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2-set-based extended functions

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Abstract

We generalize set-based extended functions, recently introduced by Mesiar et al., into 2-set-based extended functions. In both cases, an efficient reduction of repeated data to be fused is considered which prepares a sound background for big data processing. We discuss and study several types of 2-set-based extended functions, in particular, 2-set-based extended aggregation functions, including t-norms, t-conorms, uninorms, nullnorms and OWA operators.

Keywords: Data reduction, set-based extended aggregation function, set-based extended function, 2-set-based extended aggregation function, 2-set-based extended function.

1 Introduction

In big data processing the order of simple inputs does not mostly matter, i.e., such processing is symmetric (commutative). Sometimes, numerous repetition of an input information has the same influence on the final output as a substantially reduced number of such information. As an extremal tool enabling reduction of repeated data we can mention the so-called set-based extended functions on a general universe X, which were recently introduced by Mesiar et al. in [15], i.e., extended functions $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$, such that for all $\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_m) \in \bigcup_{n\in\mathbb{N}} X^n$, $F(\mathbf{x}) = F(\mathbf{y})$ whenever $\{x_1, \ldots, x_k\} = \{y_1, \ldots, y_m\}$. Consequently, then $F(\mathbf{x}) = F(\mathbf{y}) = F(z_1, \ldots, z_p)$, where $\{z_1, \ldots, z_p\} = \{x_1, \ldots, x_k\} = \{y_1, \ldots, y_m\}$ and $\operatorname{card}(\{z_1, \ldots, z_p\}) = p$.

Due to this definition, given a set-based extended function F on X and an input vector $\mathbf{x} = (x_1, \ldots, x_n)$, the only important information concerning an additional input value $x \in X$, is the truth value of the claim "$x \in \{x_1, \ldots, x_n\}$". Clearly, if $x = x_i$ for some $i \in \{1, \ldots, n\}$ then the output value depends neither on the index i nor on the fact whether there is a unique index i satisfying $x_i = x$ or there are more such indices. As typical examples of set-based extended functions we recall Zadeh's fuzzy connectives min and max [20], or extremal Yager's OWA operators [18], i.e., extended OWA operators with a fixed orness α and minimal entropy [16], in other words, extended OWA operators with weighting vectors of the form $(\alpha, 0, \ldots, 0, 1 - \alpha)$). More information on set-based extended functions will be recalled in Section 2.

The main aim of this contribution is to extend the above discussed idea considering as a relevant information concerning the cardinality of the set $\{i \mid x = x_i\}$ one of the cases: $\operatorname{card}(\{i \mid x = x_i\}) \in \{0, 1, \mathit{more}\}$, where "more" represents situations where x occurs among the values $x_1, \ldots, x_n$ more than once. In Section 3 we introduce the concept of 2-set-based extended functions and give some examples and construction methods of such functions. In Section 4 we study some particular 2-set-based extended aggregation functions as well as their constructions. At the end of the paper, a few concluding remarks will be added.

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2 Set-based extended functions

In this section we first recall the notion of set-based extended function on a general universe X as it was introduced in [15], as well as several results given there which will be important for our further discussions. Then we also recall the notion of set-based extended aggregation function and give a characterization of such functions which we have proved in [13].

Definition 2.1. *Let $X \neq \emptyset$. Any function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ will be called an extended function on X.*

Extended functions have open arity, i.e., they can work for any finite number of arguments.

Definition 2.2. *Let $X \neq \emptyset$. A function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ is called a set-based extended function on X if for any $k, m \in \mathbb{N}$ and all $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$, $\mathbf{y} = (y_1, \ldots, y_m) \in X^m$, such that $\{x_1, \ldots, x_k\} = \{y_1, \ldots, y_m\}$, we have $F(\mathbf{x}) = F(\mathbf{y})$.*

The set of all set-based extended functions on X will be denoted by $\mathcal{SBF}(X)$.

A particular case of construction of set-based extended functions is described in the following example.

Example 2.3. *Let $X \neq \emptyset$. Let $\mathcal{P} = \{E_1, \ldots, E_k\}$ be a partition of X and $a_1, \ldots, a_k \in X$. Define $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ by*

$$F(\mathbf{x}) = a_i, \quad \text{where } i = \min\{j \in \{1, \ldots, k\} \mid \{x_1, \ldots, x_n\} \cap E_j \neq \emptyset\}. \tag{1}$$

Then F is a set-based extended function on X.

Lemma 2.4. *Let $X \neq \emptyset$ and $\mathcal{H}(X) = \{\emptyset \neq E \subseteq X \mid E \text{ is finite}\}$. Then each set-based extended function F on X corresponds in a one-to-one correspondence to a set function $G\colon \mathcal{H}(X) \to X$ given, for each $E = \{x_1, \ldots, x_n\}$ in $\mathcal{H}(X)$, by*

$$G(E) = F(x_1, \ldots, x_n).$$

Clearly, $\mathcal{H}(X)$ is the power set of X except the empty set whenever X is finite, and hence if $\text{card}(X) = k$, we have $\text{card}(\mathcal{SBF}(X)) = k^{2^k-1}$. Note that if $k > 1$, there are infinitely many extended functions on X. Lemma 2.4 also justifies to call considered functions set-based functions.

The following example illustrates Lemma 2.4.

Example 2.5. *Consider $X = \{0, 1\}$. Then a function $F\colon \bigcup_{n\in\mathbb{N}} \{0,1\}^n \to \{0, 1\}$ is an extended Boolean function. The cardinality of X is $\text{card}(X) = 2$, $\mathcal{H}(X) = \{\{0\}, \{1\}, \{0, 1\}\}$, i.e., $\text{card}(\mathcal{H}(X)) = 3$, thus there are exactly $2^3 = 8$ set functions $G_i\colon \mathcal{H}(X) \to \{0, 1\}$, $i = 1, \ldots, 8$. Consequently, there are 8 set-based extended Boolean functions F_i, where F_i corresponds to G_i by Lemma 2.4. The results are summarized in Table 1.*

Table 1: Set-based extended Boolean functions

$G_i \backslash E$	{0}	{1}	{0,1}	$F_i(\mathbf{x})$
G_1	0	0	0	0
G_2	0	0	1	$\bigvee_{j,k} \lvert x_j - x_k \rvert$
G_3	0	1	0	$\bigwedge_j x_j$
G_4	0	1	1	$\bigvee_j x_j$
G_5	1	0	0	$1 - F_4(\mathbf{x})$
G_6	1	0	1	$1 - F_3(\mathbf{x})$
G_7	1	1	0	$1 - F_2(\mathbf{x})$
G_8	1	1	1	$1 - F_1(\mathbf{x})$

The following theorem shows that some algebraic properties of a function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ already imply that F is a set-based extended function on X.

Theorem 2.6. *Let $X \neq \emptyset$. If $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ is symmetric, associative and idempotent then F is a set-based extended function on X.*

By this theorem, each idempotent extended uninorm, nullnorm, t-norm or t-conorm is a set-based extended function on $[0,1]$ (but also on any bounded poset). Note that the only idempotent t-norm is the minimum t-norm T_M and in the case of t-conorms the maximum t-conorm S_M. For more information see Example 2.10.

Now, suppose X to be a (bounded) chain. Then it makes sense to define on X extended functions Min and Max,

$$Min(x_1,\dots,x_n) = \min\{x_1,\dots,x_n\}, \qquad Max(x_1,\dots,x_n) = \max\{x_1,\dots,x_n\}.$$

A total order on X has an important impact on characterization of monotone set-based extended functions on X.

Proposition 2.7. *Let X be a chain. Then $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ is a monotone non-decreasing (non-increasing) set-based extended function if and only if for each $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} X^n$ we have*

$$F(\mathbf{x}) = D(Min(\mathbf{x}), Max(\mathbf{x})), \tag{2}$$

for some monotone non-decreasing (non-increasing) function $D\colon X^2 \to X$.

Further, we provide a characterization of set-based extended aggregation functions acting on $X = [0,1]$.

Definition 2.8. *A function $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ is an extended aggregation function on $[0,1]$ if A is monotone non-decreasing and satisfies the boundary conditions, i.e.,*

(i) *for all $\mathbf{x},\mathbf{y} \in \bigcup_{n\in\mathbb{N}} [0,1]^n$ we have $A(\mathbf{x}) \leq A(\mathbf{y})$ whenever $\mathbf{x} \leq \mathbf{y}$;*

(ii) *for all elements $\mathbf{0} = (0,\dots,0), \mathbf{1} = (1,\dots,1) \in \bigcup_{n\in\mathbb{N}} [0,1]^n$, $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$.*

The set of all extended aggregation functions on $[0,1]$ will be denoted by $\mathcal{A}([0,1])$. Note that for $X = [0,1]$, the above introduced extended functions Min and Max are extended aggregation functions on $[0,1]$.

An extended aggregation function $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ which is also a set-based extended function on $[0,1]$ will be called a set-based extended aggregation function on $[0,1]$. The set of all set-based extended aggregation function on $[0,1]$ will be denoted by $\mathcal{SBAF}([0,1])$. Clearly, $\mathcal{SBAF}([0,1]) = \mathcal{A}([0,1]) \cap \mathcal{SBF}([0,1])$.

Set-based extended aggregation functions on $[0,1]$ can be completely characterized as follows [13].

Theorem 2.9. *Let $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ be an extended aggregation function on $[0,1]$. Then $A \in \mathcal{SBAF}([0,1])$ if and only if for all $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} [0,1]^n$ we have*

$$A(\mathbf{x}) = A(Min(\mathbf{x}), Max(\mathbf{x})). \tag{3}$$

For more results on set-based extended aggregation functions on $[0,1]$, see [13].

Example 2.10. *Here we summarize some distinguished set-based aggregation functions:*

- *The only t-norm in $\mathcal{SBAF}([0,1])$ is the minimum t-norm T_M,*

$$T_M(x_1,\dots,x_n) = \min\{x_1,\dots,x_n\}.$$

- *The only t-conorm in $\mathcal{SBAF}([0,1])$ is the maximum t-norm S_M,*

$$S_M(x_1,\dots,x_n) = \max\{x_1,\dots,x_n\}.$$

- *The only nullnorms in $\mathcal{SBAF}([0,1])$ are a-medians med_a, $a \in [0,1[$, given by*

$$\mathrm{med}_a(x_1,\dots,x_n) = \mathrm{med}(x_1,a,x_2,a\dots,x_{n-1},a,x_n) = \mathrm{med}\left(Min(\mathbf{x}),a,Max(\mathbf{x})\right).$$

- *A uninorm U is in $\mathcal{SBAF}([0,1])$ if and only if it is internal, i.e., $U(x_1, x_2) \in \{x_1, x_2\}$ for all $x_1, x_2 \in [0,1]$.*
- *An extended OWA operator is in $\mathcal{SBAF}([0,1])$ if and only if it is related to a weighting triangle which is formed by weighting vectors of the form $(\alpha, 0, \ldots, 0, 1-\alpha)$, for $n \geq 2$, i.e.,*

$$OWA_\alpha(\mathbf{x}) = \alpha Max(\mathbf{x}) + (1-\alpha) Min(\mathbf{x}).$$

Observe that the considered weighting triangles are induced by regular increasing monotone (RIM) quantifiers $Q_\alpha\colon [0,1] \to [0,1]$, see [18], given by

$$Q_\alpha(x) = \begin{cases} 0 & \text{if } x = 0, \\ \alpha & \text{if } x \in]0,1[, \\ 1 & \text{if } x = 1. \end{cases}$$

Then, for each $n \in \mathbb{N}$ and all $i \in \{1, \ldots, n\}$,

$$w_{i,n} = Q_\alpha\left(\frac{i}{n}\right) - Q_\alpha\left(\frac{i-1}{n}\right).$$

3 2-set-based extended functions

We start by defining the notion of 2-set-based extended function on any universe $X \neq \emptyset$.

Definition 3.1. *Let $X \neq \emptyset$ and let $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ be an extended function on X such that $F(\mathbf{x}) = F(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \bigcup_{n\in\mathbb{N}} X^n$, $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$, $\mathbf{y} = (y_1, \ldots, x_m) \in X^m$, satisfying the properties*

$$\{x_1, \ldots, x_k\} = \{y_1, \ldots, y_m\}, \tag{4}$$

, and

$$\{x \in X \mid \text{card}(\{i \mid x = x_i\}) = 1\} = \{x \in X \mid \text{card}(\{j \mid x = y_j\}) = 1\}. \tag{5}$$

Then F is called a 2-set-based extended function on X.

In what follows, the set of all 2-set-based extended functions on X will be denoted by 2-$\mathcal{SBF}(X)$.

Due to the above definition, if $F \in$ 2-$\mathcal{SBF}(X)$ and $\mathbf{x}, \mathbf{y} \in \bigcup_{n\in\mathbb{N}} X^n$ satisfy constraints (4), (5), then they also satisfy the conditions

$$\{x \in X \mid \text{card}(\{i \mid x = x_i\}) \geq 2\} = \{x \in X \mid \text{card}(\{i \mid x = y_j\}) \geq 2\},$$

and

$$\{x \in X \mid x \neq x_i \text{ for all } i\} = \{x \in X \mid x \neq y_j \text{ for all } j\}.$$

Consequently, if there is an $x \in X$ such that $\text{card}(\{i \mid x = x_i\}) = k > 2$, then $(k-2)$ elements with value x can be omitted from $\mathbf{x}$ because the set $\{x \in X \mid \text{card}(\{i \mid x = x_i\}) \geq 2\}$ remains the same (as well as both sets mentioned in Definition 3.1). This property allows us to reduce the number of data to be aggregated with $F \in$ 2-$\mathcal{SBF}(X)$, thus it might be of great importance in big data processing.

Based on Definition 3.1, we get a characterization of the set 2-$\mathcal{SBF}(X)$ which is an analogue of the characterization given in Lemma 2.4.

Lemma 3.2. *Let $X \neq \emptyset$ and*

$$\mathcal{G}(X) = \{(E, D) \mid E, D \in 2^X, E \cap D = \emptyset, E \cup D \neq \emptyset, E, D - \textit{ finite}\}.$$

Then $F \in$ 2-$\mathcal{SBF}(X)$ if and only if there is a function $G\colon \mathcal{G}(X) \to X$ such that

$$F(x_1, \ldots, x_n) = G(\{x_i \mid x_i \neq x_j \ \forall\ j \neq i\}, \{x_i \mid \text{card}(\{j \mid x_j = x_i\}) > 1\}).$$

Note that if X is a finite space with $\text{card}(X) = k$, then using notation 3^X for the set of all pairs of subsets of X with disjunctive components (used in *[8]*), we have $\mathcal{G}(X) = 3^X \setminus \{(\emptyset, \emptyset)\}$, i.e., $\text{card}(\mathcal{G}(X)) = 3^k - 1$, and thus $\text{card}(2\text{-}\mathcal{SBF}(X)) = k^{3^k-1}$.

It is obvious that each $F \in$ 2-$\mathcal{SBF}(X)$ is symmetric, i.e., for any $\mathbf{x} \in X^n$ and any permutation σ on $\{1, \ldots, n\}$ we have $F(\mathbf{x}) = F(\mathbf{x}_\sigma)$, $\mathbf{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

Now, consider an element $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} X^n$ and define the function $\varphi_{\mathbf{x}}\colon X \to \{0,1,2,\ldots\}$ by

$$\varphi_{\mathbf{x}}(x) = \mathrm{card}(\{i \mid x = x_i\}),$$

i.e., $\varphi_{\mathbf{x}}$ is a frequency function saying how many times an element $x \in X$ occurs among the components $x_1,\ldots,x_n$ of $\mathbf{x}$. Evidently, for any permutation $\mathbf{x}_\sigma$ of an n-tuple $\mathbf{x}$, we have $\varphi_{\mathbf{x}} = \varphi_{\mathbf{x}_\sigma}$. Further, denote by $\varphi(X)$ the set of all frequency functions, i.e.,

$$\varphi(X) = \{\varphi_{\mathbf{x}} \mid \mathbf{x} \in \bigcup_{n\in\mathbb{N}} X^n\}.$$

Note that a function $\tau\colon X \to \{0,1,2,\ldots\}$ is in $\varphi(X)$ if and only if $0 < \sum_{x\in X} \tau(x) < \infty$. The possibility that $\sum_{x\in X}\tau(x) = 0$ is redundant here as $\mathbf{x}$ will then have no element; clearly, $\sum_{x\in X}\tau(x)$ is indeed the arity of the input vector $\mathbf{x}$.

Each symmetric extended function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ corresponds to a function $K\colon \varphi(X) \to X$, given by

$$K(\varphi_{\mathbf{x}}) = F(\mathbf{x}). \tag{6}$$

Note that this correspondence is one-to-one due to the fact that for any $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} X^n$ and permutation $\sigma : \{1,..,n\} \to \{1,..,n\}$ it holds

$$\varphi_{\mathbf{x}} = \varphi_{\mathbf{x}_\sigma} \quad \text{and} \quad F(\mathbf{x}) = F(\mathbf{x}_\sigma).$$

Lemma 3.3. *Let $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ be a symmetric extended function on X linked to the function $K\colon \varphi(X) \to X$ via (6). Then*

(i) *F is a set-based extended function if and only if $F(\mathbf{x}) = K(\varphi_{\mathbf{x}} \wedge 1)$ for each $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} X^n$.*

(ii) *F is a 2-set-based extended function if and only if $F(\mathbf{x}) = K(\varphi_{\mathbf{x}} \wedge 2)$ for each $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} X^n$.*

By Lemma 3.3, each $F \in \mathcal{SBF}(X)$ also belongs to 2-$\mathcal{SBF}(X)$. Moreover, for each $F \in \mathcal{SBF}(X)$, the essential information concerning an input vector $\mathbf{x} = (x_1,\ldots,x_n) \in \bigcup_{n\in\mathbb{N}} X^n$ is contained in the finite set

$$H_{\mathbf{x}} = \{x \in X \mid \varphi_{\mathbf{x}}(x) \geq 1\},$$

with $\mathrm{card}(H_{\mathbf{x}}) \in \{1,\ldots,n\}$. Therefore the name "set-based extended functions" is again justified. Similarly, for 2-set-based extended functions instead of $\mathbf{x} = (x_1,\ldots,x_n) \in \bigcup_{n\in\mathbb{N}} X^n$ it is enough to deal with a couple of finite sets $(P_{\mathbf{x}}, G_{\mathbf{x}})$, where

$$P_{\mathbf{x}} = \{x \in X \mid \varphi_{\mathbf{x}}(x) = 1\} \quad \text{and} \quad G_{\mathbf{x}} = \{x \in X \mid \varphi_{\mathbf{x}}(x) > 1\}.$$

Evidently, the sets $P_{\mathbf{x}}$ and $G_{\mathbf{x}}$ are disjoint, $P_{\mathbf{x}} \cup G_{\mathbf{x}} = H_{\mathbf{x}}$ and $X \setminus (P_{\mathbf{x}} \cup G_{\mathbf{x}}) = \{x \in X \mid \varphi_{\mathbf{x}}(x) = 0\}$. This fact justifies the name "2-set-based extended functions".

Example 3.4. *Let $a, b \in X$, $a \neq b$. Define the function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ as follows:*

- *for each $x \in X$, $F(x) = x$;*
- *for each $\mathbf{x} \in \bigcup_{n\geq 2} X^n$, $F(\mathbf{x}) = \begin{cases} a & \text{if } \varphi_{\mathbf{x}}(a) \geq 2, \\ b & \text{otherwise.} \end{cases}$*

Then $F \in$ 2-$\mathcal{SBF}(X)$ but $F \notin \mathcal{SBF}(X)$.

Example 3.5. *Consider an extended function $F\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ given by*

$$F(\mathbf{x}) = \frac{\sum_{x\in H_{\mathbf{x}}} x \cdot \min\{2, \varphi_{\mathbf{x}}(x)\}}{\sum_{x\in H_{\mathbf{x}}} \min\{2, \varphi_{\mathbf{x}}(x)\}}.$$

Then $F \in$ 2-$\mathcal{SBF}([0,1])$. Observe that $F(0,0,0,1) = \frac{1}{3}$ and $F(0,0,0.2,1) = 0.3$ which contradicts the non-decreasing monotonicity of F. Note that F is weakly monotone increasing, see [17], and though F is not an extended aggregation function, for each $n \in \mathbb{N}$, $n \geq 4$, $F|_{[0,1]^n}$ is a proper pre-aggregation function (for the definition and properties see [12]).

Now we introduce a new property of extended functions which will allow us to generalize Theorem 2.6 for 2-set-based extended functions.

Definition 3.6. *Let* $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ *be an extended function. Then* F *is said to be* 2*-idempotent if for each* $\mathbf{x} = (x,\ldots,x) \in \bigcup_{n=2}^{\infty} X^n$ *we have* $F(\mathbf{x}) = F(x,x)$.

Theorem 3.7. *Let* $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ *be a symmetric, associative and* 2*-idempotent extended function. Then* F *is a* 2*-set-based extended function.*

Proof. Due to the supposed properties one can replace any constant k-tuple

$$(x_{i_1},\ldots,x_{i_k}) = (a,\ldots,a),\ k\geq 2,$$

by a tuple (a,a) with no influence on the final output which means that $F \in 2\text{-}\mathcal{SBF}(X)$. □

Example 3.8. *Consider the extended drastic product t-norm* $T_D\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$*, given by*

$$\begin{aligned} T_D(x) &= x, \\ T_D(x_1,\ldots,x_n) &= T_D(x_{(1)},x_{(2)}) = \begin{cases} 0 & \text{if } x_{(2)} < 1, \\ x_{(1)} & \text{if } x_{(2)} = 1, \end{cases} \end{aligned}$$

where $x_{(1)}$, $x_{(2)}$ *are the smallest and the second smallest elements of the input* n*-tuple* $(x_1,\ldots,x_n)$*, respectively. Thus for any* n*-tuple* $(a,\ldots,a)$, $n\geq 2$*, we have* $T_D(a,\ldots,a) = T_D(a,a)$*, hence* T_D *is* 2*-idempotent. Moreover, as being a t-norm,* T_D *is symmetric and associative [10], and so, by Theorem 3.7,* T_D *belongs to* $2\text{-}\mathcal{SBF}$*.*

Remark 3.9. *Clearly,* 2*-idempotency is a necessary condition satisfied by each* 2*-set-based extended function* F *(as well as symmetry), but this is not the case of associativity. Consider the second order statistics* $OS_2\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$*, given by*

$$OS_2(x) = x \ \text{ and } \ OS_2(x_1,\ldots,x_n) = x_{(2)}, \ \text{ if } \ n\geq 2,$$

i.e., OS_2 *is a basic OWA operator [18]. As* OS_2 *is in* $2\text{-}\mathcal{SBF}(X)$ *(see Theorem 4.1), it is symmetric and* 2*-idempotent (*OS_2 *is even idempotent), but it is not associative.*

The following useful result is straightforward.

Proposition 3.10. *Let* $k\in\mathbb{N}$*. Then, for any* $F\colon X^k \to X$ *and* $F_1,\ldots,F_k \in 2\text{-}\mathcal{SBF}(X)$*, the composite function* $G\colon \bigcup_{n\in\mathbb{N}} X^n \to X$*, given by*

$$G(\mathbf{x}) = F(F_1(\mathbf{x}),\ldots,F_k(\mathbf{x})),$$

also belongs to $2\text{-}\mathcal{SBF}(X)$*.*

Similarly, one can obtain immediately the following result.

Proposition 3.11. *Let* $\varphi\colon X \to Y$ *be a bijective mapping, and let* $F \in 2\text{-}\mathcal{SBF}(Y)$*. Then the extended function* $F_\varphi\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ *given by*

$$F_\varphi(x_1,\ldots,x_n) = \varphi^{-1}(F(\varphi(x_1),\ldots,F(\varphi(x_n)),$$

is a 2*-set-based extended function on* X*.*

The next construction is similar to the ordinal sum construction of posets introduced by Birkhoff [2] and Clifford's ordinal sum of semigroups [5].

Theorem 3.12. *Let* I *be a chain,* $(X_t)_{t\in I}$ *a disjoint system of non-empty universes, and* $(F_t)_{t\in I}$ *a system of* 2*-set-based extended functions, such that for each* $t\in I$, $F_t \in 2\text{-}\mathcal{SBF}(X_t)$*. Define the function* $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$, $X = \bigcup_{t\in I} X_t$*, by*

$$F(\mathbf{x}) = F_{t_\mathbf{x}}\left(\mathbf{x}\cap X_{t_\mathbf{x}}\right),$$

where $\mathbf{x} = (x_1,\ldots,x_n)$, $t_\mathbf{x} = \min\{t\in I \mid x_i \in X_t \text{ for some } i\}$, $\mathbf{x}\cap X_{t_\mathbf{x}} = (x_{i_1},\ldots,x_{i_k}) \in X^k_{t_\mathbf{x}}$*, and for all* $i\notin\{i_1,\ldots,i_k\}$, $x_i\in X_{t_i}$ *for some* $t_i > t_\mathbf{x}$*, i.e.,* $x_i \notin X_{t_\mathbf{x}}$*.*

Then F *is called an ordinal sum of* 2*-set-based extended functions and belongs to* $2\text{-}\mathcal{SBF}(X)$*. It will be denoted by* $F = (\langle X_t, F_t\rangle \mid t\in I)$*.*

Proof. For any $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$, the inputs x_i from the universes X_t with $t > t_{\mathbf{x}}$ have no influence on the output value $F(\mathbf{x})$. Hence, it is enough to consider the inputs from $X_{t_{\mathbf{x}}}$ only. Let us denote them by $\mathbf{x}'$. Evidently,

$$F(\mathbf{x}) = F(\mathbf{x}') = F_{t_{\mathbf{x}}}(\mathbf{x}').$$

Now, the result follows from the fact that $F_{t_{\mathbf{x}}}$ is a 2-set-based extended function. □

4 2-set-based extended aggregation functions

In this section we focus our attention on 2-set-based extended aggregation functions, i.e., we will deal with extended aggregation functions on $[0,1]$ (see Definition 2.8) which also belong to 2-$\mathcal{SBF}([0,1])$. The set of all such functions will be denoted by 2-$\mathcal{SBAF}([0,1])$, and clearly, 2-$\mathcal{SBAF}([0,1]) = \mathcal{A}([0,1]) \cap$ 2-$\mathcal{SBF}([0,1])$.

Observe that associative extended aggregation functions $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ satisfy the property $A(x) = x$ for each $x \in [0,1]$ by convention, and for arities $n \geq 2$, their values are deduced from their binary forms, e.g.,

$$A(x_1, x_2, x_3, x_4) = A(A(A(x_1, x_2), x_3), x_4).$$

Further, recall that extended weighted means or OWA operators are determined by weighting triangles Δ of the form

$$\Delta = \left(w_{i,n} \mid n \in \mathbb{N}, i \in \{1, \ldots, n\}, w_{i,n} \geq 0, \sum_{i=1}^{n} w_{i,n} = 1 \right).$$

More details can be found, e.g., in [4, 1, 9].

For 2-set-based extended aggregation functions we have an important result generalizing Theorem 2.9.

Theorem 4.1. *Let $B\colon [0,1]^4 \to [0,1]$ be a quaternary aggregation function and $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ an extended aggregation function such that, for all $n \geq 2$ and $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$, we have*

$$A(\mathbf{x}) = B(x_{(1)}, x_{(2)}, x_{(n-1)}, x_{(n)}),$$

where $(\cdot)\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that $x_{(1)} \leq \cdots \leq x_{(n)}$. Then $A \in$ 2-$\mathcal{SBAF}([0,1])$.

Proof. Clearly, $A \in \mathcal{A}([0,1])$. We prove that $A \in$ 2-$\mathcal{SBF}([0,1])$. Suppose that there is an $x \in [0,1]$ such that $k = \mathrm{card}(\{i \in \{1, \ldots, n\} \mid x_i = x\}) > 2$, and let $\mathbf{x}' = (x'_1, \ldots, x'_{n-k+2})$ be a sample coinciding with $\mathbf{x}$ with exception of $(k-2)$ omitted inputs $x_i = x$. Then $x'_{(1)} = x_{(1)}$, $x'_{(2)} = x_{(2)}$, $x'_{(n-k+1)} = x_{(n-1)}$ and $x'_{(n-k+2)} = x_{(n)}$, thus $A(\mathbf{x}') = A(\mathbf{x})$, which proves that A is a 2-set-based extended function. □

Observe that for any $A \in$ 2-$\mathcal{SBAF}([0,1])$ and $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} [0,1]^n$ we have

$$A(x_{(1)}, x_{(1)}, x_{(n)}) \leq A(\mathbf{x}) \leq A(x_{(1)}, x_{(n)}, x_{(n)}).$$

This constraint is a necessary condition for being A in 2-$\mathcal{SBAF}([0,1])$, while the previous theorem has given a sufficient condition.

In what follows, we will discuss some distinguished extended aggregation functions from the point of their 2-set-based property. We focus on extended t-norms, t-conorms, uninorms, nullnorms and OWA operators. As all these extended functions are in $\mathcal{A}([0,1])$, they belong to 2-$\mathcal{SBAF}([0,1])$ if and only if they belong to 2-$\mathcal{SBF}([0,1])$.

4.1 2-set-based extended t-norms and t-conorms

Recall that each extended t-norm $T\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ is a symmetric associative aggregation function with neutral element $e = 1$. From these properties and the fact that all 2-set based extended functions are necessarily 2-idempotent, by Theorem 3.7, it follows that T is in 2-$\mathcal{SBF}([0,1])$ if and only if it is 2-idempotent. Furthermore, due to associativity, T is 2-idempotent if and only if $T(x,x,x) = T(x,x)$ for each $x \in [0,1]$.

Now, suppose that T is in 2-$\mathcal{SBF}([0,1])$. For any $x \in [0,1]$, let $T(x,x) = a$. Then $a \leq x$ and

$$T(a,a) = T(T(x,x), T(x,x)) = T(T(x,x,x), x) = T(T(x,x), x) = T(x,x,x) = T(x,x) = a,$$

which means that a is an idempotent element of T. So, if we denote by $\mathcal{I}(T)$ the set of all idempotent elements of T, i.e., $\mathcal{I}(T) = \{x \in [0,1] \mid T(x,x) = x\}$, we have $T(x,x) \in \mathcal{I}(T)$ for each $x \in [0,1]$. Moreover, the set $\mathcal{I}(T)$ is closed under suprema.

Before giving a characterization of 2-set-based extended t-norms, recall that the diagonal section of a t-norm T is the function $\delta_T\colon [0,1] \to [0,1]$, given by $\delta_T(x) = T(x,x)$.

Theorem 4.2. *Let T be an extended t-norm. Then the following are equivalent.*

(i) $T \in 2\text{-}\mathcal{SBF}([0,1])$.

(ii) *For each $x \in [0,1]$, $\delta_T(x) = \max\{a \in \mathcal{I}(T) \mid a \le x\}$.*

(iii) $\delta_T \circ \delta_T = \delta_T$.

Proof. (i) $\Rightarrow$ (ii): Let $T \in 2\text{-}\mathcal{SBF}([0,1])$. Then, due to the previous discussion, for each $x \in [0,1]$, $\delta_T(x) = T(x,x) \in \mathcal{I}(T)$, and $\delta_T(x) \le x$ which implies $\delta_T(x) \le \max\{a \in \mathcal{I}(T) \mid a \le x\}$.

On the other hand, for any t-norm T, any $x \in [0,1]$ and all $a \in \mathcal{I}(T)$ such that $a \le x$, we have $\delta_T(x) = T(x,x) \ge T(a,a) = a$, thus $\delta_T(x) \ge \sup\{a \in \mathcal{I}(T) \mid a \le x\} = \max\{a \in \mathcal{I}(T) \mid a \le x\}$, and the result follows.

(ii) $\Rightarrow$ (iii): As for each $x \in [0,1]$, $\delta_T(x) = \max\{a \in \mathcal{I}(T) \mid a \le x\}$, $\delta_T(x) \in \mathcal{I}(T)$, thus we have $T(\delta_T(x), \delta_T(x)) = \delta_T(x)$. On the other hand,

$$T(\delta_T(x), \delta_T(x)) = T(T(x,x), T(x,x)) = \delta_T(T(x,x)) = \delta_T(\delta_T(x)) = \delta_T \circ \delta_T(x),$$

and the claim follows.

(iii) $\Rightarrow$ (i): For any t-norm T and any $x \in [0,1]$, we have

$$T(x,x,x,x) \le T(x,x,x) \le T(x,x),$$

i.e.,

$$\delta_T \circ \delta_T(x) \le T(x,x,x) \le \delta_T(x).$$

Due to (iii), we can conclude that $T(x,x,x) = \delta_T(x)$, and thus $T(x,x,x) = T(x,x)$ for each $x \in [0,1]$, which finally implies that T is 2-idempotent, and therefore $T \in 2\text{-}\mathcal{SBF}([0,1])$ by Theorem 3.7. □

Let us stress the importance of the previous theorem which has shown that the 2-set-based property of an extended t-norm depends on its diagonal section only.

The following result brings a sufficient condition ensuring the 2-set-based property of an extended t-norm.

Corollary 4.3. *Let $T\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ be an extended t-norm. If for each $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} [0,1]^n$,*

$$T(\mathbf{x}) = T(x_{(1)}, x_{(2)}),$$

where $x_{(1)}$ and $x_{(2)}$ are the smallest and the second smallest element of $\mathbf{x}$, respectively, then T is in $2\text{-}\mathcal{SBF}([0,1])$.

Proof. Clearly, for each $x \in [0,1]$,

$$\delta_T \circ \delta_T(x) = T(\delta_T(x), \delta_T(x)) = T(T(x,x), T(x,x)) = T(x,x,x,x) = T(x,x) = \delta_T(x).$$

The results now follows from Theorem 4.2. □

Both extremal t-norms, i.e., the minimum T_M and the drastic product T_D, are 2-set-based extended functions (T_M even belongs to $\mathcal{SBAF}([0,1])$). Observe that the class of all 2-set-based extended t-norms is closed under the ordinal sum construction as well as under any isomorphic transformation $\phi\colon [0,1] \to [0,1]$, ϕ being an automorphism satisfying

$$T_\phi(x_1, \ldots, x_n) = \phi^{-1}(\phi(x_1), \ldots, \phi(x_n)).$$

There are two basic types of 2-set-based extended t-norms:

(i) Let $K \subseteq [0,1[^2$ be a symmetric root set, i.e., $(x,y) \in K \Leftrightarrow (y,x) \in K$ and moreover, $(x,y) \le (u,v)$ and $(u,v) \in K$ imply $(x,y) \in K$. Then the function $T_K\colon [0,1]^2 \to [0,1]$, given by

$$T_K(x,y) = \begin{cases} 0 & \text{if } (x,y) \in K, \\ \min\{x,y\} & \text{otherwise,} \end{cases}$$

is a binary form of a 2-set-based extended t-norm. Note that $T_{[0,1[^2} = T_D$ and $T_\emptyset = T_M$. Moreover, if $K = \{(x,y) \in [0,1[^2 | \ x+y \leq 1\}$, then $T_K = T_M^n$, T_M^n being the nilpotent minimum [10]. Also observe that T_K is the greatest t-norm satisfying the property $T|_K \equiv 0$.

(ii) Let $L \subseteq [0,1]$ be a subset containing 0 and 1, closed under the suprema. Define the function $\Delta_L\colon [0,1] \to [0,1]$ by $\Delta_L(x) = \sup\{a \in L \mid a \leq x\}$. Then the function $T_{\Delta_L}: [0,1]^2 \to [0,1]$, given by

$$T_{\Delta_L}(x,y) = \begin{cases} \min\{x,y\} & \text{if } \max\{x,y\} = 1, \\ \Delta_L(\min\{x,y\}) & \text{otherwise,} \end{cases}$$

is a binary form of a 2-set-based extended t-norm. Clearly, $T_{\Delta_{[0,1]}} = T_M$ and $T_{\Delta_{\{0,1\}}} = T_D$.

Note that if L is finite, $L = \{a_0, a_1, \ldots, a_n\}$, where $0 = a_0 < a_1 < \ldots < a_n = 1$, then the previous formula will be of the form

$$T_{\Delta_L}(x,y) = \begin{cases} \min\{x,y\} & \text{if } \max\{x,y\} = 1, \\ a_i & \text{if } a_i \leq \min\{x,y\} < a_{i+1}, \end{cases}$$

with convention $a_{n+1} = 2$, see extensions of discrete t-norms discussed in [10]. Observe that T_{Δ_L} is the smallest t-norm such that $T|_{L^2} = T_M|_{L^2}$.

Note that the results for 2-set-based extended t-conorms can easily be deduced from the results for t-norms by duality. Recall that a dual aggregation function A^d to an aggregation function A is defined by

$$A^d(x_1, \ldots, x_n) = 1 - A(1-x_1, \ldots, 1-x_n),$$

and thus A^d is in 2-$\mathcal{SBAF}([0,1])$ if and only if A is in 2-$\mathcal{SBAF}([0,1])$. For example, if for any symmetric crown set $C \subseteq]0,1]^2$ (i.e., a symmetric subset such that if $(x,y) \leq (u,v)$ and $(x,y) \in C$ then also $(u,v) \in C$) we define the function $S_C\colon [0,1]^2 \to [0,1]$ by

$$S_C(x,y) = \begin{cases} 1 & \text{if } (x,y) \in C, \\ \max\{x,y\} & \text{otherwise,} \end{cases}$$

then S_C is a binary form of a 2-set-based extended t-conorm.

Due to Theorem 4.2, using the duality of t-norms and t-conorms one can derive the following characterization of 2-set-based extended t-conorms.

Theorem 4.4. *Let S be an extended t-conorm and δ_S its diagonal section. Then the following are equivalent.*

(i) $S \in 2\text{-}\mathcal{SBF}([0,1])$.

(ii) *For each $x \in [0,1]$, $\delta_S(x) = \min\{a \in \mathcal{I}(T) \mid a \geq x\}$.*

(iii) $\delta_S \circ \delta_S = \delta_S$.

Similarly, we can state the following sufficient condition for being S a 2-set-based extended t-conorm.

Corollary 4.5. *Let $S\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ be an extended t-conorm. If for each $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} [0,1]^n$,*

$$S(\mathbf{x}) = S(x_{(n-1)}, x_{(n)}),$$

where $x_{(n)}$ and $x_{(n-1)}$ are the greatest and the second greatest element of $\mathbf{x}$, respectively, then S is in 2-$\mathcal{SBF}([0,1])$.

4.2 2-set-based extended uninorms and nullnorms

Recall that an aggregation function $U\colon [0,1]^2 \to [0,1]$ is a uninorm [19] whenever it is symmetric, associative and has a neutral element $e \in [0,1]$. Clearly, if $e = 1$ then U is a t-norm, and if $e = 0$ then U is a t-conorm. Therefore, in what follows, we will only deal with neutral elements $e \in]0,1[$.

It is not difficult to check that $T_e = U|_{[0,e]^2}$ is a t-norm on $[0,e]$, i.e., the mapping $T_e^*\colon [0,1]^2 \to [0,1]$, given by

$$T_e^*(x,y) = \frac{T_e(ex,ey)}{e},$$

is a t-norm on $[0,1]$.

Similarly, $S_e = U|_{[e,1]^2}$ is a t-conorm on $[e,1]$, i.e., $S_e^*\colon [0,1]^2 \to [0,1]$, given by

$$S_e^*(x,y) = \frac{S_e(e+(1-e)x, e+(1-e)y) - e}{1-e},$$

is a t-conorm on $[0,1]$, see also [7].

By an important result for uninorms shown in [4, Section 6.2], if $U\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ is an extended uninorm then for each $n \in \mathbb{N}$ and $\mathbf{x} \in [0,1]^n$, we have

$$U(\mathbf{x}) = U(T_e(x_1 \wedge e, \dots, x_n \wedge e), S_e(x_1 \vee e, \dots, x_n \vee e).$$

Theorem 4.6. *An extended uninorm U belongs to 2-$\mathcal{SBF}([0,1])$ if and only if both T_e^* and S_e^* are in 2-$\mathcal{SBF}([0,1])$.*

Proof. From the previous facts it follows that $U \in$ 2-$\mathcal{SBF}([0,1])$ if and only if both T_e and S_e are 2-set-based extended functions, i.e., if and only if their linear transforms T_e^* and S_e^* belong to 2-$\mathcal{SBF}([0,1])$. □

Following the previous discussions and Theorems 4.2 and 4.4, we can also characterize 2-set-based extended uninorms as follows:

Corollary 4.7. *Let U be an extended uninorm and δ_U its diagonal section. Then $U \in$ 2-$\mathcal{SBF}([0,1])$ if and only if $\delta_U \circ \delta_U = \delta_U$.*

Finally, using the same notation of the extremal values of $\mathbf{x}$ as explained in Corollaries 4.3 and 4.5, we can formulate a sufficient condition for an extended uninorm U to belong to 2-$\mathcal{SBF}([0,1])$.

Corollary 4.8. *Let U be an extended uninorm. If for each $\mathbf{x} \in \bigcup_{n>1} [0,1]^n$ the values of U are given by*

$$U(\mathbf{x}) = U(x_{(1)}, x_{(2)}, x_{(n-1)}, x_{(n)}),$$

then U is in 2-$\mathcal{SBF}([0,1])$.

Similar results also hold for nullnorms. Recall that an aggregation function $V\colon [0,1]^2 \to [0,1]$ is a nullnorm whenever it is symmetric, associative and there is an element $a \in [0,1]$ such that $V(x,0) = x$ if $x \le a$, and $V(x,1) = x$ if $x \ge a$ [3]. Clearly, if $a = 1$ then V is a t-conorm, while if $a = 0$ then V is a t-norm. Therefore we will again deal with $a \in]0,1[$ only. Then a is an annihilator of V, i.e., for an extended nullnorm V we have $V(\mathbf{x}) = a$ whenever $a \in \{x_1, \dots, x_n\}$. It follows that $V|_{[0,a]^2} = S^{(a)}$ is a t-conorm on $[0,a]$, and $V|_{[a,1]^2} = T^{(a)}$ is a t-norm on $[a,1]$. Then the function $S_*^{(a)}$, which is a linear transform of $S^{(a)}$, given by

$$S_*^{(a)}(x,y) = \frac{S^{(a)}(ax, ay)}{a},$$

is a t-conorm on $[0,1]$. Similarly, $T_*^{(a)}$, given by

$$T_*^{(a)}(x,y) = \frac{T^{(a)}(a + (1-a)x, a + (1-a)y) - a}{1-a},$$

is a t-norm on $[0,1]$.

In what follows, we only summarize the results for extended nullnorms:

Theorem 4.9. *An extended nullnorm V belongs to 2-$\mathcal{SBF}([0,1])$ if and only if both $S_*^{(a)}$ and $T_*^{(a)}$ are in 2-$\mathcal{SBF}([0,1])$.*

Corollary 4.10. *Let V be an extended nullnorm and δ_V its diagonal section. Then $V \in$ 2-$\mathcal{SBF}([0,1])$ if and only if $\delta_V \circ \delta_V = \delta_V$.*

Corollary 4.11. *Let U be an extended nullnorm. If for each $\mathbf{x} \in \bigcup_{n>1} [0,1]^n$ the values of U are given by*

$$V(\mathbf{x}) = V(x_{(1)}, x_{(2)}, x_{(n-1)}, x_{(n)}),$$

then V is a 2-$\mathcal{SBF}([0,1])$.

Note that the class $\mathcal{V}_2^{(a)}$ of all 2-set-based extended nullnorms with annihilator $a \in]0,1[$ is φ-isomorphic to the set $\mathcal{T}_2 \times \mathcal{S}_2$, $\mathcal{T}_2$ being the class of all 2-set-based extended t-norms and $\mathcal{S}_2$ a class of all 2-set-based t-conorms, via $\varphi(V) = (T_*^{(a)}, S_*^{(a)})$. However, that is not the case of the class $\mathcal{U}_2^{(e)}$ of all 2-set-based extended uninorms with neutral element $e \in]0,1[$. Though to each $U \in \mathcal{U}_2^{(e)}$ we can assign the unique pair $(T_e^*, S_e^*) \in \mathcal{T}_2 \times \mathcal{S}_2$, this assignment is surjective but not injective. Indeed, for any $(T,S) \in \mathcal{T}_2 \times \mathcal{S}_2$ and $e \in]0,1[$, we have at least two uninorms $U_\wedge, U_\vee \in \mathcal{U}_2^{(e)}$, such that

$$(T_\wedge)^*_e = (T_\vee)^*_e = T \quad \text{and} \quad (S_\wedge)^*_e = (S_\vee)^*_e = S,$$

namely,

$$U_\wedge(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ x \wedge y & \text{otherwise,} \end{cases}$$

and

$$U_\vee(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ x \vee y & \text{otherwise.} \end{cases}$$

The proof that $U_\wedge$ and $U_\vee$ are uninorms can be found in [7].

As we have seen, in the case of extended t-norms, t-conorms, uninorms and nullnorms, their 2-set-based property is equivalent to the idempotency of their diagonal section, i.e., to the property $\delta \circ \delta = \delta$. In Figures 4.2, 4.2 we have visualized some possible diagonal sections for each of four discussed cases.

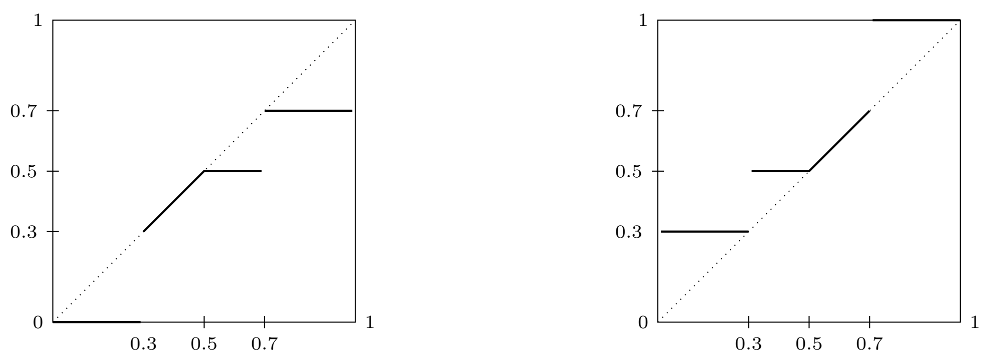

Figure 1: Possible diagonal sections of a 2-set-based t-norm (left) and a 2-set-based t-conorm (right)

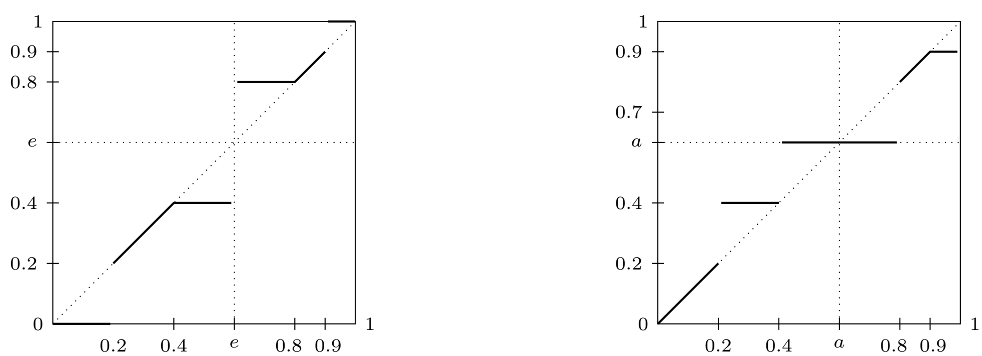

Figure 2: Possible diagonal sections of a 2-set-based uninorm (left) and a 2-set-based nullnorm (right)

4.3 2-set-based OWA operators

As already mentioned above, extended OWA operators [18] $\mathrm{OWA}_\Delta \colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ are determined by weighting triangles Δ of the form

$$\Delta = \left(w_{i,n} \mid n \in \mathbb{N}, i \in \{1,\ldots,n\}, w_{i,n} \geq 0, \sum_{i=1}^{n} w_{i,n} = 1 \right),$$

and given as follows:

$$\mathrm{OWA}_\Delta(\mathbf{x}) = \sum_{i=1}^{n} w_{i,n} x_{[i]},$$

where $\mathbf{x} = (x_1,\ldots,x_n) \in [0,1]^n$ and $[\cdot]$ is a permutation on $\{1,\ldots,n\}$ such that $x_{[1]} \geq \cdots \geq x_{[n]}$.

As shown in [13, 15], an extended OWA operator belongs to $\mathcal{SBF}([0,1])$ if and only if

$$\mathrm{OWA}_\Delta(\mathbf{x}) = \alpha Max(\mathbf{x}) + (1-\alpha)Min(\mathbf{x}) = \alpha x_{[1]} + (1-\alpha)x_{[n]},$$

i.e., if and only if its weighting triangle $\Delta = (w_{i,n})$ satisfies, for each $n > 1$ and $i \in \{1, \dots, n\}$, the properties $w_{1,n} = \alpha$, $w_{n,n} = 1 - \alpha$, otherwise $w_{i,n} = 0$.

The following theorem gives a complete characterization of 2-set-based extended OWA operators.

Theorem 4.12. *An extended OWA operator* OWA_Δ *is a 2-set-based aggregation function if and only if there exist* $\alpha_1, \dots, \alpha_4 \in [0,1]$, $\alpha_1 + \dots + \alpha_4 = 1$, *such that the weighting triangle* $\Delta = (w_{i,n})$ *satisfies the properties:*

(i) $(w_{1,3}, w_{2,3}, w_{3,3}) = (\alpha_1, \alpha_2 + \alpha_3, \alpha_4)$,

(ii) *for each* $n \geq 4$, $(w_{1,n}, w_{2,n}, \dots, w_{n,n}) = (\alpha_1, \alpha_2, 0, \dots, 0, \alpha_3, \alpha_4)$.

Proof. The sufficient condition follows from Theorem 4.1 when

$$B(x_1, \dots, x_4) = \alpha_1 x_4 + \alpha_2 x_3 + \alpha_3 x_2 + \alpha_4 x_1.$$

To prove the necessary condition, suppose that OWA_Δ is a 2-set-based aggregation function and put $\alpha_i = w_{i,4}$, $i = 1, \dots, 4$. For any $a, b \in [0,1]$, $a > b$, necessarily

$$\mathrm{OWA}_\Delta(a,a,a,b) = \mathrm{OWA}_\Delta(a,a,b) \quad \text{and} \quad \mathrm{OWA}_\Delta(a,b,b,b) = \mathrm{OWA}_\Delta(a,b,b),$$

which leads to the following equalities:

$$\begin{aligned} a(\alpha_1 + \alpha_2 + \alpha_3) + b\alpha_4 &= a(w_{1,3} + w_{2,3}) + bw_{3,3}, \\ a\alpha_1 + b(\alpha_2 + \alpha_3 + \alpha_4) &= aw_{1,3} + b(w_{2,3} + w_{3,3}). \end{aligned}$$

They are valid for all considered a, b only if $w_{3,3} = \alpha_4$, $w_{1,3} = \alpha_1$ and $w_{2,3} = \alpha_2 + \alpha_3$, which proves condition (i).

Further, from the 2-set-based property it also follows that necessarily

$$\mathrm{OWA}_\Delta(a,a,a,b,b) = \mathrm{OWA}_\Delta(a,a,b,b),$$

and

$$\mathrm{OWA}_\Delta(a,a,b,b,b) = \mathrm{OWA}_\Delta(a,a,b,b).$$

Thus

$$\begin{aligned} a(\alpha_1 + \alpha_2) + b(\alpha_3 + \alpha_4) &= a(w_{1,5} + w_{2,5} + w_{3,5}) + b(w_{4,5} + w_{5,5}), \\ a(\alpha_1 + \alpha_2) + b(\alpha_3 + \alpha_4) &= a(w_{1,5} + w_{2,5}) + b(w_{3,5} + w_{4,5} + w_{5,5}). \end{aligned}$$

These equalities are valid for all considered a, b only if

$$w_{3,5} = 0, \quad w_{1,5} + w_{2,5} = \alpha_1 + \alpha_2, \quad \text{and} \quad w_{4,5} + w_{5,5} = \alpha_3 + \alpha_4.$$

Finally, suppose that $1 \geq a > b > c \geq 0$. Then $\mathrm{OWA}_\Delta(a,a,a,b,c) = \mathrm{OWA}_\Delta(a,a,b,c)$ implies

$$b\alpha_3 + c\alpha_4 = bw_{4,5} + cw_{5,5},$$

which holds for all considered a, b, c only if $w_{4,5} = \alpha_3$ and $w_{5,5} = \alpha_4$.

Similarly, from the equality $\mathrm{OWA}_\Delta(a,b,c,c,c) = \mathrm{OWA}_\Delta(a,b,c,c)$ which has to be valid for all considered a, b, c, we obtain $w_{1,5} = \alpha_1$ and $w_{2,5} = \alpha_2$, and we can conclude that for $n = 5$,

$$(w_{1,5}, \dots, w_{5,5}) = (\alpha_1, \alpha_2, 0, \alpha_3, \alpha_4).$$

In the same way, it can be proved that for each $n > 5$, we obtain

$$(w_{1,n}, w_{2,n}, \dots, w_{n,n}) = (\alpha_1, \alpha_2, 0, \dots, 0, \alpha_3, \alpha_4),$$

so the necessity of (ii) has been proved. □

Remark 4.13. *Recall that n-ary OWA operators can be seen as the Choquet integrals with respect to a symmetric capacity $m\colon 2^{\{1,\ldots,n\}} \to [0,1]$. Then the 2-set-based extended OWA operator OWA_Δ which has been considered in Theorem 4.12 corresponds to an extended capacity $M\colon \bigcup_{n\in\mathbb{N}} 2^{\{1,\ldots,n\}} \to [0,1]$, where*

$$M|_{2^{\{1,\ldots,n\}}}(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ \alpha_1 & \text{if } \mathrm{card}(E) = 1, \\ \alpha_1+\alpha_2 & \text{if } \mathrm{card}(E) \in \{2,\ldots,n-2\}, \\ \alpha_1+\alpha_2+\alpha_3 & \text{if } \mathrm{card}(E) = n-1, \\ 1 & \text{if } \mathrm{card}(E) = n. \end{cases}$$

To introduce an integral-based 2-set-based extended aggregation function on $[0,1]$, we can consider this extended capacity M and any universal integral on $[0,1]$ [11], such as the Sugeno or Shilkret integrals. Then, for each $n \geq 4$, we have:
in the case of the Sugeno integral:
$Su_M(x_1,\ldots,x_n)$

$$= \max\{\min\{\alpha_1, x_{[1]}\}, \min\{\alpha_1+\alpha_2, x_{[2]}\}, \min\{\alpha_1+\alpha_2+\alpha_3, x_{[n-1]}\}, x_{[4]}\},$$

and in the case of the Shilkret integral:

$$Sh_M(x_1,\ldots,x_n) = \max\{\alpha_1\cdot x_{[1]}, (\alpha_1+\alpha_2)\cdot x_{[2]}, (\alpha_1+\alpha_2+\alpha_3)\cdot x_{[n-1]}, x_{[4]}\},$$

where $[\cdot]$ denotes a non-increasing permutation on $\{1,\ldots,n\}$, i.e. $x_{[1]} \geq \cdots \geq x_{[n]}$.

5 Concluding remarks

We have introduced and studied 2-set-based extended functions given on a general universe $X \neq \emptyset$, and in particular, we have focused our attention on 2-set-based extended aggregation functions acting on $[0,1]$. Theorem 4.1 shows a connection between 2-set-based extended aggregation functions and aggregation of outliers, namely aggregation of the two smallest and two greatest values of the considered samples $(x_1,\ldots,x_n)$. Though our work is rather theoretical, we expect possible applications in information sciences, in particular, in big data processing.

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