MONITORING FUZZY CAPABILITY INDEX $\widetilde{C}_{pk}$ BY USING THE EWMA CONTROL CHART WITH IMPRECISE DATA

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Abstract. A manufacturing process cannot be released to production until it has been proven to be stable. Also, we cannot begin to talk about process capability until we have demonstrated stability in our process. This means that the process variation is the result of random causes only and all assignable or special causes have been removed. In complicated manufacturing processes, such as drilling process, the natural instability of the process impedes the use of any control charts for the mean and standard deviation. However, a complicated manufacturing process can be capable in spite of this natural instability. In this paper we discuss the $\widetilde{C}_{pk}$ process capability index. We find the membership function of $\widetilde{C}_{pk}$ based on fuzzy data. Also, by using the definition of classical control charts and the method of Vännman and Castagliola, we propose new control charts that are constructed by the α-cut sets of $\widetilde{C}_{pk}$ for the natural instable manufacturing processes with fuzzy normal distributions. The results are concluded for $\alpha = 0.6$, that is chosen arbitrarily.

1. Introduction

Statistical techniques can be helpful throughout the product cycle, including activities prior to manufacturing, in quantifying process variability, in analyzing this variability relative to product requirements or specifications, and in assisting development and manufacturing in eliminating or greatly reducing this variability. This general activity is called process capability analysis. Process capability refers to the uniformity of process. Obviously, the variability in the process is a measure of the uniformity of output. There may not exist a definition of the "process capability"but in high probability the (real valued) quality characteristic X of the produced items lies between some lower and upper specification limits LSL and USL (or tolerance interval limits). Therefore the idea of process capability implies that the fraction p of produced nonconforming items should be small if the process is said to be capable. Capability indices are useful tools that can reflect how well the manufacturing process act. The simplest form of process capability index is C_p, which is introduced by Juran [16] and Kane [18] defined a modified form of C_p by C_{pk}:

$$C_p = \frac{USL - LSL}{6\sigma}, \tag{1}$$

$$C_{pk} = \frac{min(USL - \mu, \mu - LSL)}{3\sigma}, \tag{2}$$

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where μ is the process mean and σ is the process standard deviation. Since C_p and C_{pk} cannot detect the off-centering of the process, Chang et al. [7] introduced the capability index C_{pm} and Pearn et al. [25] introduced C_{pmk} as follows:

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{3}$$

$$C_{pmk} = \frac{min(USL - \mu, \mu - LSL)}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{4}$$

where T is a target value. Vänman [32] defined a class of capability indices, depending on two nonnegative parameters u and v, as:

$$C_p(u, v) = \frac{d - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, \tag{5}$$

where d is a half length of specification interval. By letting $u = 0$ or 1 and $v = 0$ or 1 we have:

$$C_p(0,0) = C_p; C_p(1,0) = C_{pk}; C_p(0,1) = C_{pm}; C_p(1,1) = C_{pmk}.$$

Because of exuberance of vague occasions, the applications of the theory of fuzzy sets which was introduced by Zadeh [37] are very wide. We can find many uses of the theory of fuzzy sets in estimating the capability index such as Kaya and Kahraman [19], Tsai and Chen [31], Parchami and Mashinchi [24], Hsu and Shu [15], Wu [36], Chen et al. [9].

The control chart was invented by Shewhart [28] while working for Bell Labs in the 1920s. His experiments showed that data from physical processes was not as tidy as first thought. He concluded that although every process displays some type of variation, some processes display controlled variation natural to the process, whereas others display uncontrolled variation, which is to say it cannot be assigned directly to definable causes as related to the process [20]. The advantage of using fuzzy quality control charts is that many intermediate states between good and bad quality can be distinguish, allowing a flexible reaction of the classifier. Moreover, the fuzzy representation of the range of the tolerance provides a possibility to handle qualitative characteristics (e.g. good, bad) in the same way as quantitative ones (represented as real numbers) [1]. First fuzzy control charts were proposed by Wang and Raz [35] in the late 1980s. This paper applies several key concepts from fuzzy set theory on the construction of attribute control charts to monitor quality data available in linguistic form. They proposed membership control limits based on the membership function opposed to traditional control limits which are based on the probability distribution function of the sample mean. Then, they proposed two different approaches for the construction of control charts for quality data that is available in linguistic form [26]. In papers that were mentioned above, in order to retain the standard format of control charts, the fuzzy sample mean is represented by a certain crisp value which is plotted on a chart whose control lines are calculated using defuzzified values of previous observations. Kanagawa

et al. [17] developed control charts for linguistic variables based on probability density functions existing behind the linguistic data in order to control the process variability as well as the process average [17]. This is different from the probability density function (normal-distribution) employed by Wang and Raz [35]. In the area of statistical process control we can refer to [2, 12, 13, 29, 30]. One of the control charts proposed in [13] and [14] is based on the concepts of fuzzy statistical confidence intervals. A cumulative sum (CUSUM) control chart for fuzzy quality data has been proposed in [34]. Based on the concept of fuzzy random variables, Wang proposed an optimal representative value for fuzzy quality data by means of a combination of a random variable with a measure of fuzziness. Cheng [10] plotted the fuzzy quality ratings on fuzzy control charts, whose construction and out-of-control conditions are developed using possibility theory. In this paper, we draw control charts that are constructed with α-cut sets. Since an α-cut set is a crisp set, we can see these control charts as special cases of classical control charts that instead of lines we have intervals. Plotting these charts without using the defuzzification methods is the advantage of using the control charts proposed in this paper. As a result, the information in data won't be lost.

The structure of this paper is as follows: in Section 2 we recall some basic concepts, and in Section 3, we place the data tables that will use in next sections. Section 4 shows that using $\overline{X}$-chart is a wrong choice. Section 5 describes the radius of gyration ranking method. Section 6 introduces the membership function for fuzzy estimator of $\widetilde{C}_{pk}$. In Section 7 we generalize the method of monitoring capability index for capable but instable processes. Finally, Section 8 presents conclusions and future lines of research.

2. Preliminaries

Definition 2.1. Let U be a universal set. A fuzzy set $\tilde{A}$ of U is defined by a membership function $\tilde{A} : U \to [0, 1]$, where $\tilde{A}(x), \forall\, x \in U$, indicates the membership degree of x in $\tilde{A}$.

Definition 2.2. A fuzzy subset $\tilde{A}$ of U is called normal if and only if

$$sup\widetilde{A}(x) = 1, \tag{6}$$

$\tilde{A}$ is convex if and only if

$$\tilde{A}(\lambda x + (1 - \lambda)y) \geq min(\tilde{A}(x), \tilde{A}(y)), \forall x, y \in U, \forall \lambda \in [0, 1], \tag{7}$$

and $\tilde{A}$ is a fuzzy number if and only if $\tilde{A}$ is normal and convex on U.

Definition 2.3. The α-cut set of a fuzzy number (which is a closed interval), denoted by $A_\alpha = [A_\alpha^-, A_\alpha^+]$, where

$$A_\alpha^- = inf\{x \in R : \tilde{A}(x) \geq \alpha\}, A_\alpha^+ = sup\{x \in R : \tilde{A}(x) \geq \alpha\}.$$

Definition 2.4. [8] A generalized fuzzy number $\tilde{A}$ is described as any fuzzy subset of the real line R, whose membership function $\tilde{A}(x)$ satisfies the following conditions:

(1) $\tilde{A}(x)$ is a continuous mapping from R to the closed interval [0, 1],
(2) $\tilde{A}(x) = 0$, $-\infty < X \leq a$,
(3) $\tilde{A}(x) = L(x)$ is strictly increasing on $[a, b]$,

(4) $\tilde{A}(x) = w$, $b \leq x \leq c$,
(5) $\tilde{A}(x) = R(x)$ is strictly decreasing on $[c, d]$,
(6) $\tilde{A}(x) = 0$, $d \leq x < \infty$,

where $0 < w \leq 1$, a, b, c, and d are real numbers. We denote this type of generalized fuzzy number as $\tilde{A} = (a, b, c, d; w)$. When $w = 1$, we denote this type of generalized fuzzy number as $\tilde{A} = (a, b, c, d)$.

Definition 2.5. [27] The $D_{p,q}$-distance, indexed by parameters $0 < p < \infty, 0 \leq q \leq 1$ between two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is a nonnegative function on $F(R)\times F(R)$ (we denote $F(R)$ as the set of all fuzzy numbers) gives as follows:

$$D_{p,q}(\tilde{A}, \tilde{B}) = \begin{cases} [(1-q)\int_0^1 |A_\alpha^- - B_\alpha^-|^p d\alpha + q\int_0^1 |A_\alpha^+ - B_\alpha^+|^p d\alpha]^{1/p} & p < \infty, \\ (1-q)sup_{0<\alpha\leq 1}(|A_\alpha^- - B_\alpha^-|) + qinf_{0<\alpha\leq 1}(|A_\alpha^+ - B_\alpha^+|) & p = \infty. \end{cases} \quad (8)$$

where $0 \leq \alpha \leq 1$.

Definition 2.6. [27] A mapping $\tilde{X} : \Omega \longrightarrow F(R)$ is said to be a fuzzy random variable associated with measurable space (Ω, A) if and only if

$$\{(\omega, x) : x \in X_\alpha(\omega)\} \in A \times B,$$

where B denotes the σ-field of Borel sets in R.

Definition 2.7. [27] The mathematical expectation of a fuzzy random variable $\tilde{X}$, for each $\alpha \in [0, 1]$, is defined as follows:

$$[\tilde{E}(\tilde{X})]_\alpha = E(X_\alpha) = [E(X_\alpha^-), E(X_\alpha^+)].$$

Definition 2.8. [27] The central $D_{2,q}$-mean square dispersion of $\tilde{X}$ about $\tilde{E}(\tilde{X})$ or $\tilde{\mu}_{\tilde{X}}$ is defined by

$$\begin{aligned} DVar(\tilde{X}) &= E([D_{2,q}(\tilde{X}, \tilde{\mu}_{\tilde{X}})]^2) \\ &= \int_\Omega [(1-q)\int_0^1 (X_\alpha^-(\omega) - (\mu_{\tilde{X}})_\alpha^-)^2 d\alpha, \\ &+ q\int_0^1 (X_\alpha^+(\omega) - (\mu_{\tilde{X}})_\alpha^+)^2 d\alpha] dP(\omega). \end{aligned}$$

Assume that $\tilde{A}$ and $\tilde{B}$ are two triangular fuzzy numbers as:

$$\tilde{A} = (a_1, a_2, a_3), \tilde{B} = (b_1, b_2, b_3),$$

then the α-cut sets of $\tilde{A}$ and $\tilde{B}$ are as follows:

$$A_\alpha = [(1-\alpha)a_1 + a_2\alpha, a_2\alpha + (1-\alpha)a_3], B_\alpha = [(1-\alpha)b_1 + b_2\alpha, b_2\alpha + (1-\alpha)b_3].$$

It can establish that:

$$D_{2,\frac{1}{2}}(\tilde{A}, \tilde{B})]^2 = \frac{1}{6}[(b_1 - a_1)^2 + 2(b_2 - a_2)^2 + (b_3 - a_3)^2 + (b_1 - a_1)(b_2 - a_2) + (b_3 - a_3)(b_2 - a_2)]. \quad (9)$$

And the estimator of $DVar(\tilde{X})$ is:

$$\hat{D}Var(\tilde{X}) = \frac{1}{n}\sum_{i=1}^{n}[D_{2,q}(\tilde{X}_i, \tilde{\bar{X}})]^2. \quad (10)$$

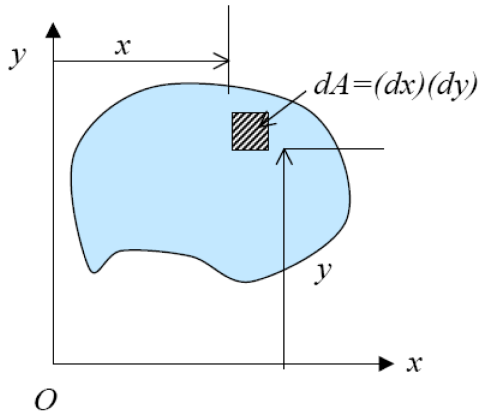


FIGURE 1. The Moment of Inertia of an Area

Definition 2.9. [21] The mass moment of inertia, I, of a body is a measure of the inertial resistance of the body to rotational acceleration and is expressed by the integral $I = \int r^2 dm$ where dm is the differential element of mass and r is the perpendicular distance from dm to the rotation axis.

Definition 2.10. [21] The area moment of inertia of a defined area about a given axis is expressed by the integral $I = \int s^2 dA$, where dA is the differential element of area and s is the perpendicular distance from dA to a defined axis either in or normal to the plane of the area. The mathematical similarity to mass moment of inertia gives rise to its name. A more fitting, but less used term is the second moment of area.

Definition 2.11. For area A in the xy plane, Figure 1, the moments of inertia of the element dA about the x and y axes are, by Definition 2.10, $dI_x = y^2 dA$ and $dI_y = x^2 dA$, respectively. The moments of inertia of A about the same axes become

$$I_x = \int y^2 dA, I_y = \int x^2 dA. \tag{11}$$

where the integration is carried out over the entire area.

Definition 2.12. The Radius of Gyration of an Area about a given axis is a distance r from the axis. At this distance r an equivalent area is thought of as a line Area parallel to the original axis. The moment of inertia of this Line Area about the original axis is unchanged. The radius of gyration of an area A with respect to the x axis is defined as the quantity that satisfies the relation,

$$I_x = r_x^2 A, \tag{12}$$

where I_x the moment of inertia of A with respect to the x axis. Solving equation for r_x, concludes that:

$$r_x = \sqrt{\frac{I_x}{A}}. \tag{13}$$

Similarly, the radius of gyration of an area A with respect to the y axis is defined as follows:

$$r_y = \sqrt{\frac{I_y}{A}}. \tag{14}$$

3. The Data

In this section we have two tables of the triangular fuzzy data from different fuzzy normal distributions that will be used.

Example 3.1. The data in Table 1 are collected from eight different fuzzy normal distributions from the manufacturing process with $\widetilde{C}_{pk} \simeq \frac{4}{3}$. Therefore, we have eight subgroups, each consists of five samples, with sample sizes equal to seven. Hence, we have forty samples. The $USL = 1$, $LSL = -1$ and $T = 0$.

$\widetilde{X}_1 = (-1.024, -.201, .622)$	$\widetilde{X}_{94} = (.263, .523, .783)$	$\widetilde{X}_{187} = (-1.380, -.459, .464)$
$\widetilde{X}_2 = (-.521, .174, .869)$	$\widetilde{X}_{95} = (-.360, .441, 1.241)$	$\widetilde{X}_{188} = (-.795, -.365, .066)$
$\widetilde{X}_3 = (-.108, .208, .529)$	$\widetilde{X}_{96} = (.176, .608, 1.039)$	$\widetilde{X}_{189} = (-.560, -.370, -.189)$
$\widetilde{X}_4 = (-1.010, -.061, .889)$	$\widetilde{X}_{97} = (-.581, .330, 1.240)$	$\widetilde{X}_{190} = (-1.490, -.581, .324)$
$\widetilde{X}_5 = (.505, .539, .573)$	$\widetilde{X}_{98} = (.375, .557, .739)$	$\widetilde{X}_{191} = (-1.450, -.468, .512)$
$\widetilde{X}_6 = (-.730, -.291, .147)$	$\widetilde{X}_{99} = (.130, .394, .658)$	$\widetilde{X}_{192} = (-1.060, -.620, -.180)$
$\widetilde{X}_7 = (-.669, -.287, .095)$	$\widetilde{X}_{100} = (.313, .458, .604)$	$\widetilde{X}_{193} = (-.780, -.670, -.554)$
$\widetilde{X}_8 = (-.739, .026, .792)$	$\widetilde{X}_{101} = (.433, .569, .705)$	$\widetilde{X}_{194} = (-.640, -.380, -.126)$
$\widetilde{X}_9 = (-.615, .181, .976)$	$\widetilde{X}_{102} = (-.240, .630, 1.499)$	$\widetilde{X}_{195} = (-.909, -.500, -.091)$
$\widetilde{X}_{10} = (.459, .646, .833)$	$\widetilde{X}_{103} = (-.219, .360, .940)$	$\widetilde{X}_{196} = (-1.102, -.507, .090)$
$\widetilde{X}_{11} = (-.657, -.167, .323)$	$\widetilde{X}_{104} = (.108, .658, 1.208)$	$\widetilde{X}_{197} = (-.650, -.386, -.120)$
$\widetilde{X}_{12} = (-.399, .047, .492)$	$\widetilde{X}_{105} = (.437, .583, .727)$	$\widetilde{X}_{198} = (-1.030, -.426, .177)$
$\widetilde{X}_{13} = (-.667, -.021, .626)$	$\widetilde{X}_{106} = (-.870, -.017, .836)$	$\widetilde{X}_{199} = (-1.167, -.456, .250)$
$\widetilde{X}_{14} = (-1.190, -.483, .226)$	$\widetilde{X}_{107} = (-.671, -.049, .574)$	$\widetilde{X}_{200} = (-.560, -.344, -.120)$
$\widetilde{X}_{15} = (-.864, -.110, .650)$	$\widetilde{X}_{108} = (-.405, -.055, .297)$	$\widetilde{X}_{201} = (-.500, -.380, -.266)$
$\widetilde{X}_{16} = (-.725, -.449, -.170)$	$\widetilde{X}_{109} = (-.589, -.076, .437)$	$\widetilde{X}_{202} = (-.767, -.470, -.170)$
$\widetilde{X}_{17} = (-.470, .210, .890)$	$\widetilde{X}_{110} = (-.396, .006, .408)$	$\widetilde{X}_{203} = (-.910, -.586, -.270)$
$\widetilde{X}_{18} = (-.877, -.220, .433)$	$\widetilde{X}_{111} = (-.063, .013, .089)$	$\widetilde{X}_{204} = (-1.010, -.580, -.157)$
$\widetilde{X}_{19} = (-.138, .025, .188)$	$\widetilde{X}_{112} = (-.033, .206, .446)$	$\widetilde{X}_{205} = (-.859, -.350, .157)$
$\widetilde{X}_{20} = (-.260, -.136, -.017)$	$\widetilde{X}_{113} = (.258, .382, .505)$	$\widetilde{X}_{206} = (-.790, -.702, -.620)$
$\widetilde{X}_{21} = (-.422, .076, .574)$	$\widetilde{X}_{114} = (-.067, .117, .301)$	$\widetilde{X}_{207} = (-.766, -.510, -.240)$
$\widetilde{X}_{22} = (-1.110, -.150, .810)$	$\widetilde{X}_{115} = (-.292, -.052, .188)$	$\widetilde{X}_{208} = (-1.55, -.744, .057)$
$\widetilde{X}_{23} = (-.218, .122, .463)$	$\widetilde{X}_{116} = (-.261, .156, .574)$	$\widetilde{X}_{209} = (-.402, -.370, -.340)$
$\widetilde{X}_{24} = (-.401, .185, .770)$	$\widetilde{X}_{117} = (-.004, .046, .095)$	$\widetilde{X}_{210} = (-1.320, -.392, .537)$
$\widetilde{X}_{25} = (.204, .428, .659)$	$\widetilde{X}_{118} = (-1.160, -.257, .645)$	$\widetilde{X}_{211} = (-1.480, -.750, -.020)$
$\widetilde{X}_{26} = (-.800, -.049, .710)$	$\widetilde{X}_{119} = (-.707, .237, 1.182)$	$\widetilde{X}_{212} = (-1.240, -.750, -.270)$
$\widetilde{X}_{27} = (-.790, -.540, -.280)$	$\widetilde{X}_{120} = (-.414, .077, .568)$	$\widetilde{X}_{213} = (-1.480, -.910, -.330)$
$\widetilde{X}_{28} = (-.720, -.210, .296)$	$\widetilde{X}_{121} = (-.455, .034, .523)$	$\widetilde{X}_{214} = (-.950, -.710, -.476)$
$\widetilde{X}_{29} = (-.361, .339, 1.038)$	$\widetilde{X}_{122} = (-.209, .129, .466)$	$\widetilde{X}_{215} = (-1.350, -.890, -.430)$
$\widetilde{X}_{30} = (-1.159, -.268, .620)$	$\widetilde{X}_{123} = (-.835, .065, .965)$	$\widetilde{X}_{216} = (-1.860, -.895, .068)$
$\widetilde{X}_{31} = (-.719, .240, 1.199)$	$\widetilde{X}_{124} = (-.605, -.235, .134)$	$\widetilde{X}_{217} = (-1.29, -.740, -.200)$
$\widetilde{X}_{32} = (-.516, .031, .578)$	$\widetilde{X}_{125} = (-.152, -.041, .071)$	$\widetilde{X}_{218} = (-1.330, -.810, -.290)$
$\widetilde{X}_{33} = (.221, .359, .498)$	$\widetilde{X}_{126} = (-.817, -.036, .740)$	$\widetilde{X}_{219} = (-.958, -.720, -.490)$
$\widetilde{X}_{34} = (-.639, -.490, -.341)$	$\widetilde{X}_{127} = (-.523, -.133, .257)$	$\widetilde{X}_{220} = (-1.200, -.708, -.220)$
$\widetilde{X}_{35} = (-.307, -.049, .208)$	$\widetilde{X}_{128} = (.179, .420, .662)$	$\widetilde{X}_{221} = (-1.320, -.700, -.070)$
$\widetilde{X}_{36} = (-.817, .024, .864)$	$\widetilde{X}_{129} = (-.623, -.219, .180)$	$\widetilde{X}_{222} = (-1.470, -.790, -.110)$
$\widetilde{X}_{37} = (.541, .795, 1.050)$	$\widetilde{X}_{130} = (-.217, -.120, -.020)$	$\widetilde{X}_{223} = (-1.120, -.720, -.330)$
$\widetilde{X}_{38} = (-.410, .405, 1.220)$	$\widetilde{X}_{131} = (-.310, -.178, -.050)$	$\widetilde{X}_{224} = (-1.110, -.740, -.380)$

Continued on Next Page

$\widetilde{X}_{39} = (.265, .509, .752)$	$\widetilde{X}_{132} = (-1.236, -.294, .650)$	$\widetilde{X}_{225} = (-1.69, -.698, .290)$
$\widetilde{X}_{40} = (-.878, .052, .981)$	$\widetilde{X}_{133} = (-1.004, -.048, .910)$	$\widetilde{X}_{226} = (-.750, -.720, -.679)$
$\widetilde{X}_{41} = (-.188, .162, .512)$	$\widetilde{X}_{134} = (-.644, -.069, .507)$	$\widetilde{X}_{227} = (-1.580, -.694, .191)$
$\widetilde{X}_{42} = (.002, .199, .395)$	$\widetilde{X}_{135} = (.323, .382, .442)$	$\widetilde{X}_{228} = (-1.670, -.758, .155)$
$\widetilde{X}_{43} = (.205, .456, .707)$	$\widetilde{X}_{136} = (-.297, -.062, .173)$	$\widetilde{X}_{229} = (-1.550, -.759, .037)$
$\widetilde{X}_{44} = (-.418, .198, .814)$	$\widetilde{X}_{137} = (-.619, -.266, .087)$	$\widetilde{X}_{230} = (-.786, -.687, -.583)$
$\widetilde{X}_{45} = (-.092, .381, .855)$	$\widetilde{X}_{138} = (-.420, .401, 1.222)$	$\widetilde{X}_{231} = (-1.145, -.883, -.621)$
$\widetilde{X}_{46} = (-.486, -.135, .217)$	$\widetilde{X}_{139} = (.293, .309, .324)$	$\widetilde{X}_{232} = (-1.117, -.780, -.446)$
$\widetilde{X}_{47} = (-.647, .184, 1.014)$	$\widetilde{X}_{140} = (-.100, .057, -.014)$	$\widetilde{X}_{233} = (-1.509, -.829, -.150)$
$\widetilde{X}_{48} = (-.490, .096, .681)$	$\widetilde{X}_{141} = (-.700, -.532, -.363)$	$\widetilde{X}_{234} = (-.911, -.774, -.637)$
$\widetilde{X}_{49} = (-.595, -.046, .504)$	$\widetilde{X}_{142} = (-.983, -.333, .316)$	$\widetilde{X}_{235} = (-1.430, -.709, .012)$
$\widetilde{X}_{50} = (-.572, .345, 1.262)$	$\widetilde{X}_{143} = (-1.011, -.279, .453)$	$\widetilde{X}_{236} = (-.805, -.698, -.592)$
$\widetilde{X}_{51} = (.017, .303, .589)$	$\widetilde{X}_{144} = (-.846, -.198, .449)$	$\widetilde{X}_{237} = (-1.470, -.810, -.160)$
$\widetilde{X}_{52} = (-.501, .256, 1.014)$	$\widetilde{X}_{145} = (-.750, -.299, .152)$	$\widetilde{X}_{238} = (-1.270, -.779, -.285)$
$\widetilde{X}_{53} = (-.754, -.0001, .754)$	$\widetilde{X}_{146} = (-.714, -.167, .380)$	$\widetilde{X}_{239} = (-1.520, -.741, .038)$
$\widetilde{X}_{54} = (.081, .461, .842)$	$\widetilde{X}_{147} = (-.473, -.176, .120)$	$\widetilde{X}_{240} = (-1.480, -.768, -.053)$
$\widetilde{X}_{55} = (-.250, .316, .883)$	$\widetilde{X}_{148} = (-1.229, -.484, .26)$	$\widetilde{X}_{241} = (-1.635, -.731, .173)$
$\widetilde{X}_{56} = (.118, .194, .270)$	$\widetilde{X}_{149} = (-.617, -.428, -.239)$	$\widetilde{X}_{242} = (-1.619, -.725, .166)$
$\widetilde{X}_{57} = (.2003, .254, .308)$	$\widetilde{X}_{150} = (-1.076, -.389, .298)$	$\widetilde{X}_{243} = (-1.140, -.808, -.474)$
$\widetilde{X}_{58} = (-.330, .201, .732)$	$\widetilde{X}_{151} = (-.529, -.345, -.162)$	$\widetilde{X}_{244} = (-1.460, -.760, -.062)$
$\widetilde{X}_{59} = (-.857, -.078, .701)$	$\widetilde{X}_{152} = (-.679, -.310, .058)$	$\widetilde{X}_{245} = (-1.080, -.880, -.685)$
$\widetilde{X}_{60} = (-.738, .196, 1.13)$	$\widetilde{X}_{153} = (-.870, -.248, .378)$	$\widetilde{X}_{246} = (-.066, -.035, -.005)$
$\widetilde{X}_{61} = (-.036, .094, .224)$	$\widetilde{X}_{154} = (-1.598, -.818, -.038)$	$\widetilde{X}_{247} = (-1.113, -.368, .376)$
$\widetilde{X}_{62} = (-.502, .066, .635)$	$\widetilde{X}_{155} = (-.417, -.336, -.255)$	$\widetilde{X}_{248} = (-.976, -.476, .024)$
$\widetilde{X}_{63} = (-.436, .033, .503)$	$\widetilde{X}_{156} = (-.946, -.017, .912)$	$\widetilde{X}_{249} = (-.778, -.298, .182)$
$\widetilde{X}_{64} = (.138, .150, .162)$	$\widetilde{X}_{157} = (-1.225, -.450, .326)$	$\widetilde{X}_{250} = (-1.424, -.518, .385)$
$\widetilde{X}_{65} = (-.463, -.125, .212)$	$\widetilde{X}_{158} = (-.562, -.075, .412)$	$\widetilde{X}_{251} = (-.864, -.254, .356)$
$\widetilde{X}_{66} = (.269, .431, .593)$	$\widetilde{X}_{159} = (-.620, -.184, .252)$	$\widetilde{X}_{252} = (-.973, -.255, .263)$
$\widetilde{X}_{67} = (-.447, .347, 1.142)$	$\widetilde{X}_{160} = (-.702, -.255, .191)$	$\widetilde{X}_{253} = (-.701, .158, 1.018)$
$\widetilde{X}_{68} = (-.065, .246, .557)$	$\widetilde{X}_{161} = (-.522, -.216, .091)$	$\widetilde{X}_{254} = (-.842, -.036, .769)$
$\widetilde{X}_{69} = (-.345, .243, .832)$	$\widetilde{X}_{162} = (-1.052, -.543, -.035)$	$\widetilde{X}_{255} = (-1.290, -.718, -.141)$
$\widetilde{X}_{70} = (-.065, .1003, .266)$	$\widetilde{X}_{163} = (-.777, -.266, .245)$	$\widetilde{X}_{256} = (-.350, -.167, .016)$
$\widetilde{X}_{71} = (.025, .627, 1.229)$	$\widetilde{X}_{164} = (-.767, .051, .868)$	$\widetilde{X}_{257} = (-.750, -.512, -.272)$
$\widetilde{X}_{72} = (.220, .483, .746)$	$\widetilde{X}_{165} = (-1.026, -.232, .563)$	$\widetilde{X}_{258} = (-1.184, -.298, .589)$
$\widetilde{X}_{73} = (-.243, .411, 1.065)$	$\widetilde{X}_{166} = (-.886, -.242, .402)$	$\widetilde{X}_{259} = (-.248, -.219, -.191)$
$\widetilde{X}_{74} = (-.020, .669, 1.358)$	$\widetilde{X}_{167} = (-.766, -.388, -.009)$	$\widetilde{X}_{260} = (-.600, -.110, .380)$
$\widetilde{X}_{75} = (-.276, .472, 1.220)$	$\widetilde{X}_{168} = (-1.067, -.256, .556)$	$\widetilde{X}_{261} = (-.469, -.300, -.133)$
$\widetilde{X}_{76} = (-.024, .426, .877)$	$\widetilde{X}_{169} = (-.739, -.206, .326)$	$\widetilde{X}_{262} = (-.933, .046, 1.024)$
$\widetilde{X}_{77} = (.379, .463, .547)$	$\widetilde{X}_{170} = (-.521, -.170, .181)$	$\widetilde{X}_{263} = (-1.053, -.340, .372)$
$\widetilde{X}_{78} = (.165, .394, .623)$	$\widetilde{X}_{171} = (-1.259, -.320, .619)$	$\widetilde{X}_{264} = (-.689, -.189, .312)$
$\widetilde{X}_{79} = (-.553, .360, 1.273)$	$\widetilde{X}_{172} = (-1.170, -.294, .582)$	$\widetilde{X}_{265} = (-.597, -.125, .346)$
$\widetilde{X}_{80} = (.663, .816, .968)$	$\widetilde{X}_{173} = (-.421, .129, .680)$	$\widetilde{X}_{266} = (.174, .234, .294)$
$\widetilde{X}_{81} = (-.119, .707, 1.530)$	$\widetilde{X}_{174} = (-1.296, -.673, -.051)$	$\widetilde{X}_{267} = (-.767, -.085, .597)$
$\widetilde{X}_{82} = (.0001, .538, 1.077)$	$\widetilde{X}_{175} = (-.419, .168, .755)$	$\widetilde{X}_{268} = (-.232, -.189, -.147)$
$\widetilde{X}_{83} = (-.653, .343, 1.339)$	$\widetilde{X}_{176} = (-.660, -.458, -.25)$	$\widetilde{X}_{269} = (-.468, -.397, -.326)$
$\widetilde{X}_{84} = (.314, .392, .470)$	$\widetilde{X}_{177} = (-.676, -.375, -.074)$	$\widetilde{X}_{270} = (-1.047, -.590, -.004)$
$\widetilde{X}_{85} = (.035, .478, .921)$	$\widetilde{X}_{178} = (-1.188, -.708, -.228)$	$\widetilde{X}_{271} = (.002, .098, .195)$
$\widetilde{X}_{86} = (.492, .599, .706)$	$\widetilde{X}_{179} = (-1.379, -.574, .309)$	$\widetilde{X}_{272} = (-1.180, -.363, .455)$
$\widetilde{X}_{87} = (-.628, .333, 1.295)$	$\widetilde{X}_{180} = (.309, -.535, -1.379)$	$\widetilde{X}_{273} = (-1.048, -.231, .587)$
$\widetilde{X}_{88} = (.204, .209, .213)$	$\widetilde{X}_{181} = (-.642, -.447, -.252)$	$\widetilde{X}_{274} = (-.867, -.144, .578)$
$\widetilde{X}_{89} = (-.460, .319, 1.094)$	$\widetilde{X}_{182} = (-.935, -.709, -.483)$	$\widetilde{X}_{275} = (-.378, -.228, -.079)$
$\widetilde{X}_{90} = (-.276, .542, 1.359)$	$\widetilde{X}_{183} = (-.612, -.440, -.270)$	$\widetilde{X}_{276} = (-1.100, -.420, .240)$
$\widetilde{X}_{91} = (-.320, .549, 1.418)$	$\widetilde{X}_{184} = (-.879, -.652, -.424)$	$\widetilde{X}_{277} = (-.856, -.338, .181)$
$\widetilde{X}_{92} = (.472, .556, .641)$	$\widetilde{X}_{185} = (-.927, -.492, -.056)$	$\widetilde{X}_{278} = (-1.246, -.270, .700)$

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$\widetilde{X}_{93} = (.084, .484, .883)$	$\widetilde{X}_{186} = (-.730, -.418, -.107)$	$\widetilde{X}_{279} = (-.622, .027, .676)$
		$\widetilde{X}_{280} = (-1.211, -.411, .389)$

TABLE 1. The Fuzzy Normal Data

Example 3.2. The data in Table 2 are collected from two different fuzzy normal distributions, from the manufacturing process with $\widetilde{C}_{pk} \simeq \frac{4}{3}$ for the first subgroup and $\widetilde{C}_{pk} \simeq 1$ for the second one. Therefore, we have two subgroups, each consist of twenty samples, with sample sizes equal to seven. Hence, we have forty samples. The $USL = 1$, $LSL = -1$ and $T = 0$.

$\widetilde{X}_1 = (-.773, .134, 1.039)$	$\widetilde{X}_{94} = (.420, .646, .873)$	$\widetilde{X}_{187} = (.473, .743, 1.012)$
$\widetilde{X}_2 = (-.151, .458, 1.068)$	$\widetilde{X}_{95} = (-.551, .167, .218)$	$\widetilde{X}_{188} = (.126, .891, 1.657)$
$\widetilde{X}_3 = (-1.180, -.565, .053)$	$\widetilde{X}_{96} = (-.536, .047, .630)$	$\widetilde{X}_{189} = (.665, .854, 1.042)$
$\widetilde{X}_4 = (-.644, .215, 1.075)$	$\widetilde{X}_{97} = (-.272, -.021, .231)$	$\widetilde{X}_{190} = (.521, .808, 1.096)$
$\widetilde{X}_5 = (-.726, .080, .885)$	$\widetilde{X}_{98} = (-.774, -.483, -.193)$	$\widetilde{X}_{191} = (.709, .800, .891)$
$\widetilde{X}_6 = (-.904, -.327, .250)$	$\widetilde{X}_{99} = (-.727, -.110, .507)$	$\widetilde{X}_{192} = (.223, .799, 1.375)$
$\widetilde{X}_7 = (-.290, -.108, .074)$	$\widetilde{X}_{100} = (-.744, -.448, -.153)$	$\widetilde{X}_{193} = (.110, .794, 1.477)$
$\widetilde{X}_8 = (-.154, .086, .326)$	$\widetilde{X}_{101} = (-.614, .210, 1.034)$	$\widetilde{X}_{194} = (.267, .814, 1.360)$
$\widetilde{X}_9 = (.008, .895, 1.780)$	$\widetilde{X}_{102} = (-1.205, -.222, .761)$	$\widetilde{X}_{195} = (.390, .816, 1.241)$
$\widetilde{X}_{10} = (.664, .692, .720)$	$\widetilde{X}_{103} = (-.705, .025, .755)$	$\widetilde{X}_{196} = (.220, .864, 1.508)$
$\widetilde{X}_{11} = (-.827, -.338, .152)$	$\widetilde{X}_{104} = (-.480, -.136, .208)$	$\widetilde{X}_{197} = (.260, .908, 1.555)$
$\widetilde{X}_{12} = (.591, .759, .927)$	$\widetilde{X}_{105} = (-.508, .076, .660)$	$\widetilde{X}_{198} = (.163, .842, 1.435)$
$\widetilde{X}_{13} = (-.797, .181, 1.160)$	$\widetilde{X}_{106} = (-.258, -.150, -.040)$	$\widetilde{X}_{199} = (.164, .799, 1.435)$
$\widetilde{X}_{14} = (-.728, -.016, .697)$	$\widetilde{X}_{107} = (-1.029, -.122, .784)$	$\widetilde{X}_{200} = (-.094, .852, 1.797)$
$\widetilde{X}_{15} = (-.322, .179, .679)$	$\widetilde{X}_{108} = (-.695, .185, 1.064)$	$\widetilde{X}_{201} = (.615, .824, 1.033)$
$\widetilde{X}_{16} = (-.522, -.051, .419)$	$\widetilde{X}_{109} = (-.390, .428, 1.246)$	$\widetilde{X}_{202} = (.039, .748, 1.457)$
$\widetilde{X}_{17} = (-.091, -.031, .029)$	$\widetilde{X}_{110} = (-.309, -.048, .210)$	$\widetilde{X}_{203} = (.636, .872, 1.108)$
$\widetilde{X}_{18} = (-.490, .372, 1.230)$	$\widetilde{X}_{111} = (-1.129, -.535, .060)$	$\widetilde{X}_{204} = (.712, .832, .951)$
$\widetilde{X}_{19} = (.310, .352, .395)$	$\widetilde{X}_{112} = (-.230, -.210, -.187)$	$\widetilde{X}_{205} = (.214, .821, 1.428)$
$\widetilde{X}_{20} = (.395, .354, .283)$	$\widetilde{X}_{113} = (-.087, .339, .764)$	$\widetilde{X}_{206} = (.395, .845, 1.295)$
$\widetilde{X}_{21} = (-.354, .168, .689)$	$\widetilde{X}_{114} = (-.581, -.268, .045)$	$\widetilde{X}_{207} = (.370, .829, 1.287)$
$\widetilde{X}_{22} = (-.399, -.302, -.205)$	$\widetilde{X}_{115} = (.079, .240, .402)$	$\widetilde{X}_{208} = (.092, .754, 1.416)$
$\widetilde{X}_{23} = (-.639, .179, .997)$	$\widetilde{X}_{116} = (-.148, .031, .210)$	$\widetilde{X}_{209} = (.032, .802, 1.573)$
$\widetilde{X}_{24} = (-.410, .408, 1.225)$	$\widetilde{X}_{117} = (-.064, .359, .782)$	$\widetilde{X}_{210} = (.453, .803, 1.154)$
$\widetilde{X}_{25} = (-.600, .122, .845)$	$\widetilde{X}_{118} = (-.584, -.490, -.396)$	$\widetilde{X}_{211} = (.117, .779, 1.440)$
$\widetilde{X}_{26} = (.109, .259, .409)$	$\widetilde{X}_{119} = (-.848, -.049, .549)$	$\widetilde{X}_{212} = (.500, .918, 1.332)$
$\widetilde{X}_{27} = (-.478, .182, .840)$	$\widetilde{X}_{120} = (-.773, -.302, .169)$	$\widetilde{X}_{213} = (-.084, .758, 1.600)$
$\widetilde{X}_{28} = (-.594, -.076, .443)$	$\widetilde{X}_{121} = (.031, .727, 1.423)$	$\widetilde{X}_{214} = (-.051, .782, 1.615)$
$\widetilde{X}_{29} = (-.899, .074, 1.046)$	$\widetilde{X}_{122} = (-.494, .206, .906)$	$\widetilde{X}_{215} = (.512, .768, 1.024)$
$\widetilde{X}_{30} = (-1.170, -.197, .776)$	$\widetilde{X}_{123} = (-.294, .345, .983)$	$\widetilde{X}_{216} = (.126, .739, 1.353)$
$\widetilde{X}_{31} = (-.578, .222, 1.022)$	$\widetilde{X}_{124} = (-.298, -.260, -.231)$	$\widetilde{X}_{217} = (.218, .801, 1.383)$
$\widetilde{X}_{32} = (-.741, -.287, .167)$	$\widetilde{X}_{125} = (-.186, -.117, -.048)$	$\widetilde{X}_{218} = (.255, .795, 1.336)$
$\widetilde{X}_{33} = (-.700, -.267, .165)$	$\widetilde{X}_{126} = (-.388, -.068, .252)$	$\widetilde{X}_{219} = (.038, .908, 1.778)$
$\widetilde{X}_{34} = (-1.028, -.202, .623)$	$\widetilde{X}_{127} = (-.256, .275, .805)$	$\widetilde{X}_{220} = (.532, .797, 1.062)$
$\widetilde{X}_{35} = (-.820, -.740, -.653)$	$\widetilde{X}_{128} = (-.724, -.069, .585)$	$\widetilde{X}_{221} = (.428, .746, 1.064)$
$\widetilde{X}_{36} = (.226, .360, .493)$	$\widetilde{X}_{129} = (-.232, .175, .583)$	$\widetilde{X}_{222} = (.793, .913, 1.032)$
$\widetilde{X}_{37} = (-.092, .081, .255)$	$\widetilde{X}_{130} = (-1.333, -.513, .307)$	$\widetilde{X}_{223} = (-.050, .890, 1.829)$
$\widetilde{X}_{38} = (-.580, -.189, .202)$	$\widetilde{X}_{131} = (-.807, -.088, .630)$	$\widetilde{X}_{224} = (.152, .798, 1.444)$

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$\widetilde{X}_{39} = (-.489, .343, 1.174)$	$\widetilde{X}_{132} = (-1.174, -.206, .763)$	$\widetilde{X}_{225} = (.239, .718, 1.198)$
$\widetilde{X}_{40} = (-1.231, -.428, .370)$	$\widetilde{X}_{133} = (-.926, -.394, .140)$	$\widetilde{X}_{226} = (.145, .785, 1.424)$
$\widetilde{X}_{41} = (-.086, -.026, .035)$	$\widetilde{X}_{134} = (-.198, .127, .452)$	$\widetilde{X}_{227} = (.258, .802, 1.348)$
$\widetilde{X}_{42} = (-.460, -.060, .339)$	$\widetilde{X}_{135} = (-.035, .071, .176)$	$\widetilde{X}_{228} = (.182, .830, 1.477)$
$\widetilde{X}_{43} = (-.447, .078, .495)$	$\widetilde{X}_{136} = (-.603, .008, .619)$	$\widetilde{X}_{229} = (.252, .796, 1.340)$
$\widetilde{X}_{44} = (-.339, .078, .495)$	$\widetilde{X}_{137} = (-1.112, -.333, .445)$	$\widetilde{X}_{230} = (.119, .840, 1.561)$
$\widetilde{X}_{45} = (-.873, -.216, .441)$	$\widetilde{X}_{138} = (-.142, .282, .705)$	$\widetilde{X}_{231} = (.314, .837, 1.359)$
$\widetilde{X}_{46} = (-.635, -.007, .620)$	$\widetilde{X}_{139} = (-.003, .087, .178)$	$\widetilde{X}_{232} = (-.259, .734, 1.728)$
$\widetilde{X}_{47} = (-.333, -.041, .250)$	$\widetilde{X}_{140} = (-.341, -.075, .192)$	$\widetilde{X}_{233} = (.535, .753, .972)$
$\widetilde{X}_{48} = (-.275, .157, .589)$	$\widetilde{X}_{141} = (.660, .814, .968)$	$\widetilde{X}_{234} = (.660, .766, .872)$
$\widetilde{X}_{49} = (.258, .273, .289)$	$\widetilde{X}_{142} = (.515, .796, 1.077)$	$\widetilde{X}_{235} = (.671, .781, .891)$
$\widetilde{X}_{50} = (-.707, .278, 1.261)$	$\widetilde{X}_{143} = (.263, .703, .143)$	$\widetilde{X}_{236} = (.729, .792, .856)$
$\widetilde{X}_{51} = (-.383, -.216, -.049)$	$\widetilde{X}_{144} = (.267, .795, 1.322)$	$\widetilde{X}_{237} = (.409, .813, 1.218)$
$\widetilde{X}_{52} = (-.087, .019, .125)$	$\widetilde{X}_{145} = (.303, .761, .218)$	$\widetilde{X}_{238} = (.175, .623, 1.072)$
$\widetilde{X}_{53} = (-.676, -.303, .069)$	$\widetilde{X}_{146} = (-.124, .751, 1.627)$	$\widetilde{X}_{239} = (.418, .784, 1.148)$
$\widetilde{X}_{54} = (-.476, -.280, -.080)$	$\widetilde{X}_{147} = (.222, .740, 1.258)$	$\widetilde{X}_{240} = (.127, .890, 1.654)$
$\widetilde{X}_{55} = (-.491, -.002, .488)$	$\widetilde{X}_{148} = (-.164, .779, 1.723)$	$\widetilde{X}_{241} = (.118, .746, 1.374)$
$\widetilde{X}_{56} = (.044, .383, .723)$	$\widetilde{X}_{149} = (.050, .687, 1.325)$	$\widetilde{X}_{242} = (.099, .871, 1.643)$
$\widetilde{X}_{57} = (-1.140, -.192, .759)$	$\widetilde{X}_{150} = (-.085, .873, 1.830)$	$\widetilde{X}_{243} = (-.098, .834, 1.767)$
$\widetilde{X}_{58} = (-.827, .093, 1.013)$	$\widetilde{X}_{151} = (.604, .845, 1.086)$	$\widetilde{X}_{244} = (-.113, .811, 1.733)$
$\widetilde{X}_{59} = (-.109, -.056, -.004)$	$\widetilde{X}_{152} = (.136, .811, 1.488)$	$\widetilde{X}_{245} = (.632, .824, 1.016)$
$\widetilde{X}_{60} = (-.459, .279, 1.017)$	$\widetilde{X}_{153} = (.521, .810, 1.094)$	$\widetilde{X}_{246} = (.576, .715, .854)$
$\widetilde{X}_{61} = (-.541, -.270, -.003)$	$\widetilde{X}_{154} = (.091, .763, 1.434)$	$\widetilde{X}_{247} = (.111, .807, 1.503)$
$\widetilde{X}_{62} = (-.414, .008, .431)$	$\widetilde{X}_{155} = (.181, .876, 1.571)$	$\widetilde{X}_{248} = (.819, .913, 1.006)$
$\widetilde{X}_{63} = (-.410, .138, .686)$	$\widetilde{X}_{156} = (.736, .804, .872)$	$\widetilde{X}_{249} = (.293, .819, 1.344)$
$\widetilde{X}_{64} = (-.667, .275, 1.218)$	$\widetilde{X}_{157} = (.513, .768, 1.023)$	$\widetilde{X}_{250} = (.285, .815, 1.345)$
$\widetilde{X}_{65} = (-.032, .386, .804)$	$\widetilde{X}_{158} = (.673, .897, 1.121)$	$\widetilde{X}_{251} = (-.094, .767, 1.628)$
$\widetilde{X}_{66} = (-.962, .022, 1.005)$	$\widetilde{X}_{159} = (.131, .798, 1.466)$	$\widetilde{X}_{252} = (.326, .811, 1.295)$
$\widetilde{X}_{67} = (-.670, -.373, -.071)$	$\widetilde{X}_{160} = (-.069, .776, 1.620)$	$\widetilde{X}_{253} = (.433, .827, 1.220)$
$\widetilde{X}_{68} = (-.887, -.186, .510)$	$\widetilde{X}_{161} = (.450, .794, 1.139)$	$\widetilde{X}_{254} = (.168, .839, 1.510)$
$\widetilde{X}_{69} = (-.932, -.265, .401)$	$\widetilde{X}_{162} = (-.021, .759, 1.540)$	$\widetilde{X}_{255} = (.048, .789, 1.530)$
$\widetilde{X}_{70} = (.048, .588, 1.127)$	$\widetilde{X}_{163} = (.067, .742, 1.418)$	$\widetilde{X}_{256} = (.278, .798, 1.318)$
$\widetilde{X}_{71} = (-.852, -.154, .544)$	$\widetilde{X}_{164} = (.964, .970, .977)$	$\widetilde{X}_{257} = (.591, .939, 1.287)$
$\widetilde{X}_{72} = (.853, .187, -.479)$	$\widetilde{X}_{165} = (.314, .916, 1.518)$	$\widetilde{X}_{258} = (.521, .671, .821)$
$\widetilde{X}_{73} = (-.226, -.048, .130)$	$\widetilde{X}_{166} = (.445, .832, 1.218)$	$\widetilde{X}_{259} = (.366, .952, 1.538)$
$\widetilde{X}_{74} = (.094, .222, .350)$	$\widetilde{X}_{167} = (-.182, .734, 1.650)$	$\widetilde{X}_{260} = (.571, .834, 1.096)$
$\widetilde{X}_{75} = (-1.190, -.191, .808)$	$\widetilde{X}_{168} = (.757, .758, .760)$	$\widetilde{X}_{261} = (.830, .875, .919)$
$\widetilde{X}_{76} = (-.522, -.351, -.179)$	$\widetilde{X}_{169} = (.339, .801, 1.264)$	$\widetilde{X}_{262} = (-.046, .708, 1.463)$
$\widetilde{X}_{77} = (-.388, -.356, -.323)$	$\widetilde{X}_{170} = (.438, .862, 1.286)$	$\widetilde{X}_{263} = (.533, .776, 1.018)$
$\widetilde{X}_{78} = (-.439, .122, .683)$	$\widetilde{X}_{171} = (.268, .729, 1.190)$	$\widetilde{X}_{264} = (.353, .795, 1.237)$
$\widetilde{X}_{79} = (-.926, -.044, .838)$	$\widetilde{X}_{172} = (-.103, .669, 1.437)$	$\widetilde{X}_{265} = (.151, .839, 1.527)$
$\widetilde{X}_{80} = (-.718, -.049, .620)$	$\widetilde{X}_{173} = (.399, .722, 1.044)$	$\widetilde{X}_{266} = (.349, .708, 1.067)$
$\widetilde{X}_{81} = (.164, .355, .545)$	$\widetilde{X}_{174} = (.049, .833, 1.618)$	$\widetilde{X}_{267} = (.106, .842, 1.578)$
$\widetilde{X}_{82} = (-.296, .073, .442)$	$\widetilde{X}_{175} = (.366, .837, 1.308)$	$\widetilde{X}_{268} = (.342, .737, 1.131)$
$\widetilde{X}_{83} = (-.411, .049, .510)$	$\widetilde{X}_{176} = (.805, .841, .876)$	$\widetilde{X}_{269} = (.133, .817, 1.500)$
$\widetilde{X}_{84} = (-.585, .397, 1.378)$	$\widetilde{X}_{177} = (.628, .804, .980)$	$\widetilde{X}_{270} = (.149, .853, 1.557)$
$\widetilde{X}_{85} = (-.358, -.201, -.045)$	$\widetilde{X}_{178} = (.102, .824, 1.546)$	$\widetilde{X}_{271} = (.391, .833, 1.275)$
$\widetilde{X}_{86} = (-.681, .170, 1.030)$	$\widetilde{X}_{179} = (.309, .783, 1.256)$	$\widetilde{X}_{272} = (.861, .880, .900)$
$\widetilde{X}_{87} = (-.436, .209, .854)$	$\widetilde{X}_{180} = (.714, .866, 1.019)$	$\widetilde{X}_{273} = (.544, .875, 1.206)$
$\widetilde{X}_{88} = (-.437, -.061, .310)$	$\widetilde{X}_{181} = (.386, .727, 1.068)$	$\widetilde{X}_{274} = (.347, .772, 1.196)$
$\widetilde{X}_{89} = (-.137, .054, .245)$	$\widetilde{X}_{182} = (.233, .841, 1.448)$	$\widetilde{X}_{275} = (.558, .829, 1.099)$
$\widetilde{X}_{90} = (-.720, -.290, .137)$	$\widetilde{X}_{183} = (.568, .759, .950)$	$\widetilde{X}_{276} = (.556, .753, .951)$
$\widetilde{X}_{91} = (-.769, -.280, .195)$	$\widetilde{X}_{184} = (.053, .792, 1.530)$	$\widetilde{X}_{277} = (-.092, .730, 1.552)$
$\widetilde{X}_{92} = (-.094, .026, .147)$	$\widetilde{X}_{185} = (.604, .847, 1.090)$	$\widetilde{X}_{278} = (.440, .870, 1.300)$

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$\tilde{X}_{93} = (-.410, .181, .770)$	$\tilde{X}_{186} = (-.040, .877, 1.795)$	$\tilde{X}_{279} = (-.075, .812, 1.700)$
		$\tilde{X}_{280} = (.418, .809, 1.200)$

TABLE 2. The Fuzzy Normal Data

4. Using $\overline{X}$-Chart

In this section we will show that using $\overline{X}$-chart that its limits are calculated based on $\overline{S}$ is not suitable for evaluating fuzzy natural instable processes that are capable and the values of capability index are not significantly different throughout the time. The control limits are calculated as follows:

$$\begin{cases} LCL_{\alpha}^{-} = \overline{\overline{X}}_{\alpha}^{-} - A_3\overline{S}_{\alpha}^{-}, LCL_{\alpha}^{+} = \overline{\overline{X}}_{\alpha}^{+} - A_3\overline{S}_{\alpha}^{+}, \\ UCL_{\alpha}^{-} = \overline{\overline{X}}_{\alpha}^{-} + A_3\overline{S}_{\alpha}^{-}, UCL_{\alpha}^{+} = \overline{\overline{X}}_{\alpha}^{+} + A_3\overline{S}_{\alpha}^{+}, \\ CL_{\alpha}^{-} = \overline{\overline{X}}_{\alpha}^{-}, CL_{\alpha}^{+} = \overline{\overline{X}}_{\alpha}^{+}, \end{cases} \tag{15}$$

the coefficient A_3 is given in [22]. Instead of crisp numbers for LCL and UCL, we have intervals. Therefore, the concept of being in control is mixed with some vagueness and we need to use the theory of fuzzy sets in order to allocate the grade of being in control to each point. We imagine the points inside control limits as a reference of a fuzzy set like D. Define d_1 as a distance from a point $\overline{X}_{i0.6}^{+}$ to the line $CL_{0.6}^{+}$ and d_2 as a distance from a point $\overline{X}_{i0.6}^{-}$ to the line $CL_{0.6}^{-}$, and d_3 as a distance between $CL_{0.6}^{+}$ and $UCL_{0.6}^{+}$, that is equal to the distance between $CL_{0.6}^{-}$ and $LCL_{0.6}^{-}$. Therefore, we have:

$$\tilde{A} : D \to [0, 1],$$

with criterion:

$$\tilde{A}(\overline{X}_{i0.6}^{+}) = \begin{cases} 1 - d_1/d_3, & d_1 < d_3 \\ 0, & otherwise \end{cases} \tag{16}$$

$$\tilde{A}(\overline{X}_{i0.6}^{-}) = \begin{cases} 1 - d_2/d_3, & d_2 < d_3 \\ 0, & otherwise \end{cases} \tag{17}$$

$$D = \{\overline{X}_{i0.6}^{+} | \overline{X}_{i0.6}^{+} \in [LCL_{0.6}^{+}, UCL_{0.6}^{+}]\} \cup \{\overline{X}_{i0.6}^{-} | \overline{X}_{i0.6}^{-} \in [LCL_{0.6}^{-}, UCL_{0.6}^{-}]\}$$

The points inside the set D will possess the membership degrees of being in control. Figures 2 and 3, display several alarms, indicating instable processes. However, these alarms will be considered as false alarms as long as the process

$\overline{X}_{i.6}^{+}$	$\overline{X}_{i.6}^{-}$	$\tilde{A}(\overline{X}_{i.6}^{+})$	$\tilde{A}(\overline{X}_{i.6}^{-})$
(1,.2196)	(1,-.1964)	0.6582	0.7193
(2,.2636)	(2,-.198)	0.5142	0.7246
(3,.1459)	(3,-.2793)	0.8992	0.9905
(4,.1773)	(4,-.2367)	0.7965	0.8512
(5,.2312)	(5,-.185)	0.6202	0.6820
(6,.5138)	(6,.099)	0	0
(7,.371)	(7,-.047)	0.1629	0.2306
(8,.4815)	(8,.0543)	0	0
(9,.3076)	(9,.3076)	0.3703	.3664
(10,.3345)	(10,.3345)	0.2823	0
(11,.7069)	(11,.7069)	0	0
(12,.7204)	(12,.7204)	0	0
(13,.6599)	(13,.6599)	0	0
(14,.675)	(14,.675)	0	0
(15,.6753)	(15,.6753)	0	0
(16,.1789)	(16,-.1705)	0.7913	0.6346
(17,.2532)	(17,-.0738)	0.5482	0.3183
(18,.1977)	(18,-.1919)	0.7298	0.7046
(19,.1493)	(19,-.0291)	0.8881	0.9352
(20,.2113)	(20,-.4832)	0.6853	0.1721
(21,-.0841)	(21,-.4832)	0.9348	0.3425
(22,-.2273)	(22,-.6362)	0.8880	0
(23,-.0235)	(23,-.4168)	0.9547	0.5597
(24,-.0128)	(24,-.01279)	0.9582	0.2116
(25,.2581)	(25,-.4832)	0.9532	0.4246
(26,-.4608)	(26,-.6219)	0.8116	0
(27,-.357)	(27,-.6106)	0.8456	0
(28,-.3208)	(28,-.7432)	0.8574	0
(29,-.2822)	(29,-.5806)	0.8700	0.0239
(30,-.3471)	(30,-.6944)	0.8488	0
(31,.3902)	(31,-.0673)	0.9100	0.2970
(32,-.5531)	(32,-.9312)	0.7812	0
(33,-.5145)	(33,-.9698)	0.7940	0
(34,-.5879)	(34,-.9481)	0.7700	0
(35,.413)	(35,-.1035)	0.0255	0.4154
(36,-.1071)	(36,-.5513)	0.2731	0.1197
(37,-.0516)	(37,-.4607)	0.4547	0.4161
(38,.0811)	(38,-.3052)	0.8888	0.9248
(39,-.0729)	(39,-.4214)	0.3850	0.5447
(40,.0002)	(40,-.511)	0.6241	0.2516

TABLE 3. Membership Grades for Points in Figure 2

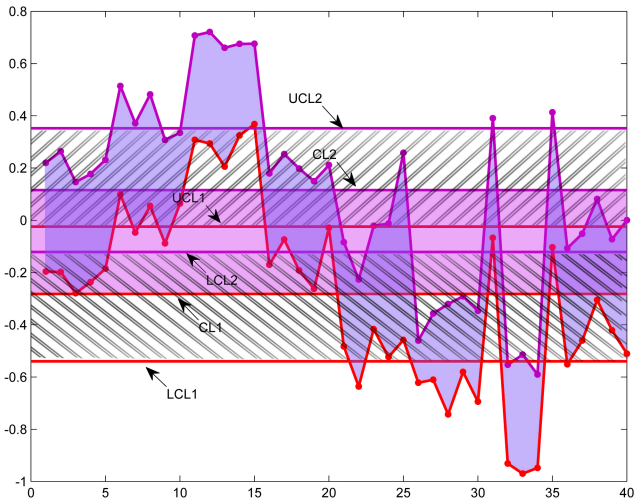


FIGURE 2. The $\bar{X}$ Control Chart when $\widetilde{C}_{pk} \simeq 4/3$

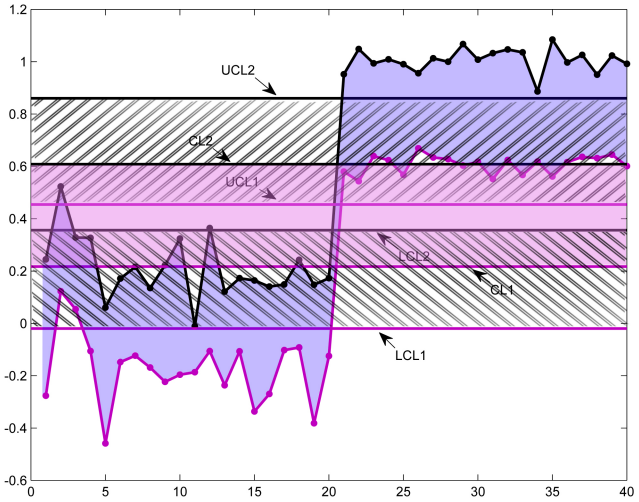


FIGURE 3. The $\overline{X}$ Control Chart when $\widetilde{C}_{pk}$ Decreases from 4/3 to 1

is able to produce conforming items and the capability index is not decreasing too drastically. In natural instable process, instability is not created because of assignable causes. Therefore, the control charts have not a true demonstration of the manufacturing process.

5. **Ranking Method**

Deng et al. [11] gave a new area method to rank fuzzy numbers with the radius of gyration (ROG) points. They mentioned when a generalized fuzzy number $\tilde{A}$ is given, the radius of gyration points of the generalized fuzzy number $\tilde{A}$ is denoted as (r_x, r_y) whose value can be obtained by equations (13) and (14). For an area made up of a number of simple shapes, the moment of inertia of the entire area is the sum of the moments of inertia of each of the individual area about the axis desired. For example, the moment of inertia of the generalized trapezoidal fuzzy number in Figure 4 can be obtained as follows:

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3, I_y = (I_y)_1 + (I_y)_2 + (I_y)_3. \tag{18}$$

Example 5.1. [11] In this example, we determine the moment of inertia and the radius of gyration of the generalized trapezoidal fuzzy number.
First, the trapezoid (a, e, f, d) can be divided into three parts, (a, e, b), (b, e, f, c) and (c, f, d). The moment of inertia of the area (aeb) with respect to x axis, and the moment of inertia of the area (aeb) with respect to y axis can be calculated as follows:

$$(I_x)_1 = \int y^2 dA = \int_0^w y^2 \frac{(b-a)(w-y)}{w} dy = \frac{(b-a)w^3}{12}, \tag{19}$$

$$(I_y)_1 = \int x^2 dA = \frac{(b-a)^3 w}{4} + \frac{(b-a)a^2 w}{2} + \frac{2(b-a)^2 aw}{3}. \tag{20}$$

The moment of inertia of area befc and cfd, with respect to x axis and y axis, can be obtained respectively, as follows:

$$(I_x)_2 = \frac{(c-b)w^3}{3}, \tag{21}$$

$$(I_x)_3 = \frac{(d-c)w^3}{12}, \tag{22}$$

$$(I_y)_2 = \frac{(c-b)w^3}{3} + (c-b)b^2 w + (c-b)^2 bw, \tag{23}$$

$$(I_y)_3 = \frac{(d-c)^3 w}{12} + \frac{(d-c)c^3 w}{2} + \frac{(d-c)^3 cw}{3}. \tag{24}$$

So, by Definition 2.12, the (ROG) point of generalized trapezoidal fuzzy number $(a, b, c, d; w)$ can be calculated as:

$$r_x = \sqrt{\frac{(I_x)_1 + (I_x)_2 + (I_x)_3}{(((c-b) + (d-a)).w)/2}}, \tag{25}$$

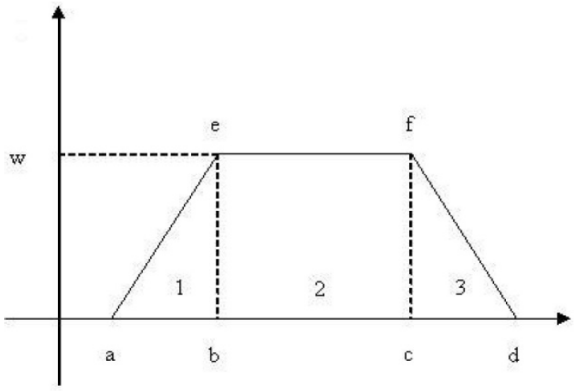


FIGURE 4. The Generalized Trapezoidal Fuzzy Number

$$r_y = \sqrt{\frac{(I_y)_1 + (I_y)_2 + (I_y)_3}{(((c-b)+(d-a)).w)/2}}, \tag{26}$$

where $(I_x)_1$, $(I_x)_2$, $(I_x)_3$, $(I_y)_1$, $(I_y)_2$ and $(I_y)_3$ are given in equations (19)-(24). The criterion for ranking the fuzzy numbers is to compute the product $r_x r_y$. Ranking of fuzzy numbers is as follows:

$$\begin{aligned}
&(1)\ r_x(\tilde{A}).r_y(\tilde{A}) > r_x(\tilde{B}).r_y(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B},\\
&(2)\ r_x(\tilde{A}).r_y(\tilde{A}) = r_x(\tilde{B}).r_y(\tilde{B}) \Rightarrow \tilde{A} = \tilde{B},\\
&(3)\ r_x(\tilde{A}).r_y(\tilde{A}) < r_x(\tilde{B}).r_y(\tilde{B}) \Rightarrow \tilde{A} < \tilde{B}.
\end{aligned} \tag{27}$$

Nasseri and Sohrabi [23] showed that according to the area between the radius and original point, this method is wrong and we should order the fuzzy numbers as follows:

$$\begin{aligned}
&(1)\ r_y(\tilde{A}) > r_y(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B}.\\
&(2) r_y(\tilde{A}) = r_y(\tilde{B}) \Rightarrow \begin{cases} r_x(\tilde{A}) > r_x(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B}.\\ r_x(\tilde{A}) < r_x(\tilde{B}) \Rightarrow \tilde{A} < \tilde{B}.\\ r_x(\tilde{A}) = r_x(\tilde{B}) \Rightarrow \tilde{A} = \tilde{B}. \end{cases}\\
&(3) r_y(\tilde{A}) < r_y(\tilde{B}) \Rightarrow \tilde{A} < \tilde{B}.
\end{aligned} \tag{28}$$

6. **Membership Function for Fuzzy Estimator of $\widetilde{C}_{pk}$**

Utilizing the fuzzy estimator for $\widetilde{C}_{pk}$ is necessary when we encounter inadequate information about the manufacturing process. In this section, we will use the $D_{p,q}$-distance in order to find the fuzzy estimator for $\widetilde{C}_{pk}$. According to the definition of Vännman [32] we have:

$$C_{pk} = \frac{d - |\mu - T|}{3\sigma}. \tag{29}$$

If we define the α-cut sets such that:

$$\bar{X}_\alpha = [\bar{X}_\alpha^-, \bar{X}_\alpha^+], \hat{C}_{pk_\alpha} = [l_{\hat{C}_{pk}}(\alpha), \psi_{\hat{C}_{pk}}(\alpha)],$$

we can obtain the membership function of $\widetilde{C}_{pk}$ as follows:

(1) If $\tilde{\bar{X}} < \tilde{T}$:

$$l_{\hat{C}_{pk}}(\alpha) = \frac{1 - (\frac{\tilde{T}_\alpha^+ - \bar{X}_\alpha^-}{d})}{3\sqrt{(\frac{\hat{D}var(\tilde{X})}{d^2})}} = \frac{1 - (\frac{\tilde{T}_\alpha^+ - \bar{X}_\alpha^-}{d})}{3\sqrt{(\frac{\frac{1}{n}\sum_{i=1}^n [D_{2,q}(\tilde{X}_i, \tilde{\bar{X}})]^2}{d^2})}}, \tag{30}$$

$$\psi_{\hat{C}_{pk}}(\alpha) = \frac{1 - (\frac{\tilde{T}_\alpha^- - \bar{X}_\alpha^+}{d})}{3\sqrt{(\frac{\hat{D}var(\tilde{X})}{d^2})}} = \frac{1 - (\frac{\tilde{T}_\alpha^- - \bar{X}_\alpha^+}{d})}{3\sqrt{(\frac{\frac{1}{n}\sum_{i=1}^n [D_{2,q}(\tilde{X}_i, \tilde{\bar{X}})]^2}{d^2})}}. \tag{31}$$

(2) If $\tilde{\bar{X}} > \tilde{T}$:

$$l_{\hat{C}_{pk}}(\alpha) = \frac{1 - (\frac{\bar{X}_\alpha^+ - \tilde{T}_\alpha^-}{d})}{3\sqrt{(\frac{\hat{D}var(\tilde{X})}{d^2})}} = \frac{1 - (\frac{\bar{X}_\alpha^+ - \tilde{T}_\alpha^-}{d})}{3\sqrt{(\frac{\frac{1}{n}\sum_{i=1}^n [D_{2,q}(\tilde{X}_i, \tilde{\bar{X}})]^2}{d^2})}}, \tag{32}$$

$$\psi_{\hat{C}_{pk}}(\alpha) = \frac{1 - (\frac{\bar{X}_\alpha^- - \tilde{T}_\alpha^+}{d})}{3\sqrt{(\frac{\hat{D}var(\tilde{X})}{d^2})}} = \frac{1 - (\frac{\bar{X}_\alpha^- - \tilde{T}_\alpha^+}{d})}{3\sqrt{(\frac{\frac{1}{n}\sum_{i=1}^n [D_{2,q}(\tilde{X}_i, \tilde{\bar{X}})]^2}{d^2})}}. \tag{33}$$

As we mentioned earlier in Section 3, we have forty samples in each table. First, the $D_{p,q}$ distances between sample points and the mean of each sample, were calculated five hundred and sixty times for both data tables according to Definition 2.8. In variance formula, $X - \overline{X}$ is a distance between X and $\overline{X}$, and we can measure this distance by using any other distance measure. Because we used $D_{p,q}$ distance as a distance measure, we could replace standard deviation with $\sqrt{DVar}$. Furthermore, for comparing the mean of each sample with the target value, the radiuses of gyration were calculated eighty times for both data tables according to Definition 2.12. Figure 5, displays the membership functions of $\widetilde{\hat{C}}_{pk}$ that were obtained by calculating the equations (30)-(33) for different values of α and placing these intervals one on top of another. By Definition 2.2, $\widetilde{\hat{C}}_{pk}$ is a fuzzy number.

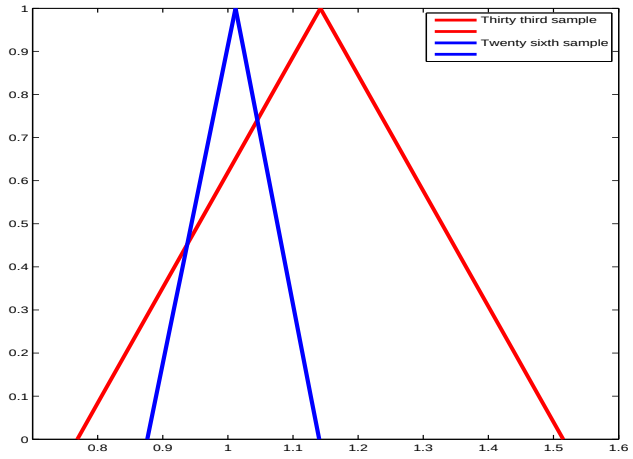


FIGURE 5. The Membership Functions of $\widetilde{\hat{C}}_{pk}$

We calculated eighty membership functions of $\widetilde{\hat{C}}_{pk}$. The membership functions of $\widetilde{\hat{C}}_{pk}$ in Figure 5, were chosen arbitrarily from samples in Table 1.

7. **Monitoring Capability Index**

Process stability refers to the consistency of the process with respect to important process characteristics such as the average value of a key dimension or the variation in that key dimension. If the process behaves consistently over time, then we say that the process is stable or in control. One of the most important issues in quality control is determining a capability of either stable or instable processes. Plotting the control charts is the best way for concluding about capability in stable processes. V*ä*nnman and Castagliola [33] proposed a capability analysis procedure for processes that are not stable, i.e. they cannot be controlled by the use of control charts. This method derived from the approach Castagliola [3, 4, 5, 6] suggested for the monitoring of the process variability. There are many machines capable and used to drill. Drilling is a manufacturing process where a round hole is created within a work piece or enlarged by rotating an end cutting tool, a drill. As an example, consider a manufacturing situation where thousands of holes, with hundreds of different diameters and drilling conditions, have to be drilled in order to assemble various components. As a result of the manufacturing complexity, the drilling process is naturally instable. However, despite this instability, the drilling process is supposed to be capable. The natural instability of this process prevents the use of any control charts for the mean and standard deviation. The alternative approach is to directly monitor the capability, so as to receive an alarm if the capability is changing and getting too low [33]. In order to monitoring capability

index, instead of directly plotting the capability index on the chart, Vännman and Castagliola [33] suggested to apply the following logarithmic transformation on $\hat{C}_{pk}$. The membership function for $\widetilde{\hat{C}}_{pk}$ was proposed in Section 6. Now, we apply this method on $\widetilde{\hat{C}}_{pk}$:

$$\widetilde{Y}_i = a_i + b_i \ln(\widetilde{\hat{C}}_{pk_i}), \tag{34}$$

where i=1, 2, ..., 40, and the α-cut sets of $\widetilde{Y}_i$ are computed as follows:

$$Y_{i_\alpha}^- = a_i + b_i \ln(\hat{C}_{pk_i\,\alpha}^-), Y_{i_\alpha}^+ = a_i + b_i \ln(\hat{C}_{pk_i\,\alpha}^+),$$
$$Y_{i\alpha} = [Y_{i\alpha}^-, Y_{i\alpha}^+].$$

Theorem 7.1. [32] *If the characteristic of the process is normally distributed the probability density function of C_{pk} can be expressed as follows:*

when u=1 and v=0,

$$f_{C_p(1,0)} = \begin{cases} \int_0^\infty I_1(z,x)dz, & x \le 0 \\ \int_0^{\frac{\sqrt{n}}{3x\sigma_t}} I_1(z,x)dz, & x > 0 \end{cases} \tag{35}$$

where,

$$I_1(z,x) = \frac{2^{1-n/2}}{\Gamma(\frac{n-1}{2})}.\frac{3z^{n-1}}{1.7725}.exp(\frac{-z^2}{2})$$

$$.(exp(-\frac{1}{2}(\frac{\sqrt{n}}{\sigma_t}(1-\mu_t) - 3xz)^2) + exp(-\frac{1}{2}(\frac{\sqrt{n}}{\sigma_t}(1+\mu_t) - 3xz)^2)),$$

and μ_t and σ_t are defined as follows:

$$\mu_t = \frac{\mu - T}{USL - T}, \sigma_t = \frac{\sigma}{USL - T}. \tag{36}$$

The parameters a_i and b_i (used in [33]) are given as follows:

$$a_i = -b_i \ln(E(\hat{C}_{pk_i})(\frac{V(\hat{C}_{pk_i})}{E^2(\hat{C}_{pk_i})} + 1)^{-\frac{1}{2}}, b_i = (\ln(\frac{V(\hat{C}_{pk_i})}{E^2(\hat{C}_{pk_i})} + 1))^{-\frac{1}{2}}. \tag{37}$$

The EWMA statistic was defined as $Z_i = (1-\lambda)Z_{i-1} + \lambda Y_i$ for crisp data [33], but for the cases where the data are not exactly known, by using the α-cut sets of $\widetilde{Z}_i$ we have:

$${Z_{i\alpha}}^- = (1-\lambda){Z_{i-1\alpha}}^- + \lambda {Y_{i\alpha}}^-, {Z_{i\alpha}}^+ = (1-\lambda){Z_{i-1\alpha}}^+ + \lambda {Y_{i\alpha}}^+.$$

In Figure 6, the violet points represent the values of ${Z_{i0.6}}^+$ and the red points represent the values of ${Z_{i0.6}}^-$. We can define the violet region mathematically like:

$$V = \{Z_i | Z_i \in [{Z_{i0.6}}^-, {Z_{i0.6}}^+]\}, \tag{38}$$

$Z_{i\,0.6}^{-}$ Fig.6	$Z_{i\,0.6}^{+}$ Fig.6	$Z_{i\,0.6}^{+}$ Fig.7	$Z_{i\,0.6}^{-}$ Fig.7
-.114	-.0269	-.0199	-.1258
-.0102	.1673	-.1756	-.3939
-.0137	.2284	.1176	-.148
-.1087	.1923	.122	-.2171
-.2239	.1352	.1249	-.2691
-.417	.0338	.0938	-.3265
-.9431	.0442	.1643	-.2866
-.9680	.0449	.1317	-.3348
-.9229	.0822	.1541	-.3572
-.9107	.0185	.0971	-.4783
-1.0096	.0042	.0475	-.5028
-1.251	-.1471	.0822	-.5249
-1.3671	-.1982	.1263	-.4891
-1.4233	-.22	.0736	-.5391
-1.1291	.0916	.1221	-.524
-.9742	.1973	.0885	-.5717
-.9623	.1667	.0517	-.5945
-.8518	.247	-.0324	-.6771
-.8363	.2338	.0084	-.6693
-.8388	.179	.0503	-.6216
-.5992	.3476	.0533	-.9366
-.4029	.4774	.0278	-1.265
-.2361	.5884	-.0394	-1.9137
-.0805	.7017	-.1406	-2.5001
-.0108	.7595	-.1013	-2.8941
.01	.7248	-.1575	-3.0293
.1256	.8043	-.2258	-3.4093
.2028	.8711	-.2015	-4.2774
.3428	.9861	-.214	-4.1994
.3619	.9884	-.1871	-4.4752
-.072	.6007	-.1432	-4.4711
.2129	.8633	-.2017	-4.4688
.2781	.9178	-.061	-4.2734
.4393	1.0569	-.1336	-4.1402
-.0021	.6842	-.1434	-4.0433
.0366	.7241	-.1779	-4.5416
-.0354	.6513	-.1972	-4.5976
-.0687	.622	-.2208	-4.5383
-.0898	.59	-.2459	-4.6186
-.057	.6402	-.2514	-4.8764

TABLE 4. The Values of $Z_{i\,0.6}^{+}$ and $Z_{i\,0.6}^{-}$ for Figures 6 and 7

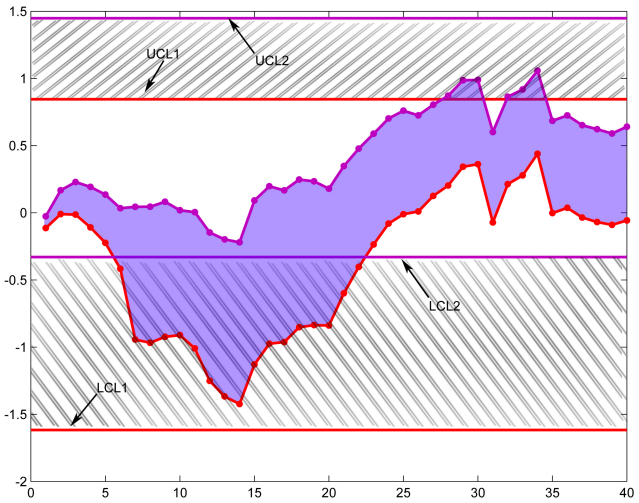


FIGURE 6. The EWMA Control Chart when $\widetilde{C}_{pk} \simeq 4/3$

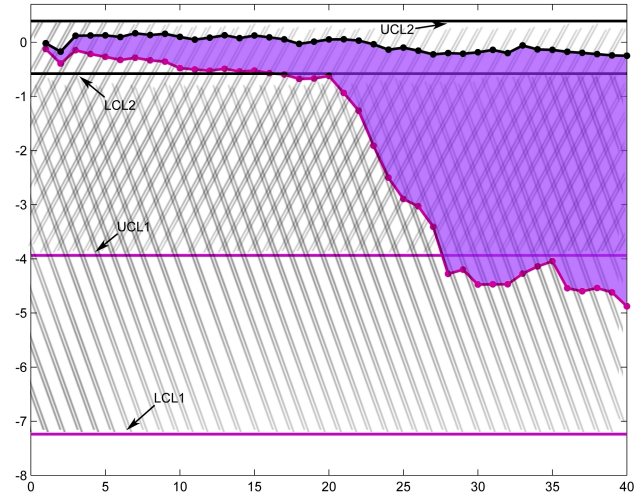


FIGURE 7. The EWMA Control Chart when $\widetilde{C}_{pk}$ Decreases from 4/3 to 1

on the other hand, the violet region shows all possible values of Z_i that lie between ${Z_{i0.6}}^-$ and ${Z_{i0.6}}^+$. Also, the EWMA control chart limits are computed as follows where λ is a smoothing constant satisfying $0 < \lambda \leq 1$:

$$\begin{cases} LCL_\alpha^- = E({Y_{i_\alpha}}^-) - K(\frac{\lambda}{2-\lambda})^{\frac{1}{2}}\sigma({Y_{i_\alpha}}^-), LCL_\alpha^+ = E({Y_{i_\alpha}}^+) - K(\frac{\lambda}{2-\lambda})^{\frac{1}{2}}\sigma({Y_{i_\alpha}}^+), \\ UCL_\alpha^- = E({Y_{i_\alpha}}^-) + K(\frac{\lambda}{2-\lambda})^{\frac{1}{2}}\sigma({Y_{i_\alpha}}^-), UCL_\alpha^+ = E({Y_{i_\alpha}}^+) + K(\frac{\lambda}{2-\lambda})^{\frac{1}{2}}\sigma({Y_{i_\alpha}}^+). \end{cases} \quad (39)$$

Figures 6 and 7 were plotted for investigating the variations in capability of fuzzy instable processes. Figure 6 displays a manufacturing process with $\widetilde{C}_{pk} \simeq 4/3$ and since it is a good level of capability, none of the points are out of control limits. In Figure 7, we can observe a decreasing trend in plot that actually displays the decrease in capability index from $\widetilde{C}_{pk} \simeq 4/3$ to $\widetilde{C}_{pk} \simeq 1$.

8. **Conclusion**

In this paper, we proposed the membership function for $\widetilde{C}_{pk}$ by using the $D_{p,q}$-distance and defined the logarithmic transformation for α-cut sets of $\widetilde{C}_{pk}$ according to the proposed method of Vännman and Castagliola. Also, we generalized the method of monitoring capability index for capable but instable processes for fuzzy normal data. For future research, we will apply this method to other fuzzy capability indices like $\widetilde{C}_{pm}$ and $\widetilde{C}_{pmk}$.

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