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Idempotent uninorms and nullnorms on bounded posets

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Abstract

The paper deals with uninorms and nullnorms as basic semi-group operations which are commutative and monotone (increasing). These operations were first introduced on the unit interval and later generalized to bounded lattices. In [Kalina 2019] they were introduced on bounded posets. This contribution is a generalization and extension of the results in [Kalina 2019]. Some necessary and some sufficient conditions for the existence of idempotent uninorms and idempotent nullnorms on bounded posets are studied. Finally, some application examples are provided.

Keywords: Associative operation, idempotent semi-group, commutative monoids, fuzzy connectives.

1 Introduction

Uninorms and nullnorms on the unit interval are special types of aggregation functions since, due to their associativity they can be straightforwardly extended to n-ary operations for arbitrary $n \in \mathbb{N}$. There are two different common generalizations of both, t-norms and t-conorms. Uninorms and nullnorms are important in various fields of applications, e.g., neuron nets, fuzzy decision making and fuzzy modeling, e.t.c. Also from the theoretical point of view they attract attention. Interesting is, e.g., a research by Devillet et al. [20]. Important is, among others, the class of idempotent uninorms and nullnorms since, for arbitrary idempotent monotone operation $*$ and for any $x, y \in [0, 1]$,

$$x \wedge y \leq x * y \leq x \vee y \tag{1}$$

holds. Because of (1), the idempotent monotone operations are called also averaging. Idempotent uninorms are known also under the name unanimous (see, e.g., Calvo et al. [6]). Citing [6], 'idempotency, in multicriteria decision making reads as follows: if all criteria are satisfied in the same degree x, then also the global score should be x'.

Uninorms were introduced by Yager and Rybalov [36]. Special types of associative, commutative and monotone operations with neutral elements had already been studied in [15, 21, 22]. Deschrijver [18, 19] showed that on the lattice $L^{[0,1]}$ of interval-valued membership grades, i.e., on the lattice of closed sub-intervals of the unit interval there exist uninorms which are neither conjunctive nor disjunctive (i.e., whose annihilator is different from both, $\mathbf{0}$ and $\mathbf{1}$). Particularly, he constructed uninorms having the neutral element $\mathbf{e} = [e, e]$, where $e \in]0, 1[$. In [28] a construction of uninorms on $L^{[0,1]}$ was presented such that, choosing arbitrary pair (e, a) of incomparable elements, $\mathbf{e}$ plays the role of the neutral element and $\mathbf{a}$ that of the annihilator.

Following further the study of uninorms on bounded lattices, in [30] the authors demonstrated that on arbitrary bounded lattice L it is possible to construct a uninorm regardless which element of L is chosen to be the neutral one. A different way of how uninorms on bounded lattices can be constructed, was presented in [2, 11, 13].

Ones uninorms which are neither conjunctive nor disjunctive are constructed, a natural question in this context arises whether the existence of incomparable elements on a bounded lattice L is sufficient for the existence of a uninorm U which is neither conjunctive nor disjunctive. This question was solved in [3] negatively. In [26] the author showed that, on some special bounded lattices, one can construct operations which are both, proper uninorms and nullnorms, and such that their neutral element, as well as annihilator, is different from both, $\mathbf{0}$ and $\mathbf{1}$.

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T-operators were defined on $[0,1]$ by M. Mas et al. [32] and further studied in [33], in 1999. In 2001, T. Calvo et al. [5] introduced the notion of nullnorms, also on $[0,1]$. Both of these operations were defined as generalizations of t-norms and t-conorms. As Mas et al. [34] in 2002 pointed out, t-operators and nullnorms coincide on $[0,1]$. Particularly, under the constraint that $\mathrm{Op} : [0,1]^2 \to [0,1]$ is a commutative, associative and monotone operation, then properties

(a) 0 and 1 are idempotent elements of Op and functions $\mathrm{Op}(0,\cdot)$ and $\mathrm{Op}(1,\cdot)$ are continuous,

(b) there exists $a \in [0,1]$ such that 0 is a partial neutral element of Op on $[0,a]$, and 1 is a partial neutral element of Op on $[a,1]$,

are equivalent to each other. Afterwards, only nullnorms were studied and later generalized as operations on bounded lattices [29]. In [4], the authors pointed out some differences between nullnorms and t-operators on bounded lattices. In this paper, the notion of t-operators will not be used.

Coming back to idempotent uninorms and nullnorms, the situation on the unit interval is solved. It is well-known that min is the unique idempotent t-norm and max the unique idempotent t-conorm. The block-wise layout of nullnorms implies directly that, for a fixed value of the annihilator, there is the unique idempotent nullnorm having the given annihilator. In the case of idempotent uninorms there is no uniqueness given by the choice of the neutral element, but this case is solved by De Baets [16], and generalized for finite chains by De Baets et al. [17]. When we go to bounded lattices possessing incomparable elements, in general we loss any uniqueness of the existence of idempotent nullnorms and some new possibilities for constructing of idempotent uninorms appear. However, in some bounded posets, choosing an annihilator (a neutral element), the idempotent nullnorms (idempotent uninorms) do not exist. Important for the existence is the set of all elements which are incomparable with the annihilator (with the neutral element). Some particular cases have already been solved, see, e.g., [7, 10, 14] for idempotent uninorms and [8, 9, 12] for idempotent nullnorms. The intention of the authors of the present paper is to look for necessary conditions and for sufficient conditions for the existence of idempotent uninorms and/or idempotent nullnorms. Since, obviously, we do not need the join and meet to exist for arbitrary pair of elements, we abandon the area of bounded lattices and study the situation in bonded posets, instead. The present paper is an extension of the conference paper [27].

The paper is organized as follows. In Section 2 some notions and results will be repeated that are important for the considerations on idempotent uninorms and nullnorms. The main body of the paper is in Section 3. Section 4 provides some applications. Finally, Section 5 contains conclusions.

2 Preliminaries

In this section, some basic notions and known facts will be recalled to make the paper self-contained. We will use the notation introduced in the monograph [1] also for bounded posets.

In the whole paper, $(P, \mathbf{0}, \mathbf{1}, \leq)$ will denote a bounded poset, where $P \neq \emptyset$ is a given set and $\mathbf{0}, \mathbf{1}$ its two distinguished elements such that $\mathbf{0} \leq x \leq \mathbf{1}$ for all $x \in P$. If it will cause no confusion, P will denote also the poset itself.

For arbitrary $x_1, x_2 \in P$, $x_1 \leq x_2$,

(a) if $x_1 \neq x_2$, then this will be denoted by $x_1 < x_2$,

(b) $[x_1, x_2]$ will denote the closed interval,

(c) $]x_1, x_2[$ will denote the open interval,

(d) $[x_1, x_2[$ and $]x_1, x_2]$ will denote the two kinds of the semi-open intervals.

Further, if x_1 and x_2 are incomparable, then this will be denoted by $x_1 \parallel x_2$. The set of all elements which are incomparable with a given one, $x \in P$, will be denoted by $\parallel_x$. If x_1 and x_2 are comparable, this will be denoted by $x_1 \nparallel x_2$.

2.1 Basic definitions and known facts on posets and on commutative monoidal operations on $[0,1]$

Well-known representatives of associative monotone and commutative operations are triangular norms introduced by Schweizer and Sklar [35] (t-norm for brevity), and their duals, t-conorms (see, e.g., [31]). The reader is assumed to be familiar with these notions, as well as with their generalizations, uninorms (introduced by Yager and Rybalov [36]) and nullnorms (introduced by Calvo et al. [5]). More on these operations the reader can find, e.g., in [6]. Just briefly,

- a uninorm $U : [0,1]^2 \to [0,1]$ is an associative, commutative and monotone binary operation with a neutral element $e \in [0,1]$. If $e = 1$, U is a t-norm, if $e = 0$, U becomes a t-conorm,
- a nullnorm $V : [0,1]^2 \to [0,1]$ is an associative, commutative and monotone binary operation with $a \in [0,1]$ is annihilator. If $a = 0$, V becomes a t-norm, if $a = 1$, V becomes a t-conorm.

For more information on associative (and monotone) operations on $[0,1]$ refer to the monographs [6, 23, 31]. Here, we present only two important results on monotone, commutative and associative operations.

Lemma 2.1. [24] *Let $*$ be a monotone, commutative and associative operation on L. Further, let c be an idempotent element. Assume that there exist elements $x, y \in L$ such that $x * c = y$. Then $y * c = y$.*

Lemma 2.1 was formulated in [24] for conjunctive uninorms and a special version was proved in [25].

Proposition 2.2. [8] *Let $(L, \leq, \mathbf{0}, \mathbf{1})$ be a bounded lattice, $a \in L \setminus \{\mathbf{0}, \mathbf{1}\}$ and V be an idempotent nullnorm on L with the annihilator a. Then the following hold:*

(a) $V(x,y) = (x \wedge a) \vee (y \wedge a)$ *if* $(x,y) \in [\mathbf{0}, a]^2 \cup [\mathbf{0}, a] \times \|_a \cup \|_a \times [\mathbf{0}, a]$,

(b) $V(x,y) = (x \vee a) \wedge (y \vee a)$ *if* $(x,y) \in [a, \mathbf{1}]^2 \cup [a, \mathbf{1}] \times \|_a \cup \|_a \times [a, \mathbf{1}]$,

(c) $V(x,y) = a$ *if* $(x,y) \in [\mathbf{0}, a] \times [a, \mathbf{1}] \cup [a, \mathbf{1}] \times [\mathbf{0}, a]$,

(d) $V(x,y) = (x \wedge a) \vee (y \wedge a)$ *or* $V(x,y) = (x \vee a) \wedge (y \vee a)$ *or* $V(x,y) \in \|_a$ *if* $(x,y) \in (\|_a)^2$.

2.2 Existence of uninorms and nullnorms on bounded posets

As it was shown in [27], on every bounded poset P possessing at least 3 elements, it is possible to construct a proper uninorm and a proper nullnorm. Important for this construction is that on every bounded poset P there exists at least one t-norm T_D and one t-conorm S_D, namely

$$T_D(x,y) = \begin{cases} x & \text{if } y = \mathbf{1}, \\ y & \text{if } x = \mathbf{1}, \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad S_D(x,y) = \begin{cases} x & \text{if } y = \mathbf{0}, \\ y & \text{if } x = \mathbf{0}, \\ \mathbf{1} & \text{otherwise.} \end{cases} \tag{2}$$

We are going to study conditions under which it is possible to define an idempotent uninorm and/or an idempotent nullnorm on P. For this reason we provide the following results.

Lemma 2.3. [27] *Let P be a bounded poset with at least three elements and $e \notin \{\mathbf{0}, \mathbf{1}\}$. There exists an idempotent t-norm on $[\mathbf{0}, e]$ if and only if $([\mathbf{0}, e], \leq)$ is a meet semi-lattice.*
There exists an idempotent t-conorm on $[e, \mathbf{1}]$ if and only if $([e, \mathbf{1}], \leq)$ is a join semi-lattice.

Lemma 2.4. [27] *Let P be a bounded poset with at least three elements and $a \notin \{\mathbf{0}, \mathbf{1}\}$. There exists the idempotent t-conorm on $[\mathbf{0}, a]$ if and only if $([\mathbf{0}, a], \leq)$ is a join semi-lattice.*
There exists the idempotent t-norm on $[a, \mathbf{1}]$ if and only if $([a, \mathbf{1}], \leq)$ is a meet semi-lattice.

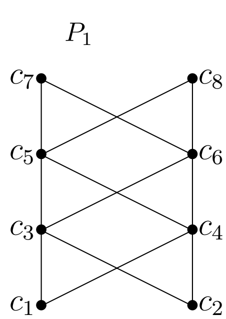

Figure 1: Hasse diagram of P_1

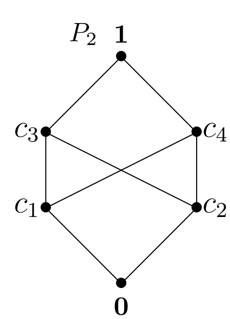

Figure 2: Hasse diagram of P_2

The next two claims from [27] state some constraints for the non-existence of idempotent uninorms and idempotent nullnorms, respectively, on bounded posets.

Proposition 2.5. [27] *Let* $(P, \mathbf{0}, \mathbf{1}, \leq)$ *be a bounded poset and* $e \notin \{\mathbf{0}, \mathbf{1}\}$*. Assume there exists eight-element set* $\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\} \subset \|_e$ *having Hasse diagram depicted in Figure 1, such that the join of* c_1, c_2*, of* c_3, c_4*, of* c_5, c_6*, and the meet of* c_3, c_4*, of* c_5, c_6*, and of* c_7, c_8*, are not defined. Then there exists no idempotent uninorm* U *on* P *whose neutral element is* e*.*

Lemma 2.6. [27] *Let* P *be the poset whose Hasse diagram is sketched in Fig. 2. There exists no idempotent nullnorm on* P*.*

In this section, beside other results, we provide some generalizations of Proposition 2.5 and Lemma 2.6. The next proposition we need for technical reasons. In fact, Proposition 2.7 modifies inequality (1) for bounded posets.

Proposition 2.7. *Let* $(P, \mathbf{0}, \mathbf{1}, \leq)$ *be a bounded poset,* $\{z_1, z_2, z_3, z_4\} \subset P$*. Assume* $z_1 \leq z_2 \leq z_4$ *and* $z_1 \leq z_3 \leq z_4$*. If* $\mathcal{O} : P^2 \to P$ *is an idempotent and monotone operation, then*

$$\mathcal{O}(z_2, z_3) \in [z_1, z_4].$$

Proof. Directly by monotonicity and idempotency we get

$$z_1 = \mathcal{O}(z_1, z_1) \leq \mathcal{O}(z_2, z_3) \leq \mathcal{O}(z_4, z_4) = z_4.$$

□

The following result is a corollary to Proposition 2.7.

Corollary 2.8. *Let* $(P, \mathbf{0}, \mathbf{1}, \leq)$ *be a bounded poset,* $\{z_1, z_2\} \subset P$*. Assume there exists neither meet nor join of elements* z_1, z_2*. If* $\mathcal{O} : P^2 \to P$ *is an idempotent and monotone operation, then either* $\mathcal{O}(z_1, z_2) \in \|_{z_1}$*, or* $\mathcal{O}(z_1, z_2) \in \|_{z_2}$*.*

Proof. Directly by Proposition 2.7 we get for arbitrary lower bound, ℓ, and upper bound u, of z_1 and z_2 that $\mathcal{O}(z_1, z_2) \in [\ell, u]$. Since there exists neither meet nor join of z_1 and z_2, the value $\mathcal{O}(z_1, z_2)$ cannot be comparable with z_1 and at the same time also with z_2. □

The next proposition is a generalization of Proposition 2.5.

Proposition 2.9. *Let* $(P, \mathbf{0}, \mathbf{1}, \leq)$ *be a bounded poset. Assume there exists a quadruple* $\{z_1, z_2, z_3, z_4\} \in P$ *such that the pairs of elements,* z_1, z_2 *and* z_3, z_4*, have neither meet nor join. Moreover, assume the following two constraints*

(a) $z_1 \leq z_3,\ z_1 \leq z_4,\ z_2 \leq z_3,\ z_2 \leq z_4,$

(b) *if* $x \in [z_i, z_j]$ *for* $i \in \{1, 2\}$ *and* $j \in \{3, 4\}$*, then* $x \in \{z_1, z_2, z_3, z_4\}$*.*

Then there exists no idempotent and monotone operation $\mathcal{O}$ *on* P*.*

Proof. Assume there exists an idempotent and monotone operation $\mathcal{O}$. By Corollary 2.8 we have $\mathcal{O}(z_1, z_2) \in \|_{z_1}$ or $\mathcal{O}(z_1, z_2) \in \|_{z_2}$ and $\mathcal{O}(z_3, z_4) \in \|_{z_3}$ or $\mathcal{O}(z_3, z_4) \in \|_{z_4}$. Without loss of generality, assume $\mathcal{O}(z_1, z_2) \ \|_{z_2}$. Then by monotonicity

$$\begin{aligned} \mathcal{O}(z_1, z_2) \leq \mathcal{O}(z_2, z_4) \leq \mathcal{O}(z_3, z_4) \\ \mathcal{O}(z_1, z_2) \leq \mathcal{O}(z_2, z_3) \leq \mathcal{O}(z_3, z_4). \end{aligned} \tag{3}$$

Then we get

$$z_2 \neq \mathcal{O}(z_2, z_4) \leq z_4, \quad z_2 \neq \mathcal{O}(z_2, z_3) \leq z_3,$$

which, due to Constraint (b), is in contradiction with the fact that $\mathcal{O}(z_3, z_4)$ is an upper bound of both, $\mathcal{O}(z_2, z_3)$ as well as $\mathcal{O}(z_2, z_4)$. □

Remark 2.10. Hasse diagrams of some examples of the quadruple $\{z_1, z_2, z_3, z_4\}$ from Proposition 2.9 are depicted in Fig. 1 – 4. In the poset P_4 (Fig. 4) the chains $b_1 \geq b_2 \geq b_3 \geq b_4 \geq \dots$ and $a_1 \leq a_2 \leq a_3 \leq \dots$ are infinite.

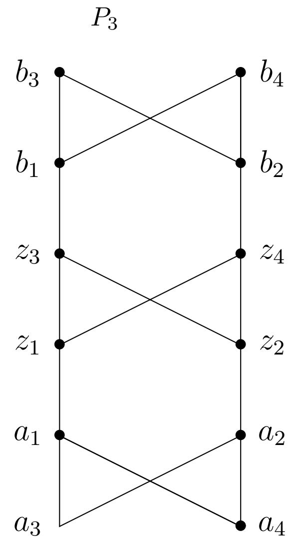

Figure 3: Hasse diagram of P_3

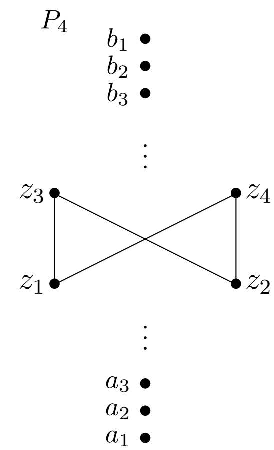

Figure 4: Hasse diagram of P_4

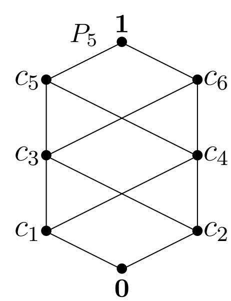

Figure 5: Hasse diagram of P_5

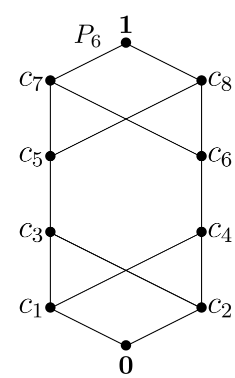

Figure 6: Hasse diagram of P_6

U	$\mathbf{0}$	c_1	c_2	c_3	c_4	c_5	c_6	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
c_1	$\mathbf{0}$	c_1	$\mathbf{0}$	c_1	c_1	c_1	c_1	c_1
c_2	$\mathbf{0}$	$\mathbf{0}$	c_2	c_2	c_2	c_2	c_2	c_2
c_3	$\mathbf{0}$	c_1	c_2	c_3	c_4	c_5	c_6	$\mathbf{1}$
c_4	$\mathbf{0}$	c_1	c_2	c_4	c_4	c_4	c_4	c_4
c_5	$\mathbf{0}$	c_1	c_2	c_5	c_4	c_5	$\mathbf{1}$	$\mathbf{1}$
c_6	$\mathbf{0}$	c_1	c_2	c_6	c_4	$\mathbf{1}$	c_6	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	c_1	c_2	$\mathbf{1}$	c_4	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

Table 1: The idempotent conjunctive uninorm $U : P_5 \times P_5 \to P_5$ with c_3 as the neutral element

3 Idempotent uninorms and nullnorms on bounded posets

At the beginning of this section an example is given illustrating two special cases of posets when it is possible to construct an idempotent uninorm.

Example 3.1. Now, we present two posets, P_5 and P_6, whose Hasse diagrams are lied out in Fig. 5 and 6, respectively. For the pair of elements $c_3, c_4 \in P_5$ there exists neither meet nor join. Further, $c_1 \wedge c_2 = \mathbf{0}$ and $c_5 \vee c_6 = \mathbf{1}$. This means, the constraints of Proposition 2.9 are not fulfilled. Tab. 1 presents an idempotent uninorm defined on the poset P_5.

For the poset P_6 whose Hasse diagram is sketched in Fig. 6, the pairs c_3, c_4 and c_5, c_6 have neither meet nor join, however, $c_5 \parallel c_4$ and $c_6 \parallel c_3$. This means, the constraint (a) from Proposition 2.9 is violated. Tab. 2 presents an idempotent uninorm defined on P_6.

U	$\mathbf{0}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	c_4	$\mathbf{0}$	c_4	c_4	c_4	c_4
c_1	$\mathbf{0}$	c_1	$\mathbf{0}$	c_1	c_4	c_1	c_4	c_4	c_4	c_4
c_2	$\mathbf{0}$	$\mathbf{0}$	c_2	c_2	c_4	c_2	c_4	c_4	c_4	c_4
c_3	$\mathbf{0}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	$\mathbf{1}$
c_4	c_4	c_4	c_4	c_4	c_4	c_4	c_4	c_4	c_4	c_4
c_5	$\mathbf{0}$	c_1	c_2	c_5	c_4	c_5	c_6	c_7	c_8	$\mathbf{1}$
c_6	c_4	c_4	c_4	c_6	c_4	c_6	c_6	c_6	c_6	c_6
c_7	c_4	c_4	c_4	c_7	c_4	c_7	c_6	c_7	$\mathbf{1}$	$\mathbf{1}$
c_8	c_4	c_4	c_4	c_8	c_4	c_8	c_6	$\mathbf{1}$	c_8	$\mathbf{1}$
$\mathbf{1}$	c_4	c_4	c_4	$\mathbf{1}$	c_4	$\mathbf{1}$	c_6	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

Table 2: The idempotent uninorm $U : P_6 \times P_6 \to P_6$ with c_3 as the neutral element and c_4 as the annihilator

The next proposition is a reformulation of the Proposition 2.2 (c). Since we work on bounded posets instead of lattices, we find important to formulate and prove this assertion.

Proposition 3.2. *Let P be a bounded poset and $V : P^2 \to P$ be a nullnorm. Assume $a \in P$ be an element fulfilling*

(a) *$V(\mathbf{0}, x) = x$ for all $x \in [\mathbf{0}, a]$,*

(b) *$V(\mathbf{1}, x) = x$ for all $x \in [a, \mathbf{1}]$.*

Then $V(x, y) = a$ for all $(x, y) \in [\mathbf{0}, a] \times [a, \mathbf{1}] \cup [a, \mathbf{1}] \times [\mathbf{0}, a]$.

Proof. By the definition of nullnorms and monotonicity we get

$$\begin{aligned} V(\mathbf{0}, x) &\geq a \quad \text{for } x \geq a, \\ V(\mathbf{1}, x) &\leq a \quad \text{for } x \leq a. \end{aligned}$$

These imply on the one hand $V(\mathbf{0}, \mathbf{1}) \geq a$, on the other hand $V(\mathbf{0}, \mathbf{1}) \leq a$, hence $V(\mathbf{0}, \mathbf{1}) = a$. By monotonicity, also $V(\mathbf{0}, x) = a$ for $x \geq a$ and $V(\mathbf{1}, x) = a$ for $x \leq a$. Finally,

$$a = V(\mathbf{0}, y) \leq V(x, y) \leq V(x, \mathbf{1}) = a,$$

holds for all $(x, y) \in [\mathbf{0}, a] \times [a, \mathbf{1}]$. Commutativity of V implies the assertion in question. □

The following proposition presents necessary conditions that $\parallel_a$ has to fulfill if there exists an idempotent nullnorm. It is a generalization of items (a), (b) and (d) in Proposition 2.2

Proposition 3.3. *Let $(P, \mathbf{0}, \mathbf{1}, \leq)$ be a bounded poset. Assume there exists an idempotent proper nullnorm V with an annihilator a. Then*

(a) *for every $x \in \parallel_a$ there exist $x \wedge a$ and $x \vee a$, and $V(\mathbf{0}, x) = x \wedge a$, $V(\mathbf{1}, x) = x \vee a$,*

(b) *$V(x, y) = (x \wedge a) \vee (y \wedge a)$ for $(x, y) \in [\mathbf{0}, a] \times \parallel_a \cup \parallel_a \times [\mathbf{0}, a]$,*

(c) *$V(x, y) = (x \vee a) \wedge (y \vee a)$ for $(x, y) \in [a, \mathbf{1}] \times \parallel_a \cup \parallel_a \times [a, \mathbf{1}]$,*

(d) *if $V(x, y) = z$ for $(x, y) \in (\parallel_a)^2$ then one of the following is satisfied:*

- *(α) for every lower bound, $\ell_{x,y}$, of x and y, and every upper bound, $u_{x,y}$, of x and y, $z \in [\ell_{x,y}, u_{x,y}]$,*
- *(β) $[z \wedge a, z \vee a] \subset [x \wedge a, x \vee a]$, $[z \wedge a, z \vee a] \subset [y \wedge a, y \vee a]$,*
- *(γ) if $z \leq x$ and $z \leq y$ then there exists $x \wedge y$ and $z = x \wedge y$, if $z \geq x$, $z \geq y$ then there exists $x \vee y$ and $z = x \vee y$,*

(δ) *if* $z \not\parallel a$ *then*

$$\begin{aligned} z \le a &\Rightarrow (x \vee a) \wedge (y \vee a) = a, \\ z \ge a &\Rightarrow (x \wedge a) \vee (y \wedge a) = a, \\ z = a &\Rightarrow (x \wedge a) \vee (y \wedge a) = a = (x \vee a) \wedge (y \vee a). \end{aligned}$$

Remark 3.4. Before proving Proposition 3.3, let us make a parallel between Propositions 3.3 and 2.2.

(a) Proposition 3.3 (a) states that $x \wedge a$ and $x \vee a$ do exist for all $x \in \parallel_a$. This part is something that obviously holds when dealing with bounded lattices. The other part of Proposition 3.3 (a) is a special case of Proposition 2.2 (a) and (b).

(b) Proposition 3.3 (b) reformulates 2.2 (a), and Proposition 3.3 (c) reformulates 2.2 (b).

(c) The case when $(x, y) \in (\parallel_a)^2$, is formulated in Proposition 3.3 (d) in a different way than in Proposition 2.2 (d).

Proof of Proposition 3.3. **(a)** Assume there exists an $x \in \parallel_a$ such that $x \wedge a$ does not exist. Denote $V(x, \mathbf{0}) = z_1$. Then, since by Proposition 3.2 $V(\mathbf{1}, \mathbf{0}) = a$, we get $z_1 \le a$, and, by idempotency of V, $x = V(x, x) \ge V(x, \mathbf{0}) = z_1$. Since $x \wedge a$ does not exist, there exists z_2 such that $z_2 \le a$, $z_2 \le x$ and either $z_2 \ge z_1$ or $z_2 \parallel z_1$. By monotonicity of V we get

$$V(\mathbf{0}, z_2) = z_2 \le V(\mathbf{0}, x) = z_1,$$

and this contradicts the assumption that either $z_1 \le z_2$ or $z_1 \parallel z_2$. Thus, $V(x, \mathbf{0}) = x \wedge a$.
Similarly we could prove the existence of $x \vee a$ and that $V(x, \mathbf{1}) = x \vee a$.

(b) By item (a) $V(x, \mathbf{0}) = a \wedge x$ for arbitrary $x \in \parallel_a$. By monotonicity, $V(x, \mathbf{0}) \le V(x, y)$ for $(x, y) \in \parallel_a \times [\mathbf{0}, a]$. By Proposition 3.2 and monotonicity,

$$a = V(\mathbf{1}, y) \ge V(x, y) \ge V(x \wedge a, y).$$

Hence, by associativity and commutativity we get

$$V(x, y) = V(V(x, y), \mathbf{0}) = V(V(x, \mathbf{0}), y) = V(x \wedge a, y).$$

Since $V \restriction [\mathbf{0}, a]^2 = S_M$, we get immediately $V(x \wedge a, y) = (x \wedge a) \vee y = (x \wedge a) \vee (y \wedge a)$,

(c) could be proved similarly to the case (b).

(d) Let $(x, y) \in (\parallel_a)^2$.

(α) Assume $\ell_{x,y}$ and $u_{x,y}$ are a lower and an upper bound of x and y, respectively. Then by idempotency and monotonicity the following holds

$$V(\ell_{x,y}, \ell_{x,y}) = \ell_{x,y} \le V(x, y), \quad V(x, y) \le V(u_{x,y}, u_{x,y}) = u_{x,y}.$$

(β) Assume $V(x, y) = z$. Then

$$z \wedge a = V(\mathbf{0}, V(x, y)) = V(V(\mathbf{0}, x), y) = V(x \wedge a, y) = (x \wedge a) \vee (y \wedge a),$$

which implies $z \wedge a \ge x \wedge a$ and $z \wedge a \ge y \wedge a$. Similarly we could prove $z \vee a \le x \vee a$ and $z \vee a \le y \vee a$.

(γ) Let $z \le x$ and $z \le y$. Assume z_1 is a lower bound of x and y. Then by monotonicity and idempotency of V we have

$$z_1 = V(z_1, z_1) \le V(x, y) = z.$$

This implies $z = x \wedge y$. Similarly we could prove that if $V(x, y) = z$ and $z \ge x$, $z \ge y$ then $z = x \vee y$.

(δ) Assume $z < a$ Then by Proposition 2.2 (c), $a = V(z, \mathbf{1})$. Since by notation of Proposition 3.3 (d), $z = V(x, y)$. Dually to the proof of part (a) we could show that, for $y \in \parallel_a$, $V(y, \mathbf{1}) \ge a$ and $V(y, \mathbf{1}) \ge y$. Since $V \restriction [a, \mathbf{1}]^2$ is the idempotent t-norm, we get $V(y, \mathbf{1}) = y \vee a$. Moreover, for arbitrary $s \in [a, \mathbf{1}]$, $a = V(\mathbf{0}, \mathbf{1}) \le V(x, s) = V(x \vee a, s)$. Using this consideration and the fact that $V \restriction [a, \mathbf{1}]^2$ is the idempotent t-norm we get the following equality

$$a = V(z, \mathbf{1}) = V(V(x, y), \mathbf{1}) = V(x, V(y, \mathbf{1})) = V(x, y \vee a) = (x \vee a) \wedge (y \vee a).$$

Similarly, by direct computation we could show the cases when $z > a$ and $z = a$.

□

Remark 3.5. Consider the bounded posets P_5 and P_6 whose Hasse diagrams are sketched in Fig. 5 and 6, respectively. Checking Proposition 3.3, we see that neither on P_5 nor on P_6 it is possible to define any idempotent nullnorm.

Before turning our attention to idempotent uninorms on a bounded $(P, \leq)$, let us introduce a partial relation $\preceq$.

Lemma 3.6. *Let $e \in P$ be a chosen element such that $\|_e \neq \emptyset$, and assume there exists a relation $\preceq$ on the set $\|_e$, fulfilling*

1. *$\preceq$ is antisymmetric,*
2. *$x \leq y$ implies $x \preceq y$ for $(x, y) \in (\|_e)^2$,*
3. *$x \preceq y$ and $y \preceq z$ implies $x \preceq z$,*
4. *let for any $(x, y) \in (\|_e)^2$ and any $z \in [\mathbf{0}, e]$, if $z \leq x$ and $x \preceq y$ then $z \leq y$,*
5. *there exists $x \wedge y \in \|_e$ for any $x \|_{\preceq} y$, where $\|_{\preceq}$ means the incomparability with respect to $\preceq$,*
6. *there exists the operation meet, $\curlywedge : (\|_e)^2 \to \|_e$, with respect to $\preceq$ fulfilling*

$$x \curlywedge y = \begin{cases} x & \text{for } x \preceq y, \\ y & \text{for } y \preceq x, \\ x \wedge y & \text{otherwise,} \end{cases}$$

and $\curlywedge$ is monotone with respect to the poset order $\leq$.

Then $\preceq$ is a partial order on $\|_e$ and $(\|_e, \preceq)$ is a meet semi-lattice.

Proof. Property 1) implies the anti-symmetry and 30 the transitivity of $\preceq$. Property 2) implies the reflexivity of $\preceq$. Properties 5) and 6) imply that $(\|_e, \preceq)$ is a meet semi-lattice. Moreover, property 4) together with 2) imply that the relation $\preceq$ does not violate the monotonicity with respect to the poset order $\leq$. □

The following propositions state some sufficient conditions for the existence of an idempotent uninorm on a bounded poset and constructions of such uninorms.

Proposition 3.7. *Let P be a bounded poset and $e \in P \setminus \{\mathbf{0}, \mathbf{1}\}$. Denote $D_e = [\mathbf{0}, e[\times [e, \mathbf{1}] \cup [e, \mathbf{1}] \times [\mathbf{0}, e[$. Assume $[\mathbf{0}, e]$ is a meet semi-lattice, $[e, \mathbf{1}]$ is a join semi-lattice. Further, let $\|_e$ be (partially) ordered by $\preceq$, introduced in Lemma 3.6 and assume $(\|_e, \preceq)$ fulfills all assumptions of Lemma 3.6. Then the function, $U : P \times P \to P$, defined by*

$$U(x, y) = \begin{cases} x \vee y & \text{for } (x, y) \in [e, \mathbf{1}]^2, \\ x \wedge y & \text{for } (x, y) \in [\mathbf{0}, e]^2 \cup D_e, \\ & \text{for } (x, y) \in [\mathbf{0}, e] \times \|_e \text{ such that } x \leq y, \\ & \text{and for } (x, y) \in \|_e \times [\mathbf{0}, e] \text{ such that } y \leq x, \\ x & \text{for } (x, y) \in \|_e \times [e, \mathbf{1}], \\ & \text{and for } (x, y) \in \|_e \times [\mathbf{0}, e] \text{ such that } y \not\leq x, \\ y & \text{for } (x, y) \in [e, \mathbf{1}] \times \|_e, \\ & \text{and for } (x, y) \in [\mathbf{0}, e] \times \|_e \text{ such that } x \not\leq y, \\ x \curlywedge y & \text{for } (x, y) \in (\|_e)^2, \end{cases} \tag{4}$$

is a conjunctive uninorm whose neutral element is e provided the interval $[\mathbf{0}, e[$ can be split into two disjoint parts I_1, I_2 in the following way

$$(\forall x \in I_2)(\forall y \in I_1)(\forall z \in \|_e)(x \leq y,\ x \leq z), \tag{5}$$

$$(\forall y_1, y_2 \in I_1)(\nexists z \in \|_e)(y_1 \wedge y_2 \leq z). \tag{6}$$

Proof. First, the idempotency, commutativity and monotonicity of U follow directly by the definition of U. The fact that e is the neutral element, is also directly due to the definition of U.

To ensure the associativity of U, realize that, by (5) and (6), $U(x,y) \in I_1$ for all pairs $(x,y) \in I_1$. By constraints 3. and 4. of the proposition, and $U(x,y) \in \|_e$ for all pairs $(x,y) \in \|_e$. Further, the whole poset P is split into 4 disjoint parts, particularly into $P_1 = [e,\mathbf{1}]$, $P_2 = I_1$, $P_3 = \|_e$ and $P_4 = I_2$. There is a dominance among these parts in the following sense

$$U(x,y) = x \quad \text{for } (x,y) \in P_i \times P_j, \text{ if } i > j.$$

Since the uninorm is, within particular parts P_1 up to P_4 associative, the proof is completed. □

Remark 3.8. By Proposition 2.9, the existence of a quadruple $\{c_1, c_2, c_3, c_4\} \subset \|_e$ such that $x_3 \leq x_1, x_3 \leq x_2, x_4 \leq x_1, x_4 \leq x_2$ and there exists no meet of x_3, x_4 violates the monotonicity of $\preceq$ defined in Proposition 3.7.

Dually to Lemma 3.6 we formulate the following.

Lemma 3.9. *Let $e \in P$ be a chosen element such that $\|_e \neq \emptyset$, and assume there exists a relation $\preceq$ on the set $\|_e$, fulfilling*

1. *$\preceq$ is antisymmetric,*
2. *$x \leq y$ implies $x \preceq y$ for $(x,y) \in (\|_e)^2$,*
3. *$x \preceq y$ and $y \preceq z$ implies $x \preceq z$,*
4. *let for any $(x,y) \in (\|_e)^2$ and any $z \in [e,\mathbf{1}]$, if $x \leq z$ and $y \preceq x$ then $y \leq z$,*
5. *there exists $x \vee y \in \|_e$ for any $x \|_{\preceq} y$,*
6. *there exists the operation join, $\curlyvee : (\|_e)^2 \to \|_e$, with respect to $\preceq$ fulfilling*

$$x \curlyvee y = \begin{cases} y & \text{for } x \preceq y, \\ x & \text{for } y \preceq x, \\ x \vee y & \text{otherwise,} \end{cases}$$

and $\curlyvee$ is monotone with respect to the poset order $\leq$.

Then $\preceq$ is a partial order on $\|_e$ and $(\|_e, \preceq)$ is a join semi-lattice.

Dually, reverting the poset-order in Proposition 3.7 we get the following assertion.

Proposition 3.10. *Let P be a bounded poset and $e \in P \setminus \{\mathbf{0}, \mathbf{1}\}$. Denote $D_e = [\mathbf{0}, e] \times]e, \mathbf{1}] \cup]e, \mathbf{1}] \times [\mathbf{0}, e]$. Assume $[\mathbf{0}, e]$ is a meet semi-lattice, $[e, \mathbf{1}]$ is a join semi-lattice. Further, let $\|_e$ be (partially) ordered by $\preceq$ introduced in Lemma 3.9 and assume that $(\|_e, \preceq)$ fulfills all assumptions of Lemma 3.9. Then the function, $U : P \times P \to P$, defined by*

$$U(x,y) = \begin{cases} x \vee y & \text{for } (x,y) \in [e,\mathbf{1}]^2 \cup D_e, \\ & \text{for } (x,y) \in [e,\mathbf{1}] \times \|_e \text{ such that } x \geq y, \\ & \text{and for } (x,y) \in \|_e \times [e,\mathbf{1}] \text{ such that } y \geq x, \\ x \wedge y & \text{for } (x,y) \in [\mathbf{0},e]^2, \\ x & \text{for } (x,y) \in \|_e \times [\mathbf{0},e], \\ & \text{and for } (x,y) \in \|_e \times [e,\mathbf{1}] \text{ such that } y \ngeq x, \\ y & \text{for } (x,y) \in [\mathbf{0},e] \times \|_e, \\ & \text{and for } (x,y) \in [e,\mathbf{1}] \times \|_e \text{ such that } x \ngeq y, \\ x \curlyvee y & \text{for } (x,y) \in (\|_e)^2, \end{cases} \tag{7}$$

is a disjunctive uninorm whose neutral element is e provided the interval $]e, \mathbf{1}]$ can be split into two disjoint parts, J_1, J_2 in the following way

$$(\forall x \in J_2)(\forall y \in J_1)(\forall z \in \|_e)(x \geq y,\ x \geq z), \tag{8}$$

$$(\forall y_1, y_2 \in J_1)(\nexists z \in \|_e)(y_1 \vee y_2 \geq z). \tag{9}$$

Similarly to Lemmas 3.6 and 3.9, we introduce yet another kind of order.

Lemma 3.11. *Let $e, a \in P$ be a chosen elements such that $a \parallel e$, and assume there exists a relation $\preceq$ on the set $\parallel_e$, fulfilling*

1. *$\preceq$ is antisymmetric,*
2. *$x \le y$ implies $x \preceq y$ for $(x,y) \in (\parallel_e)^2$,*
3. *$x \preceq y$ and $y \preceq z$ implies $x \preceq z$,*
4. *let for any $(x,y) \in (\parallel_e \cap [\mathbf{0}, a])^2$ and $x \parallel_{\preceq} y$ there exists $x \vee y$,*
5. *let for any $(x,y) \in (\parallel_e \cap [a, \mathbf{1}])^2$ and $x \parallel_{\preceq} y$ there exists $x \wedge y$,*
6. *there exists the operations $\curlyvee : (\parallel_e \cap [\mathbf{0}, a])^2 \to [\mathbf{0}, a]$ and $\curlywedge : (\parallel_e \cap [a, \mathbf{1}])^2 \to [a, \mathbf{1}]$ defined respectively by*

$$x \curlyvee y = \begin{cases} y & \text{for } x \preceq y, \\ x & \text{for } y \preceq x, \\ x \vee y & \text{otherwise,} \end{cases} \qquad x \curlywedge y = \begin{cases} x & \text{for } x \preceq y, \\ y & \text{for } y \preceq x, \\ x \wedge y & \text{otherwise,} \end{cases}$$

which are monotone with respect to the poset order $\le$.

Then $\preceq$ is a partial order on $\parallel_e$.

Proposition 3.12. *Let P be a bounded poset and $e, a \in P \setminus \{\mathbf{0}, \mathbf{1}\}$ such that $e \parallel a$. Assume $[\mathbf{0}, e]$ is a meet semi-lattice, $[e, \mathbf{1}]$ is a join semi-lattice. Further, let $\parallel_e$ be (partially) ordered by $\preceq$ introduced in Lemma 3.11. Moreover, assume $\parallel_e \cap \parallel_a = \emptyset$ and $x \parallel y$ for all $(x,y) \in \parallel_e \times \parallel_a$. Then the function, $U : P \times P \to P$, defined by*

$$U(x,y) = \begin{cases} x \vee y & \text{for } (x,y) \in [e, \mathbf{1}]^2, \\ x \wedge y & \text{for } (x,y) \in [\mathbf{0}, e]^2, \\ a & \text{for } (x,y) \in [\mathbf{0}, a] \times [a, \mathbf{1}] \cup [a, \mathbf{1}] \times [\mathbf{0}, a], \\ x & \text{for } (x,y) \in \parallel_e \times \parallel_a, \\ & \text{for } (x,y) \in [e, \mathbf{1}] \times (\parallel_a \cap [\mathbf{0}, e]), \\ & \text{for } (x,y) \in ([a, \mathbf{1}] \cap \parallel_e) \times ([a, \mathbf{1}] \cap [e, \mathbf{1}]), \\ & \text{for } (x,y) \in ([\mathbf{0}, a] \cap \parallel_e) \times ([\mathbf{0}, a] \cap [\mathbf{0}, e]), \\ & \text{and for } (x,y) \in ([\mathbf{0}, e] \cap [\mathbf{0}, a]) \times (\parallel_a \cap [e, \mathbf{1}]), \\ y & \text{for } (x,y) \in \parallel_a \times \parallel_e, \\ & \text{for } (x,y) \in (\parallel_a \cap [\mathbf{0}, e]) \times [e, \mathbf{1}], \\ & \text{for } (x,y) \in ([a, \mathbf{1}] \cap [e, \mathbf{1}]) \times ([a, \mathbf{1}] \cap \parallel_e), \\ & \text{for } (x,y) \in ([\mathbf{0}, a] \cap [\mathbf{0}, e]) \times ([\mathbf{0}, a] \cap \parallel_e), \\ & \text{and for } (x,y) \in (\parallel_a \cap [e, \mathbf{1}] \times ([\mathbf{0}, e] \cap [\mathbf{0}, a]), \\ x \curlyvee y & \text{for } (x,y) \in (\parallel_e \cap [\mathbf{0}, a])^2, \\ x \curlywedge y & \text{for } (x,y) \in (\parallel_e \cap [a, \mathbf{1}])^2 \ , \end{cases} \tag{10}$$

is a uninorm whose neutral element is e and annihilator is a, provided

$$\big(\forall (x,y) \in ([e, \mathbf{1}] \cap \parallel_a)^2\big)(x \vee y \in \parallel_a), \tag{11}$$

$$\big(\forall (x,y) \in ([\mathbf{0}, e] \cap \parallel_a)^2\big)(x \wedge y \in \parallel_a), \tag{12}$$

$$\big(\forall (x,y) \in ([\mathbf{0}, e] \cap \parallel_a) \times ([\mathbf{0}, e] \cap [\mathbf{0}, a])\big)(x \ge y). \tag{13}$$

Proof. Realize that the last two items in formula (10) define on $\parallel_e$, together with the partial order $\preceq$, an idempotent nullnorm-like operation, more precisely, that operation has all the properties of an idempotent nullnorm, possibly without the existence of the partial neutral elements. Otherwise the proof of the proposition in question copies the proof of Proposition 3.7. □

To illustrate Propositions 3.7, 3.10 and 3.12, we provide the following example.

Example 3.13. **(a)** Consider the poset P_a depicted in Fig. 7. The set $\|_e = \{z_1, z_2, z_3, z_4\}$ can be linearly ordered by $\preceq_a$ in the following way

$$z_1 \preceq_a z_2 \preceq_a z_3 \preceq_a z_4.$$

This means, $(P_a, \leq_{P_a})$ fulfills all the constraints of Proposition 3.7. Hence, formula (4) defines an idempotent conjunctive uninorm on P_a. When we revert the poset order $\leq_{P_a}$ into $\leq_{P_\alpha}$ and revert also the linear order $\preceq_a$ on $\|_e$ into $\preceq_\alpha$, we get a poset $(P_a, \leq_\alpha)$ that fulfills all the constraints of Proposition 3.10 and hence, formula (7) defines an idempotent disjunctive uninorm on $(P_a, \leq_\alpha)$.

(b) Consider the poset $(P_b, \leq_b)$ depicted in Fig. 8. In this case, it is possible to define $x \preceq_b y \Leftrightarrow x \leq_b y$ for $(x, y) \in \|_b$. P_b fulfills all the constraints of Proposition 3.12, hence, formula (10) defines an idempotent uninorms whose annihilator is a.

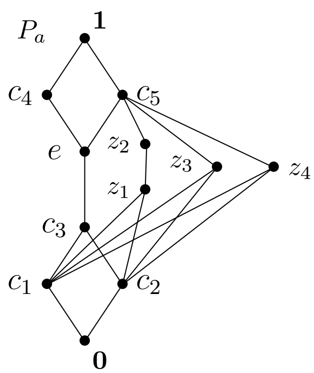

Figure 7: Hasse diagram of $(P_a, \leq_a)$

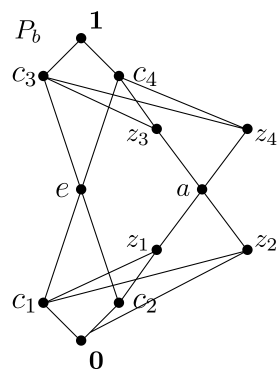

Figure 8: Hasse diagram of $(P_b, \leq_b)$

In the following propositions some sufficient conditions for the existence of idempotent nullnorms, as well as constructions of such nullnorms, are presented.

Proposition 3.14. *Let P be a bounded poset, $a \in P \setminus \{\mathbf{0}, \mathbf{1}\}$ be such that $[\mathbf{0}, a]$ is a join semi-lattice and $[a, \mathbf{1}]$ is a meet semi-lattice. Further, assume that all constraints of Proposition 3.3 are fulfilled for $\|_a$. If, moreover,*

(a) *there exists $\mathbf{0}_a < a$ and $\mathbf{1}_a > a$ such that $x \wedge a = y \wedge a = \mathbf{0}_a$ and $x \vee a = y \vee a = \mathbf{1}_a$ for all $(x, y) \in (\|_a)^2$,*

(b) *there exists an idempotent uninorm $U_a : (\|_a \cup \{\mathbf{0}_a, \mathbf{1}_a\})^2 \to (\|_a \cup \{\mathbf{0}_a, \mathbf{1}_a\})$ without zero and one divisors,*

then the following function $V : P \times P \to P$ defined by

$$V(x,y) = \begin{cases} U_a(x,y) & \text{for } (x,y) \in (\|_a)^2, \\ x \vee y & \text{for } (x.y) \in [\mathbf{0}, a]^2, \\ x \wedge y & \text{for } (x,y) \in [a, \mathbf{1}]^2, \\ (x \wedge a) \vee (y \wedge a) & \text{for } (x,y) \in [\mathbf{0}, a] \times \|_a \cup \|_a \times [\mathbf{0}, a], \\ (x \vee a) \wedge (y \vee a) & \text{for } (x,y) \in [a, \mathbf{1}] \times \|_a \cup \|_a \times [a, \mathbf{1}], \\ a & \text{otherwise}, \end{cases} \tag{14}$$

is an idempotent nullnorm.

Proof. To ensure the associativity, by constraints (a) and (b),

$$U(U(x_1, x_2), y) = x_1 \vee x_2 \vee \mathbf{0}_a = U(x_1, U(x_2, y)),$$
$$U(x, U(y_1, y_2)) = x \vee \mathbf{0}_a = U(U(x, y_1), y_2),$$

for arbitrary $(x_1, x_2, y) \in [\mathbf{0}, a]^2 \times \|_a$ and arbitrary $(x, y_1, y_2) \in [\mathbf{0}, a] \times (\|_a)^2$.

In a similar way we could prove the associativity when $(x_1, x_2, y) \in [a, \mathbf{1}]^2 \times \|_a$ and arbitrary $(x, y_1, y_2) \in [a, \mathbf{1}] \times (\|_a)^2$. All other properties of an idempotent nullnorm follow directly by the formula (14). □

Proposition 3.15. *Let P be a bounded poset, $a \in P \setminus \{\mathbf{0}, \mathbf{1}\}$ be such that $[\mathbf{0}, a]$ is a join semi-lattice and $[a, \mathbf{1}]$ is a meet semi-lattice. Further, assume that all constraints of Proposition 3.3 are fulfilled for $\|_a$. If, moreover, $\|_a$ is partially ordered by $\preceq$ fulfilling*

(a) *$x \le y$ implies $x \preceq y$ for all $(x, y) \in (\|_a)^2$,*

(b) *$x \wedge a \ge y \wedge a$ and $x \vee a \le y \vee a$ for all $(x, y) \in (\|_a)^2$ such that $x \preceq y$,*

(c) *for $(x, y) \in (\|_a)^2$ such that $x \|_{\preceq} y$, there exists $x \wedge y$ and $x \wedge y \in \|_a$,*

(d) *the operation $\curlywedge$ defined on $\|_a$ by* $x \curlywedge y = \begin{cases} x & \text{for } x \preceq y, \\ y & \text{for } y \preceq x, \\ x \wedge y & \text{otherwise}, \end{cases}$

is monotone with respect to the poset order $\le$,

then the function $V : P \times P \to P$ defined by

$$V(x, y) = \begin{cases} x \curlywedge y & \text{for } (x, y) \in (\|_a)^2, \\ x \vee y & \text{for } (x, y) \in [\mathbf{0}, a]^2, \\ x \wedge y & \text{for } (x, y) \in [a, \mathbf{1}]^2, \\ (x \wedge a) \vee (y \wedge a) & \text{for } (x, y) \in [\mathbf{0}, a] \times \|_a \cup \|_a \times [\mathbf{0}, a], \\ (x \vee a) \wedge (y \vee a) & \text{for } (x, y) \in [a, \mathbf{1}] \times \|_a \cup \|_a \times [a, \mathbf{1}], \\ a & \text{otherwise}, \end{cases} \tag{15}$$

is an idempotent nullnorm.

Proposition 3.16. *Let P be a bounded poset, $a \in P \setminus \{\mathbf{0}, \mathbf{1}\}$ be such that $[\mathbf{0}, a]$ is a join semi-lattice and $[a, \mathbf{1}]$ is a meet semi-lattice. Further, assume that all constraints of Proposition 3.3 are fulfilled for $\|_a$. If, moreover, $\|_a$ is partially ordered by $\preceq$ fulfilling*

(a) *$x \le y$ implies $x \preceq y$ for all $(x, y) \in (\|_a)^2$,*

(b) *$x \wedge a \le y \wedge a$ and $x \vee a \ge y \vee a$ for all $(x, y) \in (\|_a)^2$ such that $x \preceq y$,*

(c) *for $(x, y) \in (\|_a)^2$ such that $x \|_{\preceq} y$, there exists $x \vee y$ and $x \vee y \in \|_a$,*

(d) *the operation $\curlyvee$ defined on $\|_a$ by* $x \curlyvee y = \begin{cases} x & \text{for } x \preceq y, \\ y & \text{for } y \preceq x, \\ x \vee y & \text{otherwise}, \end{cases}$

is monotone with respect to the poset order $\le$,

then the function $V : P \times P \to P$ defined by

$$V(x, y) = \begin{cases} x \curlyvee y & \text{for } (x, y) \in (\|_a)^2, \\ x \vee y & \text{for } (x, y) \in [\mathbf{0}, a]^2, \\ x \wedge y & \text{for } (x, y) \in [a, \mathbf{1}]^2, \\ (x \wedge a) \vee (y \wedge a) & \text{for } (x, y) \in [\mathbf{0}, a] \times \|_a \cup \|_a \times [\mathbf{0}, a], \\ (x \vee a) \wedge (y \vee a) & \text{for } (x, y) \in [a, \mathbf{1}] \times \|_a \cup \|_a \times [a, \mathbf{1}], \\ a & \text{otherwise}, \end{cases} \tag{16}$$

is an idempotent nullnorm.

Finally, we provide illustrative examples to Propositions 3.14, 3.15 and 3.16.

Example 3.17. (α) Consider the poset $(\overline{P}_5, \le_{\overline{P}_5})$ where $\overline{P}_5 = P_5 \cup \{a, \mathbf{0}, \mathbf{1}, \mathbf{1}_a\}$ and $P_5 = [\mathbf{0}_{P_5}, \mathbf{1}_{P_5}]$ is the poset whose Hasse diagram is depicted in Fig. 5. The order $\le_{\overline{P}_5}$ is an extension of $\le_{P_5}$ as Hasse diagram shows in Fig. 9. On $P_5 \cup \{\mathbf{1}_a, \mathbf{0}\}$ it is possible to define an idempotent uninorm U_a without $\mathbf{0}$ and $\mathbf{1}_a$ divisors by the following

$$U_a(x, y) = \begin{cases} x \wedge y & \text{for } (x, y) \in [\mathbf{0}, c_3]^2, \\ x \vee y & \text{for } (x, y) \in [c_3, \mathbf{1}_a]^2, \\ c_4 & \text{otherwise.} \end{cases}$$

By means of formula (14) and the uninorm U_a we get an idempotent nullnorm.

(β) Consider the poset $(P_\beta, \leq_\beta)$ whose Hasse diagram is depicted in Fig. 10. In P_β, there are infinite chains of elements $s_1 \geq_\beta s_2 \geq_\beta s_3 \geq_\beta \ldots$ and of elements $t_1 \geq_\beta t_2 \geq_\beta t_3 \geq_\beta \ldots$, where all elements s_i are upper bounds of b_1, b_2 and t_i are lower bounds of b_1, b_2.
On the set $\|_a$ we define a partial order $\preceq_\beta$ by $x \preceq_\beta y$ if one of the following is fulfilled

1. $x \leq_\beta y$
2. $x \in \{s_1, s_2, s_3, \ldots\} \cup \{t_1, t_2, t_3, \ldots\} \cup \{b_1, b_2\}$ and $y \in \{d_1, d_2, d_3\}$.

Formula (15) defines an idempotent nullnorm V on P_β.

(γ) Reverting the poset order $\leq_\beta$ of P_β another poset, $(P_\beta, \leq_\gamma)$, is defined. It is possible to to define an idempotent uninorm on P_γ using formula (16) provided the set $\|_a$ is equipped with a partial order $\preceq_\gamma$ fulfilling $x \preceq_\gamma y$ if one of the following properties holds

1. $x \leq_\gamma y$
2. $x \in \{d_1, d_2, d_3\}$ and $y \in \{s_1, s_2, s_3, \ldots\} \cup \{t_1, t_2, t_3, \ldots\} \cup \{b_1, b_2\}$.

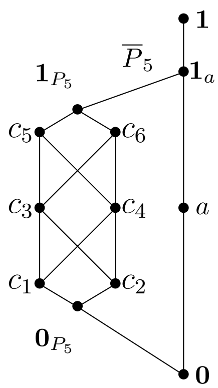

Figure 9: Hasse diagram of $\overline{P}_5$

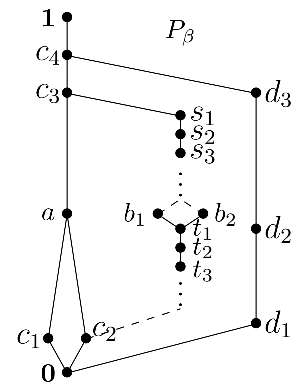

Figure 10: Hasse diagram of P_β

Remark 3.18. The poset P_5 (see Fig. 5) does not fulfill the necessary conditions of Proposition 3.3, this means, there exists no idempotent nullnorm on P_5. However, as Example 3.17(α) illustrates, if P_5 is just a sub-poset, still it may be possible to define an idempotent nullnorm provided $P_5 \subset \|_a$.

4 Application examples

In the following examples, we illustrate how idempotent uninoms and idempotent nullnorm can be applied for a construction of fuzzy regulators.

Example 4.1. Assume a set of several automata with two parallel-working parts (left and right) each of them. The left/right part of every automaton is in one of three possible states: $0 - \mathrm{m} - 1$ where m means middle. Altogether, every particular automaton is in one of 8 possible states. The set of states and the relationship among them (partial order) create the poset P_5 sketched in Fig. 5. The interpretation is the following: elements $\mathbf{0}, c_4, \mathbf{1}$ correspond to states $(0,0), (\mathrm{m},\mathrm{m}), (1,1)$, respectively, of the left/right part. Further, c_1, c_2, c_5, c_6 correspond to $(0,\mathrm{m}), (\mathrm{m},0), (\mathrm{m},1), (1,\mathrm{m})$, respectively, and finally, c_3 corresponds to both $(0,1), (1,0)$. We denote $c_3 = a$ and $c_4 = e$.
If an automaton is in the state a, a manual adjustment (maintenance) is necessary. The whole set of automata works properly if the left and right parts of different automata from the set are in the same state and if none of the automata is in the state a. The set of automata is able to make a self-adjustment unless one automaton is in a state $s_1 \in \{a, c_5, c_6, \mathbf{1}\}$ and another automaton is in $s_2 \in \{a, c_1, c_2, \mathbf{0}\}$. The adjustment (manual as well as self-adjustment) of the whole set works according the uninorm U whose layout is in Table 3.

Example 4.2. Now, we slightly modify the model from Example 4.1. We consider a set of electric carriages where particular carriages have to cooperate among themselves. Let $S = \{\mathbf{0}, c_\perp, c_1, c_2, c_3, c_4, c\top, a, \mathbf{1}\}$ be the set of states of a carriage, where states $\mathbf{0}$, $c_\perp$, $c_\top$, $\mathbf{1}$, a mean – the engine is off, minimal speed, medium speed, maximal speed, emergency

U	$\mathbf{0}$	c_1	c_2	a	e	c_5	c_6	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	a	$\mathbf{0}$	a	a	a
c_1	$\mathbf{0}$	c_1	$\mathbf{0}$	a	c_1	a	a	a
c_2	$\mathbf{0}$	$\mathbf{0}$	c_2	a	c_2	a	a	a
a	a	a	a	a	a	a	a	a
e	$\mathbf{0}$	c_1	c_2	a	e	c_5	c_6	$\mathbf{1}$
c_5	a	a	a	a	c_5	c_5	$\mathbf{1}$	$\mathbf{1}$
c_6	a	a	a	a	c_6	$\mathbf{1}$	c_6	$\mathbf{1}$
$\mathbf{1}$	a	a	a	a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

Table 3: The idempotent uninorm $U : P_5 \times P_5 \to P_5$ with e as the neutral element and a as the annihilator (see Fig. 5)

stop. States c_1, c_2, c_3, c_4 respectively mean the carriage turns slowly left, turns slowly right, turns left at medium speed, and finally, turns right at medium speed, respectively. Two or more carriages in a set are able of a self-adjustment that is given by the operation V laid out in Table 4. The set of states S is partially ordered and thus creates the poset P_7 depicted in Fig. 11.

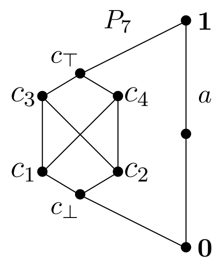

Figure 11: Hasse diagram of P_7

The operation $V : P_7 \times P_7 \to P_7$ defined in Table 4 is an idempotent nullnorm.

V	$\mathbf{0}$	$c_\perp$	c_1	c_2	c_3	c_4	$c_\top$	a	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	a	a
$c_\perp$	$\mathbf{0}$	$c_\perp$	$c_\perp$	$c_\perp$	c_3	c_4	$c_\top$	a	$\mathbf{1}$
c_1	$\mathbf{0}$	$c_\perp$	c_1	$c_\perp$	c_3	c_4	$c_\top$	a	$\mathbf{1}$
c_2	$\mathbf{0}$	$c_\perp$	$c_\perp$	c_2	c_3	c_4	$c_\top$	a	$\mathbf{1}$
c_3	$\mathbf{0}$	c_3	c_3	c_3	c_3	$c_\top$	$c_\top$	a	$\mathbf{1}$
c_4	$\mathbf{0}$	c_4	c_4	c_4	$c_\top$	c_4	$c_\top$	a	$\mathbf{1}$
$c_\top$	$\mathbf{0}$	$c_\top$	$c_\top$	$c_\top$	$c_\top$	$c_\top$	$c_\top$	a	$\mathbf{1}$
a	a	a	a	a	a	a	a	a	a
$\mathbf{1}$	a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	a	$\mathbf{1}$

Table 4: The idempotent nullnorm V with a as the annihilator (see Fig. 11)

5 Conclusions

In the paper we have found some necessary and some sufficient conditions for the existence of idempotent uninorms and idempotent nullnorms on bounded posets. The main results are exemplified. Proposition 2.7, Corollary 2.8 and Proposition 2.9 state some general conditions for the existence of monotone and idempotent operations on bounded posets.

The layout of idempotent nullnorms is formulated in Propositions 3.2 and 3.3. These two propositions are, in fact, reformulations (and some necessary extensions) of Proposition 2.2 which gives their layout on bounded lattices. Sufficient conditions for the existence of idempotent nullnorms on bounded posets are formulated in Propositions 3.14

– 3.16. Some necessary condition for the existence of idempotent uninorms on bounded posets are formulated in Propositions 3.7, 3.10 and 3.12.

Finally, in Section 4, we have provided some application examples.

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