that \(x \sim y, \) so by (3), we have 
\[ x - [x] = y - [y]. \] 
Hence \(\psi([x]) = \psi([y]). \) Therefore \(\psi\) is well-defined. We show that \(\psi\) is injective: 
\[ \psi([x]) = \psi([y]) \Rightarrow x - [x] = y - [y] \Rightarrow y - x = [y] - [x] \in \mathbb{Z} \Rightarrow x \sim y \Rightarrow [x] = [y]. \] 
Since \(\forall t \in (0, 1), \) \([t] = 0, \) so \(\psi([t]) = t - |t| = t. \) Therefore \(\psi\) is surjective. So by Definition 2.11 we have 
\[ A \sim([x]) = B(\psi([x])). \] 
On the other hand for each LG-open subset \(H\) of \(1_R\) with \(H \leq B, \) we have 
\[ \frac{\Sigma_{11}}{\sim} (\psi^{-1}(H)) = \frac{\Sigma_{11}}{\sim} (H) = \Sigma_{11}(H) = 1. \] 
Hence \(\psi \in IGPRf(A \sim, B). \) Thus we have a single IG-local coordinate neighborhood \((A \sim, \psi)\) for all points of \(1_R \sim. \)

Proposition 4.6. Consider all of the hypotheses of Theorem 4.4. Let \(R = \{(x, y) \mid x \sim y\}. \) Then \((X, \Sigma)\) is a Hausdorff \(L\)-gft if and only if \(\chi_{R,x,x} \) is an LG-closed subset of \(X \times X. \)

Proof. Let \((X, \Sigma)\) be a Hausdorff \(L\)-gft, then for each \([x], \ [y] \in X \sim\) where \([x] \neq [y], \) there exist \(U \sim, \ V \sim \in \text{supp} \sim\) such that \([x] \in U \sim, \ [y] \in V \sim\) and \(U \sim \cap V \sim = \phi. \) So for each \((x, y) \in (X \times X)\) where \((x, y) \notin R\) there exist \(U, V \in \text{supp} \Sigma, \) such that \(U \cap V = \phi\) and \((x, y) \in (U \times V). \) We show that \(\text{supp}(U \times V) \cap R = \phi. \)
\[ (a, b) \in \text{supp}(U \times V) \cap R \Rightarrow a \in U, \ b \in V, \ a \sim b \Rightarrow [a] \in U \sim, \ [b] \in V \sim, \ [a] = [b] \Rightarrow U \sim \cap V \sim \neq \phi, \] that is a contradiction. It means that for each \((x, y) \in \text{supp}(X \times X - \chi_{R,x,x})\) there exists \(U \times V \in \text{supp}(\Sigma \times \Sigma)\) such that \((x, y) \in U \times V \leq (X \times X - \chi_{R,x,x}). \) Therefore, \((\Sigma \times \Sigma)(X \times X - \chi_{R,x,x}) > \Sigma(U) \wedge \Sigma(V) > 0. \) Hence \(\chi_{R,x,x}\) is an LG-closed subset of \(X \times X.\)
Conversely, suppose that $\chi_{r,x} \times \chi_{r,x}$ is an LG-closed subset of $X \times X$. Then $(X \times X - \chi_{r,x} \times \chi_{r,x})$ is an LG-open subset. By Theorem 3.11, $\text{supp}(X \times X - \chi_{r,x} \times \chi_{r,x}) \in (\mathcal{I} \times \mathcal{I})^d$.

Hence $\text{supp}(X \times X) - R$ is an ordinary open subset. So for each $(x, y) \in (\text{supp}(X \times X) - R)$, there exists an open subsets $U \times V \in (\mathcal{I} \times \mathcal{I})^d$ such that $(x, y) \in U \times V \subseteq (\text{supp}(X \times X) - R)$. We show that $U \cap V = \emptyset$

$$a \in U \cap V \implies a \sim a \text{ and } (a, a) \in U \times V \implies (a, a) \in (U \times V) \cap R,$$

that is a contradiction. It means that for each $(x, y) \in \text{supp}(X \times X)$ where $(x, y) \notin R$, there exist $U, V \in \text{supp}\mathcal{I}^r$, such that $U \cap V = \emptyset$ and $(x, y) \in U \times V$. Since $\sim$ is an LG-open relation, then we have $\mathcal{I}(U) \geq 0$ and $\mathcal{I}(V) \geq 0$. Therefore for each $[x], [y] \in \mathcal{I}$ where $[x] \neq [y]$, there exist $U \sim, V \sim \in \text{supp}\mathcal{I}$ such that $[x] \in U \sim, [y] \in V \sim$ and $U \sim \cap V \sim = \emptyset$. So $(\mathcal{I} \sim, \mathcal{I} \sim)$ is a Hausdorff LG-fs.

5 \textbf{LG-fuzzy product manifolds}

The concept of the product of fuzzy topological spaces was introduced by C. K. Wong [19] and later by Hutton [8]. We define and investigate LG-fuzzy product manifolds by the following theorem:

\textbf{Theorem 5.1.} Let $X \in L^{M_1}$, $X_2 \in L^{M_2}$ and $(X_1, \mathcal{I}_1), (X_2, \mathcal{I}_2)$ be two LG-fuzzy manifolds of dimensions $m$, $n$ and with the LG-fuzzy structures $\mathcal{I}_1 = \{(A_{\alpha_1}, \psi_{\alpha_1})| \alpha_i \in K_i\}, i = 1, 2$ respectively. Then $(X_1 \times X_2, \mathcal{I}_1 \times \mathcal{I}_2)$ is an LG-fuzzy manifold of dimension $m + n$.

\textbf{Proof.} We define for all $A_1 \in L^{M_1}_{X_1}$, $A_2 \in L^{M_2}_{X_2}$:

$$A_1 \times A_2 \in L^{M_1 \times M_2}_{X_1 \times X_2}, \quad (A_1 \times A_2)(x, y) = A_1(x) \wedge A_2(y),$$

and

$$(\mathcal{I}_1 \times \mathcal{I}_2)(A_1 \times A_2) = \mathcal{I}_1(A_1) \wedge \mathcal{I}_2(A_2).$$

It can be verified that $\mathcal{I}_1 \times \mathcal{I}_2$ is an $L$-gradation of openness on $X_1 \times X_2$. Now let $p_1 \in X_1$, $p_2 \in X_2$. Then there exist two LG-open subsets $A_i$ of $X_i$ containing $p_i$, for $i = 1, 2$ and two LG-open subsets $B_i$ of LG-fuzzy Euclidean spaces of dimension $m, n$, respectively together with two LGP-homeomorphisms $\psi_i = (A_i, B_i)$. Therefore for any $(p_1, p_2) \in X_1 \times X_2$, there exists an LG-open subset $A_1 \times A_2$ of $X_1 \times X_2$ containing $(p_1, p_2)$ and an LG-open subset $B_1 \times B_2$ of LG-fuzzy Euclidean space of dimension $m + n$, together with an LGP-homeomorphism $\psi_1 \times \psi_2 = (A_1 \times A_2, B_1 \times B_2)$ such that for each LG-open subsets $H_1 \leq B_1, H_2 \leq B_2$ we have

$$\mathcal{I}_{(m+n)}(H_1 \times H_2) = (\mathcal{I}_1 \times \mathcal{I}_2)(\psi_1^{-1}(H_1) \times \psi_2^{-1}(H_2)),$$

where $(\psi_1 \times \psi_2)(x_1, x_2) = (\psi_1(x_1), \psi_2(x_2)) \in \mathbb{R}^{m+n}$. One can prove easily that

$$\mathcal{I}_1 \times \mathcal{I}_2 = \{(A_{\alpha_1} \times A_{\alpha_2}, (\psi_{\alpha_1} \times \psi_{\alpha_2})| \alpha_1 \in K_1, \alpha_2 \in K_2\},$$

is an LG-fuzzy structure on $X_1 \times X_2$. □

\textbf{Example 5.2.} Let $M = \mathbb{R}^2, X = \chi_{S^1}$. One can easily prove that $(X, \mathcal{I}_{13})$ is an LG-fuzzy manifold of dimension 1, similarly to Example 3.6. Then $(X \times X, \mathcal{I}_{13} \times \mathcal{I}_{13})$ is an LG-fuzzy manifold of dimensions 2 and $\text{supp}(X \times X) = S^1 \times S^1$.

6 \textbf{$C^\infty$ LG-fuzzy mappings of LG-fuzzy manifolds}

The concept of the fuzzy vector space $(V, \eta)$ over a field $F$ was defined in [18]. We extend this definition by L-fuzzification:

\textbf{Definition 6.1.} An L-fuzzy vector space $(V, \eta)$ or $\eta V$ over a field $F$ is an ordinary vector space $V$ over the field $F$, with a map $\eta : V \rightarrow L$ satisfying the following conditions for all $a, b \in V$ and $r \in F$:

1) $\eta(a + b) \geq \min\{\eta(a), \eta(b)\}$,

2) $\eta(-a) = \eta(a)$,
3) \( \eta(0) = 1 \),
4) \( \eta(ra) \geq \eta(a) \),

**Definition 6.2.** Let \((X, \mathcal{X})\) be an LG-fuzzy manifold of dimension \( n \), \( U \in \text{supp}\mathcal{X} \) and \( V \in \text{supp}\mathcal{X}_{L1} \). The LG-related function \( f \) from \( U \) to \( V \), is called a \( C^\infty \) LG-fuzzy mapping, if for every \( p \in U \),
\[
\hat{f} = f \circ \psi^{-1} : \psi(\text{supp}(A \cap U)) \to \text{supp}V,
\]
is \( C^\infty \) where \((A, \psi)\) is an LG-local coordinate neighborhood of \( p \).

We denote the set of all \( C^\infty \) LG-fuzzy mappings from an LG-open subset \( U \) of \( X \), containing \( p \) to 1, by \( C^\infty_L(p) \).

If we define \( \eta : C^\infty_L(p) \to L \), \( \eta(f) = A(p) \), where \((A, \psi)\) is an LG-coordinate neighborhood of \( p \), then \( C^\infty_L(p) \) may be considered as an \( L \)-fuzzy vector space \((C^\infty_L(p), \eta)\). Let \( \psi(q) = (x_1, \ldots, x_n) \), \( \forall q \in \text{supp}(A \cap U) \). Then \( \hat{f}(x_1, \ldots, x_n) = y(q) \), and since \( \hat{f} \) is \( C^\infty \), there exist all partial derivatives of any order of \( y \).

**Example 6.3.** In Example 3.8, if we define \( f : M_{m \times n} \to \mathbb{R}, f((a_{ij})) = \det((a_{ij})) \), then using the single IG-local coordinate neighborhood \((Z, \psi|_Z)\), we have \( \hat{f} = f \circ \psi \) is \( C^\infty \). Hence \( f \in C^\infty(p) \) for all \( p \in Z \).

From now on, we suppose that \( M_1, M_2 \) are two crisp sets, \( X \in L^{M_1}, Y \in L^{M_2} \) such that \((X, \mathcal{X}), (Y, \mathcal{R})\) are two LG-fuzzy manifolds of dimension \( n, m \) and LG-fuzzy structures \( \mathfrak{A} = \{(A_i, \psi_i), i \in K\} \) and \( \mathfrak{D} = \{(D_j, \varphi_j), j \in J\} \) respectively and \( U \in \text{supp}\mathcal{X}, V \in \text{supp}\mathcal{D} \).

**Definition 6.4.** An LG-fuzzy function \( F \in \text{LGR}f(U, V) \) is a \( C^\infty \) LG-fuzzy mapping if for every \( p \in U \),
\[
\hat{F} = \varphi \circ f \circ \psi^{-1} : \psi(\text{supp}(A \cap U)) \to \varphi(\text{supp}(B \cap V)),
\]
is \( C^\infty \) where \((A, \psi), (B, \varphi)\) are LG-local coordinate neighborhoods of \( p \) and \( F(p) \) respectively. \( F \in \text{LGR}f(U, V) \) is called a LG-diffeomorphism if it is an LG-homeomorphism and \( F, F^{-1} \) are \( C^\infty \).

More precisely, if \( \psi(q) = (x_1, \ldots, x_n) \), \( \forall q \in \text{supp}(A \cap U) \) and \( \varphi(w) = (y_1, \ldots, y_m) \), \( \forall w \in B \), then
\[
\hat{F}(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)),
\]
and each \( y_i = f_i(x_1, \ldots, x_n) \) is \( C^\infty \) on \( \psi(A) \).

**Definition 6.5.** The rank of \( F \in \text{LGR}f(X, Y) \) at \( p \) is equal to the rank at \( x = \psi(q) \) of the Jacobian matrix:
\[
\left( \begin{array}{ccc}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{array} \right) .
\]

**Example 6.6.** Let \( M_1 = M_2 = \mathbb{R}^2 \), \( L = I \) and \( X : M_1 \to I, Y : M_2 \to I \) be defined by:
\[
X(x_1, x_2) = \begin{cases} 
1 & \|x\| = 1, \\
0 & \|x\| \neq 1,
\end{cases}
\quad \text{and} \quad
Y(y_1, y_2) = \begin{cases} 
1 & \|y\| = 1, \\
0 & \|y\| \neq 1.
\end{cases}
\]
If we define \( \mathcal{T} : I^{M_1}_X \to I \), and \( \mathcal{R} : I^{M_2}_Y \to I \), by
\[
\mathcal{T}(A) = \begin{cases} 
1 & A \in \text{supp}\mathcal{X}, A \leq X, \\
0 & \text{elsewhere}.
\end{cases}
\quad \text{and} \quad
\mathcal{R}(D) = \begin{cases} 
1 & D \in \text{supp}\mathcal{X}, D \leq Y, \\
0 & \text{elsewhere}.
\end{cases}
\]
In a similar manner to the Example 3.3, we can prove that \((X, \mathcal{X}), (Y, \mathcal{R})\) are IG-fuzzy manifolds. Let \( F : M_1 \to M_2, F(x_1, x_2) = (x_1 - x_2, \sqrt{2x_1x_2}) \).

We prove that \( F \in \text{IGR}f(X, Y) \) is a \( C^\infty \) IG-fuzzy mapping. First we show \( F|\text{supp}X : \text{supp}X \to \text{supp}Y \) is well-defined and \( F[X] = Y \):
\[
(x_1, x_2) \in S^1 \Rightarrow x_1^2 + x_2^2 = 1 \Rightarrow (x_1 - x_2)^2 + (\sqrt{2x_1x_2})^2 = 1 \Rightarrow F(x_1, x_2) \in S^1,
\]
\[
F[X](y_1, y_2) = \sqrt{(x_1(x_1, x_2) : (x_1, x_2) \in F^{-1}(y_1, y_2))} = 1 = Y(y_1, y_2).
\]
Let \((A_1^+, \psi_1^+), (D_2^+, \varphi_2^+)\) be IG-local coordinate neighborhoods on \( X, Y \) respectively, then we see:
\[
\varphi_2^+ \circ F \circ \psi_1^{-1}(y) = \varphi_2^+ \circ F(\sqrt{1-y^2}, y) = \varphi_2^+ (\sqrt{1-y^2} - y, \sqrt{2y\sqrt{1-y^2}}) = \sqrt{1-y^2} - y,
\]
is \( C^\infty \). Similarly, one can show that \( \varphi_i^+ \circ F \circ \psi_j^{-1} \) is \( C^\infty \) for all \( i, j = 1, 2 \).
Example 6.7. Let \( F : \mathbb{R} \to \mathbb{R}^2 \), \( F(t) = (\cos(t - \frac{\pi}{2}), \sin(t - \frac{\pi}{2})) \). Then \( F \in \text{IGPRf}(1_{\mathbb{R}}, 1_{\mathbb{R}^2}) \) and rank \( F = 1 \) at every point of \( X \).

Theorem 6.8. (LG-fuzzy rank theorem) Let \( F \in \text{LGRf}(U,Y) \) be a \( C^\infty \) fuzzy mapping and rank \( F = k \) at every point of \( X \). If \( p \in X \), then there exist LG-local coordinate neighborhoods \((A,\psi), (B,\varphi)\) such that

\[
\psi(p) = (0,\ldots,0) \in \mathbb{R}^n, \quad \varphi(F(p)) = (0,\ldots,0) \in \mathbb{R}^m,
\]

and \( \hat{F} = \varphi \circ F \circ \psi^{-1} \) is given by:

\[
\hat{F}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0).
\]

Proof. Using Theorem 3.11, we see that \((\text{supp}X, \mathcal{T}^q)\) and \((\text{supp}Y, \mathcal{R}^q)\) are two topological manifolds of dimension \( n, m \) with the structures \( \mathcal{A}^q = \{(\text{supp}A_i, \psi_i|_{\text{supp}A_i}) | i \in K\} \) and \( \mathcal{D}^q = \{(\text{supp}D_j, \varphi_j|_{\text{supp}D_j}) | j \in J\} \) respectively. Also \( F|_{\text{supp}X} : \text{supp}X \to \text{supp}Y \) is a \( C^\infty \) mapping and rank \( F = k \) at every point of \( X \). Fix \( p \in \text{supp}X \), then by the rank theorem, there exist coordinate neighborhoods \((\text{supp}A, \psi|_{\text{supp}A}), (\text{supp}D, \varphi|_{\text{supp}D})\) of \( p \) and \( F(p) \) respectively such that

\[
\psi|_{\text{supp}A}(p) = (0,\ldots,0) \in \mathbb{R}^n, \quad \varphi|_{\text{supp}D}(F|_{\text{supp}X}(p)) = (0,\ldots,0) \in \mathbb{R}^m,
\]

and \( \tilde{F}|_{\psi(\text{supp}A)} = \varphi|_{\text{supp}D} \circ F|_{\text{supp}A} \circ \psi^{-1}|_{\psi(\text{supp}A)} \) is given by:

\[
\tilde{F}|_{\psi(\text{supp}A)}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0).
\]

Therefore the LG-fuzzy rank theorem holds for LG-fuzzy manifolds.

Remark 6.9. We can cover \( X \) and \( \tilde{X} = F[X] \) by these LG-local coordinate neighborhoods \( \mathfrak{A} = \{(A_s, \psi_s) | s \in S\} \), and \( \mathfrak{D} = \{(D_s, \varphi_s) | s \in S\} \) respectively where \( S \subseteq K \). Since \( \mathfrak{A} \) is an LG-structure of \( X \), one can show that \( \mathfrak{D} \) is an LG-structure of \( F[X] \). If \( F \) is an LG-diffeomorphism, then we have rank \( F = \dim X = \dim Y \).

Definition 6.10. The \( C^\infty \) \( L \)-related function \( F \in \text{LGRf}(X,Y) \) is an LG-fuzzy immersion (submersion) if rank \( F = \dim X = \dim Y \) at every point of \( X \).

Theorem 6.11. Let \( F \in \text{LGRf}(X,Y) \) be a \( C^\infty \) LG-fuzzy mapping. If \( F \) is an injective LG-fuzzy immersion, then \((\tilde{X}, \tilde{\mathcal{T}})\) is an LG-fuzzy submanifold of dimension \( n \), called an LG-fuzzy immersed submanifold and \( F \in \text{LGRf}(X, \tilde{X}) \) is an LG-diffeomorphism.

Proof. \( F \) establishes a one-to-one correspondence between \( \text{supp}X \) and \( \text{F}(\text{supp}X) \). Thus, \( F \in \text{LGRf}(X, \tilde{X}) \) is one-to-one and onto. Since for each \( q \in \text{F}(\text{supp}X) \), there exists only one \( p \in \text{supp}X \) such that \( F^{-1}(q) = \{p\} \), hence

\[
\hat{X}(q) = F[X](q) = \text{sup}(X(a)|F(a) = q) = X(p).
\]

Since \( F \in \text{LGRf}(X,Y) \), we have \( F[X] \leq Y \); therefore \( \tilde{X} \leq Y \). Hence \( \tilde{X} \) is an LG-fuzzy subset of \( Y \). We use \( F \) to endow \( \tilde{X} \) with an LG-structure

\[
\tilde{\mathcal{D}} = \{(D_s|_{\tilde{X}}, \pi \circ \varphi_s|_{\tilde{X}}) | (D_s, \varphi_s) \in \mathfrak{D}, \forall s \in S\},
\]

where \( \pi(y_1, \ldots, y_m) = (y_1, \ldots, y_n) \) is the projection and an LG-fuzzy topology

\[
\tilde{\mathcal{T}} : L^{M_2}_{\tilde{X}} \to L, \quad \tilde{\mathcal{T}}(H) = \mathcal{T}(F^{-1}[H]).
\]

Then \((\tilde{X}, \tilde{\mathcal{T}})\) is an LG-fuzzy manifold of dimension \( n \), called an LG-fuzzy immersed submanifold and \( F \in \text{LGRf}(X, \tilde{X}) \) is an LG-gradation-preserving. Therefore \( F \) is an LG-diffeomorphism. \( \square \)

In general, gradation of openness \( \tilde{\mathcal{T}} \) and the LG-fuzzy structure on \( \tilde{X} \) depend on \( F \) as well as \( X \), i.e. \((\hat{X}, \tilde{\mathcal{T}})\) is not a submanifold of \((Y, \mathcal{T})\). So we add the condition of LG-continuity of \( F, F^{-1} \) in the following definition:

Definition 6.12. An LG-fuzzy imbedding is a one-to-one LG-fuzzy immersion \( F \in \text{LGRf}(X,Y) \) with \( F \in \text{LGRf}(X, \tilde{X}) \) is an LG-homeomorphism from \( X \) to \( \tilde{X} = F[X] \) as an LG-fuzzy subspace of \( Y \). The image of an LG-fuzzy imbedding is called an LG-fuzzy imbedded submanifold.

Theorem 6.13. Let \( F \in \text{LGRf}(X,Y) \) be an LG-fuzzy immersion. Then for each \( p \in X \), exists an LG-neighborhood \( A \) of \( p \) such that \( F|_{\text{supp}A} \) is an LG-fuzzy imbedding.
Proof. According to Theorem 6.8, we may choose \((A, \psi)\) and \((D, \varphi)\), the LG-local coordinate neighborhoods of \(p\) and \(F(p)\), respectively, such that (4) holds. Since \(F|A| = D\) and \(D\) is an LG-open subset of \(Y\), hence \(\mathcal{L}\)-gradation of openness \(\mathfrak{Z}\) of \(F|A|\), is the same as its \(\mathcal{L}\)-gradation of openness \(\mathfrak{D}|D\) as an \(\mathcal{L}\)-fuzzy of \(Y\), i.e. \(\mathfrak{Z}(H) = \mathfrak{D}(F^{-1}(H)) = \mathfrak{D}(H)\), for all \(\mathcal{L}\)-open subset \(H\) of \(D\). On the other hand, \(\psi\) and \(\varphi\) are \(\mathcal{L}\)-homeomorphisms, hence \(\hat{F}\) is an \(\mathcal{L}\)-homeomorphism of \(\psi|A|\) and \(\varphi|D|\). Therefore \(F|_{\text{supp}A}\) is a homeomorphism, and thus the theorem holds.

Example 6.14. Let \(Z = \chi_{(1, +\infty)}\). Then \(Z\) is an \(\mathcal{L}\)-open subset of \(1_{\mathbb{R}}\). If \(\mathfrak{Z} = \mathfrak{Z}_{11}\), then \((Z, \mathfrak{Z}_Z)\) is an \(\mathcal{L}\)-fuzzy submanifold of \((1_{\mathbb{R}}, \mathfrak{Z}_{11})\). Let \(W = B((0, 0), 1, 1)\), then \(W\) is an \(\mathcal{L}\)-open subset of \(1_{\mathbb{R}^2}\). Consider \(F : \mathbb{R} \to \mathbb{R}^2\), \(F(t) = (\frac{1}{t} \cos 2\pi t, \frac{1}{t} \sin 2\pi t)\). Then \(F \in IGRf(Z, W)\) and rank \(F = 1\) at every point of \(Z\). We see \(F^{-1}(x, y) = \frac{1}{\sqrt{x^2 + y^2}}\), so \(F\) is a one-to-one \(\mathcal{L}\)-fuzzy immersion. Since \(F \in IGRf(Z, \mathfrak{Z})\) is an \(\mathcal{L}\)-homeomorphism, \(F\) is an \(\mathcal{L}\)-fuzzy imbedding.

7 \(\mathcal{L}\)-fuzzy submanifolds of \(\mathcal{L}\)-fuzzy manifolds

Definition 7.1. An \(\mathcal{L}\)-fuzzy subset \(N\) of an \(\mathcal{L}\)-fuzzy manifold \((X, \mathfrak{Z})\), is said to have the \(\mathcal{L}\)-fuzzy \(k\)-submanifold property if each \(p \in N\) has an \(\mathcal{L}\)-local coordinate neighborhood \((A, \psi)\) on \(X\) with \(\mathcal{L}\)-local coordinates \(x_1, x_2, \ldots, x_n\) such that \(\psi(p) = (0, \ldots, 0) \in \mathbb{R}^n\), and

\[
\psi(\text{supp}A \cap \text{supp}N) = \{(x_1, x_2, \ldots, x_n) \in \psi(A) \mid x_{k+1} = \ldots = x_n = 0\}.
\]

If \(N\) has this property, \(\mathcal{L}\)-coordinate neighborhoods of this type are called preferred \(\mathcal{L}\)-local coordinates.

Denote by \(\pi : \mathbb{R}^n \to \mathbb{R}^k\), \(k \leq n\), the projection to the first \(k\) coordinates. Using the notation above, we may state the following proposition:

Theorem 7.2. Let \(N \subseteq X\) have the \(\mathcal{L}\)-fuzzy \(k\)-submanifold property. Then each \(\mathcal{L}\)-preferred \(\mathcal{L}\)-local coordinate system \((A, \psi)\) of \(X\) defines an \(\mathcal{L}\)-local coordinate neighborhood \((A', \psi')\) on \(X\) where \(A' = A \cap N\), \(\psi' = \pi \circ \psi|A'\). Therefore the inclusion \(i \in IGRf(N, X)\) is an \(\mathcal{L}\)-fuzzy imbedding.

Proof. Since \(N\) is an \(\mathcal{L}\)-open subset of \(X\), thus \((N, \mathfrak{Z}_N)\) is an \(\mathcal{L}\)-fuzzy topological subspace of \(X\). Then \((A', \psi')\) are \(\mathcal{L}\)-coordinate neighborhoods covering \(N\), where \(A' = A \cap N\) is an \(\mathcal{L}\)-open subset of \(N\) and \(\psi' = \pi \circ \psi|A'\) is an \(\mathcal{L}\)-homeomorphism. Suppose that for two preferred neighborhoods \((A'_1, \psi'_1)\) and \((A'_2, \psi'_2)\), \(A'_1, A'_2\) have a nonempty intersection. We know that the change of \(\mathcal{L}\)-local coordinates is given by \(\mathcal{L}\)-homeomorphisms \(\psi'_1 \circ \psi'^{-1}_2\) and \(\psi'_2 \circ \psi'^{-1}_1\) which we must show to be \(C^\infty\). Let

\[
\gamma(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbb{R}^n,
\]

so that \(\pi \circ \gamma\) is the identity on \(\mathbb{R}^k\). This map \(\gamma\) is \(C^\infty\). Hence its restriction to \(\psi'(A')\), an \(\mathcal{L}\)-open subset of \(\mathbb{R}^k\), is \(C^\infty\); thus \(\psi'^{-1} = \psi \circ \gamma\) is \(C^\infty\), since it is a composition of \(C^\infty\) maps. On the other hand, \(\psi' = \pi \circ \psi\) so \(\psi'\) is a \(C^\infty\) map on \(A'\). Hence \(\psi'_1 \circ \psi'^{-1}_2\) is \(C^\infty\). If \(y_i = f_i(x_1, \ldots, x_k), i = 1, \ldots, k\), are the functions giving \(\psi'_1 \circ \psi'^{-1}_2\), which we know to be \(C^\infty\), then it can easily be checked that \(\psi'_1 \circ \psi'^{-1}_2\) is given by \(y_i = f_i(x_1, \ldots, x_k, 0, \ldots, 0), i = 1, \ldots, k\). Therefore \(\psi'_1 \circ \psi'^{-1}_2\) is \(C^\infty\) by Definition 3.2. Thus the totality of these \(\mathcal{L}\)-neighborhoods define a unique differentiable structure on \(N\). In preferred \(\mathcal{L}\)-local coordinates \((A', \psi')\), \(\in IGRf(N, X)\) is given on \(V\) by

\[
\psi \circ i \circ \psi'^{-1}(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0).
\]

So the map \(i\) is clearly an \(\mathcal{L}\)-fuzzy immersion. Because we have taken the relative \(\mathcal{L}\)-fuzzy topology on \(N\), the fuzzy map \(i\) is by Definition 2.13 (iii) an \(\mathcal{L}\)-homeomorphism to its image \(i(N)\), with the \(\mathcal{L}\)-fuzzy subspace topology, that is, \(i\) is an \(\mathcal{L}\)-fuzzy imbedding.

Definition 7.3. A regular \(\mathcal{L}\)-fuzzy submanifold of an \(\mathcal{L}\)-fuzzy manifold \((X, \mathfrak{Z})\) is any \(\mathcal{L}\)-fuzzy topological subspace \(N\) with the \(\mathcal{L}\)-fuzzy submanifold property and with the structure that the corresponding preferred \(\mathcal{L}\)-local coordinate neighborhoods determine on it.

Example 7.4. Let \(M = \mathbb{R}^3\), \(X : \mathbb{R}^3 \to I\), \(X(x) = \begin{cases} 1 & \|x\| = 1, \\ 0 & \|x\| \neq 1. \end{cases}\). Then \(\text{supp}X = S^2\), the unit sphere. Let \(\mathfrak{Z} : \gamma_x \to I\), \(\mathfrak{Z}(A) = \begin{cases} 1 & A \in \tau_{13}, A \leq X \\ 0 & \text{elsewhere.} \end{cases}\). We shall see that \(X\) is an \(\mathcal{L}\)-fuzzy submanifold of \((1_{\mathbb{R}^3}, \mathfrak{Z}_{13})\). If

\[
\mathfrak{Z} : \gamma_x \to I, \mathfrak{Z}(A) = \begin{cases} 1 & A \in \tau_{13}, A \leq X \\ 0 & \text{elsewhere.} \end{cases}\.
\]
q = (x_1, x_2, x_3) is an arbitrary point in suppX, it cannot lie on more than one coordinate axis. For convenience, we assume that it does not lie on the x_3-axis. We introduce the spherical LG-local coordinates (ρ, θ, φ); they are defined on _R^3_ \setminus \{x_3 = 0\} and if (1, θ_0, φ_0) are the LG-coordinates of q, we may change them a little so that it is replaced by \( \rho = \rho - 1 \), \( \theta = \theta - \theta_0 \), and \( \phi = \phi - \phi_0 \). Then it defines an LG-coordinate neighborhood of q, with q having LG-coordinates (0, 0, 0) and with the LG-open subset V of X.

**Remark 7.5.** So far, we have defined three classes of LG-fuzzy manifolds of an LG-fuzzy n-manifold (X, S). The first of these, which we usually simply call an LG-fuzzy submanifold, was defined (in 6.11) as the image N = F[N'] of an LG-fuzzy immersion F of N' into X. Since F : N' \to N \subseteq X is one-to-one and onto, we conduct (as part of the definition) carry over to N the LG-fuzzy topology and LG-fuzzy structure of N'. LG-open subsets of N are the images of LG-open sets of N' and LG-coordinate neighborhoods (A, ψ) of N are of the form A = F[A'], \( \psi = \psi \circ F^{-1} \), where (A', ψ') is an LG-local coordinate neighborhood of N'. The fact that F is LG-continuous shows that the LG-fuzzy topology of N gained in this way is in general finer than its relative LG-fuzzy topology as an LG-fuzzy subspace of X, that is, if D is LG-open subset of X, then D \cap N is LG-open subset of N, but there may be LG-open subsets of N which are not of this form.

An LG-fuzzy imbedding is a particular type of LG-fuzzy immersion, one in which A is LG-open subset of N if and only if \( A = F[U'] = D \cap N \) for some LG-open subset D of X so that the LG-fuzzy topology of the submanifold N = F[N'] is exactly its relative LG-fuzzy topology as an LG-fuzzy topological subspace of X. An LG-fuzzy imbedded submanifold is so a special type of (immersed) LG-fuzzy submanifold.

Ultimately, if N \subseteq X is an LG-fuzzy regular submanifold, then it is also an LG-fuzzy imbedded submanifold since the inclusion \( i : N \to M \) is an LG-fuzzy imbedding as we proved in 7.2.

**Theorem 7.6.** Let \( F \in LGRf(N', X) \) be an LG-fuzzy imbedding of an LG-fuzzy manifold N' of dimension k in an LG-fuzzy manifold of dimension n. Then N = F[N'] has the LG-fuzzy k-submanifold property and thus N is an LG-fuzzy regular submanifold. As such, it is LG-diffeomorphic to N' with respect to the LG-fuzzy mapping \( F \in LGRf(N', N) \).

**Proof.** Let q = F(p) be any point of N. According to Theorem 7.2 (and its proof), there are (A, ψ) and (B, φ), LG-local coordinate neighborhoods of p and F(p), respectively, such that (\ref{eq:coordinate_neighborhood}) holds. If F[A] = V \subseteq N, then the LG-neighborhood V would be a preferred LG-local coordinate neighborhood relative to N. To deduce this result, we should use the fact that F is an LG-imbedding. This denotes at least that F[A] is a relatively LG-open subset of N, that is, F[A] = W \cap N, where W is LG-open subset of X. Since F[A] \subseteq V, we can suppose W \subseteq V. Thus \( \psi[W] \) is an LG-open subset of \( \varphi[B] \) containing the origin in \( \mathbb{R}^n \) and \( \varphi[F[A]] \subseteq \psi[W] \), which is a slice S of \( \varphi[V] \), \( S = \{ x \in \varphi[V] | x_k + 1 = \ldots = x_m = 0 \} \). Hence we may select an (smaller) LG-open subset \( \varphi[V'] \subseteq \psi[W] \) and \( \varphi' = \varphi|_{\text{supp}V'} \). This is an LG-local coordinate neighborhood of q for which \( F[A] \cap V' = V' \cap N \); furthermore, taking \( A' = F^{-1}[V'] \), we see that (A', ψ'), with \( \psi' = \psi|_{\text{supp}A'} \), is an LG-local coordinate neighborhood of p and the pair \( (A', \psi') \) and \( (V', \varphi') \) have exactly the properties needed in 7.1 and \( F[A'] = V' \subseteq N \). This proves at the same time, that N has the LG-fuzzy k-submanifold property.

This is true since the inverse of F in \( LGRf(N', N) \) is given in the preferred LG-local coordinates \( (V', \pi \circ \varphi') \) and \( (A', \psi') \) by \( F^{-1}(x_1, \ldots, x_k) = (x_1, \ldots, x_k) \), which is \( C^\infty \).

**Remark 7.7.** Suppose that N \subseteq X is an LG-fuzzy immersed submanifold and that q \in N. Then there is an LG-neighborhood (V, ψ) of q, with \( \psi(p) = (0, \ldots, 0) \) such that the slice S' \subseteq \text{supp}V, consisting of all points of V whose last \( n - k \) coordinates vanish, is an LG-open set and an LG-local coordinate neighborhood of the LG-fuzzy submanifold structure of N is given by LG-local coordinate map

\[
\psi'(q) = \pi \circ \psi(q) = (x_1(q), \ldots, x_k(q)).
\]

**Theorem 7.8.** If \( F \in LGRf(N, X) \) is a one-to-one LG-fuzzy immersion and N is a compact L-gfts, then F is an LG-fuzzy imbedding and \( \bar{N} = F[N] \) an LG-fuzzy regular submanifold.

**Proof.** Since F is LG-continuous and both N and \( \bar{N} \) are Hausdorff L-gfts’s, we have an LG-continuous (one-to-one) mapping from a compact L-gfts to a Hausdorff L-gfts. Since an LG-closed subset K of N is compact, so F(K) is compact and therefore LG-closed. Thus F takes LG-closed subsets of N to LG-closed subsets of X, and since F is one-to-one and onto, it takes LG-open subsets to LG-open subsets as well. It follows that \( F^{-1} \) is LG-continuous, so \( F \in LGRf(N, \bar{N}) \) is an LG-homeomorphism and therefore an LG-imbedding.

**Theorem 7.9.** Let \( F \in LGRf(X, Y) \) be a \( C^\infty \) LG-fuzzy mapping. Suppose that F has constant rank k on X and that \( q \in F(X) \). Let \( D \) denote \( F^{-1}(q) \); then \( \chi_{L} \) is an LG-closed, LG-fuzzy regular submanifold of X of dimension \( n - k \).
Proof. Let \( p \in D \); since \( F \) has constant rank \( k \) on an \( LG \)-neighborhood of \( p \), we may find \( LG \)-local coordinate neighborhoods \( (A, \psi), (B, \varphi) \) such that (1) holds. By Example 2.9 the fuzzy point \( 0_{1} \) is an \( LG \)-closed subset of \( \mathbb{R}^{n} \), then \( \chi_{(q)} \) is an \( LG \)-closed subset of \( Y \). Hence \( \chi_{(q)} \) is an \( LG \)-closed subset since the inverse image of \( \chi_{(q)} \), under a continuous map, is \( LG \)-closed. We shall show that \( \chi_{(q)} \) has the \( LG \)-fuzzy \( n - k \) submanifold property. This means that the only points of \( D \) mapped onto \( q \) are those whose first \( k \) coordinates are zero, that is,

\[
\text{supp}A \cap D = \psi^{-1}(\psi \circ F^{-1} \circ \varphi^{-1}(0)) = \psi^{-1}(\tilde{F}^{-1}(0)) = \psi^{-1}\{x \in \psi(A) | x_{1} = \ldots = x_{k} = 0\}.
\]

Hence \( \chi_{(q)} \) is a regular \( LG \)-fuzzy \( (n - k) \)-submanifold since it has the \( LG \)-fuzzy submanifold property.

\[\square\]

**Corollary 7.10.** If \( F \in LGRf(X,Y) \) is a \( C^{\infty} \) \( LG \)-fuzzy mapping of \( LG \)-fuzzy manifolds, \( \dim X = n \leq m = \dim Y \), and rank \( F = n \) at every point of \( D = F^{-1}(q) \), then \( \chi_{(q)} \) is an \( LG \)-closed, regular \( LG \)-fuzzy submanifold of \( X \). The corollary holds because at \( p \in A \), \( F \) has the maximum rank possible, namely \( m \). It follows from the independence of rank on \( LG \)-local coordinates that, in some \( LG \)-neighborhood of \( p \) in \( N \), \( F \) also has this rank; thus the rank of \( F \) is \( m \) on an \( LG \)-open subset of \( N \) containing \( A \). But such an \( LG \)-fuzzy subset is itself an \( LG \)-fuzzy \( n \)-manifold (an \( LG \)-open submanifold) to which we may apply the theorem.

### 8 Conclusion

In this paper, we generalize all of the fuzzy structures which we have discussed in [13] to \( L \)-fuzzy set theory, where \( L = \langle L, \leq, \wedge, \vee, ' \rangle \) denotes a complete distributive lattice with at least two elements. We define the concept of an \( LG \)-fuzzy topological space \((X, \mathcal{T})\) which \( X \) is itself an \( L \)-fuzzy subset of a crisp set \( M \) and \( \mathcal{T} \) is an \( LG \)-gradation of openness of \( L \)-fuzzy subsets of \( M \) which are less than or equal to \( X \). Then we define \( C^{\infty} \) \( L \)-fuzzy manifolds with \( L \)-gradation of openness and \( C^{\infty} \) \( LG \)-fuzzy mappings of them such as \( LG \)-fuzzy immersions and \( LG \)-fuzzy imbeddings. We fuzzify the concept of the product manifolds with \( L \)-gradation of openness and define \( LG \)-fuzzy quotient manifolds when we have an equivalence relation on \( M \) and investigate the conditions of the existence of the quotient manifolds. We also introduce \( LG \)-fuzzy immersed, imbedded and regular submanifolds.

### References


