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Original paper

F-transforms determined by implicators

A. Tripathi[1], S. P. Tiwari[2] and I. Perfilieva[3]

[1,2] *Department of Mathematics & Computing, Indian Institute of Technology (ISM) Dhanbad-826004, India*
[3] *University of Ostrava, Institute for Research and Applications of Fuzzy Modeling, NSC IT4Innovations, 30. dubna 22, 701 03, Ostrava 1, Czech Republic*

tripathiabha29@gmail.com, sptiwarimaths@gmail.com, Irina.Perfilieva@osu.cz

Abstract

This work aims to study F-transforms based on general implicators and to investigate their basic properties. Interestingly, we show that some of the properties of F-transforms fail to hold in the case of implicators, such as S- and QL-implicators. Further, we establish an equivalence between L-fuzzy transformation systems and F-transforms.

Keywords: Implicator, L-fuzzy partition, F-transform, L-fuzzy transformation system.

1 Introduction

Since the inception of the notion of fuzzy transform (F-transform), by Perfilieva [18], the theory attracted interest of many researchers. It has now been significantly developed and opened a new page in the theory of semi-linear spaces. The main idea of the F-transform is to factorize (or fuzzify) the precise values of independent variables by a closeness relation, and precise values of dependent variables are averaged to an approximate value. The theory of F-transform has already been elaborated and extended from real valued to lattice-valued functions (cf., [18, 19]), from fuzzy sets to parametrized fuzzy sets [28] and from the single variable to the two (or more variables) (cf., [3, 4, 5, 30]). Recently, a number of researchers have initiated the study of F-transforms based on an arbitrary L-fuzzy partition of an arbitrary universe (cf., [8, 14, 15, 16, 21, 22, 27, 31]), where L is a complete residuated lattice. Among these studies, the relationships between F-transforms and semimodule homomorphisms were investigated in [14], a categorical study of L-fuzzy partitions of an arbitrary universe was presented in [15], while, the relationships between F-transforms and similarity relations were established in [16]. Further, in [21], an interesting relationship among F-transforms, L-fuzzy topologies/co-topologies and L-fuzzy approximation operators (which are concepts used in the study of an operator-oriented view of rough set theory) was established, while in [22], the relationship between fuzzy pretopological spaces and spaces with L-fuzzy partition was established. Also, in a different direction, a generalization of F-transforms was presented in [27] by considering the so called Q-module transforms, where Q stands for a unital quantale, while F-transforms based on a generalized residuated lattice were studied in [31]. The various studies carried out in the line of applications of F-transforms, e.g., trend-cycle estimation [6], data compression [7], numerical solution of partial differential equations [9], scheduling [12], time series [17], data analysis [20], denoising [23], face recognition [25], neural network approaches [29] and trading [32].

It is well-known that a fuzzy implicator is a generalization of the classical one to fuzzy logic, much the same way as a t-norm and a t-conorm are generalizations of the classical conjunction and disjunction, respectively. In literature, there exist many families of fuzzy implicators (cf., [24, 33]). As, the theoretical study of F-transforms carried out in [14, 15, 16, 18, 21, 22, 27, 31] is based on R-implicators only, it will be interesting to explore the theory of F-transforms based on a general fuzzy implicator, which is the theme of this paper. Specifically, we propose a broad class of F-transforms, each one of which called $(\mathcal{I}, \mathcal{T})$ F-transforms, that is represented by an implicator $\mathcal{I}$ and triangular norm $\mathcal{T}$. Also, we define three classes of F-transforms taking into account three well-known classes of implicators, namely

Corresponding Author: S. P. Tiwari
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$R-, S-, QL-$ implicators. Our interest is to investigate the properties of such F-transforms. Interestingly, we show that some of the properties of F-transforms studied in [18] do not hold in the case of certain implicators. Finally, an exciting relationship of F-transform with L-fuzzy transformation system is established. The remainder of the paper is structured as follows. In Section 2, we recall some basic notions and results of fuzzy logical operators. The pair of upper and lower F-transforms based on an L-fuzzy partition, are defined, and their properties are then examined in Section 3. In Section 4, the concepts of inverse F-transforms based on implicators are studied. Finally, Section 5 is towards a relationship between L-fuzzy transformation systems and F-transforms.

2 Preliminaries

In this section, we recall some basic fuzzy logic operators, for details, we refer [1, 2, 10, 11, 26]. Throughout this paper, $L \equiv (L, \vee, \wedge, 0, 1)$ denotes a complete lattice with the smallest element 0 and the largest element 1. We begin with the following.

Definition 2.1. *Let X be a nonempty set. Then an L-fuzzy set in X is a mapping $A : X \to L$.*

The family of all L-fuzzy sets in X is denoted by L^X. Throughout the paper, for all $a \in L, \mathbf{a} \in L^X$ such that $\mathbf{a}(x) = a,\ x \in X$ denotes constant L-fuzzy set.

Remark 2.2. *The core of an L-fuzzy set A is defined as:*

$$core(A) = \{x \in X : A(x) = 1\}.$$

If $core(A) \neq \emptyset$, then A is called a normal L-fuzzy set. For $A \subseteq X$, the characteristic function of A is a mapping $1_A : X \to L$ such that

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.3. *For $A, B \in L^X$, $A \leq B$ if $A(x) \leq B(x)$, for all $x \in X$. Also, for a given system $\{A_j : j \in J \neq \emptyset\} \subseteq L^X$,*

$$(\bigwedge_{j\in J} A_j)(x) = \bigwedge_{j\in J} A_j(x) \text{ and } (\bigvee_{j\in J} A_j)(x) = \bigvee_{j\in J} A_j(x), \forall x \in X.$$

Definition 2.4. *A triangular norm (t-norm) on L is an increasing, associative and commutative mapping $\mathcal{T} : L \times L \to L$ such that $\mathcal{T}(a, 1) = a$, for all $a \in L$. Further, the t-norm $\mathcal{T}$ is continuous, if for all $a_j, b \in L, j \in J$*

$$\mathcal{T}(\bigvee_{j\in J} a_j, b) = \bigvee_{j\in J} \mathcal{T}(a_j, b) \text{ and } \mathcal{T}(\bigwedge_{j\in J} a_j, b) = \bigwedge_{j\in J} \mathcal{T}(a_j, b).$$

Following are some well-known t-norms on $L = [0, 1]$:

(i) The standard *min* operator $\mathcal{T}_M(a, b) = \min\{a, b\}$.

(ii) The Łukasiewicz t-norm $\mathcal{T}_L(a, b) = \max\{0, a + b - 1\}$.

Definition 2.5. *A triangular conorm (t-conorm) on L is an increasing, associative and commutative mapping $\mathcal{S}$: $L \times L \to L$ such that $\mathcal{S}(a, 0) = a$, for all $a \in L$.*

Following are some well-known t-conorms on $L = [0, 1]$:

(i) The standard *max* operator $\mathcal{S}_M(a, b) = \max\{a, b\}$.

(ii) The bounded sum t-conorm $\mathcal{S}_L(a, b) = \min\{1, a + b\}$.

Definition 2.6. *A negator on L is a decreasing mapping $\mathcal{N} : L \to L$ such that $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. Further, a negator is called involutive (strong), if $\mathcal{N}(\mathcal{N}(a)) = a$, for all $a \in L$.*

The negator $\mathcal{N}_S(a) = 1 - a$ on $L = [0, 1]$ is usually referred to as the standard negator. Given a negator $\mathcal{N}$, t-norm $\mathcal{T}$ and t-conorm $\mathcal{S}$ are called dual with respect to $\mathcal{N}$ if

$$\mathcal{S}(\mathcal{N}(a), \mathcal{N}(b)) = \mathcal{N}(\mathcal{T}(a, b)) \text{ and } \mathcal{T}(\mathcal{N}(a), \mathcal{N}(b)) = \mathcal{N}(\mathcal{S}(a, b)), \forall a, b \in L.$$

For an involutive (strong) negator $\mathcal{N}$ and a t-conorm $\mathcal{S}$, the mapping $\mathcal{T}(a, b) = \mathcal{N}(\mathcal{S}(\mathcal{N}(a), \mathcal{N}(b))), \forall a, b \in L$, is a t-norm such that $\mathcal{T}$ and $\mathcal{S}$ are dual to each other with respect to the negator $\mathcal{N}$.

Definition 2.7. *An implicator on* L *is a mapping* $\mathcal{I} : L \times L \to L$ *such that* $\mathcal{I}(1,0) = 0$ *and* $\mathcal{I}(0,0) = \mathcal{I}(0,1) = \mathcal{I}(1,1) = 1$.

Following are some well-known implicators on $L = [0,1]$:

(i) Łukasiewicz implicator $\mathcal{I}_L(a,b) = \min\{1, 1-a+b\}$.

(ii) Kleene-Dienes implicator $\mathcal{I}_{KD}(a,b) = \max\{b, 1-a\}$.

(iii) Zadeh implicator $\mathcal{I}_{ZD}(a,b) = \max\{1-a, \min\{a,b\}\}$.

(iv) Gödel implicator

$$\mathcal{I}_G(a,b) = \begin{cases} 1, & a \leq b, \\ b, & a > b. \end{cases}$$

An implicator is called left monotonic and right monotonic if for all $a \in L$, $\mathcal{I}(-,a)$ is decreasing and $\mathcal{I}(a,-)$ is increasing, respectively. If $\mathcal{I}$ is both left and right monotonic, then it is called hybrid monotonic. An implicator $\mathcal{I}$ is right continuous if

$$\mathcal{I}(a, \bigwedge_{j \in J} b_j) = \bigwedge_{j \in J} \mathcal{I}(a, b_j).$$

For a left monotonic implicator $\mathcal{I}$, $\mathcal{N}(a) = \mathcal{I}(a,0), \forall a \in L$ called negator induced by implicator $\mathcal{I}$. Further, implicator $\mathcal{I}$ is said to be a border implicator if $\mathcal{I}(1,a) = a$, $\forall a \in L$. Also, an implicator $\mathcal{I}$ is said to be EP- implicator if $\mathcal{I}(a, \mathcal{I}(b,c)) = \mathcal{I}(b, \mathcal{I}(a,c)), \forall a,b,c \in L$. An implicator $\mathcal{I}$ is said to be CP- implicator if $a \leq b \Leftrightarrow \mathcal{I}(a,b) = 1$, $\forall a,b \in L$.

Now, we recall the definitions of three well-known classes of implicators. For which, let $\mathcal{T}, \mathcal{S}, \mathcal{N}$ be a t-norm, a t-conorm and a negator, respectively.

- An implicator $\mathcal{I}$ is called R-implicator (residual implicator) based on $\mathcal{T}$ if for all $a,b \in L$,

$$\mathcal{I}(a,b) = \sup\{c \in L : \mathcal{T}(a,c) \leq b\},$$

provided $\mathcal{T}$ is left continuous, i.e., $\mathcal{T}(\bigvee_{j \in J} a_j, b) = \bigvee_{j \in J} \mathcal{T}(a_j, b), a_j, b \in L, j \in J$.

Following are well-known R-implicators on $L = [0,1]$:

(i) Łukasiewicz implicator $\mathcal{I}_L$ based on $\mathcal{T}_L$,

(ii) Gödel implicator $\mathcal{I}_G$ based on $\mathcal{T}_M$.

- An implicator $\mathcal{I}$ is called S-implicator based on $\mathcal{S}$ and $\mathcal{N}$ if for all $a,b \in L$,

$$\mathcal{I}(a,b) = \mathcal{S}(\mathcal{N}(a), b).$$

Following are well-known S-implicators on $L = [0,1]$:

(i) Łukasiewicz implicator $\mathcal{I}_L$ based on $\mathcal{S}_L$ and $\mathcal{N}_S$,

(ii) Kleene-Dienes implicator $\mathcal{I}_{KD}$ based on $\mathcal{S}_M$ and $\mathcal{N}_S$.

- An implicator $\mathcal{I}$ is called QL-implicator based on $\mathcal{T}, \mathcal{S}$ and $\mathcal{N}$ if for all $a,b \in L$,

$$\mathcal{I}(a,b) = \mathcal{S}(\mathcal{N}(a), \mathcal{T}(a,b)),$$

provided $\mathcal{T}$ and $\mathcal{S}$ are dual to each other with respect to $\mathcal{N}$.

Following are the QL-implicators on $L = [0,1]$:

(i) Kleene-Dienes implicator $\mathcal{I}_{KD}$ based on $\mathcal{T}_L$, $\mathcal{S}_L$ and $\mathcal{N}_S$,

(ii) Zadeh implicator $\mathcal{I}_{ZD}$ based on $\mathcal{T}_M$, $\mathcal{S}_M$ and $\mathcal{N}_S$.

Remark 2.8. *If not stated otherwise, we would assume that* t*-norms,* t*-conorms, negators, and implicators are on* L.

Following are some well-known results about implicators.

Proposition 2.9. *R-implicator and S-implicator are hybrid monotonic, while QL-implicator is right monotonic.*

Proposition 2.10. *R-implicator, S-implicator and QL-implicator are border implicators.*

Proposition 2.11. *Let $\mathcal{T}$ be a left continuous t-norm and $\mathcal{I}$ be an R-implicator based on $\mathcal{T}$. Then for all $a, b, c \in L$,*

$$\mathcal{T}(\mathcal{I}(a,b),c) \leq \mathcal{I}(a,\mathcal{T}(b,c)) \text{ and } \mathcal{T}(a,\mathcal{I}(b,c)) \leq \mathcal{I}(\mathcal{I}(a,b),c).$$

We close this section by introducing the following notations which are going to be used in subsequent sections.

Given a t-norm $\mathcal{T}$, an implicator $\mathcal{I}$, a negator $\mathcal{N}$, and L-fuzzy sets $A, B \in L^X$, we define L-fuzzy sets $A \Rightarrow_{\mathcal{I}} B, A \otimes_{\mathcal{T}} B$ and $\neg_{\mathcal{N}}(A)$ as follows:

$$\begin{aligned}(A \Rightarrow_{\mathcal{I}} B)(x) &= \mathcal{I}(A(x), B(x)), x \in X,\\ (A \otimes_{\mathcal{T}} B)(x) &= \mathcal{T}(A(x), B(x)), x \in X,\\ (\neg_{\mathcal{N}}(A))(x) &= \mathcal{N}(A(x)), x \in X.\end{aligned}$$

3 Direct F-transforms

In this section, we study the concepts of direct F-transforms and their properties based on a t-norm $\mathcal{T}$ and an implicator $\mathcal{I}$. We begin with the definition of L-fuzzy partition from [21].

Definition 3.1. *A collection $\mathcal{P}$ of normal L-fuzzy sets $\{A_j : j \in J \neq \emptyset\}$ is called an L-fuzzy partition of a nonempty set X if the corresponding collection of ordinary sets $\{core(A_j) : j \in J\}$ is partition of X. The pair $(X, \mathcal{P})$ is called a space with L-fuzzy partition.*

Example 3.2. *Let $L = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ and $X = \{x_1, x_2, x_3\}$. Then $\mathcal{P} = \{A_1, A_2, A_3\}$ is an L-fuzzy partition of X, where*

$$\begin{aligned}A_1 &= \frac{1}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3},\\ A_2 &= \frac{0.4}{x_1} + \frac{1}{x_2} + \frac{0.6}{x_3},\\ A_3 &= \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{1}{x_3}.\end{aligned}$$

Let $\mathcal{I}$ be an implicator, $\mathcal{T}$ be a t-norm and $\mathcal{N}$ be a negator. Then we have the following.

Definition 3.3. *Let $\mathcal{P} = \{A_j : j \in J \neq \emptyset\}$ be an L-fuzzy partition of a nonempty set X and $f \in L^X$. Then*

(i) the (direct upper) $F^{\uparrow}$-transform of f computed with a t-norm $\mathcal{T}$ over the L-fuzzy partition $\mathcal{P}$ is a collection of lattice elements $\{F_j^{\uparrow}[f] : j \in J\}$ and the j-th component of (direct upper) $F^{\uparrow}$-transform is given by

$$F_j^{\uparrow}[f] = \bigvee_{x \in X} \mathcal{T}(A_j(x), f(x)),$$

(ii) the (direct lower) $F^{\downarrow}$-transform of f computed with an implicator $\mathcal{I}$ over the L-fuzzy partition $\mathcal{P}$ is a collection of lattice elements $\{F_j^{\downarrow}[f] : j \in J\}$ and the j-th component of (direct lower) $F^{\downarrow}$-transform is given by

$$F_j^{\downarrow}[f] = \bigwedge_{x \in X} \mathcal{I}(A_j(x), f(x)).$$

Remark 3.4. *In the paper, when a space with L-fuzzy partition $(X, \mathcal{P})$ is specified, we will assume that F-transforms are computed over the L-fuzzy partition $\mathcal{P}$. The arrow in denotation specifies the direction (upper, lower), thus we will shorten the formulation and refer to them as a (direct upper) $F^{\uparrow}$-transform of f computed with a t-norm $\mathcal{T}$ or a (direct lower) $F^{\downarrow}$-transform of f computed with an implicator $\mathcal{I}$. Moreover, if a t-norm or an implicator is specified, then we will further shorten formulations to a (direct upper) $F^{\uparrow}$-transform of f / a (direct lower) $F^{\downarrow}$-transform of f.*

Throughout the paper, if not specified otherwise, we will assume that by specifying an *R*- (*S*-, *QL*-) implicator $\mathcal{I}$ we have a related *t*-norm $\mathcal{T}$ to $\mathcal{I}$ with desired properties.

Example 3.5. *In continuation to Example 3.2, let* $f \in L^X$ *such that* $f = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.8}{x_3}$. *Then*

- *for R-implicator* $\mathcal{I}_L$ *and t-norm* $\mathcal{T}_L$ *(given in following table),*

$\mathcal{T}_L$	0	0.2	0.4	0.6	0.8	1
0	0	0	0	0	0	0
0.2	0	0	0	0	0	0.2
0.4	0	0	0	0	0.2	0.4
0.6	0	0	0	0.2	0.4	0.6
0.8	0	0	0.2	0.4	0.6	0.8
1	0	0.2	0.4	0.6	0.8	1

$\mathcal{I}_L$	0	0.2	0.4	0.6	0.8	1
0	1	1	1	1	1	1
0.2	0.8	1	1	1	1	1
0.4	0.6	0.8	1	1	1	1
0.6	0.4	0.6	0.8	1	1	1
0.8	0.2	0.4	0.6	0.8	1	1
1	0	0.2	0.4	0.6	0.8	1

  the $F^{\uparrow}$*-transform of* f *is the collection of lattice elements* $\{F_1^{\uparrow}[f] = 0.6, F_2^{\uparrow}[f] = 0.4, F_3^{\uparrow}[f] = 0.8\}$ *and the* $F^{\downarrow}$*-transform of* f *is the collection of lattice elements* $\{F_1^{\downarrow}[f] = 0.6, F_2^{\downarrow}[f] = 0.4, F_3^{\downarrow}[f] = 0.6\}$.

- *For S-implicator* $\mathcal{I}_L$ *based on t-conorm* $\mathcal{S}_L$ *and an involutive negator* $\mathcal{N}_S$*, the* $F^{\uparrow}$*-transform of* f *is the collection of lattice elements* $\{F_1^{\uparrow}[f] = 0.6, F_2^{\uparrow}[f] = 0.4, F_3^{\uparrow}[f] = 0.8\}$ *and the* $F^{\downarrow}$*-transform of* f *is the collection of lattice elements* $\{F_1^{\downarrow}[f] = 0.6, F_2^{\downarrow}[f] = 0.4, F_3^{\downarrow}[f] = 0.6\}$.

- *For QL-implicator* $\mathcal{I}_{KD}$ *(given below) determined by t-norm* $\mathcal{T}_L$*, t-conorm* $\mathcal{S}_L$ *and an involutive negator* $\mathcal{N}_S$*,*

$\mathcal{I}_{KD}$	0	0.2	0.4	0.6	0.8	1
0	1	1	1	1	1	1
0.2	0.8	0.8	0.8	0.8	0.8	1
0.4	0.6	0.6	0.6	0.6	0.8	1
0.6	0.4	0.4	0.4	0.6	0.8	1
0.8	0.2	0.2	0.4	0.6	0.8	1
1	0	0.2	0.4	0.6	0.8	1

  the $F^{\uparrow}$*-transform of* f *is the collection of lattice elements* $F_1^{\uparrow}[f] = 0.6, F_2^{\uparrow}[f] = 0.4, F_3^{\uparrow}[f] = 0.8$ *and the* $F^{\downarrow}$*-transform of* f *is the collection of lattice elements* $\{F_1^{\downarrow}[f] = 0.6, F_2^{\downarrow}[f] = 0.4, F_3^{\downarrow}[f] = 0.4\}$.

The following is towards some properties of direct *F*-transforms.

Proposition 3.6. *Let* $(X, \mathcal{P})$ *be a space with L-fuzzy partition* $\mathcal{P} = \{A_j : j \in J \neq \emptyset\}$ *and* $\mathcal{T}$ *be a t-norm. Then for* $\boldsymbol{a}, f, f_j \in L^X$ *and* $j \in J$

(i) if $x_j \in core(A_j)$ *then* $F_j^{\uparrow}[f] \geq f(x_j)$*;*

(ii) if $f \leq g$ *then* $F_j^{\uparrow}[f] \leq F_j^{\uparrow}[g]$*; and*

(iii) $F_j^{\uparrow}[(\boldsymbol{a} \otimes_{\mathcal{T}} f) \vee (\boldsymbol{b} \otimes_{\mathcal{T}} g)] = \mathcal{T}(a, F_j^{\uparrow}[f]) \vee \mathcal{T}(b, F_j^{\uparrow}[g])$*, for a left continuous t-norm* $\mathcal{T}$.

Proof. (i) Let $j \in J$ and $x_j \in core(A_j)$. It follows from Definition 3.3(i) that

$$F_j^{\uparrow}[f] = \bigvee_{x \in X} \mathcal{T}(A_j(x), f(x)) \geq \mathcal{T}(A_j(x_j), f(x_j)) = f(x_j),$$

Thus $F_j^{\uparrow}[f] \geq f(x_j)$, if $x_j \in core(A_j)$.

(ii) Let $f \leq g$ and $\mathcal{I}$ be right monotonic. Then

$$F_j^{\uparrow}[f] = \bigvee_{x \in X} \mathcal{T}(A_j(x), f(x)) \leq \bigvee_{x \in X} \mathcal{T}(A_j(x), g(x)) = F_j^{\uparrow}[g].$$

(iii) It follows from Definition 3.3(i) that

$$\begin{aligned}
& F_j^{\uparrow}[(\mathbf{a} \otimes_{\mathcal{T}} f) \vee (\mathbf{b} \otimes_{\mathcal{T}} g)] \\
= & \bigvee_{x \in X} \mathcal{T}(A_j(x), ((\mathbf{a} \otimes_{\mathcal{T}} f) \vee (\mathbf{b} \otimes_{\mathcal{T}} g))(x)) \\
= & \bigvee_{x \in X} \mathcal{T}(A_j(x), \mathcal{T}(a, f(x)) \vee \mathcal{T}(b, g(x))) \\
= & \bigvee_{x \in X} \mathcal{T}(A_j(x), \mathcal{T}(a, f(x))) \vee \bigvee_{x \in X} \mathcal{T}(A_j(x), \mathcal{T}(b, g(x))) \\
= & \mathcal{T}(a, \bigvee_{x \in X} \mathcal{T}(A_j(x), f(x))) \vee \mathcal{T}(b, \bigvee_{x \in X} \mathcal{T}(A_j(x), g(x))) \\
= & \mathcal{T}(a, F_j^{\uparrow}[f]) \vee \mathcal{T}(b, F_j^{\uparrow}[g]).
\end{aligned}$$

Thus $F_j^{\uparrow}[(\mathbf{a} \otimes_{\mathcal{T}} f) \vee (\mathbf{b} \otimes_{\mathcal{T}} g)] = \mathcal{T}(a, F_j^{\uparrow}[f]) \vee \mathcal{T}(b, F_j^{\uparrow}[g])$. □

Proposition 3.7. *Let $(X, \mathcal{P})$ be a space with L-fuzzy partition $\mathcal{P} = \{A_j : j \in J \neq \emptyset\}$ and $\mathcal{I}$ be an implicator. Then for $\boldsymbol{a}, f, f_j \in L^X$ and $j \in J$*

(i) if $x_j \in core(A_j)$ then $f(x_j) \geq F_j^{\downarrow}[f]$, for a border implicator $\mathcal{I}$;

(ii) if $f \leq g$ then $F_j^{\downarrow}[f] \leq F_j^{\downarrow}[g]$, for a right monotonic implicator $\mathcal{I}$; and

(iii) $F_j^{\downarrow}[(\boldsymbol{a} \Rightarrow_{\mathcal{I}} f) \wedge (\boldsymbol{b} \Rightarrow_{\mathcal{I}} g)] = \mathcal{I}(a, F_j^{\downarrow}[f]) \wedge \mathcal{I}(b, F_j^{\downarrow}[g])$, for a right continuous and EP implicator $\mathcal{I}$.

Proof. (i) Let $j \in J$ and $x_j \in core(A_j)$. It follows from Definition 3.3(ii) that

$$F_j^{\downarrow}[f] = \bigwedge_{x \in X} \mathcal{I}(A_j(x), f(x)) \leq \mathcal{I}(A_j(x_j), f(x_j)) = f(x_j).$$

Thus $f(x_j) \geq F_j^{\downarrow}[f]$, if $x_j \in core(A_j)$.

(ii) Let $f \leq g$ and $\mathcal{I}$ be right monotonic. Then

$$F_j^{\downarrow}[f] = \bigwedge_{x \in X} \mathcal{I}(A_j(x), f(x)) \leq \bigwedge_{x \in X} \mathcal{I}(A_j(x), g(x)) = F_j^{\downarrow}[g].$$

(iii) Let $\mathcal{I}$ be an EP-implicator and right continuous. It follows from Definition 3.3(ii) that

$$\begin{aligned}
& F_j^{\downarrow}[(\mathbf{a} \Rightarrow_{\mathcal{I}} f) \wedge (\mathbf{b} \Rightarrow_{\mathcal{I}} g)] \\
= & \bigwedge_{x \in X} \mathcal{I}(A_j(x), ((\mathbf{a} \Rightarrow_{\mathcal{I}} f) \wedge (\mathbf{b} \Rightarrow_{\mathcal{I}} g))(x)) \\
= & \bigwedge_{x \in X} \mathcal{I}(A_j(x), \mathcal{I}(a, f(x)) \wedge \mathcal{I}(b, g(x))) \\
= & \bigwedge_{x \in X} \mathcal{I}(A_j(x), \mathcal{I}(a, f(x))) \wedge \bigwedge_{x \in X} \mathcal{I}(A_j(x), \mathcal{I}(b, g(x))) \\
= & \mathcal{I}(a, \bigwedge_{x \in X} \mathcal{I}(A_j(x), f(x))) \wedge \mathcal{I}(b, \bigwedge_{x \in X} \mathcal{I}(A_j(x), g(x))) \\
= & \mathcal{I}(a, F_j^{\downarrow}[f]) \wedge \mathcal{I}(b, F_j^{\downarrow}[g]).
\end{aligned}$$

Thus $F_j^{\downarrow}[(\mathbf{a} \Rightarrow_{\mathcal{I}} f) \wedge (\mathbf{b} \Rightarrow_{\mathcal{I}} g)] = \mathcal{I}(a, F_j^{\downarrow}[f]) \wedge \mathcal{I}(b, F_j^{\downarrow}[g])$. □

Proposition 3.8. *Let $\mathcal{P} = \{A_j : j \in J \neq \emptyset\}$ be an L-fuzzy partition of a nonempty set X and $f \in L^X$. Then for every t-norm $\mathcal{T}$, the j-th component of (direct upper) $F^{\uparrow}$-transform of f is the least element of the set*

$$U_j = \{a \in L : \mathcal{T}(A_j(x), f(x)) \leq a, \forall x \in X\},\ j \in J.$$

Proof. To prove this, we need to show that $F_j^{\uparrow}[f] \in U_j$ and $F_j^{\uparrow}[f] \leq a$. It follows from Definition 3.3(i) that

$$F_j^{\uparrow}[f] \quad = \quad \bigvee_{x \in X} \mathcal{T}(A_j(x), f(x)) \geq \mathcal{T}(A_j(x), f(x)).$$

Thus $F_j^{\uparrow}[f] \in U_j$. Now, let $a \in L, x \in X$. Then from the given condition

$$\mathcal{T}(A_j(x), f(x)) \leq a \Rightarrow \bigvee_{x \in X} \mathcal{T}(A_j(x), f(x)) \leq a \Rightarrow F_j^{\uparrow}[f] \leq a.$$

Thus the j-th component of $F^{\uparrow}$-transform is the least element of the set U_j. □

Proposition 3.9. *Let assumptions from Proposition 3.8 hold, $\mathcal{T}$ be a t-norm and $\mathcal{I}_2$ be an CP-implicator. Then for all $a \in U_j$,*

$$\bigwedge_{x \in X} \mathcal{I}_2(\mathcal{T}(A_j(x), f(x)), a)) = 1,$$

and j-th component of (direct upper) $F^{\uparrow}$-transform is the smallest such a.

Proof. Let $j \in J$. Then for all $x \in X$, $\mathcal{T}(A_j(x), f(x)) \leq a$, or that, $\bigwedge_{x \in X} \mathcal{I}_2(\mathcal{T}(A_j(x), f(x)), a) = 1$, as $\mathcal{I}_2$ is an CP-implicator. □

In the following, we construct examples to show that the above proposition may not hold in case of some of the implicators.

Remark 3.10. (i) *If $\mathcal{I}_2$ is an S-implicator then Proposition 3.9 may not hold, For example, let an S-implicator $\mathcal{I}_{KD}$ based on t-conorm $\mathcal{S}_M$ and an involutive negator $\mathcal{N}_S$, where t-norm $\mathcal{T}_M$ is as under:*

$\mathcal{T}_M$	*0*	*0.2*	*0.4*	*0.6*	*0.8*	*1*
0	*0*	*0*	*0*	*0*	*0*	*0*
0.2	*0*	*0.2*	*0.2*	*0.2*	*0.2*	*0.2*
0.4	*0*	*0.2*	*0.4*	*0.4*	*0.4*	*0.4*
0.6	*0*	*0.2*	*0.4*	*0.6*	*0.6*	*0.6*
0.8	*0*	*0.2*	*0.4*	*0.6*	*0.8*	*0.8*
1	*0*	*0.2*	*0.4*	*0.6*	*0.8*	*1*

In continuation to Examples 3.2 and 3.5, let $f \in L^X$ such that $f = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.8}{x_3}$. Then $F_1^{\uparrow}[f] = 0.6, F_2^{\uparrow}[f] = 0.6, F_3^{\uparrow}[f] = 0.8$ and $U_1 = \{0.6, 0.8, 1\}$, $U_2 = \{0.6, 0.8, 1\}$, $U_3 = \{0.8, 1\}$.

Now, for $U_1 = \{0.6, 0.8, 1\}$,

$$\mathcal{I}_{KD}(\mathcal{T}_M(1, 0.6), 0.6) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(0.2, 0.4), 0.6) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(0.4, 0.8), 0.6) = 0.6 \neq 1;$$

$$\mathcal{I}_{KD}(\mathcal{T}_M(1, 0.6), 0.8) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(0.2, 0.4), 0.8) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(0.4, 0.8), 0.8) = 0.8 \neq 1;$$

$$\mathcal{I}_{KD}(\mathcal{T}_M(1, 0.6), 1) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(0.2, 0.4), 1) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(0.4, 0.8), 1) = 1.$$

Thus for $a = 0.6, 0.8$, Proposition 3.9 does not hold.

Also, for $U_2 = \{0.6, 0.8, 1\}$,

$$\mathcal{I}_{KD}(\mathcal{T}_M(0.4, 0.6), 0.6) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(1, 0.4), 0.6) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(0.6, 0.8), 0.6) = 0.6 \neq 1;$$
$$\mathcal{I}_{KD}(\mathcal{T}_M(0.4, 0.6), 0.8) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(1, 0.4), 0.8) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(0.6, 0.8), 0.8) = 0.8 \neq 1;$$
$$\mathcal{I}_{KD}(\mathcal{T}_M(0.4, 0.6), 1) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(1, 0.4), 1) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(0.6, 0.8), 1) = 1.$$

Thus for $a = 0.6, 0.8$, Proposition 3.9 does not hold.

Finally, for $U_3 = \{0.8, 1\}$,

$$\mathcal{I}_{KD}(\mathcal{T}_M(0.6, 0.6), 0.8) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(0.8, 0.4), 0.8) \wedge \mathcal{I}_{KD}(\mathcal{T}_M(1, 0.8), 0.8) = 0.8 \neq 1;$$

$$\mathcal{I}_{KD}(\mathcal{T}_M(0.6,0.6),1)\wedge\mathcal{I}_{KD}(\mathcal{T}_M(0.8,0.4),1)\wedge\mathcal{I}_{KD}(\mathcal{T}_M(1,0.8),1)=1.$$

Hence for $a=0.8$, Proposition 3.9 does not hold.

(ii) If $\mathcal{I}_2$ is an QL-implicator then Proposition 3.9 may not hold. For example, let QL-implicator $\mathcal{I}_{KD}$ determined by t-norm $\mathcal{T}_L$, t-conorm $\mathcal{S}_L$ and an involutive negator $\mathcal{N}_S$. In continuation to Examples 3.2 and 3.5, $F_1^{\uparrow}[f]=0.6$, $F_2^{\uparrow}[f]=0.4$, $F_3^{\uparrow}[f]=0.8$ and $U_1=\{0.6,0.8,1\}$, $U_2=\{0.4,0.6,0.8,1\}$, $U_3=\{0.8,1\}$. Now, if $U_1=\{0.6,0.8,1\}$, then

$$\mathcal{I}_{KD}(\mathcal{T}_L(1,0.6),0.6)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.2,0.4),0.6)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.4,0.8),0.6)=0.6\neq1;$$

$$\mathcal{I}_{KD}(\mathcal{T}_L(1,0.6),0.8)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.2,0.4),0.8)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.4,0.8),0.8)=0.8\neq1;$$

$$\mathcal{I}_{KD}(\mathcal{T}_L(1,0.6),1)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.2,0.4),1)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.4,0.8),1)=1.$$

Thus for $a=0.6,0.8$, Proposition 3.9 does not hold.

For $U_2=\{0.4,0.6,0.8,1\}$,

$$\mathcal{I}_{KD}(\mathcal{T}_L(0.4,0.6),0.4)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(1,0.4),0.4)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.6,0.8),0.4)=0.6\neq1;$$
$$\mathcal{I}_{KD}(\mathcal{T}_L(0.4,0.6),0.6)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(1,0.4),0.6)\wedge\mathcal{I}(\mathcal{T}_L(0.6,0.8),0.6)=0.6\neq1;$$
$$\mathcal{I}_{KD}(\mathcal{T}_L(0.4,0.6),0.8)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(1,0.4),0.8)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.6,0.8),0.8)=0.8\neq1;$$
$$\mathcal{I}_{KD}(\mathcal{T}_L(0.4,0.6),1)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(1,0.4),1)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.6,0.8),1)=1.$$

Thus for $a=0.4,0.6,0.8$, Proposition 3.9 does not hold.

Finally, for $U_3=\{0.8,1\}$,

$$\mathcal{I}_{KD}(\mathcal{T}_L(0.6,0.6),0.8)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.8,0.4),0.8)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(1,0.8),0.8)=0.8\neq1,$$

$$\mathcal{I}_{KD}(\mathcal{T}_L(0.6,0.6),1)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(0.8,0.4),1)\wedge\mathcal{I}_{KD}(\mathcal{T}_L(1,0.8),1)=1.$$

Hence for $a=0.8$, Proposition 3.9 does not hold.

(iii) If $\mathcal{I}_2$ is a non-left monotonic implicator then Proposition 3.9 may not hold. For example, let $\mathcal{I}_{ZD}$ be non-left monotonic QL-implicator based on $\mathcal{T}_M$, where $\mathcal{I}_{ZD}$ is given as under:

$\mathcal{I}_{ZD}$	0	0.2	0.4	0.6	0.8	1
0	1	1	1	1	1	1
0.2	0.8	0.8	0.8	0.8	0.8	0.8
0.4	0.6	0.6	0.6	0.6	0.6	0.6
0.6	0.4	0.4	0.4	0.6	0.6	0.6
0.8	0.2	0.2	0.4	0.6	0.8	0.8
1	0	0.2	0.4	0.6	0.8	1

In continuation to Examples 3.2 and Remark 3.10(ii), let $f\in L^X$ such that $f=\frac{0.6}{x_1}+\frac{0.4}{x_2}+\frac{0.8}{x_3}$. Then $F_1^{\uparrow}[f]=0.6$, $F_2^{\uparrow}[f]=0.6$, $F_3^{\uparrow}[f]=0.8$ and $U_1=\{0.6,0.8,1\}$, $U_2=\{0.6,0.8,1\}$, $U_3=\{0.8,1\}$.

Now, for $U_1=\{0.6,0.8,1\}$,

$$\mathcal{I}_{ZD}(\mathcal{T}_M(1,0.6),0.6)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(0.2,0.4),0.6)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(0.4,0.8),0.6)=0.6\neq1;$$
$$\mathcal{I}_{ZD}(\mathcal{T}_M(1,0.6),0.8)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(0.2,0.4),0.8)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(0.4,0.8),0.8)=0.6\neq1;$$
$$\mathcal{I}_{ZD}(\mathcal{T}_M(1,0.6),1)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(0.2,0.4),1)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(0.4,0.8),1)=0.6\neq1.$$

Thus for $a=0.6,0.8,1$, Proposition 3.9 does not hold.

Also, for $U_2=\{0.6,0.8,1\}$,

$$\mathcal{I}_{ZD}(\mathcal{T}_M(0.4,0.6),0.6)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(1,0.4),0.6)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(0.6,0.8),0.6)=0.6\neq1;$$
$$\mathcal{I}_{ZD}(\mathcal{T}_M(0.4,0.6),0.8)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(1,0.4),0.8)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(0.6,0.8),0.8)=0.6\neq1;$$
$$\mathcal{I}_{ZD}(\mathcal{T}_M(0.4,0.6),1)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(1,0.4),1)\wedge\mathcal{I}_{ZD}(\mathcal{T}_M(0.6,0.8),1)=0.6\neq1.$$

Thus for $a = 0.6, 0.8, 1$, Proposition 3.9 does not hold.

Finally, for $U_3 = \{0.8, 1\}$,

$$\mathcal{I}_{ZD}(\mathcal{T}_M(0.6, 0.6), 0.8) \wedge \mathcal{I}_{ZD}(\mathcal{T}_M(0.8, 0.4), 0.8) \wedge \mathcal{I}_{ZD}(\mathcal{T}_M(1, 0.8), 0.8) = 0.6 \neq 1;$$
$$\mathcal{I}_{ZD}(\mathcal{T}_M(0.6, 0.6), 1) \wedge \mathcal{I}_{ZD}(\mathcal{T}_M(0.8, 0.4), 1) \wedge \mathcal{I}_{ZD}(\mathcal{T}_M(1, 0.8), 1) = 0.6 \neq 1.$$

Hence for $a = 0.8$ and 1 also, Proposition 3.9 does not hold.

Proposition 3.11. *Let $\mathcal{P} = \{A_j : j \in J \neq \emptyset\}$ be an L-fuzzy partition of a nonempty set X and $f \in L^X$. Then for every implicator $\mathcal{I}$, the j-th component of (direct lower) $F^{\downarrow}$-transform of f is the greatest element of the set*

$$V_j = \{a \in L : a \leq \mathcal{I}(A_j(x), f(x)), \forall x \in X\},\ j \in J.$$

Proof. To prove this, we need to show that $F_j^{\downarrow}[f] \in V_j$ and $F_j^{\downarrow}[f] \geq a$. It follows from Definition 3.3(ii) that

$$F_j^{\downarrow}[f] = \bigwedge_{x \in X} \mathcal{I}(A_j(x), f(x)) \leq \mathcal{I}(A_j(x), f(x)).$$

Thus $F_j^{\downarrow}[f] \in V_j$. Now, let $a \in L, x \in X$. Then from the given condition

$$\mathcal{I}(A_j(x), f(x)) \geq a \Rightarrow \bigwedge_{x \in X} \mathcal{I}(A_j(x), f(x)) \geq a \Rightarrow F_j^{\downarrow}[f] \geq a.$$

Thus the j-th component of $F^{\downarrow}$-transform is the greatest element of the set V_j. □

Proposition 3.12. *Let assumptions from Proposition 3.8 hold and $\mathcal{I}_2$ be an CP-implicator. Then for all $a \in V_j$,*

$$\bigwedge_{x \in X} \mathcal{I}_2(a, \mathcal{I}_2(A_j(x), f(x))) = 1,$$

and j-th component of (direct lower) $F^{\downarrow}$-transform is the greatest such a.

In the following, we construct examples to show that the above proposition may not hold in case of some of the implicators.

Remark 3.13. *(i) If $\mathcal{I}_2$ is an S-implicator then Proposition 3.12 may not hold. For example, let $\mathcal{I}_{KD}$ be an S-implicator and in continuation to Examples 3.2 and 3.5, let $f \in L^X$ such that $f = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.8}{x_3}$. Then $F_1^{\downarrow}[f] = 0.6$, $F_2^{\downarrow}[f] = 0.4, F_3^{\downarrow}[f] = 0.4$ and $V_1 = \{0, 0.2, 0.4, 0.6\}$, $V_2 = \{0, 0.2, 0.4\}$, $V_3 = \{0, 0.2, 0.4\}$.*

Now, for $V_1 = \{0, 0.2, 0.4, 0.6\}$,

$$\mathcal{I}_{KD}(0, \mathcal{I}_{KD}(1, 0.6)) \wedge \mathcal{I}_{KD}(0, \mathcal{I}_{KD}(0.2, 0.4)) \wedge \mathcal{I}_{KD}(0, \mathcal{I}_{KD}(0.4, 0.8)) = 1;$$

$$\mathcal{I}_{KD}(0.2, \mathcal{I}_{KD}(1, 0.6)) \wedge \mathcal{I}_{KD}(0.2, \mathcal{I}_{KD}(0.2, 0.4)) \wedge \mathcal{I}_{KD}(0.2, \mathcal{I}_{KD}(0.4, 0.8)) = 0.8 \neq 1;$$

$$\mathcal{I}_{KD}(0.4, \mathcal{I}_{KD}(1, 0.6)) \wedge \mathcal{I}_{KD}(0.4, \mathcal{I}_{KD}(0.2, 0.4)) \wedge \mathcal{I}_{KD}(0.4, \mathcal{I}_{KD}(0.4, 0.8)) = 0.6 \neq 1;$$

$$\mathcal{I}_{KD}(0.6, \mathcal{I}_{KD}(1, 0.6)) \wedge \mathcal{I}_{KD}(0.6, \mathcal{I}_{KD}(0.2, 0.4)) \wedge \mathcal{I}_{KD}(0.6, \mathcal{I}_{KD}(0.4, 0.8)) = 0.6 \neq 1.$$

Thus for $a = 0.2, 0.4, 0.6$, Proposition 3.12 does not hold.

For $V_2 = \{0, 0.2, 0.4\}$,

$$\mathcal{I}_{KD}(0, \mathcal{I}_{KD}(0.4, 0.6)) \wedge \mathcal{I}_{KD}(0, \mathcal{I}_{KD}(1, 0.4)) \wedge \mathcal{I}_{KD}(0, \mathcal{I}_{KD}(0.6, 0.8)) = 1;$$

$$\mathcal{I}_{KD}(0.2, \mathcal{I}_{KD}(0.4, 0.6)) \wedge \mathcal{I}_{KD}(0.2, \mathcal{I}_{KD}(1, 0.4)) \wedge \mathcal{I}_{KD}(0.2, \mathcal{I}_{KD}(0.6, 0.8)) = 0.8 \neq 1;$$

$$\mathcal{I}_{KD}(0.4, \mathcal{I}_{KD}(0.4, 0.6)) \wedge \mathcal{I}_{KD}(0.4, \mathcal{I}_{KD}(1, 0.4)) \wedge \mathcal{I}_{KD}(0.4, \mathcal{I}_{KD}(0.6, 0.8)) = 0.6 \neq 1.$$

Thus for $a = 0.2, 0.4$, Proposition 3.12 does not hold.

Finally, for $V_3 = \{0, 0.2, 0.4\}$,

$$\mathcal{I}_{KD}(0, \mathcal{I}_{KD}(0.6, 0.6)) \wedge \mathcal{I}_{KD}(0, \mathcal{I}_{KD}(0.8, 0.4)) \wedge \mathcal{I}_{KD}(0, \mathcal{I}_{KD}(1, 0.8)) = 1;$$

$$\mathcal{I}_{KD}(0.2, \mathcal{I}_{KD}(0.6, 0.6)) \wedge \mathcal{I}_{KD}(0.2, \mathcal{I}_{KD}(0.8, 0.4)) \wedge \mathcal{I}_{KD}(0.2, \mathcal{I}_{KD}(1, 0.8)) = 0.8 \neq 1;$$

$$\mathcal{I}_{KD}(0.4, \mathcal{I}_{KD}(0.6, 0.6)) \wedge \mathcal{I}_{KD}(0.4, \mathcal{I}_{KD}(0.8, 0.4)) \wedge \mathcal{I}_{KD}(0.4, \mathcal{I}_{KD}(1, 0.8)) = 0.6 \neq 1.$$

Hence for $a = 0.2, 0.4$, Proposition 3.12 does not hold.

(ii) If $\mathcal{I}_2$ is an QL-implicator then Proposition 3.12 may not hold (follows from example in Remark 3.13(i)).

4 Inverse F-transforms

In this section, our motive is to introduce the concepts of inverse F-transforms and investigate their properties based on a t-norm $\mathcal{T}$ and an implicator $\mathcal{I}$. We begin with the following definition.

Definition 4.1. *Let $\mathcal{I}$ be an implicator, $\mathcal{T}$ be a t-norm related to $\mathcal{I}$ on a complete lattice L, and $(X, \mathcal{P})$ be a space with an L-fuzzy partition $\mathcal{P}$, where $\mathcal{P} = \{A_j \in L^X : j \in J \neq \emptyset\}$, $f \in L^X$. Further, let $F_j^{\uparrow}[f]$ and $F_j^{\downarrow}[f]$ be the j-th component of a (direct upper) $F^{\uparrow}$-transform of f computed with a t-norm $\mathcal{T}$ over $\mathcal{P}$ and (direct lower) $F^{\downarrow}$-transformation of f computed with an implicator $\mathcal{I}$ over $\mathcal{P}$, respectively. Then*

(i) the inverse (upper) $F^{\uparrow}$-transform of f computed with an implicator $\mathcal{I}$ over a fuzzy partition $\mathcal{P}$ is a mapping $\hat{f}^{\uparrow} : L^X \to L^X$ such that

$$\hat{f}^{\uparrow}(x) = \bigwedge_{j \in J} \mathcal{I}(A_j(x), F_j^{\uparrow}[f]),$$

(ii) the inverse (lower) $F^{\downarrow}$-transform of f computed with a t-norm $\mathcal{T}$ over a fuzzy partition $\mathcal{P}$ is a mapping $\hat{f}^{\downarrow} : L^X \to L^X$ such that

$$\hat{f}^{\downarrow}(x) = \bigvee_{j \in J} \mathcal{T}(A_j(x), F_j^{\downarrow}[f]).$$

Remark 4.2. *In the paper, when a space with L-fuzzy partition $(X, \mathcal{P})$ is specified, we will assume that the inverse F-transforms are computed over the L-fuzzy partition $\mathcal{P}$. The arrow in denotation specifies the direction (upper, lower), thus we will shorten the formulation and refer to them as an inverse (upper) $F^{\uparrow}$-transform of f computed with an implicator $\mathcal{I}$ or a inverse (lower) $F^{\downarrow}$-transform of f computed with a t-norm $\mathcal{T}$. Moreover, if an implicator or a t-norm is specified, then we will further shorten formulations to an inverse (upper) $F^{\uparrow}$-transform of f / an inverse (lower) $F^{\downarrow}$-transform of f.*

Throughout the paper, if not specified otherwise, we will assume that by specifying an R- (S-, QL-) implicator $\mathcal{I}$ we have a related t-norm $\mathcal{T}$ to $\mathcal{I}$ with desired properties, and inverse F-transforms are performed with respect to a pair $(\mathcal{I}, \mathcal{T})$.

Example 4.3. *In continuation to Examples 3.2 and 3.5,*

- *for an R-implicator $\mathcal{I}_L$ based on t-norm $\mathcal{T}_L$, $\hat{f}^{\uparrow} = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.8}{x_3}$ and $\hat{f}^{\downarrow} = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3}$,*
- *for an S-implicator $\mathcal{I}_L$ based on t-conorm $\mathcal{S}_L$ and an involutive negator $\mathcal{N}_S$, $\hat{f}^{\uparrow} = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.8}{x_3}$ and $\hat{f}^{\downarrow} = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3}$,*
- *for an QL-implicator $\mathcal{I}_{KD}$ determined by t-norm $\mathcal{T}_L$, t-conorm $\mathcal{S}_L$ and an involutive negator $\mathcal{N}_S$, $\hat{f}^{\uparrow} = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3}$ and $\hat{f}^{\downarrow} = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3}$.*

Now, we introduce the following.

Proposition 4.4. *Let $(X, \mathcal{P}), \mathcal{P} = \{A_j : j \in J \neq \emptyset\}$ be a space with L-fuzzy partition $\mathcal{P}$ and $f \in L^X$. Further, let $\mathcal{I}$ be an R-implicator and $\mathcal{T}$ be a related left continuous t-norm. Then*

(i) $\hat{f}^{\uparrow}(x) \geq f(x)$, $x \in X$, *and*

(ii) $F_j^{\uparrow}[f] = \bigvee\limits_{x \in X} \mathcal{T}(A_j(x), \hat{f}^{\uparrow}(x))$, $j \in J$.

Proof. (i) Let $x \in X, f \in L^X$. It follows from Proposition 2.11 and Definition 4.1(i) that

$$\begin{aligned} \hat{f}^{\uparrow}(x) &= \bigwedge_{j \in J} \mathcal{I}(A_j(x), F_j^{\uparrow}[f]) \\ &= \bigwedge_{j \in J} \mathcal{I}(A_j(x), \bigvee_{y \in X} \mathcal{T}(A_j(y), f(y))) \\ &\geq \bigwedge_{j \in J} \mathcal{I}(A_j(x), \mathcal{T}(A_j(x), f(x))) \\ &\geq \bigwedge_{j \in J} \mathcal{T}(\mathcal{I}(A_j(x), A_j(x)), f(x)) \\ &= \bigwedge_{j \in J} \mathcal{T}(1, , f(x) \\ &= f(x). \end{aligned}$$

(ii) From (i), $\hat{f}^{\uparrow}(x) \geq f(x)$, for any $x \in X$. It follows from Propositions 2.11, 3.6(ii) and Definition 3.3(i) that

$$\begin{aligned} F_j^{\uparrow}[f] = \bigvee_{x \in X} \mathcal{T}(A_j(x), f(x)) &\leq \bigvee_{x \in X} \mathcal{T}(A_j(x), \hat{f}^{\uparrow}(x)) \text{ and} \\ \mathcal{T}(A_j(x), \hat{f}^{\uparrow}(x)) &= \mathcal{T}(A_j(x), \bigwedge_{k \in J} \mathcal{I}(A_k(x), F_k^{\uparrow}[f]) \\ &\leq \mathcal{T}(A_j(x), \mathcal{I}(A_j(x), F_j^{\uparrow}[f]) \\ &\leq \mathcal{I}(\mathcal{I}(A_j(x), A_j(x)), F_j^{\uparrow}[f]) \\ &= \mathcal{I}(1, F_j^{\uparrow}[f]) \\ &= F_j^{\uparrow}[f]. \end{aligned}$$

Thus $\bigvee\limits_{x \in X} \mathcal{T}(A_j(x), \hat{f}^{\uparrow}(x)) \leq F_j^{\uparrow}[f]$ or $F_j^{\uparrow}[f] = \bigvee\limits_{x \in X} \mathcal{T}(A_j(x), \hat{f}^{\uparrow}(x))$. □

Remark 4.5. *In general, Proposition 4.4(i) is not satisfied for S- and QL-implicators. Counterexamples are given below.*

(i) For the case of S-implicator, let an S-implicator $\mathcal{I}_{KD}$ based on t-conorm $\mathcal{S}_M$ and an involutive negator $\mathcal{N}_S$. Further, in continuation to Examples 3.2, 3.5 and Remark 3.10(i), let $f \in L^X$ such that $f = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.8}{x_3}$. Then the (direct upper) $F^{\uparrow}$-transform of f determined by $\mathcal{T}_M$ is the collection $\{F_1^{\uparrow}[f] = 0.6, F_2^{\uparrow}[f] = 0.6, F_3^{\uparrow}[f] = 0.8\}$ while the inverse $F^{\uparrow}$-transform determined by $\mathcal{I}_{KD}$ is $\hat{f}^{\uparrow} = \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.6}{x_3}$. Thus there exists $x_3 \in X$ such that $\hat{f}^{\uparrow}(x_3) = 0.6 < 0.8 = f(x_3)$.

(ii) For the case of QL-implicator, let a QL-implicator $\mathcal{I}_{KD}$ determined by t-norm $\mathcal{T}_L$, t-conorm $\mathcal{S}_L$ and an involutive negator $\mathcal{N}_S$. Further, in continuation to Examples 3.2 and 3.5, let $f \in L^X$ such that $f = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.8}{x_3}$. Then the (direct upper) $F^{\uparrow}$-transform of f determined by $\mathcal{T}_L$ is the collection $\{F_1^{\uparrow}[f] = 0.6, F_2^{\uparrow}[f] = 0.4, F_3^{\uparrow}[f] = 0.8\}$, while the inverse $F^{\uparrow}$-transform determined by $\mathcal{I}_{KD}$ is $\hat{f}^{\uparrow} = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3}$. Thus there exists $x_3 \in X$ such that $\hat{f}^{\uparrow}(x_3) = 0.4 < 0.8 = f(x_3)$.

Proposition 4.6. *Let $(X, \mathcal{P}), \mathcal{P} = \{A_j : j \in J \neq \emptyset\}$ be a space with L-fuzzy partition $\mathcal{P}$ and $f \in L^X$. Further, let $\mathcal{I}$ be an R-implicator and $\mathcal{T}$ be a related left continuous t-norm. Then* *(i)* $\hat{f}^{\downarrow}(x) \leq f(x)$, $x \in X$, *and*

(ii) $F_j^{\downarrow}[f] = \bigwedge_{x\in X} \mathcal{I}(A_j(x), \hat{f}^{\downarrow}(x))$, $j \in J$.

Proof. (i) Let $x \in X$ and $f \in L^X$. It follows from Proposition 2.11 and Definition 4.1(i) that

$$\begin{aligned}
\hat{f}^{\downarrow}(x) &= \bigvee_{j\in J} \mathcal{T}(A_j(x), F_j^{\downarrow}[f]) \\
&= \bigvee_{j\in J} \mathcal{T}(A_j(x), \bigwedge_{y\in X} \mathcal{I}(A_j(y), f(y))) \\
&\leq \bigvee_{j\in J} \mathcal{T}(A_j(x), \mathcal{I}(A_j(x), f(x))) \\
&\leq \bigvee_{j\in J} \mathcal{I}(\mathcal{I}(A_j(x), A_j(x)), f(x)) \\
&= \bigvee_{j\in J} \mathcal{I}(1, f(x)) \\
&= f(x).
\end{aligned}$$

(ii) From (i), $\hat{f}^{\downarrow}(x) \leq f(x)$, for any $x \in X$. It follows from Propositions 2.11, 3.7(ii) and Definition 3.3(ii) that

$$\begin{aligned}
F_j^{\downarrow}[f] = \bigwedge_{x\in X} \mathcal{I}(A_j(x), f(x)) &\geq \bigwedge_{x\in X} \mathcal{I}(A_j(x), \hat{f}^{\downarrow}(x)) \text{ and} \\
\mathcal{I}(A_j(x), \hat{f}^{\downarrow}(x)) &= \mathcal{I}(A_j(x), \bigvee_{k\in J} \mathcal{T}(A_k(x), F_k^{\downarrow}[f])) \\
&\geq \mathcal{I}(A_j(x), \mathcal{T}(A_j(x), F_j^{\downarrow}[f])) \\
&\geq \mathcal{T}(\mathcal{I}(A_j(x), A_j(x)), F_j^{\downarrow}[f]) \\
&= \mathcal{T}(1, F_j^{\downarrow}[f]) \\
&= F_j^{\downarrow}[f].
\end{aligned}$$

Thus $\bigwedge_{x\in X} \mathcal{I}(A_j(x), \hat{f}^{\downarrow}(x)) \geq F_j^{\downarrow}[f]$ or $F_j^{\downarrow}[f] = \bigwedge_{x\in X} \mathcal{I}(A_j(x), \hat{f}^{\downarrow}(x))$. □

Remark 4.7. *In general, Proposition 4.6 is not satisfied for S- and QL-implicators. Counterexamples are given below.*

(i) For the case of S-implicator, let $\mathcal{I}_{KD}$ be an S-implicator and in continuation to Examples 3.2, 3.5 and Remark 3.10(i), let $f \in L^X$ such that $f = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3}$. Then the (direct lower) $F^{\downarrow}$-transform of f determined by $\mathcal{I}_{KD}$ is the collection $\{F_1^{\downarrow}[f] = 0.2,\ F_2^{\downarrow}[f] = 0.4, F_3^{\downarrow}[f] = 0.4\}$ and the inverse $F^{\downarrow}$-transform determined by $\mathcal{T}_M$ is $\hat{f}^{\downarrow} = \frac{0.4}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3}$. Thus there exists $x_1 \in X$ such that $\hat{f}^{\downarrow}(x_1) = 0.4 > 0.2 = f(x_1)$.

(ii) For the case of QL-implicator, let a QL-implicator $\mathcal{I}_{ZD}$ determined by t-norm $\mathcal{T}_M$, t-conorm $\mathcal{S}_M$ and an involutive negator $\mathcal{N}_S$. The implicator $\mathcal{I}_{ZD}$ is given as:

$\mathcal{I}_{ZD}$	0	0.2	0.4	0.6	0.8	1
0	1	1	1	1	1	1
0.2	0.8	0.8	0.8	0.8	0.8	0.8
0.4	0.6	0.6	0.6	0.6	0.6	0.6
0.6	0.4	0.4	0.4	0.6	0.6	0.6
0.8	0.2	0.2	0.4	0.6	0.8	0.8
1	0	0.2	0.4	0.6	0.8	1

In continuation to Example 3.2 and Remark 3.10(i), let $f \in L^X$ such that $f = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3}$. Then the (direct lower) $F^{\downarrow}$-transform of f determined by $\mathcal{I}_{ZD}$ is the collection $\{F_1^{\downarrow}[f] = 0.2,\ F_2^{\downarrow}[f] = 0.4, F_3^{\downarrow}[f] = 0.4\}$

and the inverse $F^{\downarrow}$-transform determined by $\mathcal{T}_M$ is $\hat{f}^{\downarrow} = \frac{0.4}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3}$. Thus there exists $x_1 \in X$ such that $\hat{f}^{\downarrow}(x_1) = 0.4 > 0.2 = f(x_1)$.

5 L-fuzzy transformation systems

The concept of L-fuzzy transformation systems based on R-implicator was carried out in [13]. In this section, we introduce the concept of L-fuzzy transformation systems based on general implicators. Further, we show that there is a close connection between L-fuzzy transformation systems and F-transforms studied in previous sections. We begin with the following.

Proposition 5.1. *Let X be a nonempty set and $f \in L^X$. Then for every hybrid monotonic implicator $\mathcal{I}$,*

$$\neg_{\mathcal{N}}(f) = \bigwedge_{x\in X} (\boldsymbol{f(x)} \Rightarrow_{\mathcal{I}} \neg_{\mathcal{N}}(1_{\{x\}})),$$

holds, where $\mathcal{N}(x) = \mathcal{I}(x, 0)$.

Proof. Let $y \in X$. Then

$$\begin{aligned}
(\bigwedge_{x\in X} (\mathbf{f(x)} \Rightarrow_{\mathcal{I}} \neg_{\mathcal{N}} 1_{\{x\}}))(y) &= \bigwedge_{x\in X} \mathcal{I}(f(x), (\neg_{\mathcal{N}}(1_{\{x\}}))(y)) \\
&= \bigwedge_{x\in X} \mathcal{I}(f(x), \mathcal{N}(1_{\{x\}}(y))) \\
&= \mathcal{I}(f(y), 0) \wedge \bigwedge_{x\neq y\in X} \mathcal{I}(f(x), 1)) \\
&= \mathcal{I}(f(y), 0) \wedge 1 \\
&= \mathcal{I}(f(y), 0) \\
&= \mathcal{I}(f, \mathbf{0})(y) \\
&= \mathcal{N}(f)(y).
\end{aligned}$$

Thus $(\bigwedge_{x\in X} (\mathbf{f(x)} \Rightarrow_{\mathcal{I}} \neg_{\mathcal{N}} 1_{\{x\}}))(y) = \mathcal{N}(f)(y)$. □

Remark 5.2. *The above proposition holds if $\mathcal{I}$ is an R-implicator or an S-implicator but may not hold for QL-implicators. For example, let $\mathcal{I}_{ZD}$ be an QL-implicator with respect to the involutive negator $\mathcal{N}_S$ (i.e., $\mathcal{I}_{ZD}(x, 0) = \mathcal{N}_S(x) = 1 - x$) and $f \in L^X$ such that $\mathcal{N}_S(f) = \frac{0.8}{x_1} + \frac{0.6}{x_2} + \frac{0.2}{x_3}$. In continuation to Remark 4.7(ii), $\mathcal{N}_S(f(x_1)) = 0.8, \mathcal{N}_S(f(x_2)) = 0.6, \mathcal{N}_S(f(x_3)) = 0.2$ and*
$\mathcal{I}_{ZD}(0.2, \mathcal{N}_S(1_{\{x_1\}}(x_1))) \wedge \mathcal{I}_{ZD}(0.4, \mathcal{N}_S(1_{\{x_2\}}(x_1))) \wedge \mathcal{I}_{ZD}(0.8, \mathcal{N}_S(1_{\{x_3\}}(x_1))) = 0.6 \neq 0.8 = \mathcal{N}_S(f(x_1))$,
$\mathcal{I}_{ZD}(0.2, \mathcal{N}_S(1_{\{x_1\}}(x_2))) \wedge \mathcal{I}_{ZD}(0.4, \mathcal{N}_S(1_{\{x_2\}}(x_2))) \wedge \mathcal{I}_{ZD}(0.8, \mathcal{N}_S(1_{\{x_3\}}(x_2))) = 0.6 = \mathcal{N}_S(f(x_2))$,
$\mathcal{I}_{ZD}(0.2, \mathcal{N}_S(1_{\{x_1\}}(x_3))) \wedge \mathcal{I}_{ZD}(0.4, \mathcal{N}_S(1_{\{x_2\}}(x_3))) \wedge \mathcal{I}_{ZD}(0.8, \mathcal{N}_S(1_{\{x_3\}}(x_3))) = 0.6 \neq 0.2 = \mathcal{N}_S(f(x_3))$.
Thus there exists $x_1, x_3 \in X$ such that

$$\neg_{\mathcal{N}_S}(f) \neq \bigwedge_{x\in X} (\boldsymbol{f(x)} \Rightarrow_{\mathcal{I}_{ZD}} \neg_{\mathcal{N}_S}(1_{\{x\}})).$$

Next, for $f \in L^X$ and for an L-fuzzy partition $\mathcal{P}$, it can be seen that the F-transform $F^{\downarrow}$ ($F^{\uparrow}$) induces a map $F^{\downarrow}_{\mathcal{P}}(F^{\uparrow}_{\mathcal{P}}) : L^X \to L^J$ such that

$$F^{\downarrow}_{\mathcal{P}}[f](j) = F^{\downarrow}_j[f] \ (F^{\uparrow}_{\mathcal{P}}[f](j) = F^{\uparrow}_j[f]).$$

Now, we introduce the following concept of lower L-fuzzy transformation system and show that it has a close connection with the F-transform $F^{\downarrow}_{\mathcal{P}}$ determined by an implicator $\mathcal{I}$ with certain properties.

Definition 5.3. *Let X be a nonempty set and $\mathcal{I}$ be an implicator on a complete lattice L such that $\mathcal{N}_{\mathcal{I}}$ be an involutive negator on L. Then system $\mathcal{H}_{\mathcal{I}} = (X, Y, v, H)$, where*

1. *Y is a set;*

2. $v : X \to Y$ *is a surjective map;*

3. $H : L^X \to L^Y$ *is a map such that*

(i) for all $\{f_k : k \in J\} \subseteq L^X$,

$$H[\bigwedge_{k\in J} f_k](y) = \bigwedge_{k\in J} H[f_k](y),$$

(ii) for all $a \in L, y \in Y, \boldsymbol{a}, f \in L^X$, $H[(\boldsymbol{a} \Rightarrow_{\mathcal{I}} f)](y) = \mathcal{I}(a, H[f](y))$,

(iii) for $y \in Y$ *and* $x \in X$, $(\neg_{\mathcal{N}_{\mathcal{I}}}(H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})]))(y) = 1$ *iff* $y = v(x)$,

is called a lower L-fuzzy transformation system on X determined by an implicator $\mathcal{I}$.

Example 5.4. *Let X be nonempty set,* $\mathcal{I}$ *be an implicator and* $id : X \to X$ *be the identity map. Define a map* $H : L^X \to L^X$ *such that* $H[f](x) = f(x), x \in X$. *Further, let* $\{f_k : k \in J\} \subseteq L^X$. *Then*

$$H[\bigwedge_{k\in J} f_k](x) = \bigwedge_{k\in J} f_k(x) = \bigwedge_{k\in J} H[f_k](x).$$

Now, let $x \in X, \boldsymbol{a}, f \in L^X$. *Then*

$$H[(\boldsymbol{a} \Rightarrow_{\mathcal{I}} f)](x) = (\boldsymbol{a} \Rightarrow_{\mathcal{I}} f)(x) = \mathcal{I}(a, f(x)) = \mathcal{I}(a, H[f](x)).$$

Finally, let $x, z \in X$ *and* $\mathcal{N}(\cdot) = \mathcal{I}(\cdot, 0)$ *be an involutive negator. Then*

$$(\neg_{\mathcal{N}}(H[\neg_{\mathcal{N}}(1_{\{x\}})]))(z) = (\neg_{\mathcal{N}}(\neg_{\mathcal{N}}(1_{\{x\}})))(z) = 1_{\{x\}}(z) = 1 \Leftrightarrow x = z.$$

Thus $(\neg_{\mathcal{N}}(H[\neg_{\mathcal{N}}(1_{\{x\}})]))(z) = 1$ *iff* $z = id(x)$. *Hence* $\mathcal{H}_{\mathcal{I}} = (X, X, id, H)$ *is a lower L-fuzzy transformation system on X determined by an implicator* $\mathcal{I}$.

Proposition 5.5. *Let* $\mathcal{I}$ *be hybrid monotonic, right continuous, EP-implicator on a complete lattice L such that* $\mathcal{N}_{\mathcal{I}}$ *is an involutive negator. Then the following statements are equivalent:*

(i) $\mathcal{H}_{\mathcal{I}} = (X, Y, v, H)$ *is a lower L-fuzzy transformation system on X determined by* $\mathcal{I}$ *and* $Y \subseteq X$,

(ii) There exists an L-fuzzy partition $\mathcal{P}$ *of X indexed by Y, such that* $v(x) = y$ *iff* $x \in core(A_y)$ *and* $H = F^{\downarrow}_{\mathcal{P}}$.

Proof. Let $\mathcal{H}_{\mathcal{I}} = (X, Y, v, H)$ be a lower L-fuzzy transformation system on X determined by $\mathcal{I}$. Also, let $\mathcal{P} = \{A_y : y \in Y\}$ such that for all $y \in Y$, $A_y \in L^X$ is given by $A_y(x) = (\neg_{\mathcal{N}_{\mathcal{I}}}(H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})]))(y)$, $x \in X$. Now, from Property (iii) of the above definition, $A_{v(x)}(x) = (\neg_{\mathcal{N}_{\mathcal{I}}}(H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})]))(v(x)) = 1$, or that, $x \in core(A_{v(x)})$. Further, for $t \in core(A_y) \cap core(A_z)$, $y, z \in Y$ and the fact that $\mathcal{N}_{\mathcal{I}}(x) = \mathcal{I}(x, 0)$, $(\neg_{\mathcal{N}_{\mathcal{I}}}(H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{t\}})]))(y) = 1 = (\neg_{\mathcal{N}_{\mathcal{I}}}(H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{t\}})]))(z)$, i.e., $A_y(t) = 1 = A_z(t)$ iff $y = v(t) = z$. Thus $\{core(A_y) : y \in Y\}$ is a partition of X and therefore $\mathcal{P}$ is an L-fuzzy partition of X. Now, for all $y \in Y$ and $f \in L^X$

$$\begin{aligned}
F^{\downarrow}_{\mathcal{P}}[f](y) &= \bigwedge_{x\in X} \mathcal{I}(A_y(x), f(x)) \\
&= \bigwedge_{x\in X} \mathcal{I}((\neg_{\mathcal{N}_{\mathcal{I}}}(H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})]))(y), f(x)) \\
&= \bigwedge_{x\in X} \mathcal{I}(\mathcal{N}_{\mathcal{I}}(H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})](y)), \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(f(x)))) \\
&= \bigwedge_{x\in X} \mathcal{I}(\mathcal{N}_{\mathcal{I}}(f(x)), \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})](y))))
\end{aligned}$$

$$\begin{aligned}
&= \bigwedge_{x\in X} \mathcal{I}((\neg_{\mathcal{N}_{\mathcal{I}}}(f))(x), H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})](y)) \\
&= \bigwedge_{x\in X} H[\neg_{\mathcal{N}_{\mathcal{I}}}(\mathbf{f(x)}) \Rightarrow_{\mathcal{I}} \neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})](y) \\
&= H[\bigwedge_{x\in X} (\neg_{\mathcal{N}_{\mathcal{I}}}(\mathbf{f(x)}) \Rightarrow_{\mathcal{I}} \neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}}))](y) \\
&= H[\neg_{\mathcal{N}_{\mathcal{I}}}(\neg_{\mathcal{N}_{\mathcal{I}}}(f))](y) = H[f](y).
\end{aligned}$$

Thus $H = F^{\downarrow}_{\mathcal{P}}$.
Conversely, let $\mathcal{P} = \{A_y \in L^X : y \in Y \neq \emptyset\}$ be an L-fuzzy partition of base set $X \neq \emptyset$. Let us define a mapping $v : X \to Y$ such that $v(x) = y$ iff $x \in core(A_y)$. Further, let $\mathcal{I}$ be an implicator with assumed properties (hybrid monotonic, right continuous, EP-property, $\mathcal{N}_{\mathcal{I}}(\cdot) = \mathcal{I}(\cdot, 0)$ is an involutive negator) and $H = F^{\downarrow}_{\mathcal{P}}$. Then (X, Y, v, H) is a lower L-fuzzy transformation system on X determined by $\mathcal{I}$, which is shown below.

(i) Let $y \in Y, \{f_k : k \in J\} \subseteq L^X$. Then

$$\begin{aligned} H[\bigwedge_{k\in J} f_k](y) &= F^{\downarrow}_{\mathcal{P}}[\bigwedge_{k\in J} f_k](y) \\ &= \bigwedge_{x\in X} \mathcal{I}(A_y(x), (\bigwedge_{k\in J} f_k)(x)) \\ &= \bigwedge_{x\in X} \mathcal{I}(A_y(x), \bigwedge_{k\in J} f_k(x)) \\ &= \bigwedge_{x\in X} \bigwedge_{k\in J} \mathcal{I}(A_y(x), f_k(x)) \, (\mathcal{I} \text{ is right continuous}) \\ &= \bigwedge_{k\in J} F^{\downarrow}_{\mathcal{P}}[f_k](y). \end{aligned}$$

Thus $H[\bigwedge_{k\in J} f_k](y) = \bigwedge_{k\in J} H[f_k](y)$.

(ii) Let $y \in Y, a \in L, \mathbf{a}, f \in L^X$. Then

$$\begin{aligned} H[(\mathbf{a} \Rightarrow_{\mathcal{I}} f)](y) &= F^{\downarrow}_{\mathcal{P}}[(\mathbf{a} \Rightarrow_{\mathcal{I}} f)](y) \\ &= \bigwedge_{x\in X} \mathcal{I}(A_y(x), (\mathbf{a} \Rightarrow_{\mathcal{I}} f)(x)) \\ &= \bigwedge_{x\in X} \mathcal{I}(A_y(x), \mathcal{I}(a, f(x))) \\ &= \bigwedge_{x\in X} \mathcal{I}(a, \mathcal{I}(A_y(x), f(x))) \, (\mathcal{I} \text{ is an } EP\text{-implicator}) \\ &= \mathcal{I}(a, \bigwedge_{x\in X} \mathcal{I}(A_y(x), f(x))) \, (\mathcal{I} \text{ is right continuous}) \\ &= \mathcal{I}(a, F^{\downarrow}_{\mathcal{P}}[f](y)). \end{aligned}$$

Thus $H[(\mathbf{a} \Rightarrow_{\mathcal{I}} f)](y) = \mathcal{I}(a, H[f](y))$.

(iii) Now, let $y \in Y, x \in X$. Then

$$\begin{aligned} (\neg_{\mathcal{N}_{\mathcal{I}}}(H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})]))(y) &= (\neg_{\mathcal{N}_{\mathcal{I}}}(F^{\downarrow}_{\mathcal{P}}[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})]))(y) = \mathcal{N}_{\mathcal{I}}(F^{\downarrow}_{\mathcal{P}}[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})](y)) \\ &= \mathcal{N}_{\mathcal{I}}(\bigwedge_{z\in X} \mathcal{I}(A_y(z), (\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}}))(z))) \\ &= \mathcal{N}_{\mathcal{I}}(\bigwedge_{z\in X} \mathcal{I}(A_y(z), \mathcal{N}_{\mathcal{I}}(1_{\{x\}}(z)))) \\ &= \mathcal{N}_{\mathcal{I}}(\mathcal{I}(A_y(x), 0)) \\ &= A_y(x). \end{aligned}$$

Thus $(\neg_{\mathcal{N}_{\mathcal{I}}}(H[\neg_{\mathcal{N}_{\mathcal{I}}}(1_{\{x\}})]))(y) = 1$ iff $A_y(x) = 1$ iff $v(x) = y$. □

Remark 5.6. *If $\mathcal{I}$ is an R-implicator or an S-implicator then all the conditions in Proposition 5.5 are satisfied.*

Next, we introduce the following.

Definition 5.7. *Let X be a nonempty set and $\mathcal{T}$ be a t-norm on a complete lattice L. Then a system $\mathcal{G}_{\mathcal{T}} = (X, Y, u, G)$, where*

1. Y is a set,

2. $u : X \to Y$ is a surjective map,

3. $G : L^X \to L^Y$ is a map such that

(i) for all $\{f_k : k \in J\} \subseteq L^X$, $y \in Y$,

$$G[\bigvee_{k\in J} f_k](y) = \bigvee_{k\in J} G[f_k](y),$$

(ii) for all $a \in L, y \in Y$ and $\boldsymbol{a}, f \in L^X$, $G[\boldsymbol{a} \otimes_{\mathcal{T}} f](y) = \mathcal{T}(a, G[f](y))$,

(iii) for all $x \in X$, $y \in Y$, $G[1_{\{x\}}](y) = 1$ iff $y = u(x)$,
is called an upper L-fuzzy transformation system on X with determined by a t-norm $\mathcal{T}$.

Example 5.8. *Let X be a nonempty set, $\mathcal{T}$ be a t-norm and $id : X \to X$ be an identity map. Define a map $G : L^X \to L^X$ such that $G[f](x) = f(x), x \in X$. Then for $\{f_k : k \in J\} \subseteq L^X$,*

$$G[\bigvee_{k\in J} f_k](x) = \bigvee_{k\in J} f_k(x) = \bigvee_{k\in J} G[f_k](x).$$

Now, let $x \in X, \boldsymbol{a}, f \in L^X$. Then

$$G[(\boldsymbol{a} \otimes_{\mathcal{T}} f)](x) = (\boldsymbol{a} \otimes_{\mathcal{T}} f)(x) = \mathcal{T}(a, f(x)) = \mathcal{T}(a, G[f](x)).$$

Finally, let $x, z \in X$. Then

$$G[1_{\{x\}}](z) = 1_{\{x\}}(z) = 1 \Leftrightarrow x = z.$$

Thus $G[1_{\{x\}}](z) = 1$ iff $z = id(x)$. Hence $\mathcal{G}_{\mathcal{T}} = (X, X, id, G)$ is an upper L-fuzzy transformation system on X determined by a t-norm $\mathcal{T}$.

We close this section by introducing the following.

Proposition 5.9. *Let $\mathcal{T}$ be a left continuous t-norm on a complete lattice L. Then the following statements are equivalent:*

1. $\mathcal{G}_{\mathcal{T}} = (X, Y, u, G)$ is an upper L-fuzzy transformation system on X determined by a t-norm $\mathcal{T}$ and $Y \subseteq X$,

2. There exists an L-fuzzy partition $\mathcal{P}$ of X indexed by Y, such that $u(x) = y$ iff $x \in core(A_y)$ and $G = F^{\uparrow}_{\mathcal{P}}$.

Proof. Follows from Theorem 2.2 of [13]. □

6 Conclusions

In this paper, we have studied the concepts of F-transforms determined by implicators. Further, the properties of direct and inverse F-transforms have been investigated. Among them, the interesting results are Propositions 3.9, 3.12, 4.4 and 4.6, which hold for R-implicator but not in case of S and QL-implicators. Lastly, we studied the relationships between L-fuzzy transformation systems and F-transforms.

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