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Modal operators on pseudo-BE algebras

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Abstract

In this paper, we define and study the modal operators on pseudo-BE algebras as special cases of closure operators on these structures. We prove that the composition of two modal operators is a modal operator if and only if they commute. For the particular case of a good pseudo-BCK algebra an equivalent definition of the modal operators is given, and the notion of a strong modal operator is introduced and studied. We also define the notions of modal deductive systems and modal homomorphisms on pseudo-BE algebras and we investigate their properties. It is proved that, if two modal operators have the same image, then they coincide. Also, given a normal modal deductive system H of a distributive modal pseudo-BE algebra (A, f) we construct a modal operator on the quotient pseudo-BE algebra A/H.

Keywords: Non-classical logic, pseudo-BE algebra, modal operator, strong modal operator, modal deductive system, modal homomorphism.

1 Introduction

Modal logic is an important branch of logic developed firstly in the category of nonclassical logics and has now been widely used as a formalism for knowledge representation in artificial intelligence and analysis tool in computer science. Modal logics and many-valued logics were both historically introduced in order to free oneself from the rigidity of propositional logic. With many-valued logics, the logician can choose the truth values of the propositions in a set with more than two elements. With modal logics, the logician introduces a new connector whose aim is, for instance, to model the possibility. Many systems with various kind of modal operators have been constructed in order to provide effective formalisms for talking about time, space, knowledge, beliefs, actions, obligations, temporal, spatial, epistemic, dynamic, and so forth. However, modern applications often require rather complex formal models and corresponding languages that are capable of reflecting different features of the application domain (see [1], [15], [16]). Pseudo-BCK algebras were introduced by G. Georgescu and A. Iorgulescu in [17] as algebras with "two differences", a left- and right-difference, and with a constant element 0 as the least element. Nowadays pseudo-BCK algebras are used in a dual form, with two implications, $\to$ and $\rightsquigarrow$ and with one constant element 1, that is the greatest element. Thus such pseudo-BCK algebras are in the "negative cone" and are also called "left-ones". Pseudo-BCK algebras were intensively studied in [10], [21], [22], [25], [26]. The notion of a BE-algebra as a generalization of a BCK-algebra was defined and studied by H.S. Kim et al. in [23]. Then A. Walendziak investigated the relationship between BE–algebras, implication algebras and J–algebras [47]. A. Rezaei et al. studied commutative ideals on BE-algebras [42]. Pseudo-BE algebras were introduced in [4] as generalizations of BE-algebras and properties of these structures have recently been studied in [5], [11] and [43]. Macnab, in [29] introduced the notion of a modal operator on Heyting algebras as a unary operator φ satisfying the conditions: (i) $x \leq \varphi(x)$, (ii) $\varphi\varphi(x) = \varphi(x)$, (iii) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$, for all x, y, and this notion was generalized by Rachůnek in [35] to the case of ordered sets. The study of modal operators was extended to other fuzzy logic algebras such as Boolean algebras ([28], [30], [41]), Łukasiewicz-Moisil algebras ([3], [18]), MV-algebras ([19],

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[33]), MTL-algebras ([32]), bounded Rℓ-monoids ([36], [37]), residuated lattices ([38], [39], [40]), MTL-algebras ([9], [32]), pseudo-BCK algebras ([27]) and BE-algebras ([48]).
In this paper, we define the modal operators on pseudo-BE algebras and investigate their properties. The motivation of this study consists of two types of arguments:

– algebraic arguments: the modal pseudo-BE algebras are generalization of various modal algebras;

– logical arguments: the modal operators on pseudo-BE algebras are algebraic versions of some logical operators in some many valued logics.

As another motivation of this study we mention the possible applications in computer science of modal logics: computer programming, information knowledge, verification of parallel programs, ontology languages for the semantic web ([20], [34]).
For the particular case of a good pseudo-BCK algebra an equivalent definition of the modal operators is given, and the notion of a strong modal operator is introduced and studied. We also define the notions of modal deductive systems and modal homomorphisms on pseudo-BE algebras and investigate their properties. It is proved that, if two modal operators have the same image, then they coincide. Also, given a normal modal deductive system H of a distributive modal pseudo-BE algebra (A, f) we construct a modal operator on the quotient pseudo-BE algebra A/H. Given two modal pseudo-BCK algebras (A, f), (B, g) and a modal homomorphism ϕ, we prove that there exists a modal pseudo-BCK isomorphism between $(A/\mathrm{Ker}\,(\phi), f)$ and $(\mathrm{Im}\,(\phi), g)$.

2 Preliminaries

In this section we recall some basic notions and results regarding pseudo-BE algebras and pseudo-BCK algebras.

Definition 2.1. [4] *A* pseudo-BE algebra *is an algebra* $(A, \to, \rightsquigarrow, 1)$ *of the type* $(2, 2, 0)$ *such that the following axioms are fulfilled: for all* $x, y, z \in A$,
$(psBE_1)$ $x \to x = x \rightsquigarrow x = 1$,
$(psBE_2)$ $x \to 1 = x \rightsquigarrow 1 = 1$,
$(psBE_3)$ $1 \to x = 1 \rightsquigarrow x = x$,
$(psBE_4)$ $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
$(psBE_5)$ $x \to y = 1$ *iff* $x \rightsquigarrow y = 1$.

A pseudo-BE algebra is said to be proper if it is not a BE-algebra. In a pseudo-BE algebra $(A, \to, \rightsquigarrow, 1)$, one can define a binary relation " $\leq$ " by

$$x \leq y \text{ iff } x \to y = 1 \text{ iff } x \rightsquigarrow y = 1 \text{ for all } x, y \in A.$$

Let $(A, \to, \rightsquigarrow, 1)$ be a pseudo-BE algebra. Denote:

$$x \vee_1 y = (x \to y) \rightsquigarrow y \text{ and } x \vee_2 y = (x \rightsquigarrow y) \to y, \text{ for all } x, y \in A.$$

If $\to = \rightsquigarrow$, then the pseudo-BE algebra $(A, \to, \rightsquigarrow, 1)$ is a BE-algebra and $x \vee y = (x \to y) \to y$, for all $x, y \in A$. If there is an element 0 of a pseudo-BE algebra $(A, \to, \rightsquigarrow, 1)$, such that $0 \leq x$ (i.e., $0 \to x = 0 \rightsquigarrow x = 1$), for all $x \in A$, then the pseudo-BE algebra is said to be bounded and it is denoted by $(A, \to, \rightsquigarrow, 0, 1)$. In a bounded pseudo-BE algebra $(A, \to, \rightsquigarrow, 0, 1)$ we define two negations, $x^- := x \to 0$, $x^\sim := x \rightsquigarrow 0$, for all $x \in A$. Obviously, $x^{-\sim} = x \vee_1 0$ and $x^{\sim-} = x \vee_2 0$. If $(A, \to, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BE algebra we denote, $\mathrm{Reg}\,(A) = \{x \in A \mid x^{-\sim} = x^{\sim-} = x\}$, the set of all regular elements of A, and $\mathrm{Den}\,(A) = \{x \in A \mid x^{-\sim} = x^{\sim-} = 1\}$, the set of all dense elements of A.
If $\mathrm{Reg}\,(A) = A$, then A is said to be involutive. If a bounded pseudo-BE algebra A satisfies $x^{-\sim} = x^{\sim-}$, for all $x \in A$, then A is called a good pseudo-BE algebra. Obviously, if A is involutive, then A is good and $\mathrm{Den}\,(A) = \{1\}$.

Proposition 2.2. [4] *Let* $(A, \to, \rightsquigarrow, 1)$ *be a pseudo-BE algebra. Then the following hold: for all* $x, y, z \in A$,
(1) $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$;
(2) $x \to (y \rightsquigarrow x) = 1$ *and* $x \rightsquigarrow (y \to x) = 1$;
(3) $x \to (y \to x) = 1$ *and* $x \rightsquigarrow (y \rightsquigarrow x) = 1$;
(4) $x \to ((x \to y) \rightsquigarrow y) = 1$ *and* $x \rightsquigarrow ((x \rightsquigarrow y) \to y) = 1$;
(5) $1 \leq x$ *implies* $x = 1$.

We will refer to $(A, \to, \rightsquigarrow, 1)$ by its universe A.

Definition 2.3. [17] *A pseudo-BCK algebra (more precisely, reversed left-pseudo-BCK algebra) is a structure* $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$ *where* $\leq$ *is a binary relation on* A, $\to$ *and* $\rightsquigarrow$ *are binary operations on* A *and* 1 *is an element of* A

satisfying, for all $x, y, z \in A$, the axioms:
($psBCK_1$) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$, $\quad x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;
($psBCK_2$) $x \leq (x \to y) \rightsquigarrow y$, $\quad x \leq (x \rightsquigarrow y) \to y$;
($psBCK_3$) $x \leq x$;
($psBCK_4$) $x \leq 1$;
($psBCK_5$) *if* $x \leq y$ *and* $y \leq x$, *then* $x = y$;
($psBCK_6$) $x \leq y$ *iff* $x \to y = 1$ *iff* $x \rightsquigarrow y = 1$.

Lemma 2.4. [17] *Let* $(A, \to, \rightsquigarrow, 1)$ *be a pseudo-BCK algebra. Then the following hold: for all* $x, y, z \in A$,
(1) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$;
(2) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$;
(3) $x \leq y$ *implies* $z \to x \leq z \to y$ *and* $z \rightsquigarrow x \leq z \rightsquigarrow y$;
(4) $x \to y \leq (z \to x) \to (z \to y)$ *and* $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$.

Proposition 2.5. [11] *Any pseudo-BCK algebra is a pseudo-BE algebra.*

Definition 2.6. [12] *A pseudo-BE algebra with* (A) *condition or a pseudo-BE(A) algebra for short, is a pseudo-BE algebra* $(A, \to, \rightsquigarrow, 1)$ *such that the operations* $\to$, $\rightsquigarrow$ *are antitone in the first variable, that is* (A) *condition is satisfied:*
(A) *if* $x, y \in A$ *such that* $x \leq y$, *then for all* $z \in A$,

$$y \to z \leq x \to z \text{ and } y \rightsquigarrow z \leq x \rightsquigarrow z.$$

Proposition 2.7. [12] *Let* $(A, \to, \rightsquigarrow, 1)$ *be a pseudo-BE(A) algebra and* $x, y \in A$ *such that* $x \leq y$. *Then* $(x \to z) \rightsquigarrow z \leq (y \to z) \rightsquigarrow z$ *and* $(x \rightsquigarrow z) \to z \leq (y \rightsquigarrow z) \to z$, *for all* $z \in A$.

Definition 2.8. [5] *A pseudo-BE algebra* A *is said to be distributive if for all* $x, y, z \in A$, $x \to (y \rightsquigarrow z) = (x \to y) \rightsquigarrow (x \to z)$.

Proposition 2.9. [45] *Any bounded commutative pseudo-BE algebra is involutive.*

Proposition 2.10. [45] *Let* $(A, \to, \rightsquigarrow, 0, 1)$ *be a bounded pseudo-BE algebra. Then for all* $x, y \in X$, *the following statements hold:*
(1) $x \leq x^{-\sim}$, $x \leq x^{\sim-}$;
(2) $x \to y^{\sim} = y \rightsquigarrow x^{-}$ *and* $x \rightsquigarrow y^{-} = y \to x^{\sim}$;
(3) $x^{\sim} \to y^{-\sim} = y^{-} \rightsquigarrow x^{\sim-}$ *and* $x^{-} \rightsquigarrow y^{\sim-} = y^{\sim} \to x^{-\sim}$.

Proposition 2.11. [45] *Let* $(A, \to, \rightsquigarrow, 0, 1)$ *be a bounded pseudo-BE(A) algebra. If for all* $x, y \in A$, $x \leq y$, *then* $y^{-} \leq x^{-}$, $y^{\sim} \leq x^{\sim}$, $x^{-\sim} \leq y^{-\sim}$ *and* $x^{\sim-} \leq y^{\sim-}$.

Proposition 2.12. [10] *Let* $(A, \to, \rightsquigarrow, 0, 1)$ *be a bounded pseudo BCK-algebra. Then for all* $x, y \in A$, *the following hold:*
(1) $x^{-\sim-} = x^{-}$ *and* $x^{\sim-\sim} = x^{\sim}$;
(2) $x \to y^{-\sim} = y^{-} \rightsquigarrow x^{-} = x^{-\sim} \to y^{-\sim}$ *and* $x \rightsquigarrow y^{\sim-} = y^{\sim} \to x^{\sim} = x^{\sim-} \rightsquigarrow y^{\sim-}$;
(3) $x \to y^{\sim} = y^{\sim-} \rightsquigarrow x^{-} = x^{-\sim} \to y^{\sim}$ *and* $x \rightsquigarrow y^{-} = y^{-\sim} \to x^{\sim} = x^{\sim-} \rightsquigarrow y^{-}$;
(4) $(x \to y^{\sim-})^{\sim-} = x \to y^{\sim-}$ *and* $(x \rightsquigarrow y^{-\sim})^{-\sim} = x \rightsquigarrow y^{-\sim}$.

Definition 2.13. [11]*A good pseudo-BE algebra* A *has the Glivenko properties if for all* $x, y \in A$, *it satisfies the following conditions:*

$$(x \to y)^{-\sim} = x \to y^{-\sim}, \text{and } (x \rightsquigarrow y)^{-\sim} = x \rightsquigarrow y^{-\sim}.$$

Obviously, any involutive pseudo-BE algebra has the Glivenko property.

Proposition 2.14. [45] *Let* $(A, \to, \rightsquigarrow, 0, 1)$ *be a good pseudo-BCK algebra. Define a binary operation* $\oplus$ *on* A *by* $x \oplus y := y^{\sim} \to x^{-\sim}$. *Then for all* $x, y, z \in A$, *the following hold:*
(1) $x \oplus y = x^{-} \rightsquigarrow y^{\sim-}$;
(2) $x, y \leq x \oplus y$;
(3) $x \oplus 0 = x^{-\sim}$ *and* $0 \oplus x = x^{\sim-}$;
(4) $x \oplus 1 = 1 \oplus x = 1$;
(5) $x \oplus x^{-} = 1$ *and* $x^{\sim} \oplus x = 1$;
(6) $(x \oplus y)^{-\sim} = x \oplus y = x^{-\sim} \oplus y = x \oplus y^{-\sim} = x^{-\sim} \oplus y^{-\sim}$;
(7) $\oplus$ *is associative;*
(8) $(x \oplus y) \oplus 0 = x \oplus y$;
(9) $x \leq y$ *implies* $z \oplus x \leq z \oplus y$ *and* $x \oplus z \leq y \oplus z$.

Definition 2.15. [8] *Let A be a partially ordered set (poset). A mapping $\tau : A \longrightarrow A$ is called a closure operator on A if for all $x, y \in A$, it satisfies the following conditions:*
(C_1) $x \leq \tau(x)$; *(increasing)*
(C_2) $x \leq y$ *implies* $\tau(x) \leq \tau(y)$; *(monotone)*
(C_3) $\tau(\tau(x)) = \tau(x)$. *(idempotent)*

Denote $\mathcal{CL}(A)$ the set of all closure operators on A.

A subset D of a pseudo-BE algebra A is called a deductive system of A if it satisfies the following axioms:
(ds_1) $1 \in D$,
(ds_2) $x \in D$ and $x \to y \in D$ imply $y \in D$.
A subset D of A is a deductive system if and only if it satisfies (ds_1) and the axiom:
(ds_2') $x \in D$ and $x \rightsquigarrow y \in D$ imply $y \in D$.
Denote by $\mathcal{DS}(A)$ the set of all deductive systems of A. A deductive system D of A is proper if $D \neq A$.
A deductive system D of a pseudo-BE algebra A is said to be normal if it satisfies the condition:
(ds_3) for all $x, y \in A$, $x \to y \in D$ if and only if $x \rightsquigarrow y \in D$.
Denote by $\mathcal{DS}_n(A)$ the set of all normal deductive systems of A.
For details regarding deductive systems and congruence relations on a pseudo-BE algebra we refer the reader to [4], [43].
Let A be a distributive pseudo-BE algebra. Denote by $\mathcal{CON}(A)$ the set of all congruences on A. If $\theta \in \mathcal{CON}(A)$, then $H_\theta = \{x \in A \mid (x, 1) \in \theta\} \in \mathcal{DS}_n(A)$.
Given $H \in \mathcal{DS}_n(A)$, the relation Θ_H on A defined by $(x, y) \in \Theta_H$ iff $x \to y \in H$ and $y \to x \in H$ is a congruence on A. We write $x/H = [x]_{\Theta_H}$ for every $x \in A$ and we have $H = [1]_{\Theta_H}$. Then $(A/\Theta_H, \to, [1]_{\Theta_H}) = (A/\Theta_H, \rightsquigarrow, [1]_{\Theta_H})$ is a BE-algebra called quotient pseudo-BE algebra via H and denoted by A/H (see [43]).
The function $\pi_H : A \longrightarrow A/H$ defined by $\pi_H(x) = x/H$ for any $x \in A$ is a surjective homomorphism which is called the *canonical projection* from A to A/H. One can easily prove that $\mathrm{Ker}\,(\pi_H) = H$.
Let A, B be two pseudo-BE algebras. A map $f : A \longrightarrow B$ is called a pseudo-BE homomorphism if $f(x \to y) = f(x) \to f(y)$ and $f(x \rightsquigarrow y) = f(x) \rightsquigarrow f(y)$, for all $x, y \in A$.
(We use the same notations for the operations in both pseudo-BE algebras, but the reader must be aware that they are different). If $B = A$, then f is called a pseudo-BE endomorphism. Denote $\mathcal{HOM}(A, B)$ the sets of all pseudo-BE homomorphisms from A to B.
One can easily check that, if f is a pseudo-BE homomorphism, then (i) $f(1) = 1$; (ii) $x \leq y$ implies $f(x) \leq f(y)$.
If f is a bounded pseudo-BE homomorphism such that $f(0) = 0$, then (iii) $f(x^-) = f(x)^-$; (iv) $f(x^\sim) = f(x)^\sim$.
If A and B are good pseudo-BCK algebras and f is a pseudo-BCK homomorphism such that $f(0) = 0$, then: (v) $f(x \oplus y) = f(x) \oplus f(y)$.

3 Modal operators on pseudo-BE algebras and pseudo-BCK algebras

In this section, we define the modal operators on pseudo-BE algebras and we investigate their properties. We prove that the composition of two modal operators is also a modal operator if and only if they commute. For the particular case of a good pseudo-BCK algebra an equivalent definition of the modal operators is given, and the notion of a strong modal operator is introduced and studied. We also define the modal upper set of two elements of a modal pseudo-BE algebra and we investigate some properties of this set. If $\mathrm{Fix}\,(f)$ is the set of all fix elements of a strong modal operator on an involutive pseudo-BCK algebra $(A, \to, \rightsquigarrow, 0, 1)$, then we prove that $(\mathrm{Fix}\,(f), \to, \rightsquigarrow, f(0), 1)$ is also an involutive pseudo-BCK algebra.

Definition 3.1. *Let A be a pseudo-BE algebra. A mapping $f : A \longrightarrow A$ is called a* modal operator *on A if it satisfies the following conditions: for all $x, y \in A$,*
(M_1) $x \leq f(x)$;
(M_2) $f(f(x)) = f(x)$;
(M_3) $f(x \to y) \leq f(x) \to f(y)$ *and* $f(x \rightsquigarrow y) \leq f(x) \rightsquigarrow f(y)$.
The pair (A, f) is called a modal pseudo-BE algebra.

Denote $\mathcal{MOD}(A)$ the set of all modal operators on A.

Proposition 3.2. *Let A be a pseudo-BE algebra and let $f : A \longrightarrow A$ be a modal operator on A. Then the following hold: for all $x, y \in A$,*
(1) $f(1) = 1$;
(2) $x \leq y$ *implies* $f(x) \leq f(y)$.

Proof. (1) By (M_1), $1 \le f(1)$, hence $f(1) = 1$.
(2) Let $x, y \in A$, $x \le y$, that is $x \to y = x \rightsquigarrow y = 1$. Applying (M_3), we have $1 = f(1) = f(x \to y) \le f(x) \to f(y)$, hence $f(x) \to f(y) = 1$, so $f(x) \le f(y)$. □

Example 3.3. [4] *Consider the set $A = \{0, a, b, 1\}$ and the operations $\to, \rightsquigarrow$ given by the following tables:*

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	b	1
b	a	1	1	1
1	0	a	b	1

$\rightsquigarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	b	1	1	1
1	0	a	b	1

Then $(A, \to, \rightsquigarrow, 1)$ is a bounded pseudo-BE algebra. Moreover it is even a pseudo-BCK algebra. Define the maps $f_i : A \longrightarrow A$, $i = 1, 2, 3, 4, 5$, given in the table below:

x	0	a	b	1
$f_1(x)$	1	1	1	1
$f_2(x)$	0	a	b	1
$f_3(x)$	a	a	a	1
$f_4(x)$	b	1	b	1
$f_5(x)$	b	a	b	1

Then $\mathcal{MOD}(A) = \{f_1, f_2, f_3, f_4, f_5\}$.

Example 3.4. [12] *Let $A = \{a, b, c, d, 1\}$. Define the operations $\to$ and $\rightsquigarrow$ on A as follows:*

$\to$	a	b	c	d	1
a	1	c	c	1	1
b	d	1	1	d	1
c	d	1	1	d	1
d	1	c	c	1	1
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	b	c	1	1
b	d	1	1	d	1
c	d	1	1	d	1
d	1	b	c	1	1
1	a	b	c	d	1

Then $(A, \to, \rightsquigarrow, 1)$ is a pseudo-BE(A) algebra. Define $f^1_{\alpha,\beta}, f^2_{\alpha,\beta}, f^3_{\alpha,\beta,\gamma,\delta} : A \longrightarrow A$ by:

$$f^1_{\alpha,\beta}(1) = f^1_{\alpha,\beta}(a) = f^1_{\alpha,\beta}(d) = 1,\ f^1_{\alpha,\beta}(b) = \alpha,\ f^1_{\alpha,\beta}(c) = \beta;$$
$$f^2_{\alpha,\beta}(1) = f^2_{\alpha,\beta}(b) = f^2_{\alpha,\beta}(c) = 1,\ f^2_{\alpha,\beta}(a) = \alpha, f^2_{\alpha,\beta}(d) = \beta;$$
$$f^3_{\alpha,\beta,\gamma,\delta}(1) = 1,\ f^3_{\alpha,\beta,\gamma,\delta}(a) = \alpha,\ f^3_{\alpha,\beta,\gamma,\delta}(b) = \beta,\ f^3_{\alpha,\beta,\gamma,\delta}(c) = \gamma,\ f^3_{\alpha,\beta,\gamma,\delta}(d) = \delta.$$

Then we have:

$$\mathcal{MOD}(A) = \{1_A\} \cup \{f^1_{\alpha,\beta} \mid \alpha, \beta \in \{b, c\}\} \cup \{f^2_{\alpha,\beta} \mid \alpha, \beta \in \{a, d\}\} \cup \{f^3_{\alpha,\beta,\gamma,\delta} \mid \alpha, \delta \in \{a, d\}, \beta, \gamma \in \{b, c\}\}.$$

Remark 3.5. (1) *Let $(A, \to, \rightsquigarrow, 1)$ be a pseudo-BE algebra and $1_A, Id_A : A \longrightarrow A$, defined by $1_A(x) = 1$ and $Id_A(x) = x$, for all $x \in A$. Then $1_A, Id_A \in \mathcal{MOD}(A)$.*
(2) *In any pseudo-BE algebra A, $\mathcal{MOD}(A) \subseteq \mathcal{CL}(A)$.*
(3) *For a modal pseudo-BE algebra (A, f), $f(A) = A$ is not true in general. Indeed, in Example 3.4, we have: $f^1_{\alpha,\beta}(A) = \{1, b, c\} \neq A$.*

Proposition 3.6. *Let A be a pseudo-BCK algebra and let $f \in \mathcal{MOD}(A)$. Then the following hold: for all $x, y \in A$,*
(1) *$f(x) \to f(y) = f(f(x) \to f(y)) = x \to f(y) = f(x \to f(y))$ and $f(x) \rightsquigarrow f(y) = f(f(x) \rightsquigarrow f(y)) = x \rightsquigarrow f(y) = f(x \rightsquigarrow f(y))$;*
(2) *if A is bounded, then $f(x) \le (x \to f(0)) \rightsquigarrow f(0)$ and $f(x) \le (x \rightsquigarrow f(0)) \to f(0)$.*

Proof. (1) From $x \le f(x)$, by Lemma 2.4(2), we have $f(x) \to f(y) \le x \to f(y)$. Then we get:

$$\begin{aligned} f(f(x) \to f(y)) &\le f(f(x)) \to f(f(y)) = f(x) \to f(y) \le x \to f(y) \\ &\le f(x \to f(y)) \le f(x) \to f(f(y)) = f(x) \to f(y) \\ &\le f(f(x) \to f(y)). \end{aligned}$$

Hence $f(x) \to f(y) = f(f(x) \to f(y)) = x \to f(y) = f(x \to f(y))$. Similarly, $f(x) \rightsquigarrow f(y) = f(f(x) \rightsquigarrow f(y)) = x \rightsquigarrow f(y) = f(x \rightsquigarrow f(y))$.
(2) Applying Proposition 2.2(4) and (1), we get: $f(x) \le (f(x) \to f(0)) \rightsquigarrow f(0) = (x \to f(0)) \rightsquigarrow f(0)$. Similarly, $f(x) \le (x \rightsquigarrow f(0)) \to f(0)$. □

Theorem 3.7. *Let (A, f) be a modal pseudo-BE algebra. Then $(f(A), f)$ is so.*

Proof. Using Proposition 3.6(1), the proof is clear. □

Proposition 3.8. *Let A be a pseudo-BCK algebra, $f, g \in \mathcal{MOD}(A)$ and $f \leq g$. Then $gf = g$.*

Proof. Assume that $f, g \in \mathcal{MOD}(A)$ such that $f \leq g$ and $x \in A$. Using (M$_1$) and Proposition 3.2(2), we have $g(x) \leq g(f(x)) = gf(x)$ and so $g \leq gf$. Moreover $g(f(x)) \leq g(g(x)) = g(x)$. Thus $gf = g$, by $(psBCK_5)$. □

The following example show that the condition $f \leq g$ from Proposition 3.8 is necessary.

Example 3.9. *Consider the modal operators f_3 and f_4 given in Example 3.3. Indeed, $f_3 \not\leq f_4$, since $f_3(0) = a \not\leq f_4(0) = b$ $(a \to b = b \neq 1$, and we have $(f_4(f_3))(0) = f_4(f_3(0)) = f_4(a) = 1 \neq f_4(0) = b$.*

Theorem 3.10. *Let A be a pseudo-BCK algebra, $f, g \in \mathcal{MOD}(A)$ and $f \leq g$. The following are equivalent:*
(1) *$fg = gf$;*
(2) *$fg, gf \in \mathcal{MOD}(A)$;*
(3) *$fgfg = fg$ and $gfgf = gf$.*

Proof. Assume that $f, g \in \mathcal{MOD}(A)$ and $x, y \in A$.
(1) ⇒ (2) Suppose that $fg = gf$. We have:
(M_1) $x \leq g(x) \leq f(g(x))$;
(M_2) $f(g(f(g(x)))) = f(f(g(g(x)))) = f(f(g(x))) = f(g(x))$;
(M_3) $fg(x \to y) = f(g(x \to y)) \leq f(g(x) \to g(y)) \leq f(g(x)) \to f(g(y))$ and
$fg(x \rightsquigarrow y) = f(g(x \rightsquigarrow y)) \leq f(g(x) \rightsquigarrow g(y)) \leq f(g(x)) \rightsquigarrow f(g(y))$.
Thus, $fg \in \mathcal{MOD}(A)$. By a similar argument we get $gf \in \mathcal{MOD}(A)$.
(2) ⇒ (3) Assume that $fg, gf \in \mathcal{MOD}(A)$. Since $fg \leq fgfg$, we have $fgfg = fgfgfg = fg$, by (M_2) and Proposition 3.8. By a similar argument we obtain $gfgf = gf$.
(3) ⇒ (1) Assume that $fgfg = fg$ and $gfgf = gf$. Then we have: for all $x \in A$, $f(g(x)) \leq f(g(f(x))) \leq g(f(g(f(x)))) = g(f(x))$; Similarly, $g(f(x)) \leq f(g(x))$. Thus, $fg = gf$. □

Let (A, f) be a modal pseudo-BE algebra and $x, y \in A$. We define the notion of a *modal upper set* $mA(x, y)$ as follows:
$$mA(x, y) = \{z \in A \mid x \to (y \rightsquigarrow f(z)) = 1\}.$$
Obviously, it is a non-empty set, since $1 \in mA(x, y)$.

Remark 3.11. *In general, the upper set $A(x, y) = \{z \in A \mid x \to (y \rightsquigarrow z) = 1\}$ is not equal to modal upper set $mA(x, y)$. Indeed, in Example 3.4, if we take $f := f^1_{\alpha,\beta}$, $\alpha := a$ and $\beta := b$, then $mA(1, a) = \{1, a, b, d\} \neq \{1, a, d\} = A(1, a)$.*

Proposition 3.12. *Let (A, f) be a modal pseudo-BE(A) algebra. Then for all $x, y \in A$:*
(1) *if $f(y) = y$, then $A(1, y) \subseteq mA(1, y)$;*
(2) *$mA(x, 1) \subseteq mA(x, y)$;*
(3) *$mA(f(x), 1) \subseteq mA(f(x), y)$;*
(4) *if $D \in \mathcal{DS}(A)$, then $f(mA(x, y)) \subseteq D$, for all $x, y \in D$;*
(5) *if $mA(f(x), 1) \in \mathcal{DS}(A)$ and $y \in mA(f(x), 1)$, then $mA(f(x), y) \subseteq mA(f(x), 1)$, and so $mA(f(x), y) = mA(f(x), 1)$;*
(6) *if A is a pseudo-BCK algebra (or (commutative) distributive pseudo-BE algebra), then $A(x, y) \subseteq mA(x, y)$, for all $x, y \in A$.*

Proof. (1) Let $y \in A$ and $z \in A(1, y)$. Then $1 \to (y \rightsquigarrow z) = y \rightsquigarrow z = 1$, and so $y \leq z$. Using Proposition 3.2, we have $f(y) \leq f(z)$. Hence $1 = f(y) \rightsquigarrow f(z) = 1 \to (f(y) \rightsquigarrow f(z)) = 1 \to (y \rightsquigarrow f(z))$. Thus, $z \in mA(1, y)$.
(2) Let $z \in mA(x, 1)$. Then $1 = x \to (1 \rightsquigarrow f(z)) = x \to f(z)$. Now, we get $x \to (y \rightsquigarrow f(z)) = y \rightsquigarrow (x \to f(z)) = 1$. Therefore, $z \in mA(x, y)$.
(3) Let $z \in mA(f(x), 1)$. Then $f(x) \to (1 \rightsquigarrow f(z)) = 1$, i.e., $f(x) \rightsquigarrow f(z) = 1$. Hence $f(x) \to (y \rightsquigarrow f(z)) = y \rightsquigarrow (f(x) \to f(z)) = y \rightsquigarrow 1 = 1$, i.e., $z \in mA(f(x), y)$.
(4) Assume that $D \in \mathcal{DS}(A)$ and $z \in f(mA(x, y))$. Then there exists $a \in mA(x, y)$ such that $z = f(a)$. Hence $x \to (y \rightsquigarrow f(a)) = 1 \in D$. Since D is a deductive system and $x, y \in D$, we get $y \rightsquigarrow f(a) \in D$, and so $z = f(a) \in D$. Therefore, $f(mA(x, y)) \subseteq D$.
(5) Since $f(x) \to (1 \rightsquigarrow f(x)) = f(x) \to (1 \rightsquigarrow f(f(x))) = 1$, we get $f(x) \in mA(f(x), 1)$. Now, let $y \in mA(f(x), 1)$. Since $1 = f(x) \to f(y) = f(x) \to (1 \rightsquigarrow f(y)) \in mA(f(x), 1)$, we have $f(y) \in mA(f(x), 1)$. Let $z \in mA(f(x), y)$. Then by using $(psBE_4)$
$$1 = f(x) \to (y \rightsquigarrow f(z)) = y \rightsquigarrow (f(x) \to f(z)).$$

Now, by using Proposition 3.2(1), (M_2) and (M_3) we get

$$1 = f(1) = f(y \rightsquigarrow (f(x) \to f(z)))(y) \rightsquigarrow f(f(x) \to f(z))(y) \rightsquigarrow (f(f(x)) \to f(f(z))) = f(y) \rightsquigarrow (f(x) \to f(z)) \in mA(f(x), 1).$$

Now, since $mA(f(x), 1)$ is a deductive system and $f(x), f(y) \in mA(f(x), 1)$, we get $f(z) \in mA(f(x), 1)$, and so $1 = f(x) \to (1 \to f(f(z))) = f(x) \to (1 \rightsquigarrow f(z))$. Hence $z \in mA(f(x), 1)$. Thus, $mA(f(x), y) \subseteq mA(f(x), 1)$. Using (3), we get $mA(f(x), y) = mA(f(x), 1)$.
(6) Let $z \in A(x, y)$. Then $x \to (y \rightsquigarrow z) = 1$. Using (M_1), we have $z \le f(z)$. Applying two times Lemma 2.4(2), we get $x \to (y \rightsquigarrow z) \le x \to (y \rightsquigarrow f(z))$, and so $x \to (y \rightsquigarrow f(z)) = 1$. Thus, $z \in mA(x, y)$. Similarly, if A is a (commutative) distributive pseudo-BE algebra, then (6) holds. □

The following examples show that the converse of the statements in Proposition 3.12, may not be true in general.

Example 3.13. (1) *In Example 3.3, $f_4(b) = b$ and $1 \to (b \rightsquigarrow f_4(0)) = b \rightsquigarrow b = 1$ (i.e., $0 \in mA(1, b)$), but $0 \notin A(1, b)$, since $1 \to (b \rightsquigarrow 0) = b \rightsquigarrow 0 = b \neq 1$. Thus, $mA(1, b) \not\subseteq A(1, b)$.*
(2) *Consider the modal operator f_2 given in Example 3.3. Then $mA(a, 1) = \{a, 1\}$ and $mA(a, b) = \{a, b, 1\}$. Thus, $mA(a, b) \not\subseteq mA(a, 1)$. If $x := a$, then $mA(f_2(a), 1) = mA(a, 1) = \{a, 1\}$. Applying* (2)*, we get the converse* (3) *in not valid. Take $D = \{0, a, 1\}$, we get $mA(a, 1) = \{a, 1\} \subseteq \{0, a, 1\}$, but $D \notin \mathcal{DS}(A)$. Since $0 \to b = 1 \in D$ and $0 \in D$, but $b \notin D$. Then the converse* (4) *is not valid.*

Proposition 3.14. *Let (A, f) be a modal pseudo-BE algebra and $D \in \mathcal{DS}(A)$. Then $f(D) = \bigcup_{x,y \in D} f(mA(f(x), y))$.*

Proof. Assume that $D \in \mathcal{DS}(A)$ and consider $f(z)$, for $z \in D$. Since $f(z) \to (1 \rightsquigarrow f(z)) = f(z) \to (1 \rightsquigarrow f(f(z))) = 1$, by $(psBE_3)$ and (M_2), we have $f(z) \in mA(f(z), 1)$. Now, by Proposition 3.12(3), we have $f(z) \in mA(f(z), 1) \subseteq mA(f(z), y)$. Thus, $f(z) = f(f(z)) \in f(mA(f(z), 1)) \subseteq f(mA(f(z), y))$. Therefore, $f(D) \subseteq f(mA(f(z), y)) \subseteq \bigcup_{y \in D} f(mA(f(z), y))$. On the other hand, by Proposition 3.12(5), $f(mA(x, y)) \subseteq D$, for all $x, y \in D$. Thus, $f(mA(f(x), y)) \subseteq f(D)$, for all $x, y \in D$. Therefore, $\bigcup_{x,y \in D} f(mA(f(x), y)) \subseteq f(D)$. □

The following examples shows that the condition $D \in \mathcal{DS}(A)$ is necessary, in Proposition 3.14.

Example 3.15. *Consider the pseudo-BE algebra is given in Example 3.3. Taking $D = \{0\}$, and $f_3 \in \mathcal{MOD}(A)$, and so*

$$f_3(\{0\}) = \{a\} \neq \bigcup_{x,y \in \{0\}} f(mA(f(x), y)) = mA(f(0), 0) = \{0, a, b, 1\}.$$

L.C. Ciungu proved that every finite commutative pseudo BE-algebra is a commutative BE-algebra (see [11, Remark 3.6(2)]). So, if (A, f) is a finite commutative modal pseudo-BE algebra, then it is a modal BE-algebra.

Proposition 3.16. *Let (A, f) be a modal pseudo-BE algebra and $D \subseteq A$ containing* 1*. $f(D) \in \mathcal{DS}(A)$ if and only if $x \le y \rightsquigarrow z$ (or equivalently, $x \le y \to z$) implies $z \in f(D)$, for all $x, y \in f(D)$.*

Proof. Let $f(D) \in \mathcal{DS}(A)$ and $x \le y \rightsquigarrow z$, for all $x, y \in f(D)$. Since $x, y \in f(D)$ and $f(D)$ is a deductive system, we have $y \rightsquigarrow z \in f(D)$, and so $z \in f(D)$.
Conversely, $1 \in f(D)$, since $1 \in D$. Let $x, x \rightsquigarrow y \in f(D)$. Since $x \rightsquigarrow y \le x \rightsquigarrow y$ we can see that by hypothesis $y \in f(D)$. □

Theorem 3.17. *Let (A, f) be a modal pseudo-BE algebra and $D \subseteq A$ containing* 1*. Then $f(D) \in \mathcal{DS}(A)$ if and only if $x \in f(D)$, $y \in X \setminus f(D)$ imply $x \to y, x \rightsquigarrow y \in X \setminus f(D)$.*

Proof. Assume that $f(D) \in \mathcal{DS}(A)$ and let $x, y \in A$ be such that $x \in f(D)$ and $y \in X \setminus f(D)$. If $x \to y \notin X \setminus f(D)$. Then $x \to y \in f(D)$, i.e., $y \in f(D)$, which is a contradiction. Thus $x \to y \in X \setminus f(D)$.
Conversely, $1 \in D$ by hypothesis. Let $x, x \to y \in f(D)$. If $y \notin f(D)$, then $x \to y \in X \setminus f(D)$ by assumption. This is a contradiction and so $y \in f(D)$. Hence there exists $z \in D$ such that $y = f(z)$. Thus, $f(y) = f(f(z)) = f(z) = y \in f(D)$. By a similar argument $x \rightsquigarrow y \in X \setminus f(D)$. □

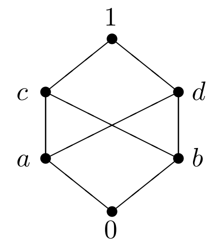
For any non-empty subset S of A, we define a subset $S^{-\sim}$ as follows:

$$S^{-\sim} := \{x \in A \mid x \to a \in S \text{ and } x \rightsquigarrow a \in S \text{ for some } a \in S\}.$$

Example 3.18. [13] *Consider the structure* $(A, \to, \rightsquigarrow, 0, 1)$*, where the operations* $\to$ *and* $\rightsquigarrow$ *on* $A = \{0, a, b, c, d, 1\}$ *are defined as follows:*

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	c	c	1	1	1	1
c	a	a	d	1	d	1
d	b	c	b	c	1	1
1	0	a	b	c	d	1

$\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	c	1	1	1
b	d	d	1	1	1	1
c	b	d	b	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1



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Then $(A, \to, \rightsquigarrow, 0, 1)$ *is a good pseudo-BE algebra. Then* $\{b\}^{-\sim} = \{1\}$ *and* $\{b, c\}^{-\sim} = \{1, d\}$*.*

Proposition 3.19. *Let* A *be a pseudo-BE algebra and* $\emptyset \neq S \subseteq A$*. Then*
(1) $A^{-\sim} = A$*;*
(2) $\{1\}^{-\sim} = A$*;*
(3) $1 \in S^{-\sim}$*;*
(4) *if* $1 \in S$*, then* $S \subseteq S^{-\sim}$*;*
(5) $S_1 \subseteq S_2$ *implies* $S_1^{-\sim} \subseteq S_2^{-\sim}$*;*
(6) *if* $D \in \mathcal{DS}(A)$*, then* $D^{-\sim} = (D^{-\sim})^{-\sim}$*;*
(7) *if* A *is a pseudo-BE(A) algebra and* $D \in \mathcal{DS}(A)$ *(resp.,* $D \in \mathcal{DS}_n(A)$*) , then* $D^{-\sim} \in \mathcal{DS}(A)$ *(resp.,* $D \in \mathcal{DS}_n(A)$*).*

Proof. We prove only (6), the other are straightforward. Let $D \in \mathcal{DS}(A)$. Since $1 \in D^{-\sim}$, we get $D^{-\sim} \subseteq (D^{-\sim})^{-\sim}$, by (3), (4) and (5).
Conversely, let $x \in (D^{-\sim})^{-\sim}$. Then there exists $a \in D^{-\sim}$ such that $x \to a \in D^{-\sim}$ and $x \rightsquigarrow a \in D^{-\sim}$. Also, there exists $b \in D$ such that $a \to b \in D$ and $a \rightsquigarrow b \in D$. Since $b \leq x \to (a \to b)$ and $b \in D$, we get $x \to (a \to b) \in D$ (note that, since $b \leq a \to b$ ($b \leq a \rightsquigarrow b$) and $b \in D$, we have $a \to b \in D$ ($a \rightsquigarrow b \in D$)). Similarly, $x \rightsquigarrow (a \to b) \in D$, and so $x \in D^{-\sim}$. Thus, $(D^{-\sim})^{-\sim} \subseteq D^{-\sim}$. Therefore, $D^{-\sim} = (D^{-\sim})^{-\sim}$. □

The following example show that the condition $D \in \mathcal{DS}(A)$ in Proposition 3.19(6) is necessary.

Example 3.20. *Consider the pseudo-BE algebra given in Example 3.18,* $\{b, c\} \notin \mathcal{DS}(A)$*, since* $1 \notin \{b, c\}$*. Then* $\{b, c\}^{-\sim} = \{1, d\}$ *and* $\{1, d\}^{-\sim} = A$*. Thus,* $\{b, c\}^{-\sim^{-\sim}} = A \neq \{1, d\} = \{b, c\}^{-\sim}$*.*

Theorem 3.21. *Let* (A, f) *be a modal pseudo-BE algebra and* $D \in \mathcal{DS}(A)$*. Then* $f(D^{-\sim}) \subseteq D^{-\sim}$*.*

Proof. Let $D \in \mathcal{DS}(A)$ and let $x \in D^{-\sim}$, that is there exists $a \in D$ such that $x \to a, x \rightsquigarrow a \in D$. It follows that $x \to a \leq f(x \to a) \leq f(x) \to f(a)$ and $x \rightsquigarrow a \leq f(x \rightsquigarrow a) \leq f(x) \rightsquigarrow f(a)$, hence $f(x) \to f(a)$, $f(x) \rightsquigarrow f(a) \in D$. Since $a \leq f(a)$, then $f(a) \in D$, so $f(x) \in D^{-\sim}$. Thus, $f(D^{-\sim}) \subseteq D^{-\sim}$. □

Proposition 3.22. *Let* $(A, \to, \rightsquigarrow, 0, 1)$ *be a bounded pseudo-BCK algebra and let* $f \in \mathcal{MOD}(A)$*. Then the following are equivalent:*
(a) $f(0) = 0$;
(b) $f(x^-) = x^-$*, for all* $x \in A$;
(c) $f(x^\sim) = x^\sim$*, for all* $x \in A$*.*

Proof. $(a) \Rightarrow (b)$ Suppose that $f(0) = 0$ and let $x \in A$. Applying Proposition 3.6(2), we have,

$$f(x) \leq (x \rightsquigarrow f(0)) \to f(0) = (x \rightsquigarrow 0) \to 0 = x^{\sim -}.$$

Hence $f(x) \leq x^{\sim -}$, so $f(x^-) \leq (x^-)^{\sim -} = x^{-\sim -} = x^-$. On the other hand, $x^- \leq f(x^-)$, hence $f(x^-) = x^-$.
$(b) \Rightarrow (a)$ If $f(x^-) = x^-$, for all $x \in A$, then $f(0) = f(1^-) = 1^- = 0$.
$(a) \Leftrightarrow (c)$ Similarly as $(a) \Leftrightarrow (b)$. □

Corollary 3.23. *Let* $(A, \to, \rightsquigarrow, 0, 1)$ *be an involutive pseudo-BCK algebra and let* $f \in \mathcal{MOD}(A)$ *such that* $f(0) = 0$*. Then* $f = 1_A$*.*

The following example shows that in Corollary 3.23, condition involutive is necessary.

Example 3.24. [26] *Consider the structure $(A, \to, \rightsquigarrow, 0, 1)$, where the operations $\to$ and $\rightsquigarrow$ on $A = \{0, a, b, c, d, 1\}$ are defined as follows:*

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	c	1
b	0	b	1	1	c	1
c	0	b	b	1	c	1
d	0	b	b	1	1	1
1	0	a	b	c	d	1

$\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	c	1
b	0	c	1	1	c	1
c	0	a	b	1	c	1
d	0	a	b	1	1	1
1	0	a	b	c	d	1

Then $(A, \to, \rightsquigarrow, 0, 1)$ is a good pseudo-BCK algebra, but it is not involutive, since $b^{-\sim} = b^{\sim -} = 1 \neq b$. Define $f : A \longrightarrow A$ by $f(0) = 0$, $f(a) = f(b) = b$ and $f(c) = f(d) = f(1) = 1$. Then $f \in \mathcal{MOD}^s(A)$, while $f \neq 1_A$.

Proposition 3.25. *Let $(A, \to, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra with the Glivenko property. Then the maps $f_1, f_2 : A \longrightarrow A$ defined by $f_1(x) = x^{-\sim}$, $f_2(x) = x^{\sim -}$, for all $x \in A$ are modal operators on A.*

Proof. By Propositions 2.10(1) and 2.12(1), we have $x \leq f_1(x)$, $x \leq f_2(x)$ and $f_1(f_1(x)) = f_1(x)$, $f_2(f_2(x)) = f_2(x)$, hence f_1 and f_2 satisfy conditions (M_1) and (M_2). Applying Glivenko property and Proposition 2.12(2), we have:

$$f_1(x \to y) = (x \to y)^{-\sim} = x \to y^{-\sim} = x^{-\sim} \to y^{-\sim} = f_1(x) \to f_1(y).$$

Since A is good, we get similarly, $f_1(x \rightsquigarrow y) = f_1(x) \rightsquigarrow f_1(y)$. Similarly, $f_2(x \to y) = f_2(x) \to f_2(y)$ and $f_2(x \rightsquigarrow y) = f_2(x) \rightsquigarrow f_2(y)$, hence condition (M_3) is satisfied. We conclude that f_1 and f_2 are modal operators on A. □

The following example show that each of condition in Proposition 3.25 is necessary.

Example 3.26. *In Example 3.3, $f_2(x) = x^{\sim -}$ is given in the following table:*

x	0	a	b	1
$f_2(x)$	0	a	a	1

Since $a^{-\sim} = 1 \neq a^{-\sim} = a$, we get it is not good and has not Glivenko property (since, $(b \rightsquigarrow 0)^{-\sim} = b^{\sim -} = a \neq b \rightsquigarrow 0^{\sim -} = b \rightsquigarrow 0 = b$). We can see that $f_2 \notin \mathcal{MOD}(A)$, since $f_2(b \rightsquigarrow 0) = f_2(b) = a \not\leq f_2(b) \rightsquigarrow f_2(0) = b \rightsquigarrow 0 = b$.

Theorem 3.27. *Let A be a pseudo-BCK algebra and let $f : A \longrightarrow A$ be a mapping on A. Then $f \in \mathcal{MOD}(A)$ if and only if it satisfies condition (M_3) and the condition:*
(M_4) $f(x) \to f(y) = x \to f(y)$ and $f(x) \rightsquigarrow f(y) = x \rightsquigarrow f(y)$, for all $x, y \in A$.

Proof. Consider $f : A \longrightarrow A$ satisfying (M_3) and (M_4). By (M_4), we get $1 = f(x) \to f(x) = x \to f(x)$, that is $x \leq f(x)$, hence (M_1) is verified. We also have $1 = f(x) \to f(x) = f(f(x)) \to f(x)$, so $f(f(x)) \leq f(x)$. Since by (M_1), $f(x) \leq f(f(x))$, it follows that $f(f(x)) = f(x)$, that is (M_2). Hence $f \in \mathcal{MOD}(A)$.
Conversely, if $\in \mathcal{MOD}(A)$, then it verifies (M_3) and by Proposition 3.6(1), (M_4) is also satisfied. □

Definition 3.28. *Let A be a good pseudo-BCK algebra and $f \in \mathcal{MOD}(A)$. If f satisfies the condition:*
(M_5) $f(x \oplus y) = f(x \oplus f(y)) = f(f(x) \oplus y)$,
for all $x, y \in A$, then f is called strong.

Denote $\mathcal{MOD}^s(A)$ the set of all strong modal operators on A.

Example 3.29. *Let $(A, \to, \rightsquigarrow, 0, 1)$ be the good pseudo-BCK algebra from Example 3.18, and the maps $f_i : A \longrightarrow A$, $i = 1, 2, 3, 4$, given in the table below:*

x	0	a	b	c	d	1
$f_1(x)$	1	1	1	1	1	1
$f_2(x)$	0	a	b	c	d	1
$f_3(x)$	c	c	c	c	1	1
$f_4(x)$	d	d	d	1	d	1

Then $\mathcal{MOD}(A) = \{f_1, f_2, f_3, f_4\}$ and $\mathcal{MOD}^s(A) = \{f_1, f_2\}$.

Remark 3.30. *Let A be a good pseudo-BCK algebra. Then:*
(1) *$1_A, Id_A \in \mathcal{MOD}^s(A)$.*
(2) *if $f \in \mathcal{MOD}^s(A)$, then $f(x \oplus y) = f(f(x) \oplus f(y))$, for all $x, y \in A$.*

Proposition 3.31. *Let $(A, \to, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra and let $f \in \mathcal{MOD}(A)$. Then the following hold: for all $x \in A$,*
(1) $x \oplus f(0) \geq f(x^{-\sim}) \geq f(x)$ and $f(0) \oplus x \geq f(x^{\sim -}) \geq f(x)$;
(2) if $f \in \mathcal{MOD}^s(A)$, then $x \oplus f(0) = f(0) \oplus f(x) = f(x^{-\sim})$.

Proof. (1) Since $f(0) \geq 0$, applying Proposition 3.6 and properties of pseudo-BCK algebras we get:

$$x \oplus f(0) = x^- \rightsquigarrow f(0)^{-\sim} \geq x^- \rightsquigarrow f(0) = f(x^- \rightsquigarrow f(0)) \geq f(x^- \rightsquigarrow 0) = f(x^{-\sim}) \geq f(x).$$

Similarly, $f(0) \oplus x = f(x^{\sim -}) \geq f(x)$.
(2) Since f is strong we have: $f(x \oplus f(0)) = f(x \oplus 0) = f(x^{-\sim})$. Hence by (1), $f(x^{-\sim}) = f(x \oplus f(0)) \geq x \oplus f(0) \geq f(x^{-\sim})$, that is $x \oplus f(0) = f(x^{-\sim})$. Similarly, $f(0) \oplus f(x) = f(x^{-\sim})$. □

Proposition 3.32. *Let $(A, \to, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra and let $f \in \mathcal{MOD}^s(A)$. Then, for all $x \in A$,*
(1) $x \oplus f(0) = f(x \oplus 0)$ implies $f(x) \oplus f(0) = x \oplus f(0)$ and $f(0) \oplus f(x) = f(0) \oplus x$;
(2) $f(0) \oplus x = f(0 \oplus x)$ implies $f(x) \oplus f(0) = f(0) \oplus x$ and $f(0) \oplus f(x) = x \oplus f(0)$.

Proof. (1) By Proposition 3.31(1), $f(x) \leq x \oplus f(0)$, and applying Proposition 2.14(9), we get $f(x) \oplus f(0) \leq x \oplus f(0) \oplus f(0)$. By hypothesis $f(0) \oplus f(0) = f(f(0) \oplus 0) = f(0 \oplus 0) = f(0)$, hence $f(x) \oplus f(0) \leq x \oplus f(0)$. On the other hand, $x \leq f(x)$ implies $x \oplus f(0) \leq f(x) \oplus f(0)$, thus $f(x) \oplus f(0) = x \oplus f(0)$. Similarly, $f(0) \oplus f(x) = f(0) \oplus x$.
(2) Similarly as (1). □

Theorem 3.33. *Let $(A, \to, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra and let $f \in \mathcal{MOD}(A)$. Then $f \in \mathcal{MOD}^s(A)$ if and only if $x \oplus f(0) = f(0) \oplus x = f(x^{-\sim})$, for all $x \in A$.*

Proof. If $f \in \mathcal{MOD}^s(A)$, then by Proposition 3.31(2), $x \oplus f(0) = f(0) \oplus x = f(x^{-\sim})$, for all $x \in A$.
Conversely, suppose that $f \in \mathcal{MOD}(A)$ such that $x \oplus f(0) = f(0) \oplus x = f(x^{-\sim}) = f(x \oplus 0)$, for all $x \in A$. Consider $x, y \in A$. Applying Proposition 3.32, we have:

$$\begin{aligned} f(x \oplus f(y)) &= f((x \oplus f(y)) \oplus 0) = (x \oplus f(y)) \oplus f(0) = x \oplus (f(y) \oplus f(0)) = x \oplus (y \oplus f(0)) = (x \oplus y) \oplus f(0) \\ &= f((x \oplus y) \oplus 0) = f(x \oplus y). \end{aligned}$$

Similarly, $f(f(x) \oplus y) = f(x \oplus y)$, hence $f \in \mathcal{MOD}^s(A)$. □

If $(A, \to, \rightsquigarrow, 0, 1)$ is a good pseudo-BCK algebra we denote, $\mathrm{Id}\,(A) = \{a \in A \mid a \oplus a = a^{-\sim}\}$, the set of all pseudo-idempotent elements of A. Obviously, $0, 1 \in \mathrm{Id}\,(A)$ in any good pseudo-BCK algebra A.

Example 3.34. *If $(A, \to, \rightsquigarrow, 0, 1)$ is the good pseudo-BCK algebra from Example 3.18, then* $\mathrm{Id}\,(A) = \{0, a, b, 1\}$.

Let $(A, \to, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra, $a \in A$ and let the maps $\varphi_a, \phi_a : A \longrightarrow A$ defined by $\varphi_a(x) = a \oplus x$, $\phi_a(x) = x \oplus a$, for all $x \in A$.

Proposition 3.35. *Let $(A, \to, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra and let $a \in \mathrm{Id}\,(A)$. The following hold:*
(1) φ_a, ϕ_a satisfy (M_1) and (M_2);
(2) $\varphi_a(x \oplus y) = \varphi_a(\varphi_a(x) \oplus y)$ and $\phi_a(x \oplus y) = \phi_a(x \oplus \phi_a(y))$, for all $x, y \in A$;
(3) if a commutes with every $x \in A$, then φ_a, ϕ_a satisfy (M_5).

Proof. (1) By Proposition 2.14(2), $x \leq a \oplus x = \varphi_a(x)$ and $x \leq x \oplus a = \phi_a(x)$, that is (M_1). Applying again Proposition 2.14(6), we have:

$$\varphi_a(\varphi_a(x)) = a \oplus \varphi_a(x) = a \oplus a \oplus x = a^{-\sim} \oplus x = a \oplus x = \varphi_a(x).$$

Similarly, $\phi_a(\phi_a(x)) = \phi_a(x)$, hence (M_2) is satisfied.
(2) For all $x, y \in A$ we have:

$$\varphi_a(\varphi_a(x) \oplus y) = a \oplus (\varphi_a(x) \oplus y) = a \oplus a \oplus x \oplus y = a^{-\sim} \oplus x \oplus y = a \oplus x \oplus y = \varphi_a(x \oplus y).$$

Similarly, $\phi_a(x \oplus \phi_a(y)) = \phi_a(x \oplus y)$.
(3) If a commutes with every $x \in A$, then, for all $x, y \in A$ we have:

$$\varphi_a(x \oplus \varphi_a(y)) = a \oplus x \oplus \varphi_a(y) = a \oplus x \oplus a \oplus y = a \oplus a \oplus x \oplus y = a^{-\sim} \oplus x \oplus y = a \oplus x \oplus y = \varphi_a(x \oplus y).$$

Similarly, $\phi_a(\phi_a(x) \oplus y) = \phi_a(x \oplus y)$. Taking into consideration (2) it follows that φ_a, ϕ_a satisfy (M_5). □

Proposition 3.36. *Let $(A, \rightarrow, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra. The following hold:*
(1) if $\varphi_a(x \rightarrow y) \leq \varphi_a(x) \rightarrow \varphi_a(y)$, for all $x, y \in A$, then $a \in \mathrm{Id}\,(A)$;
(2) if $\phi_a(x \rightsquigarrow y) \leq \phi_a(x) \rightsquigarrow \phi_a(y)$, for all $x, y \in A$, then $a \in \mathrm{Id}\,(A)$.

Proof. (1) Suppose that $\varphi_a(x \rightarrow y) \leq \varphi_a(x) \rightarrow \varphi_a(y)$, for all $x, y \in A$. Taking $x := a$ and $y := 0$ and applying Proposition 2.14(5), we get $1 = a \oplus a^- \leq a \oplus a \rightarrow a^{-\sim}$, that is $a \oplus a \rightarrow a^{-\sim} = 1$. Hence $a \oplus a \leq a^{-\sim}$. On the other hand, $a^{-\sim} \leq a^\sim \rightarrow a^{-\sim} = a \oplus a$. It follows that $a \oplus a = a^{-\sim}$, that is $a \in \mathrm{Id}\,(A)$.
(2) The proof is similar to (1). □

Corollary 3.37. *Let $(A, \rightarrow, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra. If $\varphi_a \in \mathcal{MOD}(A)$ or $\phi_a \in \mathcal{MOD}(A)$, then $a \in \mathrm{Id}\,(A)$.*

Proposition 3.38. *Let $(A, \rightarrow, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra. The following hold:*
(1) if φ_a satisfies (M_5), then $a \in \mathrm{Id}\,(A)$;
(2) if ϕ_a satisfies (M_5), then $a \in \mathrm{Id}\,(A)$.

Proof. (1) Suppose that φ_a satisfies (M_5). It follows that, $a \oplus x \oplus y = a \oplus \varphi_a(x) \oplus y = a \oplus x \oplus \varphi_a(y)$, hence $a \oplus x \oplus y = a \oplus a \oplus x \oplus y = a \oplus x \oplus a \oplus y$. Taking $x = y := 0$ and applying Proposition 2.14(3),(6), we get $a^{-\sim} = a \oplus a$, that is $a \in \mathrm{Id}\,(A)$.
(2) Similarly as (1). □

Proposition 3.39. *Let $(A, \rightarrow, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra. If $f \in \mathcal{MOD}^s(A)$ such that $f(x) = f(x^{-\sim})$, for all $x \in A$. Then $f = \varphi_{f(0)} = \phi_{f(0)}$ and $f(0) \in \mathrm{Id}\,(A)$.*

Proof. Let $f \in \mathcal{MOD}^s(A)$ such that $f(x) = f(x^{-\sim})$, for all $x \in A$. By Theorem 3.33, we have $f(x) = f(x^{-\sim}) = f(0) \oplus x = x \oplus f(0)$, that is $f = \varphi_{f(0)} = \phi_{f(0)}$. Moreover, according to Proposition 3.38, it follows that $f(0) \in \mathrm{Id}\,(A)$. □

Remark 3.40. *Since the operation $\oplus$ was defined for the case of good pseudo-BE algebras, Propositions 3.36, 3.38-3.39 and Corollary 3.37 have been proved for the case of good pseudo-BCK algebras. We show that this condition is essential for the above mentioned results. Let $A = \{1, a, b, c, 0\}$. Define the operations $\rightarrow$ and $\rightsquigarrow$ on A as follows:*

$\rightarrow$	1	a	b	c	0
1	1	a	b	c	0
a	1	1	b	1	0
b	1	a	1	1	a
c	1	a	b	1	0
0	1	1	1	1	1

$\rightsquigarrow$	1	a	b	c	0
1	1	a	b	c	0
a	1	1	b	1	b
b	1	a	1	1	0
c	1	a	b	1	0
0	1	1	1	1	1

Then $(A, \rightarrow, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK algebra, but it is not good, since $a^{-\sim} = 1 \neq a = a^{\sim -}$. Based on this example, we have the following remarks:
(1) Since $\phi_b x = x \oplus b = b^\sim \rightarrow x^{-\sim} = 0 \rightarrow x^{-\sim} = 1$, for all $x \in A$, it follows that ϕ_b satisfies the condition (2) from Proposition 3.36. On the other hand, $b \oplus b = 1 \neq b = b^{-\sim}$, so that $b \notin \mathrm{Id}\,(A)$. Hence, if A is not good, the assertion stated in Proposition 3.36 is not always valid.
(2) Obviously, ϕ_b satisfies condition (M_5), but $b \notin \mathrm{Id}\,(A)$. It follows that, if the pseudo-BCK algebra A is not good, then the result from Proposition 3.38 is not valid.
(3) The proof of Proposition 3.39 is based on Theorem 3.33. One can see that $f = \mathrm{Id}_A \in \mathcal{MOD}^s(A)$. Since $f(0) = 0$, the condition $x \oplus f(0) = f(0) \oplus x = f(x^{-\sim})$ leads to $x^{-\sim} = x^{\sim -}$, for all $x \in A$, so that A is good, a contradiction. Thus, the assertion stated in Theorem 3.33 does not hold. It follows that, if A is not good, then Proposition 3.39 is not valid.

Theorem 3.41. *Let $(A, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BE algebra and $f_1, f_2 \in \mathcal{MOD}(A)$. Then:*
(1) $f_1 \circ f_2 \in \mathcal{MOD}(A)$ if and only if $f_1 \circ f_2 = f_2 \circ f_1$;
(2) if A is a good pseudo-BCK algebra and $f_1, f_2 \in \mathcal{MOD}^s(A)$, then $f_1 \circ f_2 \in \mathcal{MOD}^s(A)$ if and only if $f_1 \circ f_2 = f_2 \circ f_1$.

Proof. (1) Let $f_1, f_2 \in \mathcal{MOD}(A)$ and let $f = f_1 \circ f_2$. According to [35, Th. 6], $f \in \mathcal{CL}(A)$ if and only if $f_1 \circ f_2 = f_2 \circ f_1$. If $f \in \mathcal{MOD}(A)$, then by Proposition 3.5, we have $f \in \mathcal{CL}(A)$, hence $f_1 \circ f_2 = f_2 \circ f_1$.
Conversely, let $f_1, f_2 \in \mathcal{MOD}(A)$ such that $f_1 \circ f_2 = f_2 \circ f_1$. Hence $f \in \mathcal{CL}(A)$, that is f satisfies (M_1) and (M_2). Since f_1 and f_2 satisfy (M_3), we have:

$$f(x \rightarrow y) = f_1(f_2(x \rightarrow y)) \leq f_1(f_2(x) \rightarrow f_2(y)) \leq f_1(f_2(x)) \rightarrow f_1(f_2(y)) = f(x) \rightarrow f(y).$$

Similarly, $f(x \rightsquigarrow y) \leq f(x) \rightsquigarrow f(y)$, that is f satisfies (M_3). Hence $f \in \mathcal{MOD}(A)$.
(2) Let $f_1, f_2 \in \mathcal{MOD}^s(A)$ and let $f = f_1 \circ f_2$. Similarly as in (1), if $f \in \mathcal{MOD}(A)$, then $f_1 \circ f_2 = f_2 \circ f_1$.
Conversely, let $f_1, f_2 \in \mathcal{MOD}^s(A)$ such that $f_1 \circ f_2 = f_2 \circ f_1$. According to (1), $f \in \mathcal{MOD}(A)$.

$$f(x\oplus y) = f_1\circ f_2(x\oplus y) = f_1\circ f_2(x\oplus f_2(y)) = f_2\circ f_1(x\oplus f_2(y)) = f_2\circ f_1(x\oplus f_1(f_2(y))) = f_1\circ f_2(x\oplus f_1\circ f_2(y)) = f(x\oplus f(y)).$$

Similarly, $f(x \oplus y) = f(f(x) \oplus y)$, hence f satisfies (M_5). Thus, $f \in \mathcal{MOD}^s(A)$. □

Example 3.42. *Let $(A, \rightarrow, \rightsquigarrow, 0, 1)$ be the good pseudo-BCK algebra from Example 3.29 with $\mathcal{MOD}(A) = \{f_1, f_2, f_3, f_4\}$ and $\mathcal{MOD}^s(A) = \{f_1, f_2\}$.*
One can easily check that, for $i = 1, 2, 3, 4$; $f_1 \circ f_i = f_i \circ f_1 = f_1$, $f_2 \circ f_i = f_i \circ f_2 = f_i$ and $f_3 \circ f_4 = f_4 \circ f_3 = f_1$. We can also see that, for $i, j = 1, 2, 3, 4$; $f_i \circ f_j = f_j \circ f_i$ and $f_i \circ f_j \in \mathcal{MOD}(A)$, $f_1 \circ f_2 = f_2 \circ f_1$ and $f_1 \circ f_2 \in \mathcal{MOD}^s(A)$.

Remark 3.43. *If the composition of two modal operators is not commutative, then it need not be a modal operator. Indeed, in Example 3.3 $f_3 \circ f_4$ and $f_4 \circ f_3$ are given in the following table:*

x	a	b	c	1
$(f_3 \circ f_4)(x)$	b	1	b	1
$(f_4 \circ f_3)(x)$	1	1	1	1

We can see that $f_3 \circ f_4 \neq f_4 \circ f_3$ and $f_3 \circ f_4 \notin \mathcal{MOD}(A)$.

Let A be a pseudo-BCK algebra and let $f \in \mathcal{MOD}(A)$. Denote by $\mathrm{Fix}\,(f) = \{x \in A \mid f(x) = x\}$.
By the definition of a modal operator it follows that $\mathrm{Fix}\,(f) = \mathrm{Im}\,(f)$.

Theorem 3.44. *Let $(A, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCK algebra and $f, g \in \mathcal{MOD}(A)$. If $\mathrm{Fix}\,(f) = \mathrm{Fix}\,(g)$, then $f = g$.*

Proof. Assume that $x \in A$. Using (M_2), we have $f(f(x)) = f(x)$, and so $f(x) \in \mathrm{Fix}\,(f) = \mathrm{Fix}\,(g)$. Thus, $g(f(x)) = f(x)$. Therefore, $gf = f$. By a similar argument $fg = g$. Now, since $x \leq f(x)$ and $x \leq g(x)$, we get:

$$g(x) \leq g(f(x)) = f(x) \text{ and } f(x) \leq f(g(x)) = g(x).$$

Thus, $f(x) = g(x)$, and so $f = g$. □

Proposition 3.45. *Let $(A, \rightarrow, \rightsquigarrow, 0, 1)$ be an involutive pseudo-BCK algebra and let $f \in \mathcal{MOD}^s(A)$. Then $(\mathrm{Fix}\,(f), \rightarrow, \rightsquigarrow, f(0), 1)$ is an involutive pseudo-BCK algebra, where $\rightarrow$ and $\rightsquigarrow$ are the restrictions of operations from A to $\mathrm{Fix}\,(f)$.*

Proof. Applying Propositions 2.14(2) and 3.31(2), for all $x \in \mathrm{Fix}\,(f)$, we have $f(0) \leq x \oplus f(0) = f(x^{-\sim}) = f(x) = x$. By (M_3), for all $x, y \in \mathrm{Fix}\,(f)$ we have, $f(0) \leq f(x \rightarrow y) \leq f(x) \rightarrow f(y) = x \rightarrow y$. Similarly, $f(0) \leq x \rightsquigarrow y$. Moreover, by Proposition 3.6, we get $x \rightarrow y = f(x) \rightarrow f(y) = f(f(x) \rightarrow f(y)) = f(x \rightarrow y)$, and by a similar argument $x \rightsquigarrow y = f(x \rightsquigarrow y)$, for all $x, y \in \mathrm{Fix}\,(f)$. It follows that $\mathrm{Fix}\,(f)$ is closed under operations $\rightarrow$ and $\rightsquigarrow$. We conclude that $(\mathrm{Fix}\,(f), \rightarrow, \rightsquigarrow, f(0), 1)$ is an involutive pseudo-BCK algebra. □

Remark 3.46. *If $(A, \rightarrow, \rightsquigarrow, 0, 1)$ is not involutive, then, in general, $(\mathrm{Fix}\,(f), \rightarrow, \rightsquigarrow, f(0), 1)$ is not involutive. Indeed, if f is the modal operator from Example 3.24, then we have $\mathrm{Fix}\,(f) = \{0, b, 1\}$ and $b^{-\sim} = b^{\sim -} = 1 \neq b$, so $(\mathrm{Fix}\,(f), \rightarrow, \rightsquigarrow, 0, 1)$ is not involutive.*

4 On modal pseudo-BE homomorphisms

In this section we define the notions of modal deductive systems and modal homomorphisms on pseudo-BE algebras and we investigate their properties. It is proved that, if two modal operators have the same image, then they coincide. Also, given a normal modal deductive system H of a distributive modal pseudo-BE algebra (A, f) we construct a modal operator on the quotient pseudo-BE algebra A/H. Given two modal pseudo-BCK algebras (A, f), (B, g) and a modal homomorphism ϕ, we prove that there exists a modal pseudo-BCK isomorphism between $(A/\mathrm{Ker}\,(\phi), f)$ and $(\mathrm{Im}\,(\phi), g)$.

Let A be a pseudo-BE algebra. For $f \in \mathcal{MOD}(A)$, $\mathcal{K}_f = \mathrm{Ker}\,(f) = \{x \in A \mid f(x) = 1\}$ is called the *kernel* of f.

Proposition 4.1. *Let (A, f) be a modal pseudo-BE algebra. Then the following hold:*
(1) $\mathrm{Ker}\,(f) \in \mathcal{DS}(A)$;
(2) $\mathrm{Im}\,(f)$ *is a modal subalgebra of* A;
(3) $\mathrm{Im}\,(f) = \{x \in A \mid x = f(x)\}$;
(4) $\mathrm{Ker}\,(f) \cap \mathrm{Im}\,(f) = \{1\}$.

Proof. Obviously, (1) $1 \in \mathrm{Ker}\,(f)$. Let $x, y \in A$ such that $x, x \to y \in \mathrm{Ker}\,(f)$, that is $f(x) = f(x \to y) = 1$. Since $1 = f(x \to y) \leq f(x) \to f(y) = 1 \to f(y) = f(y)$, we get $f(y) = 1$, that is $y \in \mathrm{Ker}\,(f)$. Hence $\mathrm{Ker}\,(f) \in \mathcal{DS}(A)$.
(2) Since by Proposition 3.2(1), $1 = f(1) \in \mathrm{Im}\,(f)$, we have $1 \in \mathrm{Im}\,(f)$.
If $x, y \in \mathrm{Im}\,(f)$, then from $f(x) \to f(y) = f(f(x) \to f(y))$ and $f(x) \rightsquigarrow f(y) = f(f(x) \rightsquigarrow f(y))$, it follows that $f(x) \to f(y), f(x) \rightsquigarrow f(y) \in \mathrm{Im}\,(f)$ (by Proposition 3.6).
Thus, $\mathrm{Im}\,(f)$ is a subalgebra of A. Obviously, $f_{|\mathrm{Im}\,(f)} \in \mathcal{MOD}(\mathrm{Im}\,(f))$.
(3) Clearly, $\{x \in A \mid x = f(x)\} \subseteq \mathrm{Im}\,(f)$. Let $x \in \mathrm{Im}\,(f)$, that is there exists $x_1 \in A$ such that $x = f(x_1)$. Applying (M_2), we have $x = f(x_1) = f(f(x_1)) = f(x)$, that is $x \in \{x \in A \mid x = f(x)\}$.
Thus, $\mathrm{Im}\,(f) \subseteq \{x \in A \mid x = f(x)\}$ and we conclude that $\mathrm{Im}\,(f) = \{x \in A \mid x = f(x)\}$.
(4) Let $y \in \mathrm{Ker}\,(f) \cap \mathrm{Im}\,(f)$, so $f(y) = 1$ and there exists $x \in A$ such that $f(x) = y$. It follows that $y = f(x) = f(f(x)) = f(y) = 1$, thus, $\mathrm{Ker}\,(f) \cap \mathrm{Im}\,(f) = \{1\}$. □

Theorem 4.2. *Let $(A, \to, \rightsquigarrow, 1)$ be a pseudo-BE algebra and $f, g \in \mathcal{MOD}(A)$. If $\mathrm{Im}\,(f) = \mathrm{Im}\,(g)$, then $f = g$.*

Proof. From Proposition 4.1(3), we have $\mathrm{Fix}\,(f) = \mathrm{Fix}\,(g)$ and applying Theorem 3.44, it follows that $f = g$. □

Definition 4.3. *Let A be a pseudo-BE algebra and let $D \in \mathcal{DS}(A)$. Then D is called a* modal deductive system *of A if there exists $f \in \mathcal{MOD}(A)$ such that $D = \mathrm{Ker}\,(f)$.*

Denote $\mathcal{MDS}(A)$ the set of all modal deductive systems of A.

Remark 4.4. *Since $\{1\} = \mathcal{K}_{Id_A}$ and $A = \mathcal{K}_{1_A}$, then $\{1\}, A \in \mathcal{MDS}(A)$.*

Example 4.5. *Let $(A, \to, \rightsquigarrow, 0, 1)$ be the good pseudo-BCK algebra from Example 3.29, with $\mathcal{MOD}(A) = \{f_1, f_2, f_3, f_4\}$. We have:*

$$\mathcal{DS}(A) = \{\{1\}, \{c, 1\}, \{d, 1\}, A\}.$$

One can easily see that $\{1\} = \mathcal{K}_{Id_A} = \mathcal{K}_{f_2}$, $A = \mathcal{K}_{1_A} = \mathcal{K}_{f_1}$, $\{c, 1\} = \mathcal{K}_{f_4}$, $\{d, 1\} = \mathcal{K}_{f_3}$. Hence $\mathcal{MDS}(A) = \mathcal{DS}(A)$.

Lemma 4.6. *Let (A, f) be a modal pseudo-BE algebra and let $D \in \mathcal{MDS}(A)$. Then $f(D) \subseteq D$.*

Proof. Let $D \in \mathcal{MDS}(A)$ and let $y \in f(D)$, that is there exists $x \in D$ such $f(x) = y$. Since by (M_1) $x \to f(x) = 1 \in D$, it follows that $y = f(x) \in D$. Hence $f(D) \subseteq D$. □

Corollary 4.7. *Let (A, f) be a modal pseudo-BE algebra and let $D \in \mathcal{MDS}(A)$. Then $f(D) \subseteq D^{-\sim}$.*

Proof. Since $1 \in D$, by Proposition 3.19(4), we have $D \subseteq D^{-\sim}$. Applying Lemma 4.6, we get $f(D) \subseteq D^{-\sim}$. □

Proposition 4.8. *The congruences on a modal pseudo-BE algebra (A, f) coincide with the congruences of A.*

Proof. Let $\theta \in \mathcal{CON}(A)$. There is a one-to-one correspondence between $\mathcal{CON}(A)$ and $\mathcal{DS}_n(A)$. Let $H \in \mathcal{DS}_n(A)$ and let $(x, y) \in \theta_H$, that is $x \to y, y \to x \in H$. It follows that $f(x \to y), f(y \to x) \in H$. Since $f(x \to y) \leq f(x) \to f(y)$ and $f(y \to x) \leq f(y) \to f(x)$, we get $f(x) \to f(y), f(y) \to f(x) \in H$, that is $(f(x), f(y)) \in \theta_H$. Hence $\theta_H \in \mathcal{CON}(A, f)$. □

Theorem 4.9. *Let (A, f) be a modal pseudo-BE algebra and let $\theta \in \mathcal{CON}(A)$. Define $\hat{f} : A/\theta \longrightarrow A/\theta$ by $\hat{f}(x/\theta) = f(x)/\theta$, for all $x \in A$. Then $\hat{f} \in \mathcal{MOD}(A/\theta)$. If A is a good pseudo-BCK algebra and $f \in \mathcal{MOD}^s(A)$, then $\hat{f} \in \mathcal{MOD}^s(A/\theta)$.*

Definition 4.10. *Let (A, f) and (B, g) be modal pseudo-BE algebras and $\phi : A \longrightarrow B$ be a pseudo-BE homomorphism. Then ϕ is called a modal pseudo-BE homomorphism if $\phi(f(x)) = g(\phi(x))$, for all $x \in A$.*

Denote $\mathcal{MHOM}((A, f), (B, g))$ the sets of all modal pseudo-BE homomorphisms from (A, g) to (B, g). For $\phi \in \mathcal{MHOM}((A, f), (B, g))$, $\mathrm{Ker}\,(\phi) = \{x \in A \mid \phi(x) = 1\}$ is called the kernel of ϕ.

Remark 4.11. *If (A, f) is a modal pseudo-BE algebra, then $1_A, Id_A \in \mathcal{MHOM}((A, f), (A, f))$.*

Example 4.12. *Let $(A, \to, \rightsquigarrow, 0, 1)$ be the good pseudo-BCK algebra from Example 3.29 with $\mathcal{MOD}(A) = \{f_1, f_2, f_3, f_4\}$. Consider the maps $\phi_i : A \longrightarrow A$, $i = 1, 2, 3$ given in the table below:*

x	0	a	b	c	d	1
$\phi_1(x)$	1	1	1	1	1	1
$\phi_2(x)$	0	a	b	c	d	1
$\phi_3(x)$	0	b	a	d	c	1

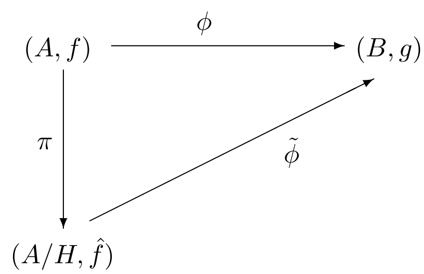
Then $\mathcal{HOM}(A, A) = \{\phi_1, \phi_2, \phi_3\}$ and $\mathcal{MHOM}((A, f_3), (A, f_3)) = \{\phi_1, \phi_2\}$.

Proposition 4.13. *Let (A, f) and (B, g) be modal pseudo-BE algebras and let $\phi \in \mathcal{MHOM}((A, f), (B, g))$. Then the following hold:*
(1) *if A' is a modal subalgebra of A, then $\phi(A')$ is a modal subalgebra of B;*
(2) $\mathrm{Ker}\,(\phi) \in \mathcal{MDS}(A)$;
(3) *if ϕ is surjective and $D \in \mathcal{MDS}(A)$, then $\phi(D) \in \mathcal{MDS}(B)$;*
(4) *if $G \in \mathcal{MDS}(B)$, then $\phi^{-1}(G) \in \mathcal{MDS}(A)$.*

Proposition 4.14. *Let (A, f) be a modal distributive pseudo-BE algebra and let $H \in \mathcal{DS}_n(A)$. Define $\pi : A \longrightarrow A/H$ by $\pi(x) = [x]_H$, for all $x \in A$. Then $\pi \in \mathcal{MHOM}((A, f), (A/H, \hat{f}))$ and $\mathrm{Ker}\,(\pi) = H$, where $\hat{f} : A/H \longrightarrow A/H$ by $\hat{f}([x]_H) = [f(x)]/H$.*

Proof. According to Theorem 4.9, $\hat{f} \in \mathcal{MOD}(A/H)$. Obviously, $\pi \in \mathcal{HOM}(A, A/H)$. We have $\pi(f(x)) = [f(x)]_H = \hat{f}([x]_H) = \hat{f}(\pi(x))$, for all $x \in A$. Hence $\pi \in \mathcal{MHOM}((A, f), (A/H, \hat{f}))$. Obviously, $\mathrm{Ker}\,(\pi) = H$. □

Theorem 4.15. *Let (A, f) and (B, g) be modal pseudo-BCK algebras, and $\phi \in \mathcal{MHOM}((A, f), (B, g))$. Consider $H \in \mathcal{DS}_n(A)$ such that $H \subseteq \mathrm{Ker}\,(\phi)$, and let $\pi \in \mathcal{MHOM}((A, f), (A/H, \hat{f}))$, $\hat{f} \in \mathcal{MOD}(A/H)$, defined in Proposition 4.14. Then there exists a unique $\tilde{\phi} \in \mathcal{MHOM}((A/H, \hat{f}), (B, g))$ such that:*
(1) *the following diagram is commutative:*



(2) $\mathrm{Im}\,(\tilde{\phi}) = \mathrm{Im}\,(\phi)$;
(3) $\mathrm{Ker}\,(\tilde{\phi}) = \mathrm{Ker}\,(\phi)/H$.

Proof. (1) Define $\tilde{\phi} : A/H \longrightarrow B$, by $\tilde{\phi}([x]_H) = \phi(x)$. We show that $\tilde{\phi}$ is well defined on A/H. Indeed, if $y \in [x]_H$, then $x \to y, y \to x \in H \subseteq \mathrm{Ker}\,\phi$, that is $1 = \phi(x \to y) = \phi(x) \to \phi(y)$ and $1 = \phi(y \to x) = \phi(y) \to \phi(x)$. Thus, $\phi(x) \leq \phi(y)$ and $\phi(y) \leq \phi(x)$, that is $\phi(x) = \phi(y)$. Since $\phi \in \mathcal{MHOM}((A, f), (B, g))$, we have: $\tilde{\phi}(\hat{f}([x]_H)) = \phi(f(x)) = g(\phi(x)) = g(\tilde{\phi}([x]_H))$, for all $x \in A$. Hence $\tilde{\phi} \in \mathcal{MHOM}((A/H, \hat{f}), (B, g))$ and obviously, $\tilde{\phi} \circ \pi = \phi$.
Suppose that there exists $\mu \in \mathcal{MHOM}((A/H, \hat{f}), (B, g))$ with $\tilde{\phi} \circ \pi = \mu \circ \pi$. It follows that $\tilde{\phi}(\pi(x)) = \mu(\pi(x))$, for all $x \in A$. Since π is surjective, for any element $y \in A/H$ there exists $x \in A$ such that $y = \pi(x)$, hence $\mu = \tilde{\phi}$.
(2) We can see that

$$\mathrm{Im}\,(\tilde{\phi}) = \{y \in B \mid \text{ there exists } [x]_H \in A/H, \tilde{\phi}([x]_H) = y\} = \{y \in B \mid \text{ there exists } x \in A, \phi(x) = y\} = \mathrm{Im}\,(\phi).$$

(3) For all $x \in A$ we have $[x]/H \in \mathrm{Ker}\,(\tilde{\phi})$ if and only if $\tilde{\phi}([x]_H) = 1$ if and only if $\phi(x) = 1$ if and only if $x \in \mathrm{Ker}\,(\phi)$. Hence $\mathrm{Ker}\,(\tilde{\phi}) = \{[x]_H \in A/H \mid x \in \mathrm{Ker}\,(\phi)\} = \mathrm{Ker}\,(\phi)/H$. □

Corollary 4.16. *Let (A, f) and (B, g) be modal pseudo-BCK algebras, and let $\phi \in \mathcal{MHOM}((A, f), (B, g))$. Then there exists a modal pseudo-BCK isomorphism between $(A/\mathrm{Ker}\,(\phi), f)$ and $(\mathrm{Im}\,(\phi), g)$.*

Proof. Taking $H = \mathrm{Ker}\,(\phi)$ and $B = \mathrm{Im}\,(\phi)$ in Theorem 4.15, it follows that $\tilde{\phi}$ is a modal isomorphism between $(A/\mathrm{Ker}\,(\phi), f)$ and $(\mathrm{Im}\,(\phi), g)$. □

5 Conclusions

The modal operators have been studied on many fuzzy logic algebras such as MV-algebras ([19]), bounded Rℓ-monoids ([36], [37]), residuated lattices ([24], [38], [39], [40]), MTL-algebras ([32]), BE-algebras ([48]) and pseudo-BCK algebras ([27]). In this paper, we generalize the modal operators to the case of pseudo-BE algebras and we investigate their properties. The results of this study have algebraic and logic significance, as well as possible applications in computer science. These results could be a starting point for future research:

(1) One topic of a future research could be the development of modal propositional calculus whose algebraic models are pseudo-BE algebras. The main example for various modal algebras arises from a Kripke-style frame and the representation theorem for these algebras uses this construction. This kind of example can also be given for modal pseudo-BE algebras.
(2) According to D. Scott, the modal logics should not focus only on modal operators. In order to get philosophically significant results in deontic logic or epistemic logic, the modal operators should be combined with other operators, such as the tense operators (see [46]). Classical tense logic is a logical system obtained from the bivalent logic by adding two tense operators: G, for the future tense ("it is always going to be the case that"), and H for the past tense ("it has always been the case that"). New tense logics can be obtained starting from other logical systems, such as many-valued logics, by adding appropriate tense operators. Tense operators have been studied by many authors for various classes of algebras: Basic algebras and effect algebras ([6]), MV-algebras and Łukasiewicz-Moisil algebras ([14], [7]), residuated lattices ([2]). A representation theorem for the semisimple tense MV-algebras was proved by J. Paseka in [33]. The tense operators could be also defined and studied for the case of pseudo-BE algebras.
(3) Recently, the notion of a pseudo-CI algebra was defined and studied in [44] as a generalization of CI-algebras ([31]). As another direction of research, one could introduce and investigate the modal operators on pseudo-CI algebras.

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