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Original paper

C-continuous fuzzy posets

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Abstract

In this paper, we develop the C-continuity via Scott Q-cotopological spaces, where Q is a commutative and integral quantale. The main results are: (1) The set of all C-compact elements of a complete Q-ordered set is irreducible complete; (2) The Scott Q-cotopology of a Q-ordered set under the fuzzy inclusion order is C-continuous, even C-prealgebraic; (3) We establish an adjunction between the category of irreducible complete Q-ordered sets and the category of C-prealgebraic complete Q-ordered sets.

Keywords: Category, Q-order, Scott Q-cotopology, C-continuity.

1 Introduction

In domain theory, there is an important result that a dcpo P is continuous if and only if the lattice $C(P)$ of all Scott-closed sets of P is completely distributive. In order to investigate the lattice of Scott-closed sets for a general poset P, Ho and Zhao introduced the notion of C-continuous poset and showed that P is continuous if and only if $C(P)$ is completely distributive by C-continuity (see [3]).

Developing topological spaces in quantale-valued order theory is an interesting topic. However, since the table of truth-values is usually a quantale, not a Boolean algebra, there is no natural way to switch between open sets and closed sets. So it may make a difference whether we postulate topological spaces in terms of open sets or in terms of closed sets. On one hand, many authors study fuzzy topological spaces in terms of open sets, such as Lowen [8] , Zhang and Liu [13], Kotzé [4, 5], Lai and Zhang [6], Yao [10, 12] and so on. On the other hand, Zhang (see [14]), Lai and Zhang (see [7]) study fuzzy topological spaces in terms of closed sets.

Scott topology is a key T_0 topology to connect order theory with topology. In the fuzzy setting, for a complete Heyting algebra L, Yao (see[10]) studied systemically fuzzy Scott topology on L-fuzzy dcpos. By using fuzzy Scott topology, Yao showed the category of fuzzy domains is dually equivalent to the category of strong completely distributive L-ordered sets (see [12]). Recently, making use of the fuzzy order between closed sets, for a commutative and integral quantale Q, Zhang [14] established a theory of fuzzy sobriety of Q-cotopological spaces based on irreducible closed sets. In [7], Lai and Zhang presented a comparative study of forward Cauchy ideals, flat ideals and irreducible ideals, and developed Scott Q-cotopological spaces, where Q is a commutative and unital quantale. In order to develop the methods of closed sets in fuzzy topological spaces, it is necessary to discuss the properties of fuzzy Scott closed sets. In this paper, motivated by the developments of Scott Q-cotopological spaces, we extend the beneath relation and the C-continuity (see [3]) to the quantale-valued setting.

2 Preliminaries

Throughout the paper, we refer to [1] for category theory, to [9] for quantale theory.

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Definition 2.1. [9] *A commutative and integral quantale is a triple $(Q, \&, \leq)$ such that $(Q, \leq)$ is a complete lattice with a bottom element 0 and a top element 1, $(Q, \&, 1)$ is a commutative monoid and $p\&(\bigvee_{j\in J} q_j) = \bigvee_{j\in J}(p\&q_j)$ for all $p \in Q$ and $\{q_j\}_{j\in J} \subseteq Q$.*

Since $p\&-$ preserves arbitrary sups, it has a right adjoint, which we shall denote by $p \to -$. Thus $p\&q \leq r \iff q \leq p \to r$ for all $p, q, r \in Q$.

From now on, unless otherwise stated, Q always denotes a commutative and integral quantale. In fact, a commutative and integral quantale just is a complete (commutative) residuated lattice. Let X be a set. Q^X denotes the set of all Q-subsets of X, that is, the set of all maps from X to Q. Clearly, Q^X is a complete lattice under the pointwise order.

Proposition 2.2. [9] *Let Q be a quantale. Then the following statements hold:*

(1) $1 \to p = p$;

(2) $p \leq q \iff 1 = p \to q$;

(3) $p \to (q \to r) = (p\&q) \to r$;

(4) $p\&(p \to q) \leq q$;

(5) $\Big(\bigvee_{j\in J} p_j\Big) \to q = \bigwedge_{j\in J} (p_j \to q)$;

(6) $p \to \Big(\bigwedge_{j\in J} q_j\Big) = \bigwedge_{j\in J} (p \to q_j)$.

Definition 2.3. [2] *Let X be a set. A map $R\colon X \times X \longrightarrow Q$ is called a Q-order on X if for all $x, y, z \in X$,*

(1) $R(x, x) = 1$ *(reflexivity);*

(2) $R(x, y) = R(y, x) = 1$ *implies* $x = y$ *(antisymmetry);*

(3) $R(x, y)\&R(y, z) \leq R(x, z)$ *(transitivity).*

The pair (X, R) is called a Q-ordered set. We often write simply X for a Q-ordered set (X, R) and $X(x, y)$ for $R(x, y)$ if no confusion would arise.

Let X be a set. The map $sub_X\colon Q^X \times Q^X \longrightarrow Q$ defined by $sub_X(A, B) = \bigwedge_{x\in X} (A(x) \to B(x))$ is called the *fuzzy inclusion order*. We can check that sub_X is a Q-order on Q^X. In particular, if X is a singleton set, then the Q-ordered set (Q^X, sub_X) reduces to the Q-ordered set (Q, R_Q), where $R_Q(p, q) = p \to q$. For all $p \in Q, A \in Q^X$, we write $p\&A$, $p \to A \in Q^X$ for the fuzzy sets given by $(p\&A)(x) = p\&A(x)$ and $(p \to A)(x) = p \to A(x)$, respectively. Let X be a Q-ordered set, $x_0 \in X$, $A \in Q^X$. Then x_0 is called the *supremum* (resp., *infimum*) of A, denoted by $x_0 = \sup A$(resp., $x_0 = \inf A$), if for all $y \in X$, $\bigwedge_{x\in X} (A(x) \to X(x, y)) = X(x_0, y)$(resp, $\bigwedge_{x\in X} (A(x) \to X(y, x)) = X(y, x_0)$). A Q-ordered set X is said to be complete, if $\sup A$ or $\inf A$ exists for all $A \in Q^X$.

A fuzzy upper (resp., lower) set of a Q-ordered set X is a map $\varphi\colon X \longrightarrow Q$ such that $X(x, y)\&\varphi(x) \leq \varphi(y)$ (resp., $X(x, y)\&\varphi(y) \leq \varphi(x)$) for all $x, y \in X$. A map $f\colon X \longrightarrow Y$ between Q-ordered sets is order-preserving if $X(x, y) \leq Y(f(x), f(y))$ for all $x,\ y \in X$. A pair of order-preserving maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ is called a Q-adjunction between Q-ordered sets X and Y, if $Y(f(x), y) = X(x, g(y))$ for all $x \in X,\ y \in Y$. In this case, f is called the left adjoint of g and dually g the right adjoint of f.

Let $X,\ Y$ be sets and $f\colon X \longrightarrow Y$ a map. Then the Zadeh forward power set operator $f^{\to}\colon Q^X \longrightarrow Q^Y$ and the Zadeh backward power set operator $f^{\leftarrow}\colon Q^Y \longrightarrow Q^X$ are defined, respectively, by

$$f^{\to}(A)(y) = \bigvee_{f(x)=y} A(x), \qquad f^{\leftarrow}(B) = B \circ f,$$

for all $A \in Q^X, y \in Y$ and $B \in Q^Y$. It can be easily seen that $(f^{\to}, f^{\leftarrow})$ is a Q-adjunction between (Q^X, sub_X) and (Q^Y, sub_Y).

Definition 2.4. [7] *A Q-cotopology on a set X is a subset τ of Q^X such that*

$(C1)$ $0_X, 1_X \in \tau$;

$(C2)$ $A \vee B \in \tau$ *for all* $A, B \in \tau$;

$(C3)$ $\bigwedge_{j\in J} A_j \in \tau$ *for each* $\{A_j\}_{j\in J} \subseteq \tau$.

The pair (X, τ) is called a Q-cotopological space, elements in τ are called closed sets of (X, τ). A Q-cotopology τ is stratified if it also satisfies the condition: $p \to A \in \tau$ for all $p \in Q, A \in \tau$.

We often write X for a Q-cotopological space (X, τ). A map $f\colon X \longrightarrow Y$ between Q-cotopological spaces is continuous if $f^{\leftarrow}(A) = A \circ f$ is closed in X whenever A is closed in Y.

Given a Q-cotopological space (X, τ), its closure operator $^- : Q^X \longrightarrow Q^X$ is defined by

$$\overline{A} = \bigwedge \{B \in \tau \mid A \leq B\},$$

for all $A \in Q^X$. Then X is stratified if and only if the closure operator $^- : (Q^X, sub_X) \longrightarrow (Q^X, sub_X)$ is order-preserving (see [14]).

3 C-continuity via Scott Q-cotopological spaces

Definition 3.1. [7] *Let X be a Q-ordered set and $\phi\colon X \longrightarrow Q$ a fuzzy lower set of X. ϕ is an irreducible ideal if ϕ satisfies that $\bigvee_{x\in X} \phi(x) = 1$ and ϕ is irreducible in the sense that $sub_X(\phi, \phi_1 \vee \phi_2) = sub_X(\phi, \phi_1) \vee sub_X(\phi, \phi_2)$ for all lower sets ϕ_1, ϕ_2 of X.*

Remark 3.2. *A fuzzy set $\phi\colon X \longrightarrow Q$ is inhabited in* [7] *if $\bigvee_{x\in X} \phi(x) = 1$. Inhabited fuzzy sets can be seen as a fuzzy version of non-empty subsets.*

We denote the set of irreducible ideals by $\mathcal{I}(X)$. A Q-ordered set X is irreducible complete if each irreducible ideal of X has a supremum. A order-preserving map $f\colon X \longrightarrow Y$ is fuzzy Scott continuous if $f(\sup \phi) = \sup f^{\rightarrow}(\phi)$ for all $\phi \in \mathcal{I}(X)$ whenever $\sup \phi$ exists. Let **FICPO** denote the category of irreducible complete Q-ordered sets with fuzzy Scott continuous maps.

Example 3.3. *Let X be a Q-ordered set. For each x in X, $\downarrow x = X(-, x)$ is an irreducible ideal of X.*

Definition 3.4. [7] *Let X be a Q-ordered set. A fuzzy set $\psi\colon X \longrightarrow Q$ of X is $\mathcal{I}$-closed if it is a fuzzy lower set and $sub_X(\phi, \psi) \leq \psi(\sup \phi)$ for all $\phi \in \mathcal{I}(X)$ whenever $\sup \phi$ exists.*

Proposition 3.5. [7] *Let X be a Q-ordered set. Then the set $\sigma^{co}_{\mathcal{I}}(X)$ of $\mathcal{I}$-closed fuzzy sets of X is a stratified Q-cotopology on X, called the Scott Q-cotopology.*

Let $\sigma^{co^+}_{\mathcal{I}}(X)$ denote the set of all inhabited $\mathcal{I}$-closed fuzzy sets of X for which sup exists.

Definition 3.6. *Let X be a Q-ordered set. For each x in X, we define a map $\Downarrow x : X \longrightarrow Q$ as follows:*

$$\forall\, y \in X, \quad \Downarrow x(y) = \bigwedge_{\phi \in \sigma^{co^+}_{\mathcal{I}}(X)} (X(x, \sup \phi) \to \phi(y)).$$

We say that $\Downarrow\colon X \times X \longrightarrow Q$ is a fuzzy beneath relation on X. A map $f\colon X \longrightarrow Y$ between Q-ordered sets preserves the fuzzy beneath relation if $\Downarrow x(y) \leq \Downarrow f(x)(f(y))$ for all $x, y \in X$.

The fuzzy beneath relation is different from the underlying fuzzy order in general. Please see the following example.

Example 3.7. *Let $Q = \{0, \frac{1}{2}, 1\}$ with the usual order. Define a binary operation & on Q as follows:*

&	0	½	1
0	0	0	0
½	0	0	½
1	0	½	1

Then $(Q, \&)$ is a commutative and integral quantale. Let $X = \{a, b, c, d, e\}$. The order $\leq$ on X is defined as follows:

$$a \leq b \leq d \leq e; a \leq c \leq d \leq e.$$

Then $(X, \leq)$ is a poset. We define $R : X \times X \longrightarrow Q$ by $R(x, y) = 1$ when $x \leq y$, otherwise $R(x, y) = 0$. Clearly, R is a Q-order on X. Moreover, $\mathcal{I}(X) = \{\downarrow x \mid x \in X\}$. Next we show that $\Downarrow x \neq \downarrow x$ for some $x \in X$. Define a map $\phi\colon X \longrightarrow Q$ by

$$\phi(a) = \phi(b) = \phi(c) = 1; \phi(d) = \frac{1}{2}; \phi(e) = 0.$$

Then ϕ is an inhabited $\mathcal{I}$-closed fuzzy set of X and $\sup \phi = d$. Thus $\Downarrow d(d) \leq X(d, \sup \phi) \to \phi(d) = \frac{1}{2}$. Since $\downarrow d(d) = 1$, we have that $\downarrow d \neq \Downarrow d$.

Proposition 3.8. *Let X be a Q-ordered set. Then the following statements hold:*
(1) $\Downarrow x \leq \downarrow x$;
(2) $\downarrow x(u) \& \Downarrow y(x) \& \downarrow v(y) \leq \Downarrow v(u)$;
(3) $\bigwedge_{z \in X} \downarrow z(x) \leq \Downarrow y(x)$.

Proposition 3.9. *Let X be a Q-ordered set. For each a in X, then $\Downarrow a \in \sigma_{\mathcal{I}}^{co}(X)$.*

Proof. By Proposition 3.8(2), $\Downarrow a$ is a fuzzy lower set. For each $D \in \mathcal{I}(X)$ with $\sup D$ existing, $\phi \in \sigma_{\mathcal{I}}^{co^+}(X)$ and $y \in X$, we have that

$$\begin{aligned} & D(y) \& sub_X(D, \Downarrow a) \& X(a, \sup \phi) \\ \leq\ & D(y) \& (D(y) \to \Downarrow a(y)) \& X(a, \sup \phi) \\ \leq\ & \Downarrow a(y) \& X(a, \sup \phi) \\ \leq\ & \phi(y). \end{aligned}$$

Then $sub_X(D, \Downarrow a) \& X(a, \sup \phi) \leq sub_X(D, \phi) \leq \phi(\sup D)$. Therefore,

$$sub_X(D, \Downarrow a) \leq \bigwedge_{\phi \in \sigma_{\mathcal{I}}^{co^+}(X)} (X(a, \sup \phi) \to \phi(\sup D)) = \Downarrow a(\sup D).$$

□

Proposition 3.10. *Let X be a complete Q-ordered set. For each a in X, then $\Downarrow a \in \sigma_{\mathcal{I}}^{co^+}(X)$.*

Proof. Since X is complete, $\inf 1_X$ exists. For all $y \in X$, $1 = 1_X(y) \leq X(\inf 1_X, y)$, then $X(\inf 1_X, y) = 1$. For any $\phi \in \sigma_{\mathcal{I}}^{co^+}(X)$, we have $\bigvee_{t \in X} \phi(t) = 1$. Then

$$1 = \bigvee_{t \in X} \phi(t) = \bigvee_{t \in X} \Big(X(\inf 1_X, t) \& \phi(t) \Big) \leq \phi(\inf 1_X).$$

Thus, for any $a \in X$,

$$\bigvee_{t \in X} \Downarrow a(t) \geq \Downarrow a(\inf 1_X) = \bigwedge_{\phi \in \sigma_{\mathcal{I}}^{co^+}(X)} (X(a, \sup \phi) \to \phi(\inf 1_X)) = 1.$$

By Proposition 3.9, we can conclude that $\Downarrow a \in \sigma_{\mathcal{I}}^{co^+}(X)$. □

Definition 3.11. *A Q-ordered set X is said to be C-continuous if for each x in X, $x = \sup \Downarrow x$.*

Remark 3.12. *When $Q = \mathbf{2} = (\{0, 1\}, \wedge, \leq)$, a C-continuous Q-ordered set is just a C-continuous poset defined in* [3].

Example 3.13. *The Q-ordered set (Q, R_Q) is C-continuous, where $R_Q(x, y) = x \to y$ for all $x, y \in Q$. In fact, for each fuzzy lower set ϕ of (Q, R_Q), we have:*
(1) *$\phi \colon Q \longrightarrow Q$ is decreasing;*
(2) *For all $x \in Q$, since $\phi(x) \& R_Q(1, x) \leq \phi(1)$, $x \& \phi(x) \leq \phi(1)$.*
Then, $\sup \phi = \bigvee_{x \in Q} x \& \phi(x) = \phi(1)$. By Proposition 3.8(1), for any $x \in Q$, $\sup \Downarrow x \leq \sup \downarrow x = x$. For all $\psi \in \sigma_{\mathcal{I}}^{co^+}(Q)$, we have that

$$\Downarrow x(1) = \bigwedge_{\psi \in \sigma_{\mathcal{I}}^{co^+}(Q)} (R_Q(x, \sup \psi) \to \psi(1)) = \bigwedge_{\psi \in \sigma_{\mathcal{I}}^{co^+}(Q)} ((x \to \psi(1)) \to \psi(1)) \geq x.$$

Then, $x \leq \Downarrow x(1) = \sup \Downarrow x$. So (Q, R_Q) is C-continuous.

Let Φ be a class of weights. A Q-ordered set X is Φ-continuous (see [7]) if it is Φ-complete and the left adjoint $\sup : \Phi(X) \longrightarrow X$ of $\eta : X \longrightarrow \Phi(X)$ defined by $\eta(x) = \downarrow x$ for x in X has a left adjoint. In particular, if we consider $\Phi = \mathcal{I}(X)$, then the Q-ordered set X is irreducible continuous if and only if X is irreducible complete and the left adjoint $\sup : \mathcal{I}(X) \longrightarrow X$ of $\eta : X \longrightarrow \mathcal{I}(X)$ has a left adjoint.

Remark 3.14. *When $Q = \mathbf{2} = (\{0,1\}, \wedge, \leq)$, the irreducible continuity is just the continuity in the classical setting. There exists a C-continuous but not irreducible continuous poset. For example, let $X = \mathbb{N} \cup \{\top\} \cup \{b\}$, where $\mathbb{N}$ is the set of natural numbers with its ordinary order. The order $\leq$ on X is defined as follows:*

$$\forall\ n \in \mathbb{N}, n \leq \top;\ 0 \leq b \leq \top.$$

Then $(\Gamma(X), \subseteq)$ is C-continuous but not irreducible continuous, where $\Gamma(X)$ denotes the set of all Scott-closed sets of X. Moreover, there exits an irreducible continuous but not C-continuous Q-ordered set. Please see the following example.

Example 3.15. *Let Q be the quantale as in Example 3.7. Let $X = \{a, b, c, d, e\}$. The order $\leq$ on X is defined as follows:*

$$a \leq b \leq e; a \leq c \leq e; a \leq d \leq e.$$

Then $(X, \leq)$ is a poset. We define $R : X \times X \longrightarrow Q$ by $R(x, y) = 1$ when $x \leq y$, otherwise $R(x, y) = 0$. Clearly, R is a Q-order on X. Then X is irreducible continuous but not C-continuous. Obviously, $\mathcal{I}(X) = \{\downarrow x \mid x \in X\}$. Then X is irreducible complete. $\forall \phi \in \mathcal{I}(X)$, $\forall\ x \in X$, there exists $y \in X$ such that $\phi = \downarrow y$. Then

$$sub_X(\eta(x), \phi) = sub_X(\downarrow x, \downarrow y) = X(x, y) = X(x, \sup \phi).$$

Thus $\eta : X \longrightarrow \mathcal{I}(X)$ is the left adjoint of $\sup : \mathcal{I}(X) \longrightarrow X$. Therefore, X is irreducible continuous. However, X is not C-continuous. Indeed, $\forall \psi \in \sigma_{\mathcal{I}}^{co^+}(X)$, ψ is an inhabited and fuzzy lower set of X. Then $\psi(a) = 1$. Thus,

$$\Downarrow b(a) = \bigwedge_{\psi \in \sigma_{\mathcal{I}}^{co^+}(X)} (X(b, \sup \psi) \to \psi(a)) = 1.$$

Since $\Downarrow b(c) \leq \downarrow b(c) = 0$, then $\Downarrow b(c) = 0$. Similarly, $\Downarrow b(x) = 0$ for $x = d, e$. Define a mapping $\varphi : X \longrightarrow Q$ by

$$\varphi(a) = \varphi(c) = \varphi(d) = 1; \varphi(b) = \varphi(e) = 0.$$

Then φ is an inhabited $\mathcal{I}$-closed fuzzy set of X and $\sup \varphi = e$. Thus $\Downarrow b(b) \leq X(b, \sup \varphi) \to \varphi(b) = 0$. Assume that $\sup \Downarrow b = b$. Then, $\forall\ y \in X$, $X(b, y) = \bigwedge_{x \in X} (\Downarrow b(x) \to X(x, y))$. Take $y = a$. Then $X(b, a) = 0$ and $\bigwedge_{x \in X} (\Downarrow b(x) \to X(x, a)) = 1$, which is a contradiction. So we conclude that X is not C-continuous.

Theorem 3.16. *The complete Q-ordered set X is C-continuous if and only if $(\Downarrow, \sup)$ is a Q-adjunction between $(X, X(-,-))$ and $(\sigma_{\mathcal{I}}^{co^+}(X), sub_X)$.*

Proof. **Sufficiency**. By Proposition 3.10, $\Downarrow x \in \sigma_{\mathcal{I}}^{co^+}(X)$. Since $(\Downarrow, \sup)$ is a Q-adjunction, $X(x, \sup \Downarrow x) = 1$. From Proposition 3.8(1), $1 = sub_X(\Downarrow x, \downarrow x) \leq X(\sup \Downarrow x, x)$, then $x = \sup \Downarrow x$. So X is C-continuous.
Necessity. For any $x,\ y \in X$, we have that

$$\begin{aligned}
& sub_X(\Downarrow x, \Downarrow y) \\
= & \bigwedge_{z \in X} (\Downarrow x(z) \to \Downarrow y(z)) \\
\geq & \bigwedge_{z \in X} \bigwedge_{\psi \in \sigma_{\mathcal{I}}^{co^+}(X)} \Big((X(x, \sup \psi) \to \psi(z)) \to (X(y, \sup \psi) \to \psi(z)) \Big) \\
\geq & \bigwedge_{\psi \in \sigma_{\mathcal{I}}^{co^+}(X)} (X(y, \sup \psi) \to X(x, \sup \psi)) \\
\geq & X(x, y).
\end{aligned}$$

Then $\Downarrow$ is order-preserving. For all $\psi, \phi \in \sigma_{\mathcal{I}}^{co^+}(X)$,

$$\begin{aligned}
X(\sup \psi, \sup \phi) &= \bigwedge_{x \in X} (\psi(x) \to X(x, \sup \phi)) \\
&\geq \bigwedge_{x \in X} (\psi(x) \to \phi(x)) \\
&= sub_X(\psi, \phi).
\end{aligned}$$

Then sup is order-preserving. For all $x \in X$ and $\phi \in \sigma_{\mathcal{I}}^{co^+}(X)$,

$$\begin{aligned} sub_X(\Downarrow x, \phi) &= \bigwedge_{y\in X} (\Downarrow x(y) \to \phi(y)) \\ &\geq \bigwedge_{y\in X} ((X(x, \sup\phi) \to \phi(y)) \to \phi(y)) \\ &\geq X(x, \sup\phi). \end{aligned}$$

Since X is C-continuous,

$$\begin{aligned} X(x, \sup\phi) &= X(\sup \Downarrow x, \sup\phi) \\ &= \bigwedge_{y\in X} (\Downarrow x(y) \to X(y, \sup\phi)) \\ &\geq \bigwedge_{y\in X} (\Downarrow x(y) \to \phi(y)) \\ &= sub_X(\Downarrow x, \phi). \end{aligned}$$

Then $(\Downarrow, \sup)$ is a Q-adjunction between $(X, X(-,-))$ and $(\sigma_{\mathcal{I}}^{co^+}(X), sub_X)$. □

Definition 3.17. *An element x of a Q-ordered set X is called C-compact if $\Downarrow x(x) = 1$.*

We use $K(X)$ to denote the set of all C-compact elements of X.

Remark 3.18. *Let X be a complete Q-ordered set. Then $K(X) \neq \emptyset$. Indeed, it follows from the proof of Proposition 3.10 that $\phi(\inf 1_X) = 1$ for each $\phi \in \sigma_{\mathcal{I}}^{co^+}(X)$. Then $\Downarrow \inf 1_X(\inf 1_X) = 1$, which implies that $\inf 1_X$ is a C-compact element of X.*

Proposition 3.19. *Let X be a complete Q-ordered set. Then $K(X)$ is irreducible complete.*

Proof. Suppose that D is an irreducible ideal of $K(X)$. Since X is complete, $\sup D$ exists. We first show that $\downarrow D$ is an irreducible ideal of X. Clearly, $\downarrow D$ is a fuzzy lower set of X. Let A, B be fuzzy lower sets of X,

$$\begin{aligned} & sub_X(\downarrow D, A \vee B) \\ =\ & \bigwedge_{x\in X} (\downarrow D(x) \to (A \vee B)(x)) \\ =\ & \bigwedge_{x\in X} \Big(\Big(\bigvee_{a\in K(X)} D(a) \& X(x,a)\Big) \to (A\vee B)(x)\Big) \\ =\ & \bigwedge_{a\in K(X)} \Big(D(a) \to \Big(\bigwedge_{x\in X} (X(x,a) \to (A \vee B)(x))\Big)\Big) \\ =\ & \bigwedge_{a\in K(X)} (D(a) \to sub_X(\downarrow a, A \vee B)) \\ =\ & \bigwedge_{a\in K(X)} (D(a) \to (A\vee B)(a)) \\ =\ & sub_{K(X)}(D, A\vee B) \\ =\ & sub_{K(X)}(D, A) \vee sub_{K(X)}(D, B) \\ =\ & sub_X(\downarrow D, A) \vee sub_X(\downarrow D, B). \end{aligned}$$

Next, we shall prove that $\sup D \in K(X)$. For all $E \in \sigma_{\mathcal{I}}^{co^+}(X)$ and $x \in X$,

$$\begin{aligned} & \downarrow D(x) \& X(\sup D, \sup E) \\ =\ & \bigvee_{a\in K(X)} D(a) \& X(x,a) \& X(\sup D, \sup E) \\ \leq\ & \bigvee_{a\in K(X)} X(a, \sup D) \& X(\sup D, \sup E) \& X(x,a) \\ \leq\ & \bigvee_{a\in K(X)} X(a, \sup E) \& X(x,a) \\ \leq\ & X(a, \sup E) \& \Downarrow a(x) \\ \leq\ & E(x). \end{aligned}$$

Then

$$X(\sup D, \sup E) \leq sub_X(\downarrow D, E) \leq E(\sup\downarrow D) = E(\sup D).$$

Therefore,

$$\Downarrow \sup D(\sup D) = \bigwedge_{E\in {\sigma_{\mathcal{I}}^{co}}^{+}(X)} (X(\sup D, \sup E) \to E(\sup D)) = 1.$$

So, $\sup_{K(X)} D = \sup D$, which means that $K(X)$ is irreducible complete. □

Proposition 3.20. *Let X be a Q-ordered set and $\Psi \in \sigma_{\mathcal{I}}^{co}(\sigma_{\mathcal{I}}^{co}(X))$. Then $\sup \Psi = \bigvee_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(F)\&F$.*

Proof. (1) $\bigvee_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(F)\&F$ is $\mathcal{I}$-closed.

Since F is a fuzzy lower set, it is easy to check that $\bigvee_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(F)\&F$ is a fuzzy lower set. Let D be an irreducible ideal of X with $\sup D$ existing. Define $\Psi_D \colon \sigma_{\mathcal{I}}^{co}(X) \longrightarrow Q$ as follows:

$$\forall\ A \in \sigma_{\mathcal{I}}^{co}(X), \quad \Psi_D(A) = \begin{cases} D(x), & \exists\ x \in X, s.t.\ A = \downarrow x, \\ 0, & otherwise. \end{cases}$$

We claim that $\downarrow\Psi_D$ is an irreducible ideal of $\sigma_{\mathcal{I}}^{co}(X)$. Let $\mathcal{A}, \mathcal{B}$ be fuzzy lower sets of $\sigma_{\mathcal{I}}^{co}(X)$. Then

$$\begin{aligned} & sub_{\sigma_{\mathcal{I}}^{co}(X)}(\downarrow\Psi_D, \mathcal{A}\vee\mathcal{B}) \\ = \; & sub_{\sigma_{\mathcal{I}}^{co}(X)}(\Psi_D, \mathcal{A}\vee\mathcal{B}) \\ = \; & \bigwedge_{A\in\sigma_{\mathcal{I}}^{co}(X)} (\Psi_D(A) \to (\mathcal{A}\vee\mathcal{B})(A)) \\ = \; & \bigwedge_{x\in X} (D(x) \to (\mathcal{A}\vee\mathcal{B})(\downarrow x)) \\ = \; & \bigwedge_{x\in X} (D(x) \to \mathcal{A}(\downarrow x) \vee \mathcal{B}(\downarrow x)) \\ = \; & \bigwedge_{x\in X} \Big(D(x) \to \Big(\bigvee_{\phi\in\sigma_{\mathcal{I}}^{co}(X)} \mathcal{A}(\phi)\&\phi(x)\Big) \vee \Big(\bigvee_{\varphi\in\sigma_{\mathcal{I}}^{co}(X)} \mathcal{B}(\varphi)\&\varphi(x)\Big)\Big) \\ = \; & sub_X(D, (\bigvee_{\phi\in\sigma_{\mathcal{I}}^{co}(X)} \mathcal{A}(\phi)\&\phi) \vee (\bigvee_{\varphi\in\sigma_{\mathcal{I}}^{co}(X)} \mathcal{B}(\varphi)\&\varphi)) \\ = \; & sub_X(D, \bigvee_{\phi\in\sigma_{\mathcal{I}}^{co}(X)} \mathcal{A}(\phi)\&\phi) \vee sub_X(D, \bigvee_{\varphi\in\sigma_{\mathcal{I}}^{co}(X)} \mathcal{B}(\varphi)\&\varphi) \\ = \; & sub_{\sigma_{\mathcal{I}}^{co}(X)}(\Psi_D, \mathcal{A}) \vee sub_{\sigma_{\mathcal{I}}^{co}(X)}(\Psi_D, \mathcal{B}) \\ = \; & sub_{\sigma_{\mathcal{I}}^{co}(X)}(\downarrow\Psi_D, \mathcal{A}) \vee sub_{\sigma_{\mathcal{I}}^{co}(X)}(\downarrow\Psi_D, \mathcal{B}). \end{aligned}$$

Next, we shall show that $\sup \Psi_D = \downarrow \sup D$. For all $A \in \sigma_{\mathcal{I}}^{co}(X)$,

$$\bigwedge_{B\in\sigma_{\mathcal{I}}^{co}(X)} (\Psi_D(B) \to sub_X(B, A)) = \bigwedge_{x\in X} (D(x) \to sub_X(\downarrow x, A)) = sub_X(D, A) = sub_X(\downarrow \sup D, A).$$

Then $\sup \Psi_D = \downarrow \sup D$. Thus,

$$\begin{aligned} & sub_X(D, \bigvee_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(F)\&F) \\ = \; & \bigwedge_{d\in X} \Big(D(d) \to \Big(\bigvee_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(F)\&F(d)\Big)\Big) \\ = \; & \bigwedge_{d\in X} \Big(D(d) \to \Big(\bigvee_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(F)\&sub_X(\downarrow d, F)\Big)\Big) \\ \leq \; & \bigwedge_{d\in X} (D(d) \to \Psi(\downarrow d)) \\ = \; & \bigwedge_{A\in\sigma_{\mathcal{I}}^{co}(X)} (\Psi_D(A) \to \Psi(A)) \\ = \; & sub_{\sigma_{\mathcal{I}}^{co}(X)}(\Psi_D, \Psi) \end{aligned}$$

$$
\begin{aligned}
&= sub_{\sigma_{\mathcal{I}}^{co}(X)}(\downarrow \Psi_D, \Psi) \\
&\leq \Psi(\downarrow \sup D) \\
&\leq \bigvee_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(F)\&F(\sup D).
\end{aligned}
$$

(2) $\sup\Psi = \bigvee_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(F)\&F$. For all $\phi\in\sigma_{\mathcal{I}}^{co}(X)$,

$$
\begin{aligned}
&\bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)} (\Psi(F)\to sub_X(F,\phi)) \\
=& \bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Big(\Psi(F)\to\Big(\bigwedge_{y\in X}(F(y)\to\phi(y))\Big)\Big) \\
=& \bigwedge_{y\in X}\bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)} (\Psi(F)\to(F(y)\to\phi(y))) \\
=& \bigwedge_{y\in X}\Big(\Big(\bigvee_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(F)\&F(y)\Big)\to\phi(y)\Big) \\
=& sub_X(\bigvee_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(F)\&F, \phi).
\end{aligned}
$$

□

Proposition 3.21. *Let X be a Q-ordered set. Then $F(x)\leq \Downarrow F(\downarrow x)$ for all $F\in\sigma_{\mathcal{I}}^{co}(X)$ and $x\in X$.*

Proof. For all $\Psi\in{\sigma_{\mathcal{I}}^{co}}^{+}(\sigma_{\mathcal{I}}^{co}(X))$, we have that

$$
\begin{aligned}
& sub_X(F,\sup\Psi)\&F(x) \\
=& \Big(\bigwedge_{y\in X}(F(y)\to\sup\Psi(y))\Big)\&F(x) \\
\leq& (F(x)\to\sup\Psi(x))\&F(x) \\
\leq& \sup\Psi(x) \\
=& \bigvee_{C\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(C)\&C(x) \\
=& \bigvee_{C\in\sigma_{\mathcal{I}}^{co}(X)} \Psi(C)\&sub_X(\downarrow x, C) \\
\leq& \Psi(\downarrow x).
\end{aligned}
$$

Thus, $F(x)\leq \bigwedge_{\Psi\in{\sigma_{\mathcal{I}}^{co}}^{+}(\sigma_{\mathcal{I}}^{co}(X))} (sub_X(F,\sup\Psi)\to\Psi(\downarrow x)) = \Downarrow F(\downarrow x)$. □

Corollary 3.22. *Let X be a Q-ordered set. Then $\downarrow x\in K(\sigma_{\mathcal{I}}^{co}(X))$ for each $x\in X$.*

Definition 3.23. *Let X be a Q-ordered set, $a\in X$. Define $k_a\colon X\longrightarrow Q$ as follows:*

$$
\forall\, x\in X,\quad k_a(x)=\begin{cases} X(x,a), & x\in K(X),\\ 0, & \textit{otherwise}.\end{cases}
$$

A Q-ordered set X is said to be C-prealgebraic if for each $a\in X$, $\sup k_a = a$.

Clearly, every C-prealgebraic Q-ordered set is C-continuous.

Theorem 3.24. *Let X be a Q-ordered set. Then $(\sigma_{\mathcal{I}}^{co}(X), sub_X)$ is a C-prealgebraic complete Q-ordered set.*

Proof. For all $\mathcal{D}\in Q^{\sigma_{\mathcal{I}}^{co}(X)}$, it is easy to check that

$$
\inf\mathcal{D}=\bigwedge_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\to A,\quad \sup\mathcal{D}=\overline{\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\&A},
$$

where $\overline{\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\&A}$ denotes the closure of $\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\&A$ in $\sigma_{\mathcal{I}}^{co}(X)$. Then $(\sigma_{\mathcal{I}}^{co}(X), sub_X)$ is complete. Next we shall show that $(\sigma_{\mathcal{I}}^{co}(X), sub_X)$ is C-prealgebraic. For each $E\in\sigma_{\mathcal{I}}^{co}(X)$, it suffices to show that $\sup k_E = E$. Then for all $B\in\sigma_{\mathcal{I}}^{co}(X)$, we need to prove that

$$sub_X(E,B)=\bigwedge_{F\in K(\sigma_{\mathcal{I}}^{co}(X))}(k_E(F)\to sub_X(F,B)).$$

Clearly, $sub_X(E,B)\le \bigwedge_{F\in K(\sigma_{\mathcal{I}}^{co}(X))}(k_E(F)\to sub_X(F,B))$. For all $x\in X$,

$$\bigwedge_{F\in K(\sigma_{\mathcal{I}}^{co}(X))}(k_E(F)\to sub_X(F,B))\le k_E(\downarrow x)\to sub_X(\downarrow x,B)=E(x)\to B(x).$$

Then $\bigwedge_{F\in K(\sigma_{\mathcal{I}}^{co}(X))}(k_E(F)\to sub_X(F,B))\le sub_X(E,B)$, and thus $\bigwedge_{F\in K(\sigma_{\mathcal{I}}^{co}(X))}(k_E(F)\to sub_X(F,B))=sub_X(E,B)$. □

4 An adjunction between FICPO and FCPAlg

Let **FCPAlg** denote the category whose objects are C-prealgebraic complete Q-ordered sets and morphisms are the left adjoints which preserve the fuzzy beneath relation. In this section, we establish an adjunction between **FICPO** and **FCPAlg**.

By Proposition 3.19, for each C-prealgebraic complete Q-ordered set L, $K(L)$ is irreducible complete. Let $f\colon L\longrightarrow M$ be a morphism in **FCPALg**, the restriction $K(f)\colon K(L)\longrightarrow K(M)$ of f in $K(L)$ is a morphism in **FICPO**. Then we can build a functor $K\colon$ **FCPALg**$\longrightarrow$**FICPO**.

Proposition 4.1. *Let (d,g) be a Q-adjunction between Q-ordered sets X and Y, where $d\colon X\longrightarrow Y$ and $g\colon Y\longrightarrow X$. Then $\downarrow g^{\to}(F)\in\sigma_{\mathcal{I}}^{co}(X)$ for each $F\in\sigma_{\mathcal{I}}^{co}(Y)$.*

Proof. Clearly, $\downarrow g^{\to}(F)$ is a fuzzy lower set. Let D be an irreducible ideal of X with $\sup D$ existing. We can check that $\downarrow d^{\to}(D)\in\mathcal{I}(Y)$. Since,

$$\begin{aligned}
sub_X(D,\downarrow g^{\to}(F)) &= \bigwedge_{x\in X}(D(x)\to\downarrow g^{\to}(F)(x))\\
&= \bigwedge_{x\in X}\Big(D(x)\to\Big(\bigvee_{z\in X}\bigvee_{g(y)=z}(F(y)\&X(x,z))\Big)\Big)\\
&= \bigwedge_{x\in X}\Big(D(x)\to\Big(\bigvee_{y\in Y}(F(y)\&X(x,g(y)))\Big)\Big)\\
&= \bigwedge_{x\in X}\Big(D(x)\to\Big(\bigvee_{y\in Y}(F(y)\&Y(d(x),y))\Big)\Big)\\
&\le \bigwedge_{x\in X}(D(x)\to F(d(x)))\\
&= sub_X(D,d^{\leftarrow}(F))\\
&= sub_Y(d^{\to}(D),F)\\
&= sub_Y(\downarrow d^{\to}(D),F)\\
&\le F(\sup(\downarrow d^{\to}(D)))\\
&= F(d(\sup D))\\
&\le \downarrow g^{\to}(F)(gd(\sup D))\\
&\le \downarrow g^{\to}(F)(\sup D).
\end{aligned}$$

So, $\downarrow g^{\to}(F)\in\sigma_{\mathcal{I}}^{co}(X)$. □

Remark 4.2. In Proposition 4.1, if F is inhabited, then $\downarrow g^{\to}(F)$ is inhabited.

Proposition 4.3. *Let (d,g) be a Q-adjunction between complete Q-ordered sets X and Y, where $d\colon X\longrightarrow Y$ and $g\colon Y\longrightarrow X$. If g preserves sups of inhabited $\mathcal{I}$-closed fuzzy sets, then d preserves the fuzzy beneath relation. Furthermore, if X is C-continuous, then the converse conclusion is also true.*

Proof. (1) For all $F\in\sigma_{\mathcal{I}}^{co^+}(Y)$ and $x,\ y\in X$, we have that

$$
\begin{aligned}
& Y(d(y),\sup F)\& \Downarrow y(x) \\
= \; & Y(d(y),\sup F)\& \Big(\bigwedge_{E\in \sigma_{\mathcal{I}}^{co^+}(X)} (X(y,\sup E)\to E(x))\Big) \\
\leq \; & X(y, g(\sup F))\&(X(y,\sup \downarrow g^{\to}(F)) \to \downarrow g^{\to}(F)(x)) \\
= \; & X(y, \sup g^{\to}(F))\&(X(y,\sup g^{\to}(F)) \to \downarrow g^{\to}(F)(x)) \\
\leq \; & \downarrow g^{\to}(F)(x) \\
= \; & \bigvee_{z\in X}\bigvee_{g(c)=z} F(c)\& X(x,z) \\
= \; & \bigvee_{c\in Y} F(c)\& X(x,g(c)) \\
= \; & \bigvee_{c\in Y} F(c)\& Y(d(x),c) \\
\leq \; & F(d(x)).
\end{aligned}
$$

Then $\Downarrow y(x) \leq \bigwedge_{F\in \sigma_{\mathcal{I}}^{co^+}(Y)} (Y(d(y),\sup F)\to F(d(x))) = \Downarrow d(y)(d(x))$.

(2) Conversely, for all $F \in \sigma_{\mathcal{I}}^{co^+}(Y)$, it suffices to prove that $g(\sup F) = \sup g^{\to}(F)$. So we need to prove that $X(g(\sup F),x) = \bigwedge_{z\in X}(g^{\to}(F)(z)\to X(z,x))$ for each $x\in X$.

$$
\begin{aligned}
& \bigwedge_{z\in X}(g^{\to}(F)(z)\to X(z,x)) \\
= \; & \bigwedge_{z\in X}\bigwedge_{g(y)=z}(F(y)\to X(z,x)) \\
= \; & \bigwedge_{y\in Y}(F(y)\to X(g(y),x)) \\
\geq \; & \bigwedge_{y\in Y}(Y(y,\sup F)\to X(g(y),x)) \\
\geq \; & \bigwedge_{y\in Y}(X(g(y),g(\sup F))\to X(g(y),x)) \\
\geq \; & X(g(\sup F),x).
\end{aligned}
$$

On the other hand, since X is C-continuous,

$$
\begin{aligned}
& X(g(\sup F),x) \\
= \; & X(\sup \Downarrow g(\sup F),x) \\
= \; & \bigwedge_{z\in X}(\Downarrow g(\sup F)(z)\to X(z,x)) \\
\geq \; & \bigwedge_{z\in X}(\Downarrow dg(\sup F)(d(z))\to X(z,x)) \\
= \; & \bigwedge_{z\in X}((\Downarrow dg(\sup F)(d(z))\& Y(dg(\sup F),\sup F))\to X(z,x)) \\
\geq \; & \bigwedge_{z\in X}(F(d(z))\to X(z,x)) \\
\geq \; & \bigwedge_{y\in Y}(F(y)\to X(g(y),x)) \\
= \; & \bigwedge_{z\in X}(g^{\to}(F)(z)\to X(z,x)).
\end{aligned}
$$

Therefore, $\bigwedge_{z\in X}(g^{\to}(F)(z)\to X(z,x)) = X(g(\sup F),x)$. □

For each map $f\colon X\longrightarrow Y$ between Q-ordered sets is fuzzy Scott continuous if and only if for each $\phi\in\sigma_{\mathcal{I}}^{co}(Y)$, $\phi\circ f\in\sigma_{\mathcal{I}}^{co}(X)$ (see Proposition 5.7 in [7]). Let $f\colon X\longrightarrow Y$ be a morphism in **FICPO**. Then $f\colon (X,\sigma_{\mathcal{I}}^{co}(X))\longrightarrow (Y,\sigma_{\mathcal{I}}^{co}(Y))$ is continuous. Thus, the map $f^{\leftarrow}\colon \sigma_{\mathcal{I}}^{co}(Y)\longrightarrow\sigma_{\mathcal{I}}^{co}(X)$ is well-defined and preserves arbitrary meets, so it is a right adjoint.

Proposition 4.4. *Let $f\colon X \longrightarrow Y$ be a morphism in* **FICPO** *and $h\colon \sigma_{\mathcal{I}}^{co}(X) \longrightarrow \sigma_{\mathcal{I}}^{co}(Y)$ be the left adjoint of $f^{\leftarrow}$. Then h preserves the fuzzy beneath relation.*

Proof. By Proposition 4.3, it suffices to show that $f^{\leftarrow}$ preserves sups of inhabited $\mathcal{I}$-closed fuzzy sets of $\sigma_{\mathcal{I}}^{co}(Y)$. So we need to prove that $f^{\leftarrow}(\sup\Psi) = \sup(f^{\leftarrow})^{\rightarrow}(\Psi)$ for all $\Psi \in {\sigma_{\mathcal{I}}^{co}}^{+}(\sigma_{\mathcal{I}}^{co}(Y))$. Since,

$$\begin{aligned}
& \bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)} ((f^{\leftarrow})^{\rightarrow}(\Psi)(F) \to sub_X(F,E)) \\
= & \bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)} \Big(\Big(\bigvee_{f^{\leftarrow}(C)=F} \Psi(C)\Big) \to sub_X(F,E)\Big) \\
= & \bigwedge_{C\in\sigma_{\mathcal{I}}^{co}(Y)} (\Psi(C) \to sub_X(f^{\leftarrow}(C),E)) \\
= & \bigwedge_{x\in X}\bigwedge_{C\in\sigma_{\mathcal{I}}^{co}(Y)} (\Psi(C) \to (C(f(x)) \to E(x))) \\
= & \bigwedge_{x\in X} \Big(\Big(\bigvee_{C\in\sigma_{\mathcal{I}}^{co}(Y)} \Psi(C)\& f^{\leftarrow}(C)\Big)(x) \to E(x)\Big) \\
= & \; sub_X(\bigvee_{C\in\sigma_{\mathcal{I}}^{co}(Y)} \Psi(C)\& f^{\leftarrow}(C), E) \\
= & \; sub_X(f^{\leftarrow}(\bigvee_{C\in\sigma_{\mathcal{I}}^{co}(Y)} \Psi(C)\& C), E) \\
= & \; sub_X(f^{\leftarrow}(\sup\Psi), E).
\end{aligned}$$

Therefore, $f^{\leftarrow}(\sup\Psi) = \sup(f^{\leftarrow})^{\rightarrow}(\Psi)$. □

By above Proposition 4.4, we can build a functor $\sigma_{\mathcal{I}}^{co}$: **FICPO**$\longrightarrow$**FCPAlg**, where $\sigma_{\mathcal{I}}^{co}$ sends X to the Scott Q-cotopology of X for each irreducible complete Q-ordered set X and sends each morphism $f\colon X \longrightarrow Y$ in **FICPO** to the mapping $\sigma_{\mathcal{I}}^{co}(f) : \sigma_{\mathcal{I}}^{co}(X) \longrightarrow \sigma_{\mathcal{I}}^{co}(Y)$ which is the left adjoint of $f^{\leftarrow}$.

Theorem 4.5. *$\sigma_{\mathcal{I}}^{co} \dashv K$ is an adjunction between* **FICPO** *and* **FCPAlg**.

Proof. Let X be an irreducible complete Q-ordered set. Define $\eta_X\colon X \longrightarrow K(\sigma_{\mathcal{I}}^{co}(X))$ by $\eta_X(x) = \downarrow x$. It is easy to check that η_X is a morphism in **FICPO**. Let L be a C-prealgebric complete Q-ordered set and let $g\colon X \longrightarrow K(L)$ be a morphism in **FICPO**. Define a map $\hat{g}\colon \sigma_{\mathcal{I}}^{co}(X) \longrightarrow L$ as $\hat{g}(E) = \sup g^{\rightarrow}(E)$ for all $E \in \sigma_{\mathcal{I}}^{co}(X)$. Then $\hat{g}$ is a morphism in **FCPAlg**. We divided the proof of this conclusion into two steps.

Step 1. $\hat{g}$ is a left adjoint. It suffices to show that $\hat{g}$ preserves arbitrary sups. So we need to prove that $\hat{g}(\sup\mathcal{D}) = \sup(\hat{g})^{\rightarrow}(\mathcal{D})$ for all $\mathcal{D} \in Q^{\sigma_{\mathcal{I}}^{co}(X)}$. For each $y \in L$,

$$\begin{aligned}
& \bigwedge_{x\in L} ((\hat{g})^{\rightarrow}(\mathcal{D})(x) \to L(x,y)) \\
= & \bigwedge_{x\in L} \Big(\Big(\bigvee_{\hat{g}(A)=x} \mathcal{D}(A)\Big) \to L(x,y)\Big) \\
= & \bigwedge_{A\in\sigma_{\mathcal{I}}^{co}(X)} (\mathcal{D}(A) \to L(\hat{g}(A),y)) \\
\geq & \bigwedge_{A\in\sigma_{\mathcal{I}}^{co}(X)} (sub_X(A,\sup\mathcal{D}) \to L(\hat{g}(A),y)) \\
\geq & \bigwedge_{A\in\sigma_{\mathcal{I}}^{co}(X)} (L(\hat{g}(A),\hat{g}(\sup\mathcal{D})) \to L(\hat{g}(A),y) \\
\geq & \; L(\hat{g}(\sup\mathcal{D}),y).
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
L(\hat{g}(\sup\mathcal{D}),y) &= L(\sup g^{\rightarrow}(\sup\mathcal{D}),y) \\
&= L(\sup g^{\rightarrow}(\overline{\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)} \mathcal{D}(A)\&A}),y) \\
&\geq L(\sup(g^{\rightarrow}(\overline{\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)} \mathcal{D}(A)\&A})),y)
\end{aligned}$$

$$\begin{aligned}
&= \bigwedge_{x\in L}\Big(\overline{g^{\rightarrow}\big(\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\&A\big)}(x)\rightarrow L(x,y)\Big)\\
&= sub_L\Big(\overline{g^{\rightarrow}\big(\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\&A}, \downarrow y\Big)\\
&\geq sub_L\Big(g^{\rightarrow}\big(\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\&A\big), \downarrow y\Big)\\
&\geq sub_L\Big(g^{\rightarrow}\big(\overline{\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\&A}\big), \downarrow y\Big)\\
&= sub_X\Big(\overline{\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\&A}, g^{\leftarrow}(\downarrow y)\Big)\\
&\geq sub_X\Big(\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\&A, g^{\leftarrow}(\downarrow y)\Big)\\
&= \bigwedge_{z\in X}\Big(\Big(\bigvee_{A\in\sigma_{\mathcal{I}}^{co}(X)}\mathcal{D}(A)\&A(z)\Big)\rightarrow L(g(z),y)\Big)\\
&= \bigwedge_{A\in\sigma_{\mathcal{I}}^{co}(X)}\Big(\mathcal{D}(A)\rightarrow\bigwedge_{z\in X}(A(z)\rightarrow L(g(z),y))\Big)\\
&= \bigwedge_{A\in\sigma_{\mathcal{I}}^{co}(X)}(\mathcal{D}(A)\rightarrow L(\sup g^{\rightarrow}(A),y))\\
&= \bigwedge_{A\in\sigma_{\mathcal{I}}^{co}(X)}(\mathcal{D}(A)\rightarrow L(\hat{g}(A),y))\\
&= \bigwedge_{x\in L}((\hat{g})^{\rightarrow}(\mathcal{D})(x)\rightarrow L(x,y)).
\end{aligned}$$

Then $\bigwedge_{x\in L}((\hat{g})^{\rightarrow}(\mathcal{D})(x)\rightarrow L(x,y)) = L(\hat{g}(\sup\mathcal{D}),y)$, and thus $\hat{g}(\sup\mathcal{D}) = \sup(\hat{g})^{\rightarrow}(\mathcal{D})$.

Step 2. We shall show that $\hat{g}$ preserves the fuzzy beneath relation. From Proposition 4.3, we only need to prove that the right adjoint $f\colon L\longrightarrow\sigma_{\mathcal{I}}^{co}(X)$ satisfies that $f(\sup C) = \sup f^{\rightarrow}(C)$ for all $C\in{\sigma_{\mathcal{I}}^{co}}^{+}(L)$. For each $E\in\sigma_{\mathcal{I}}^{co}(X)$,

$$\begin{aligned}
&\bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)}(f^{\rightarrow}(C)(F)\rightarrow sub_X(F,E))\\
&= \bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)}\Big(\Big(\bigvee_{f(y)=F}C(y)\Big)\rightarrow sub_X(F,E)\Big)\\
&= \bigwedge_{y\in L}(C(y)\rightarrow sub_X(f(y),E))\\
&\geq \bigwedge_{y\in L}(L(y,\sup C)\rightarrow sub_X(f(y),E))\\
&\geq \bigwedge_{y\in L}(sub_X(f(y),f(\sup C))\rightarrow sub_X(f(y),E))\\
&\geq sub_X(f(\sup C),E).
\end{aligned}$$

Then $\bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)}(f^{\rightarrow}(C)(F)\rightarrow sub_X(F,E))\geq sub_X(f(\sup C),E)$. On the other hand, for each $a\in X$,

$$\begin{aligned}
&f(\sup C)(a)\&\Big(\bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)}(f^{\rightarrow}(C)(F)\rightarrow sub_X(F,E))\Big)\\
&= f(\sup C)(a)\&\Big(\bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)}\Big(\Big(\bigvee_{f(y)=F}C(y)\Big)\rightarrow sub_X(F,E)\Big)\Big)\\
&= f(\sup C)(a)\&\Big(\bigwedge_{y\in L}(C(y)\rightarrow sub_X(f(y),E))\Big)\\
&\leq f(\sup C)(a)\&\Big(\bigwedge_{y\in L}(C(y)\rightarrow(f(y)(a)\rightarrow E(a)))\Big)\\
&= sub_X(\downarrow a, f(\sup C))\&\Big(\Big(\bigvee_{y\in L}C(y)\&f(y)(a)\Big)\rightarrow E(a)\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq L(\hat{g}(\downarrow a), \hat{g}(f(\sup C)))\&\Big(\Big(\bigvee_{y\in L} C(y)\&f(y)(a)\Big) \to E(a)\Big)\\
&\leq L(g(a), \sup C)\& \Downarrow g(a)(g(a))\&\Big(\Big(\bigvee_{y\in L} C(y)\&f(y)(a)\Big) \to E(a)\Big)\\
&\leq C(g(a))\&\Big(\Big(\bigvee_{y\in L} C(y)\&f(y)(a)\Big) \to E(a)\Big)\\
&= C(\hat{g}(\downarrow a))\&\Big(\Big(\bigvee_{y\in L} C(y)\&f(y)(a)\Big) \to E(a)\Big)\\
&\leq \downarrow f^{\to}(C)(\downarrow a)\&\Big(\Big(\bigvee_{y\in L} C(y)\&f(y)(a)\Big) \to E(a)\Big)\\
&\leq \Big(\bigvee_{y\in L} C(y)\&f(y)(a)\Big)\&\Big(\Big(\bigvee_{y\in L} C(y)\&f(y)(a)\Big) \to E(a)\Big)\\
&\leq E(a).
\end{aligned}
$$

Then $\bigwedge_{F\in\sigma_{\mathcal{I}}^{co}(X)} (f^{\to}(C)(F) \to sub_X(F, E)) = sub_X(f(\sup C), E)$, and thus $f(\sup C) = \sup f^{\to}(C)$.

Next, we shall show that $g = K(\hat{g}) \circ \eta_X$. For all $x \in X$, $g(x) = \hat{g}(\downarrow x) = \hat{g}(\eta_X(x)) = K(\hat{g}) \circ \eta_X(x)$.

Finally, if $\overline{g}\colon \sigma_{\mathcal{I}}^{co}(X) \longrightarrow L$ is also a morphism in **FCPAlg** such that $g = K(\overline{g}) \circ \eta_X$, then we claim that $\hat{g} = \overline{g}$. It suffices to show that $\bigwedge_{y\in L}((g)^{\to}(E)(y) \to L(y, x)) = L(\overline{g}(E), x)$ for all $E \in \sigma_{\mathcal{I}}^{co}(X)$ and $x \in L$. Let g' be the right adjoint of $\overline{g}$.

$$
\begin{aligned}
&\bigwedge_{y\in L}((g)^{\to}(E)(y) \to L(y, x))\\
=& \bigwedge_{y\in L}\Big(\Big(\bigvee_{g(z)=y} E(z)\Big) \to L(y, x)\Big)\\
=& \bigwedge_{y\in L}\bigwedge_{g(z)=y}(E(z) \to L(y, x))\\
=& \bigwedge_{z\in X}(E(z) \to L(g(z), x))\\
=& \bigwedge_{z\in X}(E(z) \to L(\overline{g}(\downarrow z), x))\\
=& \bigwedge_{z\in X}(E(z) \to sub_X(\downarrow z, g'(x)))\\
=& sub_X(E, g'(x))\\
=& L(\overline{g}(E), x).
\end{aligned}
$$

So $\hat{g} = \overline{g}$. □

5 Conclusions

In this paper, we develop the C-continuity via Scott Q-cotopological spaces and establish an adjunction between the category of irreducible complete Q-ordered sets and the category of C-prealgebraic complete Q-ordered sets. Sobriety as an interesting property in domain theory, Yao [12] showed that when L is a frame, the fuzzy Scott topological space on a fuzzy domain is modified L-sober in terms of open sets. A question raised here is that whether the Scott Q-cotopological space on an irreducible complete C-continuous Q-ordered set is sober in the sense of [14]. In fact, in the classical setting, this is still an open problem.

As we know that the algebra with respect to T_0 open filter monad is a continuous lattice. Recently, for many valued setting, Yao [11] gave algebraic representation of fuzzy continuous lattices via the open filter monad. Motivated by the work of Yao, Yue and Pang, another question is that whether C-continuous complete Q-ordered sets have the above corresponding property like fuzzy continuous lattices.

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