Iranian Journal of Fuzzy Systems

Volume 18, Number 3, (2021), pp. 129-141

University of Sistan and Baluchestan

Original paper

Milstein method for solving fuzzy differential equation

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Abstract

To solve fuzzy differential equations driven by Liu process, three Milstein schemes are proposed in this work, which are explicit Milstein scheme, semi-implicit Milstein scheme and improved Milstein scheme. Improved Milstein scheme is constructed by correcting the error with semi-implicit method, the error is the difference between the exact solution of fuzzy differential equations and the solution derived from Milstein scheme. These numerical methods are proved to have strong convergence with order two. Accompanying the results above, the concept of mean-stability of numerical schemes for fuzzy differential equations is put forward and analyzed. For a linear test equation, it is showed the mean-stability region of improved Milstein scheme is bigger than explicit Milstein scheme. Finally, the accuracy and effectiveness of these schemes are confirmed through numerical examples.

Keywords: Credibility, fuzzy differential equation, Milstein method, semi-implicit Milstein method, improved Milstein method, mean-stability.

1 Introduction

There are many literatures discussing ordinary or stochastic differential equation and their numerical solutions. Different numerical schemes were put forward and applied in practice. As their counterpart, fuzzy differential equations are needed in many fields and are studied by scholars. In this work, we will address how to solve fuzzy differential equations driven by Liu process, and how to obtain a relatively accurate numerical solution.

1.1 Research background

With the development of society, there exists objective or artificial uncertainty in growing number of fields. There are many forms of expressions, such as randomness and fuzziness. People have an urgent need to solve the problems coming from uncertain environment. So it not only has profound significance, but also has a wide range of application prospects to study uncertainty theory. For the moment, many scholars have studied stochastic differential equations. A lot of papers on numerical methods of stochastic differential equations have been published. However, we know the world is neither random nor vague. We also describe part of uncertain phenomenon with fuzziness. As a counterpart of stochastic differential equation driven by Brownian motion, fuzzy differential equation driven by Liu process (Liu [13]) is an important object of study. In this work, we will concentrate our attentions on the numerical schemes for fuzzy differential equations driven by Liu process. Some scholars have obtained many theorems about fuzzy differential equations. Different from former researches, we consider them in the theory of credibility. Furthermore, we discuss the effectiveness and accuracy of numerical methods through considering their convergence and stability.

1.2 Literature review

With the introduction of fuzzy set in 1965, fuzzy mathematics was created. The concept of fuzzy set was proposed by Zadeh [25], which is a generalization of classical set. After that one analyzes fuzzy phenomenon with proper membership

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Received: February 2020; Revised: November 2020; Accepted: January 2021.

function and correlation operation of fuzzy set. Zadeh [26] gave the concept of possibility measure to measure a fuzzy event. Due to the absence of self-duality in possibility measure, Liu and Liu [14] put forward credibility measure with the property of self-duality. In 2004, credibility theory was set up and was improved in 2007. Li and Liu [12] concluded a relation between possibility measure and credibility measure, and proved a sufficient and necessary condition for credibility measure. Since then, credibility theory made a steady development.

In the system of credibility theory, Liu proposed that fuzzy variable is a kind of function mapping from credibility space to the set of real numbers. After this, the concepts of fuzzy process, Liu integral and fuzzy differential equations were raised gradually. A fuzzy process was proposed by Liu [13] to deal with the evolution of fuzzy phenomena with time in 2008. Similar to Brownian motion in stochastic process, Liu process is the most common and useful fuzzy process. At the beginning, Liu process involved the field of real numbers and then was extended to complex numbers by Qin and Wen [16]. As counterparts of Ito integral and Ito formula, Liu integral and Liu formula were introduced by Liu [13]. The concepts of generalized Liu integral and some properties were subsequently given by You, Ma and Huo [22]. Fuzzy differential equations were widely developed in finance, such as in [8, 15], they were used for setting pricing models. So far two main types of fuzzy differential equations have been studied by scholars. The differences between them is the performance of the fuzziness. One is obtained by fuzzifying the coefficients or initial conditions of a classical differential equation. In the past several decades, such fuzzy differential equations have been studied in [1, 2, 7, 11]. Some numerical methods were proposed in [3, 9]. Furthermore, Jana, Maiti and Roy [10] applied these fuzzy differential equations to production-recycling system. The other one is driven by Liu process, which is presented by Liu [13]. Different from the former, the fuzziness of the fuzzy differential equations driven by Liu process not only shows in initial conditions and coefficients, but also in the driven process. We will study the second type of fuzzy differential equation in this paper. Analytic solutions of linear fuzzy differential equations and some reducible fuzzy differential equations were given in [24]. But not all fuzzy differential equations can be solved, in order to satisfy needs in applications, numerical methods were proposed in [18, 19]. After the numerical methods were proposed, we need to discuss their effectiveness through observing the convergence and stability of numerical methods. Four kinds of numerical methods with convergence analysis for fuzzy differential equations driven by Liu process were proposed by Cheng and You [5]. Some concepts of stability for fuzzy differential equations were put forward in [27], including Lyapunov stability, uniform stability, asymptotical stability and etc. In addition, [20, 21] studied the stability in mean and in credibility, respectively. However, the stability of numerical methods for fuzzy differential equations have not been involved like stochastic differential equations in [17]. So we will give the concept of mean-stability of numerical methods in this paper.

1.3 Structure arrangement

The framework of this paper is as follows: some concepts about fuzzy mathematics is recalled in the next section. In Section 3, fuzzy explicit Milstein and fuzzy semi-implicit Milstein methods are proposed and their convergence are studied. Fuzzy improved Milstein method is derived in Section 4. After these numerical methods were put forward, we discuss mean-stability of numerical methods in the following work. Finally, we make a brief conclusion.

2 Preliminaries

Definition 2.1. [14] *If ξ is a fuzzy variable, then the expected value of ξ is defined by*

$$E[\xi]=\int_0^{+\infty}\mathrm{Cr}\{\xi\geq r\}\mathrm{d}r-\int_{-\infty}^{0}\mathrm{Cr}\{\xi\leq r\}\mathrm{d}r,$$

at least one of the two integrals on the right is finite.

Definition 2.2. [13] *A fuzzy process Q_t is said to be a Liu process, it need to satisfy the following conditions at the same time.*
(i) Q_0 is equal to 0,
(ii) Q_t has independent and identical-distribution increments,
(iii) $Q_{s+t}-Q_s$ is a fuzzy variable following normally distribution, and its expected value and variance are et and σ^2t^2, respectively.

If $e=0$ and $\sigma=1$, Q_t is a standard Liu process.

Theorem 2.3. [6] *For any given θ, the path $Q_t(\theta)$ of Liu process Q_t is Lipschitz continuous, that is, there exists a Lipschitz constant $Z(\theta)$ satisfying*

$$\mid Q_m-Q_n\mid\leq Z(\theta)\mid m-n\mid,\ \ \forall m,n,$$

where $\mathrm{Cr}\{\theta\} > 0$.

Definition 2.4. [13] *Partition the closed interval* $[c, d]$ *with* $c = t_1 < t_2 < \cdots < t_{n+1} = d$. *If the limit of* $\sum_{i=1}^{n} X_{t_i}(Q_{t_{i+1}} - Q_{t_i})$ *exists and is a fuzzy variable, then the fuzzy integral of fuzzy process* X_t *with respect to* Q_t *is*

$$\int_c^d X_t \mathrm{d}Q_t = \lim_{\rho \to 0} \sum_{i=1}^{n} X_{t_i}(Q_{t_{i+1}} - Q_{t_i}),$$

where $\rho = \max_{1 \le i \le n} \mid t_{i+1} - t_i \mid$, Q_t *is a standard Liu process.*

If fuzzy process Y_t is integrable on $[a, c]$ and $[c, b]$, then Y_t is integrable on $[a, b]$. Moreover,

$$\int_a^b Y_t \mathrm{d}Q_t = \int_a^c Y_t \mathrm{d}Q_t + \int_c^b Y_t \mathrm{d}Q_t,$$

which was given by You and Wang [23].

Definition 2.5. (Fuzzy differential equation driven by Liu process, Liu [13]) *A fuzzy differential equation (FDE) driven by Liu process is represented as*

$$\mathrm{d}X_t = p(t, X_t)\mathrm{d}t + q(t, X_t)\mathrm{d}Q_t,$$

where Q_t *is a standard Liu process, and* p, q *are some given functions.*

In [24], fuzzy differential equations (FDEs) were divided into linear FDEs, generalized linear FDEs, homogeneous FDEs and reducible FDEs.

A fuzzy differential equation with the following form

$$\mathrm{d}X_t = (a + bX_t)\mathrm{d}t + (c + dX_t)\mathrm{d}Q_t,$$

is called linear FDE if a, b, c and d are constants. If $a = c = 0$, it is called linear homogeneous FDE.

Theorem 2.6. (Existence and uniqueness of solutions for fuzzy differential equations, Chen and Qin [4]) *If two given functions* $p(t, x)$ *and* $q(t, x)$ *satisfying the following conditions:*
(i)(Linear growth condition)

$$\mid p(t, x) \mid + \mid q(t, x) \mid \le L_1(1 + \mid x \mid), \forall x \in R, 0 \le t \le T,$$

(ii)(Lipschitz condition)

$$\mid p(t, x) - p(t, y) \mid + \mid q(t, x) - q(t, y) \mid \le L_2 \mid x - y \mid, \forall x, y \in R, 0 \le t \le T,$$

where L_1, L_2 *are two positive constants, then fuzzy differential equation*

$$\mathrm{d}X_t = p(t, X_t)\mathrm{d}t + q(t, X_t)\mathrm{d}Q_t,$$

has an unique solution.

Theorem 2.7. [19] *If* p *and* q *are given continuously measurable functions that satisfy Linear growth condition. For* $\forall 0 \le s \le t \le T$, X_t, X_s *are any different solutions of fuzzy differential equation*

$$\mathrm{d}X_t = p(X_t)\mathrm{d}t + q(X_t)\mathrm{d}Q_t,$$

corresponding to different times. Then there exists a constant $N(\theta)$ *such that* $(X_t - X_s)^2 \le N(\theta)(t - s)^2$, *where* $N(\theta)$ *just relies on* s, t *and initial value only.*

Definition 2.8. [18] *For fuzzy differential equation* $\mathrm{d}X_t = p(X_t)\mathrm{d}t + q(X_t)\mathrm{d}Q_t$, *if there exists a constant* $M(\theta)$ *relating to the Lipschitz constant* $Z(\theta)$ *satisfying* $\mid \delta_{t_{n+1}} \mid \le M(\theta)h^{p_1}$, *then the numerical method is called locally convergent with order* p_1, *where* $\delta_{t_{n+1}}$ *is one-step error between exact solution and numerical solution,* h *is the step size of numerical solution.*

Definition 2.9. [20] *The fuzzy differential equation*

$$\mathrm{d}X_t = p(t, X_t)\mathrm{d}t + q(t, X_t)\mathrm{d}Q_t,$$

is called stability in mean if for any two solutions X_t and Y_t corresponding to different initial values X_0 and Y_0 respectively, we have

$$\lim_{|X_0 - Y_0| \to 0} E[|X_t - Y_t|] = 0, \forall t \geq 0.$$

Theorem 2.10. [20] *Suppose that fuzzy differential equation $\mathrm{d}X_t = p(t, X_t)\mathrm{d}t + q(t, X_t)\mathrm{d}Q_t$ has a unique solution for each initial value. In addition, $p(t, x)$ and $q(t, x)$ satisfy the strong Lipschitz condition*

$$| p(t,x) - p(t,y) | + | q(t,x) - q(t,y) | \leq L(t) | x - y |, \forall x, y \in R, 0 \leq t \leq T,$$

where $L(t)$ is an integrable function on $[0, T]$, then the fuzzy differential equation is stable in mean.

3 Fuzzy explicit Milstein and fuzzy semi-implicit Milstein methods

Our work in this paper is to propose some numerical methods for fuzzy differential equation

$$\begin{cases} \mathrm{d}X_t = p(X_t)\mathrm{d}t + q(X_t)\mathrm{d}Q_t \\ X_{t_0} = X_0. \end{cases} \tag{3.1}$$

Here we denote $X(t)$ as the exact solution of equation (3.1) at the point t.

In this section, we will first put forward two simple fuzzy numerical schemes, and then discuss the corresponding convergences.

3.1 Fuzzy numerical schemes

In [19], the second-order fuzzy Taylor expansion is produced as

$$\begin{aligned} X_t = X_0 &+ p(X_0)\int_0^t \mathrm{d}s + q(X_0)\int_0^t \mathrm{d}Q_s + p'(X_0)p(X_0)\int_0^t\int_0^s \mathrm{d}s_1\mathrm{d}s + p'(X_0)q(X_0)\int_0^t\int_0^s \mathrm{d}Q_{s_1}\mathrm{d}s \\ &+ q'(X_0)p(X_0)\int_0^t\int_0^s \mathrm{d}s_1\mathrm{d}Q_s + q'(X_0)q(X_0)\int_0^t\int_0^s \mathrm{d}Q_{s_1}\mathrm{d}Q_s + \widetilde{R}, \end{aligned}$$

where $\widetilde{R}$ represents remainder term, which will not be given here.

Through truncating the fuzzy Taylor expansion mentioned above, we get the following expansion

$$X_{t_{n+1}} \approx X_{t_n} + p(X_{t_n})\int_{t_n}^{t_{n+1}} \mathrm{d}s + q(X_{t_n})\int_{t_n}^{t_{n+1}} \mathrm{d}Q_s + q'(X_{t_n})q(X_{t_n})\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} \mathrm{d}Q_{s_1}\mathrm{d}Q_s.$$

Inspired by these, a new numerical scheme for fuzzy differential equations driven by Liu process is described as follows,

$$X_{t_{n+1}} = X_{t_n} + p(X_{t_n})h + q(X_{t_n})\Delta Q_{t_n} + \frac{1}{2}q'(X_{t_n})q(X_{t_n})\Delta Q_{t_n}^2, \qquad (3.2),$$

where h is one-step size, $h = \Delta t = t_{n+1} - t_n$, $\Delta Q_{t_n} = Q_{t_{n+1}} - Q_{t_n}$. We call this method fuzzy explicit Milstein method.

Accompanying the proposing of fuzzy explicit Milstein method, we try to discuss fuzzy implicit Milstein method. Due to more information used in implicit method, we predict that the fuzzy implicit Milstein method will be more accurate and has better properties. Next, we refer to (3.3) as fuzzy semi-implicit Milstein method, i.e.

$$X_{n+1} = X_n + \{\alpha p(X_{n+1}) + (1-\alpha)p(X_n)\}h + q(X_n)\Delta Q_n + \frac{1}{2}q'(X_n)q(X_n)\Delta Q_n^2, \tag{3.3}$$

where $\alpha \in [0, 1]$.

3.2 Convergence analysis of fuzzy Milstein methods

A numerical method without convergence doesn't make any sense, so its convergence must be studied. In this section we will prove the proposed methods are convergent almost surely and give their convergence order.

Theorem 3.1. *If $p(x)$ and $q(x)$ in fuzzy differential equation (3.1) satisfy Lipschitz condition and linear growth condition, then fuzzy explicit Milstein method (3.2) is locally convergent almost surely with order two.*

Proof. In order to discuss convergence, we need to find out the local error of exact solution and numerical solution. For the convenience of calculation, we write (3.1) in integral form, that is

$$X(t) = X_0 + \int_{t_0}^{t} p(X(s))\mathrm{d}s + \int_{t_0}^{t} q(X(s))\mathrm{d}Q_s. \tag{3.4}$$

Noting that $X_{t_n} = X(t_n)$, the exact solution of (3.1) at time $t_{n+1} = (n+1)h$ is

$$\begin{aligned} X(t_{n+1}) &= X_0 + \int_{t_0}^{t_{n+1}} p(X(s))\mathrm{d}s + \int_{t_0}^{t_{n+1}} q(X(s))\mathrm{d}Q_s \\ &= X_0 + \int_{t_0}^{t_n} p(X(s))\mathrm{d}s + \int_{t_0}^{t_n} q(X(s))\mathrm{d}Q_s + \int_{t_n}^{t_{n+1}} p(X(s))\mathrm{d}s + \int_{t_n}^{t_{n+1}} q(X(s))\mathrm{d}Q_s \\ &= X(t_n) + \int_{t_n}^{t_{n+1}} p(X(s))\mathrm{d}s + \int_{t_n}^{t_{n+1}} q(X(s))\mathrm{d}Q_s. \end{aligned} \tag{3.5}$$

The local error (one-step error) is

$$\begin{aligned} \delta_{t_{n+1}} &=\mid X_{(t_{n+1})} - X_{t_{n+1}} \mid \\ &=\mid \int_{t_n}^{t_{n+1}} p(X(s)) - p(X_{t_n})\mathrm{d}s + \int_{t_n}^{t_{n+1}} q(X(s)) - q(X_{t_n})\mathrm{d}Q_s - \frac{1}{2}q'(X_n)q(X_n)\Delta Q_n^2 \mid \\ &\leq\mid \int_{t_n}^{t_{n+1}} p(X(s)) - p(X_{t_n})\mathrm{d}s \mid + \mid \int_{t_n}^{t_{n+1}} q(X(s)) - q(X_{t_n})\mathrm{d}Q_s \mid + \mid \frac{1}{2}q'(X_n)q(X_n)\Delta Q_n^2 \mid \\ &\leq L_2 \int_{t_n}^{t_{n+1}} \mid X(s) - X_{t_n} \mid \mathrm{d}s + L_2 \int_{t_n}^{t_{n+1}} \mid X(s) - X_{t_n} \mid \mathrm{d}Q_s + \mid \frac{1}{2}q'(X_n)q(X_n)\Delta Q_n^2 \mid \\ &\leq L_2\sqrt{N(\theta)}h \int_{t_n}^{t_{n+1}} \mathrm{d}s + L_2\sqrt{N(\theta)}h \int_{t_n}^{t_{n+1}} \mathrm{d}Q_s + \frac{1}{2}L_2 \mid q(X_{t_n}) \mid \Delta Q_n^2. \end{aligned}$$

According to Lipschitz continuity of Liu process,

$$\Delta Q_n^2 = (Q_{t_{n+1}} - Q_{t_n})^2 \leq [Z(\theta) \mid t_{n+1} - t_n \mid]^2 = Z^2(\theta)h^2,$$

and

$$\mid \int_{t_n}^{t_{n+1}} \mathrm{d}Q_s \mid=\mid \lim_{\delta\to 0}\sum_{i=1}^{k}(Q_{s_{i+1}} - Q_{s_i}) \mid\leq\mid Z(\theta) \mid\mid \int_{t_n}^{t_{n+1}} \mathrm{d}s \mid=\mid Z(\theta) \mid h.$$

Then, we have

$$\delta_{t_{n+1}} \leq L_2\sqrt{N(\theta)}h^2 + L_2\sqrt{N(\theta)} \mid Z(\theta) \mid h^2 + \frac{1}{2}L_2 \mid q(X_{t_n}) \mid Z^2(\theta)h^2 = M(\theta)h^2,$$

where

$$M(\theta) = L_2\sqrt{N(\theta)} + L_2\sqrt{N(\theta)} \mid Z(\theta) \mid + \frac{1}{2}L_2 \mid q(X_{t_n}) \mid Z^2(\theta).$$

It shows that explicit Milstein method is convergent almost surely and the convergence order is two. □

Theorem 3.2. *If $p(x)$ and $q(x)$ in fuzzy differential equation (3.1) satisfy Lipschitz condition and linear growth condition, then fuzzy semi-implicit Milstein method (3.3) is locally convergent almost surely with order two.*

Proof. The proof is similar as that of Theorem 3.1. □

4 Fuzzy improved Milstein method

Compared with exact solution, all the numerical methods obtained have certain errors. Thus it is crucial to reduce error in different ways. Here we take one-step error approximation for explicit Milstein method so that the numerical solution is more accurate.

4.1 Derivation of the improved Milstein method

Now we will deduce a new numerical solution for fuzzy differential equation (3.1) called fuzzy improved Milstein method. This method is obtained through approximating fuzzy explicit Milstein with one-step error. Let us examine the difference between exact solution (3.5) and numerical solution (3.2) at the point t_{n+1}, that is

$$\varphi(t_{n+1}) = X(t_{n+1}) - X_{t_{n+1}}. \tag{4.1}$$

In order to find a proper approximation of $\varphi(t_{n+1})$, denoted by $\overline{\varphi}_{n+1}$, we put forward a new one-step approximation $\overline{X}_{t_{n+1}}$ given by

$$\overline{X}_{t_{n+1}} = X_{t_{n+1}} + \overline{\varphi}_{n+1}. \tag{4.2}$$

Such a scheme can be regarded as a fuzzy improved Milstein method, which is expected to have a smaller local error compared with fuzzy explicit Milstein method. Based on this idea, we carry out the following analysis.

Note that $X_{t_n} = X(t_n)$, for $t \in (t_n, t_{n+1}]$, $\mathrm{d}X_t = p(X_{tn})\mathrm{d}t + q(X_{tn})\mathrm{d}Q_t + q'(X_{tn})q(X_{tn})(Q_t - Q_{tn})\mathrm{d}Q_t$. Differentiating both sides of $\varphi(t) = X(t) - X_t$, we have

$$\begin{aligned}
\mathrm{d}\varphi(t) &= (p(X(t)) - p(X_{t_n}))\mathrm{d}t + (q(X(t)) - q(X_{t_n}))\mathrm{d}Q_t - q'(X_{t_n})q(X_{t_n})(Q_t - Q_{t_n})\mathrm{d}Q_t \\
&= (p(X(t)) - p(X_t))\mathrm{d}t + (p(X_t) - p(X_{t_n}))\mathrm{d}t + (q(X(t)) - q(X_t))\mathrm{d}Q_t \\
&\quad + (q(X_t) - q(X_{t_n}) - q'(X_{t_n})q(X_{t_n})(Q_t - Q_{t_n}))\mathrm{d}Q_t \\
&= m(t)\varphi(t)\mathrm{d}t + h(t)\varphi(t)\mathrm{d}Q_t + F(t)\mathrm{d}t + G(t)\mathrm{d}Q_t,
\end{aligned}$$

where

$$m(t) = \int_0^1 p'(X_t + \xi\varphi(t))\mathrm{d}\xi, \quad h(t) = \int_0^1 p'(X_t + \zeta\varphi(t))\mathrm{d}\zeta,$$

and

$$F(t) = p(X_t) - p(X_{t_n}), \tag{4.3}$$

$$G(t) = q(X_t) - q(X_{t_n}) - q'(X_{t_n})q(X_{t_n})(Q_t - Q_{t_n}). \tag{4.4}$$

For $t \in (t_n, t_{n+1}]$, $\varphi(t)$ is governed by the following fuzzy differential equation:

$$\begin{cases}
\mathrm{d}\varphi(t) = [m(t)\varphi(t) + F(t)]\mathrm{d}t + [h(t)\varphi(t) + G(t)]\mathrm{d}Q_t, \\
\varphi(t_n) = 0.
\end{cases}$$

Replacing $m(t)$ and $h(t)$ with $p'(X_t)$ and $q'(X_t)$, we get

$$\begin{cases}
\mathrm{d}\overline{\varphi}(t) = [p'(X_t)\overline{\varphi}(t) + F(t)]\mathrm{d}t + [q'(X_t)\overline{\varphi}(t) + G(t)]\mathrm{d}Q_t, \\
\overline{\varphi}(t_n) = 0.
\end{cases} \tag{4.5}$$

Next we work to solve $\overline{\varphi}(t)$. Apply fuzzy semi-implicit Milstein method (3.3) to linear fuzzy differential equation (4.5) to get

$$\begin{aligned}
\overline{\varphi}_{n+1} = \overline{\varphi}_n &+ \{\alpha[p'(X_{t_{n+1}})\overline{\varphi}_{n+1} + F(t_{n+1})] + (1+\alpha)[p'(X_{t_n})\overline{\varphi}_n + F(t_n)]\}h \\
&+ [q'(X_{t_n})\overline{\varphi}_n + G(t_n)]\Delta Q_n + \frac{1}{2}q'(X_{t_n})[q'(X_{t_n})\overline{\varphi}_n + G(t_n)]\Delta Q_n^2.
\end{aligned}$$

Because $X_{t_n} = X(t_n)$, we have $\overline{\varphi}_n = \overline{\varphi}(t_n) = \varphi(t_n) = 0$, $G(t_n) = 0$ and $F(t_n) = 0$ by (4.1)-(4.4). Then $\overline{\varphi}_{n+1}$ is obtained as

$$\overline{\varphi}_{n+1} = [1 - \alpha h p'(X_{t_{n+1}})]^{-1}\alpha h[p(X_{t_{n+1}}) - p(X_{t_n})].$$

According to (4.2), the local improved Milstein method is constructed as follows,

$$\begin{cases}
\overline{X}_{t_{n+1}} = X_{t_{n+1}} + [1 - \alpha h p'(X_{t_{n+1}})]^{-1}\alpha h[p(X_{t_{n+1}}) - p(X_{t_n})] \\
X_{t_{n+1}} = X_{t_n} + p(X_{t_n})\Delta t_n + q(X_{t_n})\Delta Q_{t_n} + \frac{1}{2}q'(X_{t_n})q(X_{t_n})\Delta Q_{t_n}^2,
\end{cases} \tag{4.6}$$

where $X_{t_{n+1}}$ is given by (3.2) and $X_{t_n} = X(t_n)$. As a result, we propose the following global scheme for (3.1)

$$\begin{cases} \overline{X}_{t_{n+1}} = X_{t_{n+1}} + [1 - \alpha h p'(X_{t_{n+1}})]^{-1} \alpha h [p(X_{t_{n+1}}) - p(\overline{X}_{t_n})], \\ X_{t_{n+1}} = \overline{X}_{t_n} + p(\overline{X}_{t_n})h + q(\overline{X}_{t_n})\Delta Q_{t_n} + \frac{1}{2} q'(\overline{X}_{t_n}) q(\overline{X}_{t_n}) \Delta Q_{t_n}^2. \end{cases} \tag{4.7}$$

4.2 Convergence analysis of fuzzy improved Milstein method

Convergence of fuzzy improved Milstein method will be discussed in this section, which can be used to judge wether the method is feasible.

Theorem 4.1. *If $p(x)$ and $q(x)$ in fuzzy differential equation (3.1) satisfy Lipschitz condition and linear growth condition, then fuzzy improved Milstein method (4.7) is locally convergent almost surely with order two.*

Proof. To study local convergence, let $\overline{X}_{t_n} = X_{t_n} = X(t_n)$. Since $p(x)$ and $q(x)$ satisfy Lipschitz condition, we get $p'(X_{t_{n+1}}) < L_2$ and $q'(X_{t_n}) < L_2$. The difference δ_{n+1} between (3.5) and (4.7) is

$$\begin{aligned} \delta_{t_{n+1}} =& \mid X(t_{n+1}) - \overline{X}_{t_{n+1}} \mid \\ =& \mid \int_{t_n}^{t_{n+1}} [p(X(s)) - p(\overline{X}_{t_n})] \mathrm{d}s + \int_{t_n}^{t_{n+1}} [q(X(s)) - q(\overline{X}_{t_n})] \mathrm{d}Q_s \\ & - \frac{1}{2} q'(\overline{X}_{t_n}) q(\overline{X}_{t_n}) \Delta Q_{t_n}^2 - [1 - \alpha h p'(X_{t_{n+1}})]^{-1} \alpha h [p(X_{t_{n+1}}) - p(\overline{X}_{t_n})] \mid \\ \leq& \mid \int_{t_n}^{t_{n+1}} [p(X(s)) - p(\overline{X}_{t_n})] \mathrm{d}s \mid + \mid \int_{t_n}^{t_{n+1}} [q(X(s)) - q(\overline{X}_{t_n})] \mathrm{d}Q_s \mid \\ & + \mid \frac{1}{2} q'(\overline{X}_{t_n}) q(\overline{X}_{t_n}) \Delta Q_{t_n}^2 \mid + \mid [1 - \alpha h p'(X_{t_{n+1}})]^{-1} \alpha h [p(X_{t_{n+1}}) - p(\overline{X}_{t_n})] \mid \\ \leq& L_2 \int_{t_n}^{t_{n+1}} \mid X(s) - \overline{X}_{t_n} \mid \mathrm{d}s + L_2 \int_{t_n}^{t_{n+1}} \mid X(s) - \overline{X}_{t_n} \mid \mathrm{d}Q_s + \frac{1}{2} \Delta Q_{t_n}^2 L_2 \mid q(\overline{X}_{t_n}) \mid \\ & + \mid \frac{\alpha h L_2}{1 - \alpha h L_2} \sqrt{N(\theta)} h \mid . \end{aligned}$$

Since when $h \to 0$, $\frac{\alpha L_2}{1 - \alpha h L_2}$ is bounded, there is a constant B so that

$$\lim_{h \to 0} \frac{\alpha L_2}{1 - \alpha h L_2} \leq B.$$

Then

$$\begin{aligned} \lim_{h \to 0} \delta_{t_{n+1}} &\leq \lim_{h \to 0} [L_2 \sqrt{N(\theta)} h^2 + L_2 \mid Z(\theta) \mid \sqrt{N(\theta)} h^2 + \frac{1}{2} L_2 Z^2(\theta) h^2 \mid q(\overline{X}_{t_n}) \mid + B \sqrt{N(\theta)} h^2] \\ &= \lim_{h \to 0} M(\theta) h^2 \\ &= 0, \end{aligned}$$

where $M(\theta) = L_2 \sqrt{N(\theta)} + L_2 \mid Z(\theta) \mid \sqrt{N(\theta)} + \frac{1}{2} L_2 Z^2(\theta) \mid q(\overline{X}_{t_n}) \mid + B \sqrt{N(\theta)}$.

Thus fuzzy improved Milstein method is locally convergent in mean and the order is also two. □

5 Stability analysis

In the former study of numerical solution for fuzzy differential equation driven by Liu process, stability analysis of numerical methods has not been involved. However, considering convergence is not enough, other properties of numerical methods should be discussed. In this section, we focus on the mean-stability of the methods proposed in this paper. Considering the following fuzzy linear test equation

$$\begin{cases} \mathrm{d}X_t = \lambda X_t \mathrm{d}t + \mu X_t \mathrm{d}Q_t \\ X_{t_0} = X_0, \end{cases} \tag{5.1}$$

where $\lambda, \mu \in \mathbb{C}$, $\mathbb{C}$ denotes complex domain. The exact solution of (5.1) is given by

$$X_t = X_0 \exp(\lambda t + \mu Q_t), \tag{5.2}$$

and fuzzy differential equation (5.1) is stable in mean by Theorem 2.4. Applying a numerical method to (5.1), we obtain a one-step iteration

$$X_{t_{n+1}} = R(h, \lambda, \mu, \Delta Q_{t_n}) X_{t_n},$$

where $R(h, \lambda, \mu, \Delta Q_{t_n})$ is a function related to $h, \lambda, \mu, \Delta Q_{t_n}$.

Following this, mean-stability is defined as follows.

Definition 5.1. *The numerical method is said to be mean-stable if* $\overline{R}(h, \lambda, \mu) = \mid E[R(h, \lambda, \mu, \Delta Q_{t_n})] \mid < 1$*, where* $E[R]$ *indicates the expectation of fuzzy variable* R*,* $\overline{R}(h, \lambda, \mu)$ *is called mean-stability function,* $U = \{(h, \lambda, \mu) \mid \overline{R}(h, \lambda, \mu) < 1)\}$ *is defined as stability region of mean-stability.*

During the discussion of mean-stability of these numerical methods mentioned above, we find the mean-stability function of fuzzy semi-implicit Milstein method is the same as that of fuzzy improved Milstein method. However, through observing numerical schemes, fuzzy improved Milstein method is explicit numerical method, and explicit numerical method is easier to be carried out than semi-implicit numerical method. Thus, we will only calculate the stability functions of the following two numerical methods.

(1) Fuzzy explicit Milstein method

$$X_{t_{n+1}} = X_{t_n} + \lambda h X_{t_n} + \mu \Delta Q_{t_n} X_{t_n} + \frac{1}{2}\mu^2 \Delta Q_{t_n}^2 X_{t_n} = (1 + \lambda h + \mu \Delta Q_{t_n} + \frac{1}{2}\mu^2 \Delta Q_{t_n}^2) X_{t_n}.$$

Applying $E[\Delta Q_{t_n}] = 0$, $E[\Delta Q_{t_n}^2] = h^2$, we derive

$$\overline{R}_1(h, \lambda, \mu) = \mid E[R(h, \lambda, \mu, \Delta Q_{t_n})] \mid = \mid 1 + \lambda h + \frac{1}{2}\mu^2 h^2 \mid .$$

Theorem 5.2. *For fixed* h, λ, μ*, fuzzy explicit Milstein method is mean-stable if*

$$2h\mathrm{Re}\lambda + h^2[(\mathrm{Re}\mu)^2 - (\mathrm{Im}\mu)^2 + \mid \lambda \mid^2] + h^3[((\mathrm{Re}\mu)^2 - (\mathrm{Im}\mu)^2)\mathrm{Re}\lambda + 2\mathrm{Re}\{\mu\}\mathrm{Im}\{\mu\}\mathrm{Im}\{\lambda\}] + \frac{1}{4} \mid \mu \mid^4 h^4 < 0.$$

Proof. According to Definition 5.1, fuzzy explicit Milstein method is mean-stable if $\overline{R}_1(h, \lambda, \mu) < 1$, that is

$$\begin{aligned}
&(1 + \lambda h + \frac{1}{2}u^2 h^2)(1 + \overline{\lambda} h + \frac{1}{2}\overline{\mu}^2) \\
&= 1 + 2h\mathrm{Re}\lambda + [(\mathrm{Re}\mu)^2 - (\mathrm{Im}\mu)^2]h^2 + \mid \lambda \mid^2 h^2 + \mathrm{Re}\lambda[(\mathrm{Re}\mu)^2 - (\mathrm{Im}\mu)^2]h^3 \\
&\quad + 2\mathrm{Im}\{\lambda\}\mathrm{Re}\{\mu\}\mathrm{Im}\{\mu\}h^3 + \frac{1}{4} \mid \mu \mid^4 h^4 \\
&= 1 + 2h\mathrm{Re}\lambda + [(\mathrm{Re}\mu)^2 - (\mathrm{Im}\mu)^2 + \mid \lambda \mid^2]h^2 + \{\mathrm{Re}\lambda[(\mathrm{Re}\mu)^2 - (\mathrm{Im}\mu)^2] \\
&\quad + 2\mathrm{Im}\{\lambda\}\mathrm{Re}\{\mu\}\mathrm{Im}\{\mu\}\}h^3 + \frac{1}{4} \mid \mu \mid^4 h^4 \\
&< 1.
\end{aligned}$$

Therefore,

$$2h\mathrm{Re}\lambda + h^2[(\mathrm{Re}\mu)^2 - (\mathrm{Im}\mu)^2 + \mid \lambda \mid^2] + h^3[((\mathrm{Re}\mu)^2 - (\mathrm{Im}\mu)^2)\mathrm{Re}\lambda + 2\mathrm{Re}\{\mu\}\mathrm{Im}\{\mu\}\mathrm{Im}\{\lambda\}] + \frac{1}{4} \mid \mu \mid^4 h^4 < 0.$$

□

Corollary 5.3. *Let* λ, μ *be real numbers. Fuzzy Milstein method is mean-stable for any* h*, if*

$$2\lambda h + \mu^2 h^2 + \lambda^2 h^2 + \lambda \mu^2 h^3 + \frac{1}{4}\mu^4 h^4 < 0 \ \textit{and} \ -\frac{\mu^2}{2} - \frac{2}{h} < \lambda < -\frac{\mu^2}{2}.$$

Proof. This corollary is easy to be proved according to Definition 5.1. □

Remark 5.4. *Let* λ, μ *be real numbers. Fuzzy Milstein method must not be mean-stable if* $\lambda > 0$*.*

(2) Fuzzy improved Milstein method

Applying (4.7) to (5.1), we obtain

$$\begin{aligned}\overline{X}_{t_{n+1}} &= \overline{X}_{t_n} + \lambda h\overline{X}_{t_n} + \mu\Delta Q_{t_n}\overline{X}_{t_n} + \frac{1}{2}\Delta Q_{t_n}^2\overline{X}_{t_n} + [1-\alpha h\lambda]^{-1}\alpha h[\lambda X_{t_{n+1}} - \lambda\overline{X}_{t_n}] \\ &= \frac{1}{1-\alpha\lambda h}[1+\lambda h - \alpha\lambda h + \mu\Delta Q_{t_n} + \frac{1}{2}\mu^2\Delta Q_{t_n}^2]\overline{X}_{t_n}.\end{aligned}$$

The stability function is

$$\overline{R}_2(h,\lambda,\mu) = \left| \frac{1+\lambda h - \alpha\lambda h + \frac{1}{2}\mu^2h^2}{1-\alpha\lambda h} \right|.$$

In order to make discussion simple, here we only discuss the case $\alpha = 1$, that is

$$\overline{R}_2(h,\lambda,\mu) = \left| \frac{1+\frac{1}{2}\mu^2h^2}{1-\lambda h} \right|.$$

Theorem 5.5. *For fixed h, λ, μ, fuzzy improved Milstein method is mean-stable if*

$$[(\text{Re}\mu)^2 - (\text{Im}\mu)^2]h^2 + \frac{1}{4} \mid \mu \mid^4 h^4 + 2h\text{Re}\lambda - \mid \lambda \mid^2 h^2 < 0.$$

Proof. According to Definition 5.1, if fuzzy improved Milstein method is mean-stable, i.e. $\overline{R} = \left| \frac{1+\frac{1}{2}\mu^2h^2}{1-\lambda h} \right| < 1$, then

$$\frac{1+\frac{1}{2}\mu^2h^2}{1-\lambda h} \cdot \frac{1+\frac{1}{2}\overline{\mu}^2h^2}{1-\overline{\lambda}h} = \frac{1+[(\text{Re}\mu)^2 - (\text{Im}\mu)^2]h^2 + \frac{1}{4} \mid \mu \mid^4 h^4}{1-2h\text{Re}\lambda + \mid \lambda \mid^2 h^2} < 1.$$

Thus the condition of mean-stability for fuzzy improved Milstein method can be obtained as

$$[(\text{Re}\mu)^2 - (\text{Im}\mu)^2]h^2 + \frac{1}{4} \mid \mu \mid^4 h^4 + 2h\text{Re}\lambda - \mid \lambda \mid^2 h^2 < 0.$$

□

Corollary 5.6. *Let λ, μ be real numbers. The fuzzy improved Milstein method is mean-stable if*

$$\frac{1}{4}\mu^4h^4 + \mu^2h^2 + 2\lambda h - \lambda^2h^2 < 0.$$

Proof. Based on the proof of Theorem 5.2., the corollary is easy to be proved. □

In order to distinguish fuzzy explicit Milstein method and fuzzy improved Milstein method, we depict the stability regions in Figure 1. Clearly, we can see the mean-stability region of fuzzy improved Milstein method is bigger than that of fuzzy explicit Milstein method. Therefore, we obtain that the mean-stability of fuzzy improved Milstein is superior to fuzzy explicit Milstein method for fuzzy differential equations driven by Liu process.

6 Numerical example

To illustrate the effectiveness of the proposed numerical methods in this work, we give some examples as follows in this section. Taking

$$\begin{cases} \text{d}X_t = \lambda X_t\text{d}t + \mu X_t\text{d}Q_t \\ X_{t_0} = 1, \end{cases} \tag{6.1}$$

where λ and μ are fixed constants. Next we discuss the mean-stability regions of fuzzy explicit Milstein method and fuzzy improved Milstein method when λ and μ are given by two cases.

Case I:λ, μ are real numbers.

(i)$\lambda = -2, \mu = 1$.

Solving (6.1) in fuzzy explicit Milstein method and fuzzy improved Milstein method, respectively, we obtain two stability functions

$$\overline{R}_M = \left| 1 - 2h + \frac{1}{2}h^2 \right|, \quad \overline{R}_{IM} = \left| \frac{1+\frac{1}{2}h^2}{1+2h} \right|,$$

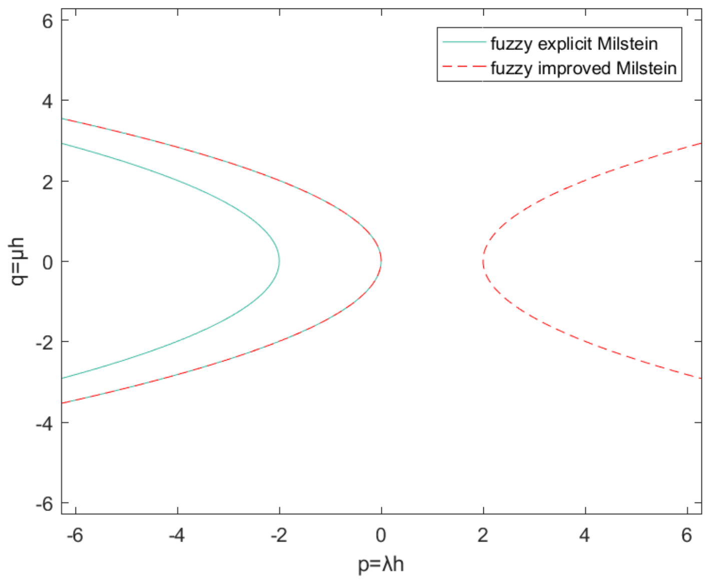

Figure 1: The mean-stability regions of fuzzy explicit Milstein method and fuzzy improved Milstein method

where $\overline{R}_M$ shows the stability function of fuzzy explicit Milstein method and $\overline{R}_{IM}$ shows the stability function of fuzzy improved Milstein method.

For a certain fuzzy differential equation, studying the stability region of its numerical scheme is to analyse value range of h. Since λ and μ are real numbers, it is easy to obtain the stability regions of two numerical schemes as Table 1. The step size h of fuzzy improved Milstein method has wider range of values. Thus the mean-stability of fuzzy improved Milstein method outperforms fuzzy explicit Milstein method.

(ii)$\lambda = -5, \mu = -1$.

Solving (6.1) in two numerical methods in line with (i), the stability functions of fuzzy explicit Milstein method and fuzzy improved Milstein method are derived as

$$\overline{R}_M = \mid 1 - 5h + \frac{1}{2}h^2 \mid, \ \ \overline{R}_{IM} = \mid \frac{1 + \frac{1}{2}h^2}{1 + 5h} \mid .$$

According to Definition 5.1, we get the stability regions of these two numerical methods, which can be seen in Table 1. Fuzzy explicit Milstein method is stable when $h \in (5 + \sqrt{23}, 10)$ and fuzzy improved Milstein method is stable when $h \in (0, 10)$. Thus fuzzy improved Milstein method can be considered more stable.

(iii)$\lambda = -1000, \mu = 300$.

For equation (6.1) with $\lambda = -1000$ and $\mu = 300$, the stability functions of fuzzy explicit Milstein method and fuzzy improved Milstein method are given as

$$\overline{R}_M = \mid 1 - 1000h + 45000h^2 \mid, \ \ \overline{R}_{IM} = \mid \frac{1 + 45000h^2}{1 + 1000h} \mid .$$

Fuzzy explicit Milstein method is stable when $h \in (0, 0.0022) \bigcup (0.02, 0.0222)$ and fuzzy improved Milstein method is stable when $h \in (0, 0.0222)$.

(iv)$\lambda = -5000, \mu = -200$.

For equation (6.1) with $\lambda = -5000$ and $\mu = -200$, the stability functions of fuzzy explicit Milstein method and fuzzy improved Milstein method are given as

$$\overline{R}_M = \mid 1 - 5000h + 20000h^2 \mid, \ \ \overline{R}_{IM} = \mid \frac{1 + 20000h^2}{1 + 5000h} \mid .$$

Fuzzy explicit Milstein method is stable when $h \in (0, 0.0004) \bigcup (0.2496, 0.25)$ and fuzzy improved Milstein method is stable when $h \in (0, 0.25)$.

More examples can be solved by the above method. To be concise, we choose only four examples to illustrate here. It is obvious from Table 1, when numerical methods are stable, there is a wider range of step size h for fuzzy improved

Milstein method. Therefore, the stability of fuzzy improved Milstein method is superior to that of fuzzy explicit Milstein method.

Table 1: Stable regions of fuzzy explicit Milstein method and fuzzy improved Milstein method (Case I)

	h	
(λ,μ)	explicit	improved
$(-2,1)$	$(1,2)\bigcup(2,4)$	$(0,4)$
$(-5,-1)$	$(5+\sqrt{23},10)$	$(0,10)$
$(-1000,300)$	$(0,0.0022)\bigcup(0.02,0.0222)$	$(0,0.0222)$
$(-5000,-200)$	$(0,0.0004)\bigcup(0.2496,0.25)$	$(0,0.25)$

Case II:$\lambda,\mu\in\mathbb{C}$.

(i) $\lambda=-20+20i,\mu=5$.

Under the circumstance, in order to simplify calculation, let $l=\lambda h, k=\frac{\mu}{\lambda}$. Since λ is a complex number, l,k are also complex numbers and $\mid k\mid=\frac{1}{8\sqrt{2}}$. By Definition 5.1, we get the mean-stability regions for two methods as Figure 2. The area surrounded by the circle in the left is the mean-stability region of fuzzy explicit Milstein method. The mean-stability region of fuzzy improved Milstein method is the whole plane except for the area enclosed by the circle on the right. Clearly, the mean-stability of fuzzy improved Milstein method is better than fuzzy explicit Milstein method.

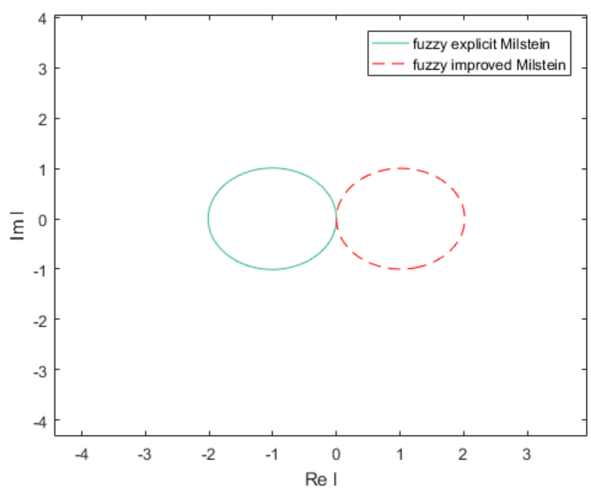

Figure 2: The mean-stability regions of Milstein method and the improved Milstein method for $\lambda=-20+20i,\mu=5$

(ii) $\lambda=-5,\mu=10+10i$.

Different from circumstance (i) in Case II, in this example, the mean-stability of two numerical methods is discussed by Theorem 5.1 and Theorem 5.2, respectively. We can summarize the results in Table 2. Thus, we concluded that the mean-stability of fuzzy improved Milstein method is superior to fuzzy explicit Milstein method.

Table 2: Stable region of fuzzy explicit Milstein method and fuzzy improved Milstein method ($\lambda=-5,\mu=10+10i$)

method	explicit	improved
h	(0, 0.09)	(0, 0.11)

All the above numerical experiments verfy that the mean-stability of fuzzy improved Milstein method is better than that of fuzzy explicit Milstein method.

7 Conclusions

Three methods for solving fuzzy differential equations driven by Liu process have been proposed in this paper. Fuzzy explicit Milstein method and fuzzy semi-implicit Milstein method were first put forward, then fuzzy improved Milstein method was constructed by error correcting. After these numerical methods were proposed, we proved that these numerical schemes have strong convergence with order two. In this work, the concept about mean-stability of numerical methods for fuzzy differential equations driven by Liu process was raised. The mean-stability analysis of proposed methods was carried out. During this process, we found that fuzzy semi-implicit Milstein method is the same as fuzzy improved Milstein method, when the value of α is equal. So we only studied mean-stability of fuzzy explicit and improved ($\alpha = 1$) Milstein methods. What's more, the mean-stability region of fuzzy improved Milstein method is clearly larger than fuzzy explicit Milstein method. Numerical test confirmed the effectiveness of these methods and fuzzy improved Milstein method is superior to fuzzy explicit Milstein method for fuzzy differential equation driven by Liu process.

Acknowledgement

This work was supported by Natural Science Foundation of China Grant No.61773150, Science and Technology Project of Hebei Education Department No. ZD2020172 and No. QN2020124.

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