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Original paper

On the properties of the fuzzy weighted average of fuzzy numbers with normalized fuzzy weights

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Abstract

Weighted average with normalized weights is a widely used aggregation operator that takes into account the varying degrees of importance of the numbers in a data set. It possesses some important properties, like monotonicity, continuity, additivity, etc., that play an important role in practical applications. The inputs of the aggregation as well as the normalized weights are usually not known precisely. In such a case, their values can be expressed by fuzzy numbers, and the fuzzy weighted average of fuzzy numbers with normalized fuzzy weights needs to be employed in the model. The aim of the paper is to reveal whether and in which way the properties of the weighted average operator can be observed also for its fuzzy extension. It is shown that it possesses three conditions characteristic for aggregation operators – identity, monotonicity and boundary conditions, and besides that, also compensation, idempotency, stability for linear transformation, 1-lipschitzianity, and continuity. Furthermore, it is proved that it preserves strict monotonicity in case of positive fuzzy weights, and symmetry in case of equal fuzzy weights, although it does not coincide with the fuzzy arithmetic mean operator in such a case. One of the most valuable result of the study is the fact that in contrast to the crisp weighted average operator, it is not additive. The importance of the obtained results is discussed and illustrated by several illustrative examples.

Keywords: Aggregation operator, fuzzy numbers, normalized fuzzy weights, fuzzy probabilities, fuzzy weighted average.

1 Introduction

One of key steps in decision making and in many other branches is the summarization of the available information into one value. This is standardly done by some kind of aggregation function. One of the most popular is the weighted average of real numbers $x_1, \dots, x_n$ with normalized weights $w_1, \dots, w_n \in [0, 1]$, satisfying $\sum_{i=1}^{n} w_i = 1$, that is given by

$$a_w(x_1, \dots, x_n; w_1, \dots, w_n) = \sum_{i=1}^{n} w_i \cdot x_i.$$

It takes into account the varying degrees of importance of inputs to aggregation and because of that it is widely used in various mathematical models and practical applications.

One of the branches, where this operation is frequently applied, are multi-criteria decision making (MCDM) models. In these models, the overall evaluations of alternatives are often calculated as weighted averages of evaluations with respect to the partial criteria; the normalized weights represent shares of the partial evaluations in the overall one (see e.g. [7, 23, 24]).

The weighted average operation with normalized weights is also used in discrete stochastic models of decision making under risk, where the evaluations of alternatives depend on the fact which of the states of the world $SoW_1, \dots, SoW_n$

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will occur (see e.g. [30]). The evaluation of any alternative is in these models a discrete random variable U that takes its values $u_1, \ldots, u_n$ with probabilities $p_1, \ldots, p_n$, $\sum_{i=1}^n p_i = 1$; the value p_i, for any $i \in \{1, \ldots, n\}$, expresses the probability of occurring of SoW_i. The expected value of evaluation of an alternative, that is usually taken into consideration when the alternative is to be compared with the others, is given by $EU = \sum_{i=1}^n p_i u_i$, i.e. $p_1, \ldots, p_n$ play the role of normalized weights.

In the above mentioned applications, input parameters of the weighted average, i.e. normalized weights and weighted values, are often uncertain. For instance, the weights of criteria or the probabilities of states of the world are often set subjectively on the basis of experts' experiences or opinions. Information about the weighted values can be incomplete, missing, or also vague, e.g. in the case of the expert evaluation of alternatives with respect to a qualitative criterion. Such kinds of uncertainty can be sufficiently modelled by means of tools of fuzzy sets theory (see an overview made in [7]). Fuzzy weighted average has been applied in many practical applications - in risk evaluation (see e.g. [18, 29]), in decision making (see e.g. [5, 11, 13, 15, 26]), in information processing (see e.g. [31]) and many other branches.

The study of possibilities of fuzzyfying the weighted average operation started in the second half of 70's in [1]. In the crisp (i.e. non-fuzzy) case, the weighted average with normalized weights represents a special case of the weighted average $x = \sum_{i=1}^n w_i x_i / \sum_{i=1}^n w_i$, where $w_i \geq 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n w_i > 0$. However, it was pointed out in [8, 21] that in fuzzy environment, it is necessary to carefully distinguish whether the fuzzy weights express the uncertain values of non-normalized weights, i.e. there is no interaction among the weights, or the uncertain values of normalized weights, i.e. the sum of the weights is assumed to be equal to one. The calculation of the fuzzy weighted average of fuzzy numbers with noninteractive fuzzy weights were studied e.g. in [1, 16, 17]. In the second situation, the assumption that the sum of $w_1, \ldots, w_n$ is equal to one implies that uncertain values of normalized weights have to be modelled only by a special structure of interactive fuzzy numbers called *a tuple of normalized fuzzy weights* (see e.g. [20, 23]). The calculation of the fuzzy weighted average of fuzzy numbers with normalized fuzzy weights is different from the previous case; it was studied e.g. in [6, 9, 20].

The weighted average belongs to the class of aggregation operators. Aggregation operators are mathematical objects that have the function of reducing a tuple of inputs into a unique representative, i.e. an output of the aggregation. An overview of definition and possible properties of aggregation operators can be found, for instance, in [2, 4, 12]. The weighted average operator possesses some properties, like monotonicity, continuity, idempotency, additivity etc., that play an important role in practical applications (see [4]). In [22], it was studied whether and in which way such properties can be observed also for the fuzzy weighted average with noninteractive (i.e. non-normalized) fuzzy weights. The aim of this paper is to make an analogous study for the fuzzy weighted average of fuzzy numbers with normalized fuzzy weights. Particularly, to reveal possessing/non-observance of the following properties - compensation, strict monotonicity, continuity, idempotency, stability for a linear transformation, lipschitzianity, symmetry and additivity. The biggest advantage of the knowledge of the fulfilment or non-observance of these properties lies in the fact that it enables to properly employ the fuzzy weighted average with normalized fuzzy weights in fuzzy models.

The paper is organized as follows. Section 2 contains preliminaries. First, the definition of an aggregation operator and its possible properties will be mentioned. Afterwards, the basic notions dealing with fuzzy numbers and a fuzzy weighted average of fuzzy numbers with normalized fuzzy weights will be recalled. In Section 3, the fuzzy weighted average operator associated with a sequence of tuples of normalized fuzzy weights will be introduced first. Subsequently, it will be shown in which way it satisfies the generalizations of properties of the weighted average operator. The importance of preserving such properties in practical applications will be illustrated by simple examples and remarks there.

2 Preliminaries

2.1 Aggregation operators

An aggregation operator is a sequence of aggregation functions. As the inputs of aggregation, usually collections of real numbers from the unit interval $[0, 1]$ are considered (see, e.g., [4]). A general case, introduced in [27] as *a classical aggregation operator*, where aggregation functions are defined on an arbitrary nonempty interval $I \subseteq (-\infty, \infty)$, will be applied in the paper (the word "classical" will be omitted as there is no reason for using it in this study).

Notation 2.1. *Throughout the paper, we will assume that I is a nonempty real interval and we set $x^- := \inf I$ and $x^+ := \sup I$. Note that x^- and x^+ might belong to I or not, possibly with $x^- = -\infty$ or $x^+ = +\infty$.*

Definition 2.2. [27] *An aggregation operator $\mathcal{A}$ on an interval I is the sequence $\{A_n\}_{n=1}^{\infty}$ of aggregation functions*

$$A_n : I^n \to I,$$

that satisfy the following conditions:

1. $A_1(x) = x$ *for each* $x \in I$.
2. *If* $x_i \leq y_i$, *for all* $i = 1, \ldots, n$, *where* $n \in \mathbb{N}$, *then* $A_n(x_1, \ldots, x_n) \leq A_n(y_1, \ldots, y_n)$.
3. *For each* $n \in \mathbb{N}$:

$$\inf_{(x_1,\ldots,x_n)\in I^n} A_n(x_1, \ldots, x_n) = x^- \quad \textit{and} \quad \sup_{(x_1,\ldots,x_n)\in I^n} A_n(x_1, \ldots, x_n) = x^+.$$

Notation 2.3. *From conditions 2 and 3 in Def. 2.2 it follows that if* $x^- \in I$, *then* $A_n(x^-, \ldots, x^-) = x^-$, *otherwise*

$$\lim_{(x_1,\ldots,x_n)\to(x^-,\ldots,x^-)} A_n(x_1, \ldots, x_n) = x^-,$$

and if $x^+ \in I$, *then* $A_n(x^+, \ldots, x^+) = x^+$, *otherwise*

$$\lim_{(x_1,\ldots,x_n)\to(x^+,\ldots,x^+)} A_n(x_1, \ldots, x_n) = x^+.$$

In the literature, many different properties of aggregation operators have been defined and studied in details (see, e.g., [2, 4, 19]). Further in the paper, the following properties of aggregation operators will be considered.

Definition 2.4. [4] *We say that an aggregation operator* $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ *on* I *is*

1. *compensative (averaging), if for each* $n \in \mathbb{N}$ *the following inequalities hold for any* n*-tuple* $(x_1, \ldots, x_n) \in I^n$:

$$\min\{x_1, \ldots, x_n\} \leq A_n(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\};$$

2. *idempotent, if for all* $x \in I$ *and all* $n \in \mathbb{N}$ *it holds that* $A_n(x, \ldots, x) = x$;
3. *symmetric (commutative), if for all* $n \in \mathbb{N}$, *each vector* $(x_1, \ldots, x_n) \in I^n$, *and any permutation* σ *of* $\{1, \ldots, n\}$:

$$A_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = A_n(x_1, \ldots, x_n);$$

4. *strictly monotonic, if for all* $n \in \mathbb{N}$, A_n *is strictly monotonic, i.e. if for all* $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in I^n$ *such that* $x_i < y_i$ *for one* $i \in \{1, \ldots, n\}$ *and* $x_j = y_j$ *for any* $j \neq i$:

$$A_n(x_1, \ldots, x_n) < A_n(y_1, \ldots, y_n);$$

5. *stable for a linear transformation, if for each* $n \in \mathbb{N}$, *all* $r, t \in \mathbb{R}$, *and all* $(x_1, \ldots, x_n) \in I^n$:

$$A_n(rx_1 + t, \ldots, rx_n + t) = rA_n(x_1, \ldots, x_n) + t;$$

6. L*-Lipschitz, where* $L \in (0, \infty)$, *if for all* $n \in \mathbb{N}$ *and all* $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in I^n$:

$$|A_n(x_1, \ldots, x_n) - A_n(y_1, \ldots, y_n)| \leq L \sum_{i=1}^{n} |x_i - y_i|;$$

7. *continuous, if for all* $n \in \mathbb{N}$, *the aggregation function* A_n *is continuous,*
8. *additive, if for all* $n \in \mathbb{N}$, *and for all* $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in I^n$ *it holds that*

$$A_n(x_1, \ldots, x_n) + A_n(y_1, \ldots, y_n) = A_n(x_1 + y_1, \ldots, x_n + y_n).$$

At the end of this section, let us define the weighted average operation as an aggregation operator and summarize its properties. For simplification of the notations, a special symbol will be used for all n-tuples of normalized weights.

Notation 2.5. *Throughout the paper, let* $\mathcal{W}_n$ *denote the set of all* n*-tuples of normalized weights, i.e.*

$$\mathcal{W}_n = \left\{ (w_1, \ldots, w_n) \in \mathbb{R}^n \mid w_i \geq 0,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} w_i = 1 \right\}.$$

Definition 2.6. [4] *Let $W = \{\mathbf{w}_n\}_{n=1}^{\infty}$, be a sequence of normalized weight vectors where*

$$\mathbf{w}_n = (w_{n1}, \ldots, w_{nn}) \in \mathcal{W}_n, \ n = 1, 2, \ldots.$$

The aggregation operator $\mathcal{A}^W = \{A_n^W\}_{n=1}^{\infty}$ where for each $n \in \mathbb{N}$ and each $(x_1, \ldots, x_n) \in I^n$:

$$A_n^W(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_{ni} \cdot x_i,$$

is called a weighted average operator associated with W on an interval I.

It is obvious that for any sequence of normalized weight vectors $W = \{\mathbf{w}_n\}_{n=1}^{\infty}$, $\mathcal{A}^W$ is compensative, idempotent, stable for a linear transformation, 1-Lipschitz, continuous, and additive. If for any $n \in \mathbb{N}$, $w_{ni} > 0$, $i = 1, \ldots, n$, then $\mathcal{A}^W$ is strictly monotonic. $\mathcal{A}^W$ is symmetric, if and only if for any $n \in \mathbb{N}$, $w_{ni} = w_{nj}$ for all $i, j \in \{1, \ldots, n\}$. In such a case, $\mathcal{A}^W$ coincides with the arithmetic mean operator, i.e. for any $n \in \mathbb{N}$, $A_n^W(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$. Possessing of these properties leads to the fact that the weighted average operator is so widely used in decision making models and other applications. Comparison of this operator with other existing operators can be found e.g. in [4].

2.2 Fuzzy numbers

Before defining a fuzzy weighted average, it will be necessary to recall basic facts concerning fuzzy numbers.

Definition 2.7. *A fuzzy number is a fuzzy set X on $\mathbb{R}$ whose membership function $\mu_X : \mathbb{R} \to [0,1]$ fulfils the following three conditions:*

1. *the set $Core\, X = \{x \in \mathbb{R} \mid \mu_X(x) = 1\}$, called the core of X, is nonempty,*
2. *for any $\alpha \in (0,1]$, the set $[X]_\alpha = \{x \in \mathbb{R} \mid \mu_X(x) \geq \alpha\}$, called the α-cut of X, is a closed interval (the 1-cut $[X]_1$ means the core of X),*
3. *the set $Supp\, X = \{x \in \mathbb{R} \mid \mu_X(x) > 0\}$, called the support of X, is bounded.*

Notation 2.8. *The set of all fuzzy numbers will be denoted by $\mathcal{F}_N(\mathbb{R})$ throughout the paper. If the closure $Cl(SuppX) \subseteq I$, we say that a fuzzy number X is defined on an interval I. The set of all fuzzy numbers defined on I will be denoted by the symbol $\mathcal{F}_N(I)$ in the paper.*

Notation 2.9. *It was shown in [25] that any fuzzy number X can be uniquely given by the pair of functions $\underline{x}$ and $\overline{x}$ defined on $[0,1]$ such that $[X]_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)]$ for all $\alpha \in (0,1]$ and that $[\underline{x}(0), \overline{x}(0)]$ is the closure of the support of X, further denoted by $[X]_0$. The functions $\underline{x}$ and $\overline{x}$ are left-continuous on $(0,1]$, right-continuous at 0, and*

$$\text{for all } 0 \leq \alpha < \beta \leq 1: \quad \underline{x}(\alpha) \leq \underline{x}(\beta) \leq \overline{x}(\beta) \leq \overline{x}(\alpha). \tag{1}$$

For such a description of a fuzzy number, the notation $X = [\underline{x}, \overline{x}]$ will be used throughout the paper.

Notation 2.10. *Let us note that a real number $x \in \mathbb{R}$ can be viewed as a fuzzy number $X = [\underline{x}, \overline{x}]$ of a special kind, where $\underline{x}(\alpha) = \overline{x}(\alpha) = x$ for all $\alpha \in [0,1]$. By this convention we can handle the cases where some input variables (weights or weighted values) are crisp and some are fuzzy.*

Notation 2.11. *By $X = \langle x_1, x_2, x_3 \rangle$, where $x_1, x_2, x_3 \in \mathbb{R}$, $x_1 \leq x_2 \leq x_3$, we will denote the triangular fuzzy number $X = [\underline{x}, \overline{x}]$ such that $\underline{x}(\alpha) = x_1 + \alpha(x_2 - x_1)$ and $\overline{x}(\alpha) = x_3 - \alpha(x_3 - x_2)$ for all $\alpha \in [0,1]$.*

In what follows, the following definition of basic ordering of fuzzy numbers will be considered.

Definition 2.12. *We say that a fuzzy number $X = [\underline{x}, \overline{x}]$ is less than or equal to a fuzzy number $Y = [\underline{y}, \overline{y}]$, denoted by $X \leq Y$, if*

$$\underline{x}(\alpha) \leq \underline{y}(\alpha) \quad \text{and} \quad \overline{x}(\alpha) \leq \overline{y}(\alpha) \quad \text{for all } \alpha \in [0,1].$$

If $X \leq Y$, but $Y \not\leq X$, then we say that a fuzzy number X is less than a fuzzy number Y, denoted by $X < Y$.

Standard fuzzy arithmetic operations with fuzzy numbers are based on interval arithmetics (see e.g. [14]). Further in the paper, definitions of the sum and multiplication of fuzzy numbers will be needed.

Definition 2.13. *Let* $X, Y \in \mathcal{F}_N(\mathbb{R})$, $X = [\underline{x}, \overline{x}]$ *and* $Y = [\underline{y}, \overline{y}]$. *A sum of fuzzy numbers* X *and* Y *is the fuzzy number* $X + Y = [\underline{x+y}, \overline{x+y}]$ *such that* $\underline{x+y} = \underline{x} + \underline{y}$ *and* $\overline{x+y} = \overline{x} + \overline{y}$. *A multiplication of fuzzy numbers* X *and* Y *is the fuzzy number* $X \cdot Y = [\underline{x \cdot y}, \overline{x \cdot y}]$ *such that* $\underline{x \cdot y} = \min\{\underline{x} \cdot \underline{y}, \underline{x} \cdot \overline{y}, \overline{x} \cdot \underline{y}, \overline{x} \cdot \overline{y}\}$ *and* $\overline{x \cdot y} = \max\{\underline{x} \cdot \underline{y}, \underline{x} \cdot \overline{y}, \overline{x} \cdot \underline{y}, \overline{x} \cdot \overline{y}\}$.

In what follows, the following metric $d_F : \mathcal{F}_N(\mathbb{R})^2 \to \mathbb{R}_0^+$ will be used: for any fuzzy numbers $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$,

$$d_F(X, Y) = \int_0^1 \left(|\underline{x}(\alpha) - \underline{y}(\alpha)| + |\overline{x}(\alpha) - \overline{y}(\alpha)| \right) d\alpha.$$

The existence of the integral follows from the fact that $\underline{x}$, $\underline{y}$, $\overline{x}$, and $\overline{y}$ are bounded and monotone on $[0, 1]$.

2.3 Fuzzy weighted average of fuzzy numbers with normalized fuzzy weights

In the crisp case, the normalized weights $w_1, \ldots, w_n$ are supposed to be non-negative real numbers whose sum is equal to 1. In the fuzzy case, the way how to express the normalized fuzzy weights is more complicated (as the following definition shows), the following n-tuple of interactive fuzzy numbers has to be applied.

Definition 2.14. [23] *An* n*-tuple of fuzzy numbers* $W_1, \ldots, W_n$ *defined on* $[0, 1]$, $W_i = [\underline{w}_i, \overline{w}_i]$, $i = 1, \ldots, n$, *is called an* n*-tuple of normalized fuzzy weights if, for all* $\alpha \in (0, 1]$ *and all* $i \in \{1, \ldots, n\}$, *the following condition is satisfied:*

$$\textit{For an arbitrary } w_i \in [W_i]_\alpha \textit{ there exist } w_j \in [W_j]_\alpha,\ j = 1, \ldots, n,\ j \neq i, \textit{ such that } w_i + \sum_{j=1, j\neq i}^{n} w_j = 1.$$

Notation 2.15. *For a given* $n \in \mathbb{N}$, *the set of all* n*-tuples of normalized fuzzy weights will be denoted by* $\mathcal{W}_{nF}$ *throughout the paper.*

After defining the normalized fuzzy weights, let us focus on the fuzzy extension of the weighted average of real numbers $x_1, \ldots, x_n$ with normalized weights $w_1, \ldots, w_n$. It is a well-known fact (see, e.g., [9, 20, 23, 24]) that the proper fuzzification of this operation cannot be calculated as $X = \sum_{i=1}^n W_i \cdot X_i$, i.e. simply by employing the standard fuzzy arithmetic operations, because there is an interaction among the normalized weights. Hence, the concept of constrained fuzzy arithmetic (see, e.g., [14]) has to be applied instead.

Definition 2.16. [20] *For a given* $n \in \mathbb{N}$, *let* $(W_1, \ldots, W_n) \in \mathcal{W}_{nF}$, *and let* $X_i \in \mathcal{F}_N(I)$, $X_i = [\underline{x}_i, \overline{x}_i]$, $i = 1, \ldots, n$. *The fuzzy weighted average of* $X_1, \ldots, X_n$ *with normalized fuzzy weights* $W_1, \ldots, W_n$ *is a fuzzy number* $a_{wF}(X_1, \ldots, X_n; W_1, \ldots, W_n) = [\underline{a}_w, \overline{a}_w]$ *defined on* I *such that for all* $\alpha \in [0, 1]$:

$$\underline{a}_w(\alpha) = \min\left\{ \sum_{i=1}^{n} w_i \cdot \underline{x}_i(\alpha) \mid w_i \in [W_i]_\alpha,\ i = 1, \ldots, n, \sum_{i=1}^{n} w_i = 1 \right\}, \tag{2}$$

$$\overline{a}_w(\alpha) = \max\left\{ \sum_{i=1}^{n} w_i \cdot \overline{x}_i(\alpha) \mid w_i \in [W_i]_\alpha,\ i = 1, \ldots, n, \sum_{i=1}^{n} w_i = 1 \right\}. \tag{3}$$

The usual approach for computing the fuzzy weighted average is the α-cut decomposition method. The input fuzzy numbers are discretized into a set of α-cuts and the boundary values of α-cuts of the fuzzy weighted average are computed applying (2) and (3). Thus, the final fuzzy weighted average is observed only approximately by connecting these α-cuts together.

Notation 2.17. *Let us note that it is not necessary to solve the linear programming problems* (2) *and* (3), *since an effective algorithm for computing the values* $\underline{a}_w(\alpha)$ *and* $\overline{a}_w(\alpha)$ *was introduced in* [23] *(it represents an extension of the algorithm for computing the expected value of a discrete random variable with interval probabilities introduced in* [6]*): For each* $\alpha \in [0, 1]$, *let* $\{i_k\}_{k=1}^n$ *be such a permutation on the index set* $\{1, \ldots, n\}$ *that* $\underline{x}_{i_1}(\alpha) \leq \underline{x}_{i_2}(\alpha) \leq \ldots \leq \underline{x}_{i_n}(\alpha)$. *For all* $k \in \{1, \ldots, n\}$, *let us denote*

$$w_{i_k}(\alpha) = 1 - \sum_{j=1}^{k-1} \overline{w}_{i_j}(\alpha) - \sum_{j=k+1}^{n} \underline{w}_{i_j}(\alpha).$$

Let $k^* \in \{1, \ldots, n\}$ *be such an index that* $\underline{w}_{i_{k^*}}(\alpha) \leq w_{i_{k^*}}(\alpha) \leq \overline{w}_{i_{k^*}}(\alpha)$. *Then*

$$\underline{a}_w(\alpha) = \sum_{j=1}^{k^*-1} \overline{w}_{i_j}(\alpha) \cdot \underline{x}_{i_j}(\alpha) + w_{i_{k^*}}(\alpha) \cdot \underline{x}_{i_{k^*}}(\alpha) + \sum_{j=k^*+1}^{n} \underline{w}_{i_j}(\alpha) \cdot \underline{x}_{i_j}(\alpha). \tag{4}$$

Let $\{i_h\}_{h=1}^n$ be such a permutation on an index set $\{1,\ldots,n\}$ that $\overline{x}_{i_1}(\alpha) \geq \overline{x}_{i_2}(\alpha) \geq \ldots \geq \overline{x}_{i_n}(\alpha)$. For all $h \in \{1,\ldots,n\}$, let us denote

$$w_{i_h}(\alpha) = 1 - \sum_{j=1}^{h-1} \overline{w}_{i_j}(\alpha) - \sum_{j=h+1}^{n} \underline{w}_{i_j}(\alpha).$$

Let $h^* \in \{1,\ldots,n\}$ be such an index that $\underline{w}_{i_{h^*}}(\alpha) \leq w_{i_{h^*}}(\alpha) \leq \overline{w}_{i_{h^*}}(\alpha)$. Then

$$\overline{a}_w(\alpha) = \sum_{j=1}^{h^*-1} \overline{w}_{i_j}(\alpha) \cdot \overline{x}_{i_j}(\alpha) + w_{i_{h^*}}(\alpha) \cdot \overline{x}_{i_{h^*}}(\alpha) + \sum_{j=h^*+1}^{n} \underline{w}_{i_j}(\alpha) \cdot \overline{x}_{i_j}(\alpha). \tag{5}$$

Notation 2.18. *Procedures how can be the fuzzy weighted average of fuzzy number with normalized fuzzy weights a_{wF} applied for solving multi-criteria decision making problems are described e.g. in* [7, 23, 24]. *Usually, the problem is to compare alternatives with respect to an n-tuple of criteria. The process is divided into the following steps:*

1. *The evaluations of alternatives with respect to particular criteria are given on some interval I, typically $I = [0, 1]$, where 0 means the worst possible and 1 the best possible evaluation. These evaluations can be set expertly, e.g. by chosen a proper linguistic term from some linguistic scale, or they can be ill-known, therefore they are modelled by n-tuples of fuzzy numbers.*

2. *The normalized fuzzy weights $W_1, \ldots, W_n$ of criteria represent shares of the particular evaluations in the overall one. They are given expertly. Practical methods how can be set an n-tuple of normalized fuzzy weights in practical applications, e.g. in the form of triangular fuzzy numbers, are described in* [23, 24].

3. *For each alternative, its overall fuzzy evaluation is computed as the fuzzy weighted average of its fuzzy evaluations with respect to the criteria with normalized fuzzy weights.*

4. *The alternatives are ranked on the basis of their overall fuzzy evaluations; some method for ranking the fuzzy numbers needs to be applied (*[3] *for an overview of the possible methods).*

3 Fuzzy weighted average operator associated with a sequence of tuples of normalized fuzzy weights and its properties

First in this section, the fuzzy weighted average operator associated with a sequence of tuples of normalized fuzzy weights will be defined. Afterwards, whether and in which way it preserves the above mentioned properties of the weighted average operator will be examined. At the end of this section, the comparison with the results concerning the fuzzy weighted average operator associated with a sequence of tuples of noninteractive weights that were presented in [22] will be made.

3.1 Definition and three basic properties

In order to study the particular properties of the fuzzy weighted average of fuzzy numbers with normalized fuzzy weights, it will be at first necessary to introduce this operator as a sequence of mappings, analogously like in the case of the weighted average operator.

Definition 3.1. *Let $W_F = \{\mathbf{W}_n\}_{n=1}^{\infty}$, where $\mathbf{W}_n = (W_{n1}, \ldots, W_{nn}) \in \mathcal{W}_{nF}$, $n = 1, 2, \ldots$, be a sequence of tuples of normalized fuzzy weights. The fuzzy weighted average operator on an interval I associated with W_F is a sequence $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, $A_n^{W_F} : \mathcal{F}_N(I)^n \to \mathcal{F}_N(I)$ is for any n-tuple of fuzzy numbers $(X_1, \ldots, X_n) \in \mathcal{F}_N(I)^n$ given by*

$$A_n^{W_F}(X_1, \ldots, X_n) = a_{wF}(X_1, \ldots, X_n; W_{n1}, \ldots, W_{nn}).$$

Let us show now that the fuzzy weighted average operator $\mathcal{A}^{W_F}$ possesses an extension of the three conditions from Definition 2.2 of an aggregation operator, namely identity, monotonicity, and the boundary condition.

Theorem 3.2. *Let $W_F = \{\mathbf{W}_n\}_{n=1}^{\infty}$ be an arbitrary sequence of tuples of normalized fuzzy weights and $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator on an interval I associated with W_F. Then $\mathcal{A}^{W_F}$ is the fuzzy aggregation operator, i.e.*

1. $A_1^{W_F}(X) = X$ *for all $X \in \mathcal{F}_N(I)$,*

2. *for any* $n \in \{2, 3, \ldots\}$*, and any fuzzy numbers* $X_1, \ldots, X_n$ *and* $Y_1, \ldots, Y_n$ *defined on* I *satisfying* $X_i \leq Y_i$ *for all* $i \in \{1, \ldots, n\}$*, it holds that*
$$A_n^{W_F}(X_1, \ldots, X_n) \leq A_n^{W_F}(Y_1, \ldots, Y_n),$$

3. *for each* $n \in \mathbb{N}$ *and any fuzzy numbers* $X_i \in \mathcal{F}_N(I)$*,* $X_i = [\underline{x}_i, \overline{x}_i]$*,* $i = 1, \ldots, n$*, if we denote* $A_n^{W_F}(X_1, \ldots, X_n) = [\underline{a}_{wn}, \overline{a}_{wn}]$*, then*
$$\lim_{(\overline{x}_1(0), \ldots, \overline{x}_n(0)) \to (x^-, \ldots, x^-)} \overline{a}_{wn}(0) = x^-, \quad \text{and} \quad \lim_{(\underline{x}_1(0), \ldots, \underline{x}_n(0)) \to (x^+, \ldots, x^+)} \underline{a}_{wn}(0) = x^+. \tag{6}$$

Proof. 1. Let $X = [\underline{x}, \overline{x}] \in \mathcal{F}_N(I)$, and let us denote $A_1^{W_F}(X) = [\underline{a}_{w1}, \overline{a}_{w1}]$. Then for all $\alpha \in [0, 1]$:
$$\underline{a}_{w1}(\alpha) = \min \{w_1 \cdot \underline{x}(\alpha) \mid w_1 \in [W_{11}]_\alpha, \, w_1 = 1\} = \underline{x}(\alpha),$$
$$\overline{a}_{w1}(\alpha) = \max \{w_1 \cdot \overline{x}(\alpha) \mid w_1 \in [W_{11}]_\alpha, \, w_1 = 1\} = \overline{x}(\alpha),$$
i.e. $A_1^{W_F}(X) = X$.

2. Let $n \in \{2, 3, \ldots\}$ be arbitrary. Let, for all $i \in \{1, \ldots, n\}$, $X_i = [\underline{x}_i, \overline{x}_i]$ and $Y_i = [\underline{y}_i, \overline{y}_i]$ be fuzzy numbers defined on I such that $\underline{x}_i(\alpha) \leq \underline{y}_i(\alpha)$ and $\overline{x}_i(\alpha) \leq \overline{y}_i(\alpha)$ for all $\alpha \in [0, 1]$. Let us denote $A_n^{W_F}(X_1, \ldots, X_n) = [\underline{a}_{wn}^X, \overline{a}_{wn}^X]$ and $A_n^{W_F}(Y_1, \ldots, Y_n) = [\underline{a}_{wn}^Y, \overline{a}_{wn}^Y]$.
For any $\alpha \in [0, 1]$, let $w_i^* \in [W_{ni}]_\alpha$, $i = 1, \ldots, n$, be such that $\sum_{i=1}^n w_i^* = 1$ and that
$$\underline{a}_{wn}^Y(\alpha) = \min \left\{ \sum_{i=1}^n w_i \cdot \underline{y}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha, \, i = 1, \ldots, n, \sum_{i=1}^n w_i = 1 \right\} = \sum_{i=1}^n w_i^* \cdot \underline{y}_i(\alpha).$$
Let us denote
$$x^* = \sum_{i=1}^n w_i^* \cdot \underline{x}_i(\alpha).$$
Since $\underline{x}_i(\alpha) \leq \underline{y}_i(\alpha)$ for all $i \in \{1, \ldots, n\}$, $x^* \leq \underline{a}_{wn}^Y(\alpha)$. Further,
$$\underline{a}_{wn}^X(\alpha) = \min \left\{ \sum_{i=1}^n w_i \cdot \underline{x}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha, \, i = 1, \ldots, n, \sum_{i=1}^n w_i = 1 \right\} \leq x^*.$$
Hence, $\underline{a}_{wn}^X(\alpha) \leq \underline{a}_{wn}^Y(\alpha)$. The proof that $\overline{a}_{wn}^X(\alpha) \leq \overline{a}_{wn}^Y(\alpha)$ would be analogous.

3. Equations (6) follow directly from the fact that for any n-tuple of normalized weights $(w_1, \ldots, w_n) \in \mathcal{W}_n$ it holds that
$$\lim_{(\overline{x}_1(0), \ldots, \overline{x}_n(0)) \to (x^-, \ldots, x^-)} \sum_{i=1}^n w_i \cdot \overline{x}_i(0) = x^-, \quad \lim_{(\underline{x}_1(0), \ldots, \underline{x}_n(0)) \to (x^+, \ldots, x^+)} \sum_{i=1}^n w_i \cdot \underline{x}_i(0) = x^+.$$

□

Importance of the monotonicity property 2. consists, e.g., in preserving *the Pareto dominance* in fuzzy MCDM models as the following example shows.

Example 3.3. *Let us consider the fuzzy MCDM model, where alternatives are evaluated with respect to* n *criteria and where the particular evaluations are expressed by fuzzy numbers. Moreover, let the vaguely given information about importance of particular criteria is modelled by an* n*-tuple of normalized fuzzy weights. The overall fuzzy evaluations of alternatives are then computed as the fuzzy weighted averages of the fuzzy evaluations with respect to the particular criteria.*

For example, let us assume the case when there exist two alternatives x_1 *and* x_2 *such that their evaluations with respect to the particular criteria are given by fuzzy numbers* $X_{11}, \ldots, X_{1n}$*, and* $X_{21}, \ldots, X_{2n}$*, respectively. If* x_1 *is Pareto dominant with respect to* x_2*, i.e. if* $X_{1i} \geq X_{2i}$ *for all* $i \in \{1, \ldots, n\}$*, then, regardless of the applied* n*-tuple of normalized fuzzy weights,* x_1 *will not be preferred by* x_2 *since the following relation holds for any* $(W_1, \ldots, W_n) \in \mathcal{W}_{nF}$*:*
$$a_{wF}(X_{11}, \ldots, X_{1n}; W_1, \ldots, W_n) \geq a_{wF}(X_{21}, \ldots, X_{2n}; W_1, \ldots, W_n).$$

If I would be a closed interval, then the boundary property 3. can be interpreted in such a way that if we aggregate by the fuzzy weighted average operator associated with a sequence of tuples of normalized fuzzy weights only the minimal (maximal) possible inputs, then we obtain the minimal (maximal) possible output, regardless of the applied tuple of normalized fuzzy weights. Significance of this property is illustrated by the following example.

Example 3.4. *Let us consider the fuzzy MCDM model described in Example 3.3. According to Theorem 3.2, if all the fuzzy evaluations with respect to the particular criteria tend to the worst (best) possible evaluation, then the overall fuzzy evaluation of an alternative also tends to the worst (best) one, regardless of the applied n-tuple of normalized fuzzy weights. For example, if the fuzzy evaluations of alternatives with respect to the particular criteria are restricted to $I = [0, 1]$, where 0 means the worst possible and 1 the best possible evaluation, for any $(W_1, \ldots, W_n) \in \mathcal{W}_{nF}$ we get*

$$a_{wF}(0, \ldots, 0; W_1, \ldots, W_n) = 0 \text{ and } a_{wF}(1, \ldots, 1; W_1, \ldots, W_n) = 1.$$

Hence, in such cases the fuzziness of the normalized weights does not affect the overall evaluation.

Summing up, we have shown that for any sequence of n-tuples of normalized fuzzy weights, the corresponding fuzzy weighted average operator $\mathcal{A}^{W_F}$ satisfies generalized conditions from the definition of an aggregation operator. In the following, the fulfilment of other fundamental properties of the weighted average operator by $\mathcal{A}^{W_F}$ will be examined.

3.2 Compensation, idempotency, stability of linear transformation, and lipschitzianity

At first, the way of possessing the compensative property, idempotency, stability for linear transformation and lipschitzianity will be shown in the following theorem.

Theorem 3.5. *Let $W_F = \{\mathbf{W}_n\}_{n=1}^{\infty}$ be an arbitrary sequence of tuples of normalized fuzzy weights and $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator on an interval I associated with W_F. Then*

1. *for each $n \in \mathbb{N}$ and any fuzzy numbers $X_i \in \mathcal{F}_N(I)$, $X_i = [\underline{x}_i, \overline{x}_i]$, $i = 1, \ldots, n$, the following inequalities hold*

$$\min{}_F(X_1, \ldots, X_n) \le A_n^{W_F}(X_1, \ldots, X_n) \le \max{}_F(X_1, \ldots, X_n),$$

where $\min_F$ and $\max_F$ represent the fuzzy extension of functions min *and* max*, i.e. $\min_F(X_1, \ldots, X_n) = [\underline{\min}, \overline{\min}]$ and $\max_F(X_1, \ldots, X_n) = [\underline{\max}, \overline{\max}]$, where for all $\alpha \in [0, 1]$:*

$$\underline{\min}(\alpha) = \min\{\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)\}, \ \overline{\min}(\alpha) = \min\{\overline{x}_1(\alpha), \ldots, \overline{x}_n(\alpha)\},$$
$$\underline{\max}(\alpha) = \max\{\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)\}, \ \overline{\max}(\alpha) = \max\{\overline{x}_1(\alpha), \ldots, \overline{x}_n(\alpha)\};$$

2. *for all $X \in \mathcal{F}_N(I)$ and all $n \in \mathbb{N}$ it holds that $A_n^{W_F}(X, \ldots, X) = X$;*

3. *for each $n \in \mathbb{N}$, all $r, t \in \mathbb{R}$, and all $(X_1, \ldots, X_n) \in \mathcal{F}_N(I)^n$ it holds that*

$$A_n^{W_F}(rX_1 + t, \ldots, rX_n + t) = rA_n^{W_F}(X_1, \ldots, X_n) + t;$$

4. *for all $n \in \mathbb{N}$ and all fuzzy numbers $X_i \in \mathcal{F}_N(I)$, $Y_i \in \mathcal{F}_N(I)$, $X_i = [\underline{x}_i, \overline{x}_i]$, $Y_i = [\underline{y}_i, \overline{y}_i]$, $i = 1, \ldots, n$, it holds that*

$$\int_0^1 \left(|\underline{a}_{wn}^X(\alpha) - \underline{a}_{wn}^Y(\alpha)| + |\overline{a}_{wn}^X(\alpha) - \overline{a}_{wn}^Y(\alpha)|\right) d\alpha \le \sum_{i=1}^{n} \int_0^1 \Big(|\underline{x}_i(\alpha) - \underline{y}_i(\alpha)| + |\overline{x}_i(\alpha) - \overline{y}_i(\alpha)|\Big) d\alpha,$$

where $A_n^{W_F}(X_1, \ldots, X_n) = [\underline{a}_{wn}^X, \overline{a}_{wn}^X]$ and $A_n^{W_F}(Y_1, \ldots, Y_n) = [\underline{a}_{wn}^Y, \overline{a}_{wn}^Y]$.

Proof. 1. Let $n \in \mathbb{N}$ be arbitrary. For any n-tuple of fuzzy numbers $X_1, \ldots, X_n$ defined on I, let us denote $A_n^{W_F}(X_1, \ldots, X_n) = [\underline{a}_{wn}, \overline{a}_{wn}]$. Since a weighted average is a compensative operator, it holds for any n-tuple of real normalized weights $w_1, \ldots, w_n$ that

$$\min\{\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)\} \le \sum_{i=1}^{n} w_i \underline{x}_i(\alpha) \le \max\{\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)\}.$$

Hence, it is clear that for all $\alpha \in [0, 1]$:

$$\min\{\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)\} \le \underline{a}_{wn}(\alpha) \le \max\{\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)\},$$

$$\min\{\overline{x}_1(\alpha), \ldots, \overline{x}_n(\alpha)\} \le \overline{a}_{wn}(\alpha) \le \max\{\overline{x}_1(\alpha), \ldots, \overline{x}_n(\alpha)\}.$$

2. Let $n \in \mathbb{N}$ be arbitrary, and let $X = [\underline{x}, \overline{x}]$ be an arbitrary fuzzy number defined on I. Let us denote $A_n^{W_F}(X, \ldots, X) = [\underline{a}_{wn}, \overline{a}_{wn}]$. Then for all $\alpha \in [0,1]$:

$$\begin{aligned}\underline{a}_{wn}(\alpha) &= \min\left\{\sum_{i=1}^{n} w_i \cdot \underline{x}(\alpha) \mid w_i \in [W_{ni}]_\alpha,\, i = 1, \ldots, n,\, \sum_{i=1}^{n} w_i = 1\right\} \\ &= \min\left\{\underline{x}(\alpha) \sum_{i=1}^{n} w_i \mid w_i \in [W_{ni}]_\alpha,\, i = 1, \ldots, n,\, \sum_{i=1}^{n} w_i = 1\right\} = \underline{x}(\alpha).\end{aligned}$$

The equality $\overline{a}_{wn}(\alpha) = \overline{x}(\alpha)$ for all $\alpha \in [0,1]$ can be derived analogously.

3. Let $n \in \mathbb{N}$ and $r, t \in \mathbb{R}$ be arbitrary. Let $X_i = [\underline{x}_i, \overline{x}_i]$, $i = 1, \ldots, n$, be arbitrary fuzzy numbers defined on I. Let us denote $A_n^{W_F}(X_1, \ldots, X_n) = [\underline{a}_{wn}, \overline{a}_{wn}]$. Then

$$rA_n^{W_F}(X_1, \ldots, X_n) + t = (r\underline{a}_{wn} + t, r\overline{a}_{wn} + t).$$

Further, let us denote $A_n^{W_F}(rX_1 + t, \ldots, rX_n + t) = [\underline{a}_{wn}^{rt}, \overline{a}_{wn}^{rt}]$. For all $\alpha \in [0,1]$, we get

$$\begin{aligned}\underline{a}_{wn}^{rt}(\alpha) &= \min\left\{\sum_{i=1}^{n} w_i(r\underline{x}_i(\alpha) + t) \mid w_i \in [W_{ni}]_\alpha,\, i = 1, \ldots, n,\, \sum_{i=1}^{n} w_i = 1\right\} \\ &= r\min\left\{\sum_{i=1}^{n} w_i \cdot \underline{x}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha,\, i = 1, \ldots, n,\, \sum_{i=1}^{n} w_i = 1\right\} + t = r\underline{a}_{wn}(\alpha) + t.\end{aligned}$$

The proof that $\overline{a}_{wn}^{rt}(\alpha) = r\overline{a}_{wn}(\alpha) + t$ for all $\alpha \in [0,1]$ would be analogous.

4. Let $n \in \mathbb{N}$ be arbitrary, and let $\alpha \in [0,1]$ be such that $\underline{a}_{wn}^{X}(\alpha) \geq \underline{a}_{wn}^{Y}(\alpha)$. Then

$$\begin{aligned}|\underline{a}_{wn}^{X}(\alpha) - \underline{a}_{wn}^{Y}(\alpha)| = \underline{a}_{wn}^{X}(\alpha) - \underline{a}_{wn}^{Y}(\alpha) &= \min\left\{\sum_{i=1}^{n} w_i \cdot \underline{x}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha,\, i = 1, \ldots, n,\, \sum_{i=1}^{n} w_i = 1\right\} \\ &\quad - \min\left\{\sum_{i=1}^{n} w_i \cdot \underline{y}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha,\, i = 1, \ldots, n,\, \sum_{i=1}^{n} w_i = 1\right\} \\ &= \min\left\{\sum_{i=1}^{n} w_i \cdot \underline{x}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha,\, i = 1, \ldots, n, \sum_{i=1}^{n} w_i = 1\right\} - \sum_{i=1}^{n} w_i^* \cdot \underline{y}_i(\alpha) \\ &\leq \sum_{i=1}^{n} w_i^* \cdot \underline{x}_i(\alpha) - \sum_{i=1}^{n} w_i^* \cdot \underline{y}_i(\alpha) \\ &= \sum_{i=1}^{n} w_i^* \left(\underline{x}_i(\alpha) - \underline{y}_i(\alpha)\right) \\ &\leq \sum_{i=1}^{n} w_i^* |\underline{x}_i(\alpha) - \underline{y}_i(\alpha)| \\ &\leq \sum_{i=1}^{n} |\underline{x}_i(\alpha) - \underline{y}_i(\alpha)|.\end{aligned}$$

If $\alpha \in [0,1]$ is such that $\underline{a}_{wn}^{X}(\alpha) < \underline{a}_{wn}^{Y}(\alpha)$, then

$$\begin{aligned}|\underline{a}_{wn}^{X}(\alpha) - \underline{a}_{wn}^{Y}(\alpha)| = -\underline{a}_{wn}^{X}(\alpha) + \underline{a}_{wn}^{Y}(\alpha) &= -\min\left\{\sum_{i=1}^{n} w_i \cdot \underline{x}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha,\, i = 1, \ldots, n,\, \sum_{i=1}^{n} w_i = 1\right\} \\ &\quad + \min\left\{\sum_{i=1}^{n} w_i \cdot \underline{y}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha,\, i = 1, \ldots, n,\, \sum_{i=1}^{n} w_i = 1\right\} \\ &= -\sum_{i=1}^{n} w_i^{**} \cdot \underline{x}_i(\alpha) + \min\left\{\sum_{i=1}^{n} w_i \cdot \underline{y}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha,\, i = 1, \ldots, n,\, \sum_{i=1}^{n} w_i = 1\right\}\end{aligned}$$

$$
\begin{aligned}
&\leq -\sum_{i=1}^{n} w_i^{**} \cdot \underline{x}_i(\alpha) + \sum_{i=1}^{n} w_i^{**} \cdot \underline{y}_i(\alpha) \\
&= \sum_{i=1}^{n} w_i^{**} \left(\underline{y}_i(\alpha) - \underline{x}_i(\alpha)\right) \\
&\leq \sum_{i=1}^{n} |\underline{y}_i(\alpha) - \underline{x}_i(\alpha)| \\
&= \sum_{i=1}^{n} |\underline{x}_i(\alpha) - \underline{y}_i(\alpha)|.
\end{aligned}
$$

Summing up, we obtain for all $n \in \mathbb{N}$ and $\alpha \in [0,1]$ that $|\underline{a}_{wn}^X(\alpha) - \underline{a}_{wn}^Y(\alpha)| \leq \sum_{i=1}^n |\underline{x}_i(\alpha) - \underline{y}_i(\alpha)|$. Analogously, we can obtain that $|\overline{a}_{wn}^X(\alpha) - \overline{a}_{wn}^Y(\alpha)| \leq \sum_{i=1}^n |\overline{x}_i(\alpha) - \overline{y}_i(\alpha)|$. Therefore,

$$
\int_0^1 \left(|\underline{a}_{wn}^X(\alpha) - \underline{a}_{wn}^Y(\alpha)| + |\overline{a}_{wn}^X(\alpha) - \overline{a}_{wn}^Y(\alpha)|\right) d\alpha \leq \sum_{i=1}^{n} \int_0^1 \left(|\underline{x}_i(\alpha) - \underline{y}_i(\alpha)| + |\overline{x}_i(\alpha) - \overline{y}_i(\alpha)|\right) d\alpha,
$$

which completes the proof.

□

The importance of the compensation property 1. from Theorem 3.5 can be illustrated by the following example.

Example 3.6. *Let us consider a MCDM problem, where an alternative x is to be evaluated with respect to four criteria. The normalized fuzzy weights of the criteria that express uncertain shares of the partial goals on the total one are given by a quadruple of triangular fuzzy numbers $W_1 = \langle 0.05, 0.1, 0.25\rangle$, $W_2 = \langle 0.1, 0.2, 0.3\rangle$, $W_3 = \langle 0.15, 0.3, 0.5\rangle$, $W_4 = \langle 0.2, 0.4, 0.6\rangle \in \mathcal{F}_N([0,1])$, and the fuzzy evaluations of x with respect to the particular criteria are expressed by fuzzy numbers $X_1 = \langle 0.1, 0.4, 0.5\rangle$, $X_2 = \langle 0.2, 0.3, 0.6\rangle$, $X_3 = \langle 0.5, 0.6, 0.9\rangle$, $X_4 = \langle 0.4, 0.7, 0.8\rangle \in \mathcal{F}_N([0,1])$, where 0 is the worst possible and 1 the best possible evaluation. The membership functions of the normalized fuzzy weights and fuzzy evaluations are shown in Fig. 1.*

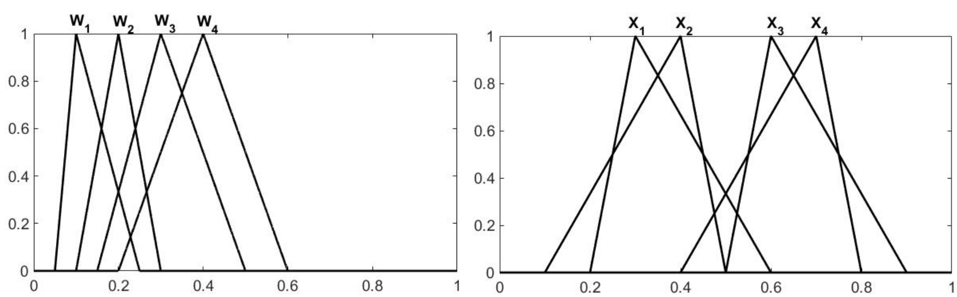

Figure 1: Normalized fuzzy weights $W_1, \ldots, W_4$ and fuzzy evaluations $X_1, \ldots, X_4$.

The overall fuzzy evaluation of x is expressed by the fuzzy number $X = a_{wF}(X_1, \ldots, X_4; W_1, \ldots, W_4)$. In Fig. 2, we can see that the fuzzy number X, obtained by the α-cut decomposition method using the formulas (4) *and* (5)*, lies between the fuzzy numbers $MIN = \min_F(X_1, \ldots, X_4)$ and $MAX = \max_F(X_1, \ldots, X_4)$, i.e. it does not exceed the boundaries given by fuzzy evaluations of x with respect to the particular criteria.*

Notation 3.7. *As has been mentioned in* [4]*, idempotency is supposed to be a genuine property of aggregation operators in many branches, e.g., in MCDM (see* [10]*). In fuzzy MCDM models, the idempotency property observed in Theorem 3.5 can be read as follows: if all criteria are satisfied in the same fuzzy degree X, then also the overall fuzzy evaluation is X, regardless of the applied tuple of normalized fuzzy weights. Thus, the fuzziness of the normalized weights affects the resulting fuzzy weighted average with normalized fuzzy weights only in the case when the fuzzy weighted values are not all the same. How the uncertainty of the fuzzy weights affects the resulting fuzzy weighted average if the fuzzy weighted values are not the same was studied in* [24]*, together with consequences in fuzzy decision making models.*

Notation 3.8. *In fuzzy models, the stability for linear transformation ensures us that a linear transformation can be applied either to the fuzzy weighted values $X_1, \ldots, X_n$, i.e. before the aggregation by the fuzzy weighted average, or*

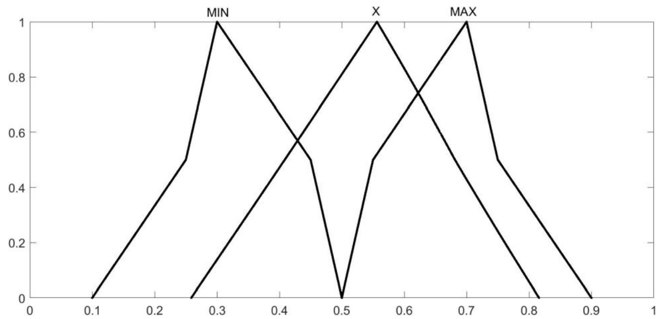

Figure 2: Fuzzy numbers MIN, X, and MAX.

to the resulting fuzzy weighted average $a_{wF}(X_1,\dots,X_n;W_1,\dots,W_n)$. The result of the aggregation will be the same, regardless of the applied n-tuple of normalized fuzzy weights.

Notation 3.9. *The property 4. can be reformulated in the sense that the fuzzy weighted average operator with normalized fuzzy weights $\mathcal{A}^{W_F}$ is 1-Lipschitz in the metric d_F. This implies that it is also continuous with respect to d_F, i.e. that for all $n\in\mathbb{N}$, all $\varepsilon>0$ and for any fuzzy numbers $X_i^0\in\mathcal{F}_N(I)$, $X_i^0=[\underline{x}_i^0,\overline{x}_i^0]$, $i=1,\dots,n$, there exists $\delta>0$ such that if $X_i\in\mathcal{F}_N(I)$, $X_i=[\underline{x}_i,\overline{x}_i]$, $i=1,\dots,n$, are fuzzy numbers satisfying*

$$\sum_{i=1}^{n}\int_0^1\left(|\underline{x}_i(\alpha)-\underline{x}_i^0(\alpha)|+|\overline{x}_i(\alpha)-\overline{x}_i^0(\alpha)|\right)d\alpha<\delta,$$

then it holds for $A_n^{W_F}(X_1^0,\dots,X_n^0)=[\underline{a}_n^0,\overline{a}_n^0]$ and $A_n^{W_F}(X_1,\dots,X_n)=[\underline{a}_n,\overline{a}_n]$ that

$$\int_0^1\left(|\underline{a}_n(\alpha)-\underline{a}_n^0(\alpha)|+|\overline{a}_n(\alpha)-\overline{a}_n^0(\alpha)|\right)d\alpha<\varepsilon.$$

In case of $\mathcal{A}^{W_F}$, the continuity ensures that (regardless of the applied tuple of normalized fuzzy weights) if the input fuzzy weighted values are changed only slightly, the resulting fuzzy weighted average with normalized fuzzy weights will be close to the original one.

3.3 Symmetry, strict monotonicity, and additivity

Let us focus now on the remaining properties of the weighted average operator − symmetry, strict monotonicity, and additivity. In order to study strict monotonicity, it is necessary to apply only positive fuzzy weights, analogously as in the crisp case.

Definition 3.10. *We say that fuzzy weights $W_i=[\underline{w}_i,\overline{w}_i]$, $i=1,\dots,n$, are positive if $\underline{w}_i(0)>0$ for all $i\in\{1,\dots,n\}$.*

Theorem 3.11. *Let $W_F^+=\{\mathbf{W}_n^+\}_{n=1}^{\infty}$ be an arbitrary sequence of tuples of positive normalized fuzzy weights and $\mathcal{A}^{W_F^+}=\{A_n^{W_F^+}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator on an interval I associated with W_F^+. Let, for all $n\in\mathbb{N}$, $X_i=[\underline{x}_i,\overline{x}_i]$ and $Y_i=[\underline{y}_i,\overline{y}_i]$, $i=1,\dots,n$, be fuzzy numbers defined on I such that $X_{i_0}<Y_{i_0}$ for one $i_0\in\{1,\dots,n\}$ and $X_j=Y_j$ for all $j\in\{1,\dots,n\}$, $j\neq i_0$. Then $A_n^{W_F^+}(X_1,\dots,X_n)<A_n^{W_F^+}(Y_1,\dots,Y_n)$.*

Proof. Let $n\in\mathbb{N}$ be arbitrary. Let us denote $A_n^{W_F^+}(X_1,\dots,X_n)=[\underline{a}_{wn}^X,\overline{a}_{wn}^X]$ and $A_n^{W_F^+}(Y_1,\dots,Y_n)=[\underline{a}_{wn}^Y,\overline{a}_{wn}^Y]$. If there exists $\alpha\in[0,1]$ such that $\underline{x}_{i_0}(\alpha)<\underline{y}_{i_0}(\alpha)$, then

$$\begin{aligned}\underline{a}_{wn}^Y(\alpha)&=\min\left\{\sum_{j=1,\,j\neq i_0}^{n}w_j\cdot\underline{y}_j(\alpha)+w_{i_0}\cdot\underline{y}_{i_0}(\alpha)\mid w_i\in[W_{ni}]_\alpha,\,i=1,\dots,n,\,\sum_{i=1}^{n}w_i=1\right\}\\&=\sum_{j=1,\,j\neq i_0}^{n}w_j^*\cdot\underline{y}_j(\alpha)+w_{i_0}^*\cdot\underline{y}_{i_0}(\alpha)>\sum_{j=1,\,j\neq i_0}^{n}w_j^*\cdot\underline{x}_j(\alpha)+w_{i_0}^*\cdot\underline{x}_{i_0}(\alpha)\\&>\min\left\{\sum_{j=1,\,j\neq i_0}^{n}w_j\cdot\underline{x}_j(\alpha)+w_{i_0}\cdot\underline{x}_{i_0}(\alpha)\mid w_i\in[W_{ni}]_\alpha,\,i=1,\dots,n,\,\sum_{i=1}^{n}w_i=1\right\}=\underline{a}_{wn}^X(\alpha).\end{aligned}$$

The proof that if there exists $\alpha \in [0,1]$ such that $\overline{x}_{i_0}(\alpha) < \overline{y}_{i_0}(\alpha)$, then $\overline{a}^X_{wn}(\alpha) < \overline{a}^Y_{wn}(\alpha)$, would be analogous. □

The importance of this property can be illustrated by the following example that is based on the fuzzy MCDM model from Example 3.3.

Example 3.12. *Let $W_1, \dots, W_n$ be positive normalized fuzzy weights expressing the importance of the particular criteria. Let the fuzzy evaluations of the two alternatives x_1 and x_2 with respect to the particular criteria be expressed by the fuzzy numbers $X_{11}, \dots, X_{1n}$, and $X_{21}, \dots, X_{2n}$, respectively. Let, for some $i \in \{1, \dots, n\}$, $X_{1i} > X_{2i}$, and $X_{1j} = X_{2j}$ for all $j \in \{1, \dots, n\} \setminus \{i\}$, i.e. the alternative x_1 is better than x_2 with respect to the i-th criterion and is equal with x_2 with respect to the rest of the criteria. Then the overall fuzzy evaluation of x_1 will be better than the overall fuzzy evaluation of x_2 because the following relation holds:*

$$a_{wF}(X_{11}, \dots, X_{1n}; W_1, \dots, W_n) > a_{wF}(X_{21}, \dots, X_{2n}; W_1, \dots, W_n).$$

The last but one property of fuzzy aggregation operators studied in this paper is symmetry. The following theorem shows that if in each tuple of normalized fuzzy weights the particular fuzzy weights are equal, then the fuzzy weighted average operator is also symmetric.

Theorem 3.13. *Let $E_F = \{\mathbf{E}_n\}_{n=1}^{\infty}$ be a sequence of tuples of normalized fuzzy weights $\mathbf{E}_n = (W_{1n}, \dots, W_{nn})$ such that $W_{ni} = W_{nj}$ for all $i, j \in \{1, \dots, n\}$. Let $\mathcal{A}^{E_F} = \{A_n^{E_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator on an interval I associated with E_F. Then for all $n \in \mathbb{N}$, each n-tuple of fuzzy numbers $(X_1, \dots, X_n)$ defined on I, and any permutation σ of $\{1, \dots, n\}$ it holds that*

$$A_n^{E_F}(X_{\sigma(1)}, \dots, X_{\sigma(n)}) = A_n^{E_F}(X_1, \dots, X_n).$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. Let $W \in \mathcal{F}_N(\mathbb{R})$ be such that $W_{ni} = W$ for $i = 1, \dots, n$, i.e. $\mathbf{E}_n = (W, \dots, W)$. Let $X_i = [\underline{x}_i, \overline{x}_i]$, $i = 1, \dots, n$, be arbitrary fuzzy numbers defined on I and let σ be arbitrary permutation of $\{1, \dots, n\}$. Let us denote $A_n^{E_F}(X_1, \dots, X_n) = [\underline{a}_{wn}, \overline{a}_{wn}]$ and $A_n^{E_F}(X_{\sigma(1)}, \dots, X_{\sigma(n)}) = [\underline{a}^{\sigma}_{wn}, \overline{a}^{\sigma}_{wn}]$. Then for all $\alpha \in [0,1]$ we get

$$\begin{aligned}
\underline{a}_{wn}(\alpha) &= \min\left\{ \sum_{i=1}^{n} w_i \cdot \underline{x}_i(\alpha) \mid w_i \in [W]_\alpha,\ i = 1, \dots, n,\ \sum_{i=1}^{n} w_i = 1 \right\} \\
&= \min\left\{ \sum_{i=1}^{n} w_i \cdot \underline{x}_{\sigma(i)}(\alpha) \mid w_i \in [W]_\alpha,\ i = 1, \dots, n,\ \sum_{i=1}^{n} w_i = 1 \right\} \\
&= \underline{a}^{\sigma}_{wn}(\alpha),
\end{aligned}$$

since the range $[W]_\alpha$ for the admissible values of the weights does not depend on i. The proof that $\overline{a}_{wn}(\alpha) = \overline{a}^{\sigma}_{wn}(\alpha)$ for all $\alpha \in [0,1]$ would be analogous. □

In practical applications, the symmetry property means that if the uncertain values of the normalized fuzzy weights are expressed by the equal fuzzy numbers, then, analogously as in the crisp case, the resulting fuzzy weighted average with normalized fuzzy weights does not depend on the sequence of the normalized fuzzy weighted values. However, in contrast to the crisp case, the fuzzy weighted average operator associated with a sequence of tuples of equal normalized fuzzy weights does not coincide with the fuzzy arithmetic mean operator $\mathcal{M}$. The fuzzy arithmetic mean operator is $\mathcal{M} = \{M_n\}_{n=1}^{\infty}$ where for any $n \in \mathbb{N}$, M_n represents the fuzzy extension of arithmetic mean operation; it is given for any fuzzy numbers $X_i = [\underline{x}_i, \overline{x}_i]$, $i = 1, \dots, n$, by

$$M_n(X_1, \dots, X_n) = \left[\frac{1}{n} \sum_{i=1}^{n} \underline{x}_i, \frac{1}{n} \sum_{i=1}^{n} \overline{x}_i \right].$$

Theorem 3.14. *Let $E_F = \{\mathbf{E}_n\}_{n=1}^{\infty}$ be a sequence of tuples of normalized fuzzy weights $\mathbf{E}_n = (W_{1n}, \dots, W_{nn})$ such that $W_{ni} = W_{nj}$ for all $i, j \in \{1, \dots, n\}$. Let $\mathcal{A}^{E_F} = \{A_n^{E_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator on an interval I associated with E_F. Let $\mathcal{M} = \{M_n\}_{n=1}^{\infty}$ be the fuzzy arithmetic mean operator. Then for all $n \in \mathbb{N}$ and for each n-tuple of fuzzy numbers $(X_1, \dots, X_n)$ defined on I:*

$$M_n(X_1, \dots, X_n) \subseteq A_n^{E_F}(X_1, \dots, X_n), \tag{7}$$

where the equality $M_n(X_1, \dots, X_n) = A_n^{E_F}(X_1, \dots, X_n)$ holds, if and only if at least one of the following three conditions is satisfied:

1. *$n = 1$,*
2. *$\mathbf{E}_n = (\frac{1}{n}, \ldots, \frac{1}{n})$, i.e. normalized weights are crisp,*
3. *$X_i = X_j$ for all $i, j \in \{1, \ldots, n\}$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary. Let $\mathbf{E}_n = (W, \ldots, W)$, where $W \in \mathcal{F}_N([0,1])$. Let $X_i = [\underline{x}_i, \overline{x}_i]$, $i = 1, \ldots, n$, be arbitrary fuzzy numbers defined on I. Let us denote $A_n^{E_F}(X_1, \ldots, X_n) = [\underline{a}_{wn}, \overline{a}_{wn}]$ and $M_n(X_1, \ldots, X_n) = [\underline{m}_n, \overline{m}_n]$. For any $\alpha \in [0,1]$:

$$\frac{1}{n} \sum_{i=1}^{n} \underline{x}_i(\alpha) \in \left\{ \sum_{i=1}^{n} w_i \cdot \underline{x}_i(\alpha) \mid w_i \in [W]_\alpha,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} w_i = 1 \right\}.$$

Hence

$$\min \left\{ \sum_{i=1}^{n} w_i \cdot \underline{x}_i(\alpha) \mid w_i \in [W]_\alpha,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} w_i = 1 \right\} \leq \frac{1}{n} \sum_{i=1}^{n} \underline{x}_i(\alpha),$$

i.e. $\underline{a}_{wn}(\alpha) \leq \underline{m}_n(\alpha)$. Analogously, it can be shown that

$$\max \left\{ \sum_{i=1}^{n} w_i \cdot \overline{x}_i(\alpha) \mid w_i \in [W]_\alpha,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} w_i = 1 \right\} \geq \frac{1}{n} \sum_{i=1}^{n} \overline{x}_i(\alpha),$$

i.e. $\overline{a}_{wn}(\alpha) \geq \overline{m}_n(\alpha)$. Therefore, $M_n(X_1, \ldots, X_n) \subseteq A_n^{E_F}(X_1, \ldots, X_n)$. Further, let us examine the cases when $M_n(X_1, \ldots, X_n) = A_n^{E_F}(X_1, \ldots, X_n)$.

For $n = 1$, the equality is obvious since $M_1(X_1) = X_1 = A_1^{E_F}(X_1)$ for any $X_1 \in \mathcal{F}_N(I)$. Let $n \geq 2$ and let $\alpha \in [0,1]$ be arbitrary. If $[W]_\alpha = \{\frac{1}{n}\}$, then $\underline{a}_{wn}(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \underline{x}_i(\alpha) = \underline{m}_n(\alpha)$ and $\overline{a}_{wn}(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \overline{x}_i(\alpha) = \overline{m}_n(\alpha)$. If $\underline{x}_i(\alpha) = \underline{x}(\alpha)$ and $\overline{x}_i(\alpha) = \overline{x}(\alpha)$ for all $i \in \{1, \ldots, n\}$, then, applying Theorem 3.5, $\underline{a}_{wn}(\alpha) = \underline{x}(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \underline{x}(\alpha) = \underline{m}_n(\alpha)$ and $\overline{a}_{wn}(\alpha) = \overline{x}(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \overline{x}(\alpha) = \overline{m}_n(\alpha)$. Thus, if $\mathbf{E}_n = (\frac{1}{n}, \ldots, \frac{1}{n})$, or if $X_i = X_j$ for all $i, j \in \{1, \ldots, n\}$, then $A_n^{E_F}(X_1, \ldots, X_n) = M_n(X_1, \ldots, X_n)$.

Finally, for $n \geq 2$ and for some $\alpha \in [0,1]$, let us assume that $[W]_\alpha = [\underline{w}(\alpha), \overline{w}(\alpha)]$, where $\underline{w}(\alpha) < \overline{w}(\alpha)$, and that $[X_i]_\alpha \neq [X_j]_\alpha$ for some $i, j \in \{1, \ldots, n\}$, $i \neq j$. If $\underline{x}_i(\alpha) < \underline{x}_j(\alpha)$, then obviously $\underline{a}_{wn}(\alpha) < \frac{1}{n} \sum_{k=1}^{n} \underline{x}_k(\alpha) = \underline{m}_n(\alpha)$, as, for instance,

$$\overline{w}(\alpha) \cdot \underline{x}_i(\alpha) + \frac{1 - \overline{w}(\alpha)}{n-1} \sum_{k=1,\, k \neq i}^{n} \underline{x}_k(\alpha) < \frac{1}{n} \sum_{k=1}^{n} \underline{x}_k(\alpha),$$

where obviously $\frac{1-\overline{w}(\alpha)}{n-1} \in [W]_\alpha$. Analogously, it can be shown that if $\overline{x}_i(\alpha) < \overline{x}_j(\alpha)$, then $\overline{a}_{wn}(\alpha) > \frac{1}{n} \sum_{k=1}^{n} \overline{x}_k(\alpha) = \overline{m}_n(\alpha)$. Hence, $[M_n(X_1, \ldots, X_n)]_\alpha$ is a strict subset of $[A_n^{E_F}(X_1, \ldots, X_n)]_\alpha$. □

The last property studied in the paper will be additivity. It will be shown that in contrast to the crisp case, the fuzzy weighted average operator associated with a sequence of tuples of normalized fuzzy weights does not possess this property.

Theorem 3.15. *Let $W_F = \{\mathbf{W}_n\}_{n=1}^{\infty}$ be an arbitrary sequence of tuples of normalized fuzzy weights and $\mathcal{A}^{W_F} = \{A_n^{W_F}\}_{n=1}^{\infty}$ be the fuzzy weighted average operator on an interval I associated with W_F. Then for any $n \in \{2, 3, \ldots\}$ and any fuzzy numbers $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ defined on I the following holds:*

$$A_n^{W_F}(X_1 + Y_1, \ldots, X_n + Y_n) \subseteq A_n^{W_F}(X_1, \ldots, X_n) + A_n^{W_F}(Y_1, \ldots, Y_n). \tag{8}$$

Proof. Let $n \in \mathbb{N}$ be arbitrary, let $X_i = [\underline{x}_i, \overline{x}_i]$, $Y_i = [\underline{y}_i, \overline{y}_i]$ $i = 1, \ldots, n$, be arbitrary fuzzy numbers, and let us denote $A_n^{W_F}(X_1, \ldots, X_n) = [\underline{x}, \overline{x}]$, $A_n^{W_F}(Y_1, \ldots, Y_n) = [\underline{y}, \overline{y}]$, and $A_n^{W_F}(X_1 + Y_1, \ldots, X_n + Y_n) = [\underline{x+y}, \overline{x+y}]$. Then for all $\alpha \in [0,1]$,

$$\begin{aligned}
(\underline{x+y})(\alpha) &= \min \left\{ \sum_{i=1}^{n} w_i \left(\underline{x}_i(\alpha) + \underline{y}_i(\alpha) \right) \mid w_i \in [W_{ni}]_\alpha,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} w_i = 1 \right\} \\
&= \min \left\{ \sum_{i=1}^{n} w_i \cdot \underline{x}_i(\alpha) + \sum_{i=1}^{n} w_i \cdot \underline{y}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} w_i = 1 \right\} \\
&\geq \min \left\{ \sum_{i=1}^{n} w_i \cdot \underline{x}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} w_i = 1 \right\}
\end{aligned}$$

$$+\min\left\{\sum_{i=1}^{n} w_i \cdot \underline{y}_i(\alpha) \mid w_i \in [W_{ni}]_\alpha,\ i=1,\dots,n,\ \sum_{i=1}^{n} w_i = 1\right\}$$
$$= \underline{x}(\alpha) + \underline{y}(\alpha).$$

The proof that, for all $\alpha \in [0,1]$, $(\overline{x+y})(\alpha) \leq \overline{x}(\alpha) + \overline{y}(\alpha)$ would be analogous. □

The lack of additivity is of great importance and needs to be considered in fuzzy models where fuzzy weighted average with normalized fuzzy weights is applied. For instance, for the expected values of random variables X and Y the well-known equality $E(X+Y) = EX + EY$ always holds. If we consider fuzzy probabilities in discrete stochastic models, this equality does not hold in general. Let us illustrate this by the following example.

Example 3.16. *Let us consider the discrete stochastic model where the investment portfolio consists of two projects X and Y. The future yields from X and Y depend solely on the fact which one of the two mutually disjunctive states of the world SoW_1 and SoW_2 will occur in the future. The yields, estimated expertly by triangular fuzzy numbers, are given in Tab. 1. The probabilities of occurring of SoW_1 and SoW_2 are also estimated expertly as "approximately half"; they are described by the triangular fuzzy numbers $P_1 = P_2 = \langle 0.4, 0.5, 0.6\rangle$.*

Table 1: Dependance of fuzzy yields from the projects X, Y, and $X+Y$ on the states of the world and corresponding expected yields

Project	SoW_1	SoW_2	Expected yield
X	$\langle -10, -5, 0\rangle$	$\langle 20, 25, 30\rangle$	$\langle 2, 10, 18\rangle$
Y	$\langle 20, 25, 30\rangle$	$\langle -10, -5, 0\rangle$	$\langle 2, 10, 18\rangle$
$X+Y$	$\langle 10, 20, 30\rangle$	$\langle 10, 20, 30\rangle$	$\langle 10, 20, 30\rangle$

The fuzzy expected yields EX and EY from the projects (see the last column in Tab. 1) are given by the fuzzy weighted averages of the particular fuzzy yields under SoW_1 and SoW_2 with the fuzzy probabilities P_1 and P_2 that represent the pair of normalized fuzzy weights here.

It can be easily seen from Tab. 1 that the portfolio is built in such a way that the overall future yield $X+Y$ is independent of the SoW_1 and SoW_2. No matter which one of the states of the world will occur, the sum of the fuzzy yields from X and Y will be equal to $\langle 10, 20, 30\rangle$, and thus (as the fuzzy weighted average operator is idempotent), also the overall fuzzy expected yield $E(X+Y)$ is equal to this fuzzy number. However, the sum of the fuzzy expected yields from X and Y is equal to $\langle 4, 20, 36\rangle$, i.e. $E(X+Y) \subset EX+EY$. The membership functions of $E(X+Y)$ and $EX+EY$ are shown in Fig. 3.

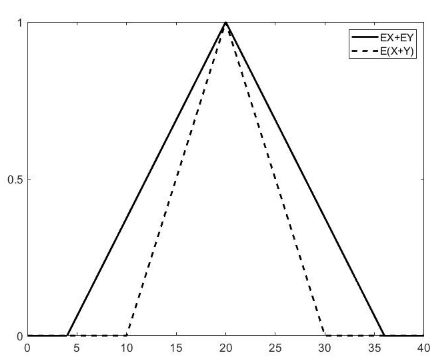

Figure 3: Membership functions of $E(X+Y)$ and $E(X)+E(Y)$.

In the subsequent part, let us show two conditions that guarantee the additivity of the fuzzy weighted average of fuzzy numbers with normalized fuzzy weights.

Notation 3.17. *It follows directly from the proof of Theorem 3.15 that if W_F is a sequence of tuples of crisp normalized weights, then for any $n \in \{2, 3, \ldots\}$ and any fuzzy numbers $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ defined on I:*

$$A_n^{W_F}(X_1 + Y_1, \ldots, X_n + Y_n) = A_n^{W_F}(X_1, \ldots, X_n) + A_n^{W_F}(Y_1, \ldots, Y_n).$$

Therefore, if the normalized weights are known precisely, then the corresponding fuzzy weighted average operator is additive. For example, if we in Example 3.16 replace the fuzzy probabilities P_1 and P_2 by the precisely known $p_1 = p_2 = 0.5$, then the additivity holds and $E(X + Y) = EX + EY = \langle 5, 10, 15 \rangle$.

Theorem 3.18. *Let $n \in \mathbb{N}$, let $X_i = [\underline{x}_i, \overline{x}_i]$, $Y_i = [\underline{y}_i, \overline{y}_i]$, $i = 1, \ldots, n$, be fuzzy numbers defined on I and let for all $\alpha \in [0, 1]$ there exist the permutations σ and ς of $\{1, \ldots, n\}$ such that*

$$\underline{x}_{\sigma(1)}(\alpha) \le \underline{x}_{\sigma(2)}(\alpha) \le \ldots \le \underline{x}_{\sigma(n)}(\alpha) \ \wedge\ \underline{y}_{\sigma(1)}(\alpha) \le \underline{y}_{\sigma(2)}(\alpha) \le \ldots \le \underline{y}_{\sigma(n)}(\alpha),$$

$$\overline{x}_{\varsigma(1)}(\alpha) \le \overline{x}_{\varsigma(2)}(\alpha) \le \ldots \le \overline{x}_{\varsigma(n)}(\alpha) \ \wedge\ \overline{y}_{\varsigma(1)}(\alpha) \le \overline{y}_{\varsigma(2)}(\alpha) \le \ldots \le \overline{y}_{\varsigma(n)}(\alpha).$$

Then

$$A_n^{W_F}(X_1 + Y_1, \ldots, X_n + Y_n) = A_n^{W_F}(X_1, \ldots, X_n) + A_n^{W_F}(Y_1, \ldots, Y_n),$$

for any n-tuple of normalized fuzzy weights $\mathbf{W}_n$.

Proof. The proof directly follows from the algorithm for calculation of the fuzzy weighted average with normalized fuzzy weights that is described in Notation 2.17. The ordering of the boundary points of α-cuts of the fuzzy numbers $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ is the same and hence, the same n-tuples of crisp normalized weights are used in formulas (4) and (5) for calculating the boundary values of α-cuts of fuzzy weighted averages $A_n^{W_F}(X_1, \ldots, X_n)$ and $A_n^{W_F}(Y_1, \ldots, Y_n)$. □

3.4 Comparison with the fuzzy weighted average operator associated with sequence of tuples of noninteractive fuzzy weights

Finally in this section, let us compare described properties of the fuzzy weighted average operator associated with a sequence of tuples of normalized fuzzy weights with the properties of the fuzzy weighted average operator associated with sequence of tuples of noninteractive (i.e. non-normalized) fuzzy weights that were presented in [22].

For both operators, the three conditions characteristic for aggregation operators – identity, monotonicity and boundary conditions, can be observed. Furthermore, both operators can be called compensative, idempotent, stable for a linear transformation, 1-Lipschitz in the metric d_F, and thus also continuous. If only the tuples of positive fuzzy weights (normalized or noninteractive) are considered, both operators fulfil strict monotonicity. Analogously, in case of tuples of equal fuzzy weights, both are symmetric. However, any of these two operators does not coincide with the fuzzy arithmetic mean operator; they coincide only if at least one of the following three conditions is satisfied: $n = 1$, the weights in all tuples are crisp, or the inputs of aggregations are the same. Thus, we have to conclude that the properties are identical.

One of the key property that is in contrast to the crisp case not possessed by the fuzzy weighted average operator associated with sequence of tuples of normalized fuzzy weights is the additivity. However, this property was not considered in [22], so the comparison cannot be made in this case.

It would be also interesting the comparison with properties of fuzzy extension of other aggregation operators, like for instance fuzzy OWA, etc. It is a possibility for future research in this field.

4 Conclusion

In the paper, the properties of the fuzzy weighted average operator that is associated with a sequence of tuples of normalized fuzzy weights were examined. First, it was shown that this operator fulfils the fuzzy extensions of identity, monotonicity and boundary conditions, i.e. that it satisfies three conditions characteristic for aggregation operators. Afterwards, it was shown in which way it preserves the main properties of the crisp weighted average operator, namely compensation, idempotency, stability for linear transformation, lipschitzianity, continuity, strict monotonicity in case of positive fuzzy weights, and symmetry in case of equal fuzzy weights (although it does not coincide with the fuzzy arithmetic mean operator in this case). One of the most valuable results of the paper is the fact that in contrast to the crisp case, the fuzzy weighted average operator associated with a sequence of tuples of normalized fuzzy weights is not additive fuzzy operator. One important consequence of this fact was illustrated by an example.

Further research in this area could be focused on the properties of the fuzzy weighted average operator in case when the weights and/or the weighted values are described by fuzzy vectors (see [20] for more details). Fuzzy vectors enable to incorporate also another kinds of interactions among the weights or among the weighted values into the fuzzy model. It would be also worth to study the properties of the fuzzy extensions of other aggregation operators, like fuzzy OWA operator, etc.

Another possible extension of this study is to focus on the other methods of ranking of fuzzy numbers than the strong one presented in Def. 2.12, because it is not total, fuzzy numbers can be incomparable with respect to this relation. It can be examined the monotonicity of the fuzzy weighted average operator with respect to ranking indices, like center of gravity, median, expected value, etc., that were introduced in the past (see [3] for an overview in this field). Other possibility in this direction is to study the properties of fuzzy extensions of aggregation operators where nondecreasing monotonicity on each argument is replaced by weaker condition. Such an operator defined on the set of nonnegative triangular fuzzy numbers was proposed recently in [28].

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