

On the symmetric quintuple implicational method of fuzzy reasoning

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Abstract

A novel fuzzy reasoning method called the SQI (symmetric quintuple implicational) method is put forward, which is a generalization of the QIP (quintuple implication principle) method. First of all, the symmetric quintuple implicational principles are presented, which are distinct from the ones of the QIP method. Then unified optimal solutions of the SQI method are obtained for FMP (fuzzy modus ponens) and FMT (fuzzy modus tollens), meanwhile corresponding reversible properties are verified. Furthermore, focusing on the case of multiple rules, optimal solutions of the SQI method are achieved, which involves two general approaches, i.e., FITA (first-infer-then-aggregate) and FATI (first-aggregate-then-infer). Equivalence relation of continuity and interpolation is analyzed for both FITA and FATI under the environment of the SQI method. Finally, one computing example arising in the field of affective computing is given for the SQI method with FATI. It is found that the SQI method preserves the same properties as the QIP method.

Keywords: Fuzzy reasoning, fuzzy implication, compositional rule of inference, quintuple implication principle, symmetric implicational method.

1 Introduction

The research on fuzzy reasoning and fuzzy systems has been widely used in industrial control, image processing and artificial intelligence, and has achieved remarkable success [1], [18], [28], [31]. The basic model of fuzzy reasoning is as follows [3], [6]:

$$\text{If } A \text{ implies } B, \text{ then } A^* \text{ implies } B^*, \quad (1)$$

Fuzzy reasoning is regarded as one of theoretical bases of fuzzy control. The main forms of fuzzy reasoning are divided into the following reasoning schemes, i.e., FMP (fuzzy modus ponens) and FMT (fuzzy modus tollens):

$$\text{FMP: Given the rule } A \rightarrow B \text{ and input } A^*, \text{ calculate the output } B^*, \quad (2)$$

$$\text{FMT: Given the rule } A \rightarrow B \text{ and input } B^*, \text{ calculate the output } A^*. \quad (3)$$

Here $A, A^* \in F(X)$ and $B, B^* \in F(Y)$, where $F(X), F(Y)$ respectively represent the set of all fuzzy subsets of universe X and Y . The classical fuzzy reasoning method is the CRI (Compositional Rule of Inference) algorithm proposed by Zadeh in 1973 [36]. In 1999, Wang [32] proposed the triple I algorithm for FMP and FMT. In [32], Wang proposed that the fact that A^* implies B^* should be taken into account (i.e., $A^* \rightarrow B^*$ should be employed) and should be completely sustained by $A \rightarrow B$. Then the key formula of the triple I algorithm was as follows ($x \in X, y \in Y, A, A^* \in F(X)$, and $B, B^* \in F(Y)$):

$$(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow B^*(y)). \quad (4)$$

The optimal solution of the triple I algorithm to FMP was the smallest B^* such that (4) got its maximum value for any $x \in X, y \in Y$ (where A, B, A^* were given in advance), while the optimal solution of the triple I algorithm to FMT was the largest A^* such that (4) got its maximum value for any $x \in X, y \in Y$ (in which A, B, B^* were provided ahead of time).

A number of research studies focused on the nature of the triple I algorithm, which gradually showed its advantages in strict logic basis, good reduction, point-by-point optimization and so on. Song et al. proposed the triple I restriction algorithm and the reverse triple I algorithm in [23], [26]. Zhang and Yang [37] analyzed the triple I algorithm in four familiar propositional logic systems, which is based upon the generalized root of theory. Pei [22] established a reasonable logical basis for the triple I algorithm from the viewpoint of the logical system with the monoidal t-norm. Zheng et al. [38] revealed a new kind of residual intuitionistic fuzzy implication, and then studied the triple I algorithm under the environment of intuitionistic fuzzy sets. Luo and Zhou [12] proposed $[\alpha, \beta]$ -triple I algorithm founded on interval-valued fuzzy sets, while verified its robustness. Luo and Liu [11] put forward new interval-valued fuzzy connectives, meanwhile analyzed the robustness of triple I algorithm with interval-valued fuzzy sets. Han et al. [4] incorporated the inconsistent bipolarity into the triple I algorithm, and researched the triple I FMT method with inconsistent bipolar information. Wang and Qin [34] put forward the λ -triple I algorithm from the viewpoints of the fuzzy soft modus ponens and fuzzy soft modus tollens, while its robustness was discussed. Li and Liu [8] studied the entire triple I algorithm for double fuzzy control systems and manifold learning of dimensionality reduction.

It is worth mentioning that Zhou et al. [39] presented the quintuple implication principle (QIP) for fuzzy reasoning as an important development of triple I algorithm. Here the QIP solution B^* to the FMP problem is the minimum fuzzy set such that

$$(A(x) \rightarrow B(y)) \rightarrow ((A^*(x) \rightarrow A(x)) \rightarrow (A^*(x) \rightarrow B^*(y))), \quad (5)$$

gets its maximum value. Meanwhile the QIP solution A^* to the FMT problem is the smallest fuzzy set that lets

$$(A(x) \rightarrow B(y)) \rightarrow ((B(y) \rightarrow B^*(y)) \rightarrow (A(x) \rightarrow A^*(x))), \quad (6)$$

take the maximum value. Following that, Luo and Zhou [13], [14] established the logical basis of the QIP reasoning method, and investigated the principle of interval-valued QIP algorithm of fuzzy reasoning. Li and Qin [9] researched the robustness and universal approximation capability of fuzzy system based on the QIP method.

There is a very serious problem with the triple I algorithm, that is, the triple I algorithm is not ideal from the perspective of some kind of fuzzy system on account of its feeblish response ability and practicability (see [5], [7], [10], [19]). Li compared the triple I algorithm with the CRI algorithm, and drew the conclusion that the triple I algorithm was basically weaker than the latter in the category of fuzzy system [7]. Li et al. [10] obtained the fact that 12 fuzzy systems can be used in 23 ones via the CRI algorithm (by discussing their response ability). However, Hou et al. [5] drew the conclusion that only 2 practicable fuzzy systems are obtained in 51 ones on the strength of triple I algorithm, while Pan et al. also gained similar results in [19]. This weakness will hinder the development of the triple I algorithm. To overcome the problem, a natural way of thinking is to generalize the triple I algorithm.

Actually, implication connective is the foundation for a logic system to carry through reasoning. The fuzzy implication in (4) of the triple I algorithm is to build a strong connection between the fuzzy reasoning and the logic system, and hence offer solid logic basis for fuzzy reasoning. The first and third fuzzy implications in (4) relate to the if-then relationship. Moreover, the second fuzzy implication in (4) can be expressed as an implication connective. Based upon such point of view, we [30] proposed the symmetric implication algorithm as a generalization of the triple I algorithm, in which (4) is promoted as follows:

$$(A(x) \rightarrow_1 B(y)) \rightarrow_2 (A^*(x) \rightarrow_1 B^*(y)). \quad (7)$$

A comprehensive study has been carried out around the definition, solution and properties of symmetric implicational algorithms [30]. We [29] extended the sustaining degree to the two-dimensional sustaining degree, and based on it a novel symmetric implicational method was presented and systemically analyzed, and this method contained related symmetric implicational algorithms and triple I algorithms as its special situations.

As far as the QIP method is concerned, the serious problem also exists. To solve such problem, we can generalize it in a similar way. In fact, the first, third, and fifth fuzzy implications in (5) and (6) can be regarded as implication connectives; while the second and fourth fuzzy implications in (5) and (6) reflects “if-then” relationship of the inference model (1). Consequently, here we put forward a new fuzzy reasoning algorithm called the symmetric quintuple implicational (SQI for short) method, in which (5) and (6) are respectively generalized as:

$$(A(x) \rightarrow_1 B(y)) \rightarrow_2 ((A^*(x) \rightarrow_1 A(x)) \rightarrow_2 (A^*(x) \rightarrow_1 B^*(y))), \quad (8)$$

$$(A(x) \rightarrow_1 B(y)) \rightarrow_2 ((B(y) \rightarrow_1 B^*(y)) \rightarrow_2 (A(x) \rightarrow_1 A^*(x))). \quad (9)$$

Here the former is for FMP and the latter is for FMT, meanwhile two fuzzy implication $\rightarrow_1, \rightarrow_2$ are employed. In this study, we investigate the fuzzy reasoning theory of symmetric quintuple implicational method for FMP and FMT.

The innovation points of the SQI method is reflected by the following aspects. To begin with, symmetric implicational idea is introduced into the quintuple implication principle, and new fuzzy reasoning principles are presented which improve the previous ones. In addition, unified structures of optimal solutions of the SQI method are established for FMP, FMT as well as the case of multiple rules. Thereafter, the equivalence relation of continuity and interpolation is revealed for both FITA and FATI under the category of the SQI method. Lastly, the SQI method can generate a wider range of strategies than the QIP method.

The study is organized as follows. The second section provides some preliminaries. In the third and fourth sections, we cover a series of studies on the SQI method for FMP and FMT. In the fifth section, we concentrate on the SQI method for multiple fuzzy rules. In the sixth section, one computing example in the field of affective computing is provided for the SQI method. The seventh section gives some conclusions.

2 Preliminaries

Definition 2.1. [6, 33] Let $\otimes : [0, 1]^2 \rightarrow [0, 1]$ satisfies the conditions: (i) $a \otimes b = b \otimes a$, (ii) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$, (iii) $1 \otimes a = a$, (iv) if $a \leq b$, then $a \otimes c \leq b \otimes c$, thus define \otimes as an t -norm. If \otimes also satisfies $a \otimes \bigvee\{x_i \mid i \in P\} = \bigvee\{a \otimes x_i \mid i \in P\}$ ($a, x_i \in [0, 1], P \neq \emptyset$), then \otimes is said to be the left continuous t -norm.

Definition 2.2. [16] Let \rightarrow be a mapping $[0, 1]^2 \rightarrow [0, 1]$ satisfying

$$0 \rightarrow 0 = 1, 0 \rightarrow 1 = 1, 1 \rightarrow 1 = 1, 1 \rightarrow 0 = 0, \quad (10)$$

then \rightarrow is called a fuzzy implication on $[0, 1]$.

Definition 2.3. [6] Assume that \otimes and \rightarrow are two $[0, 1]^2 \rightarrow [0, 1]$ mappings. Then (\otimes, \rightarrow) is called an adjunction pair, whenever the following adjunction condition holds for any $a, b, c \in [0, 1]$,

$$a \otimes b \leq c \quad \text{iff} \quad b \leq a \rightarrow c. \quad (11)$$

Definition 2.4. [39] The biresiduum operation is defined as ($a, b \in [0, 1]$)

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a). \quad (12)$$

Definition 2.5. [16] A function $\rightarrow : [0, 1]^2 \rightarrow [0, 1]$ is called an R -implication, if there exists a left-continuous t -norm \otimes such that

$$a \rightarrow b = \bigvee\{y \in [0, 1] \mid a \otimes y \leq b\}, \quad a, b \in [0, 1]. \quad (13)$$

In Definition 2.5, we also have

$$a \otimes b = \bigwedge\{x \in [0, 1] \mid b \leq I(a, x)\}, \quad a, b \in [0, 1]. \quad (14)$$

Lemma 2.6. [18, 33] Suppose that \otimes is a left-continuous t -norm on $[0, 1]$, and that \rightarrow is obtained by (13). Then (\otimes, \rightarrow) is an adjunction pair and \rightarrow satisfies these conditions:

- (C1) $a \rightarrow b$ is not reduced with respect to the second variable b ,
- (C2) $a \rightarrow b$ is right continuous with regard to b ,
- (C3) $a \rightarrow b$ does not increase with respect to the first variable a ,
- (C4) $a \leq b$ iff $a \rightarrow b = 1$,
- (C5) $1 \rightarrow a = a$,
- (C6) $a \leq b \rightarrow c$ iff $b \leq a \rightarrow c$,
- (C7) $a \rightarrow (b \rightarrow c)$ iff $b \rightarrow (a \rightarrow c)$,
- (C8) $\bigwedge\{a \rightarrow x_i \mid i \in P\} = a \rightarrow \bigwedge\{x_i \mid i \in P\}$,
- (C9) $\bigwedge\{x_i \rightarrow b \mid i \in P\} = \bigvee\{x_i \mid i \in P \rightarrow b\}$,
- (C10) $a \leftrightarrow a = 1$,
- (C11) $a \leftrightarrow b \leq a \leftrightarrow (b \otimes (b \leftrightarrow a))$,
- (C12) $(a \leftrightarrow b) \otimes (b \leftrightarrow c) \leq a \leftrightarrow c$,
- (C13) $(a \leftrightarrow b) \wedge (c \leftrightarrow d) \leq (a \wedge c) \leftrightarrow (b \wedge d)$,

$$(C14) (a \leftrightarrow b) \vee (c \leftrightarrow d) \leq (a \vee c) \leftrightarrow (b \vee d),$$

$$(C15) (a \leftrightarrow b) \otimes (c \leftrightarrow d) \leq (a \otimes c) \leftrightarrow (b \otimes d),$$

$$(C16) (a \rightarrow b) \otimes (b \rightarrow c) \leq a \rightarrow c,$$

in which $a, b, c, x_i \in [0, 1]$ and $P \neq \emptyset$.

Definition 2.7. [35] For $F(Z)$ on any set Z , a partial order relationship \leq_F is defined as $A \leq_F B \iff A(z_0) \leq_F B(z_0)$ ($A, B \in F(Z)$), where \iff means “if and only if”.

Lemma 2.8. [35] $\langle F(Z), \leq_F \rangle$ is a complete lattice.

3 The symmetric quintuple implicational method for FMP

For the FMP problem (2), being considered from the perspective of the SQI method, we can establish the following principle:

Symmetric quintuple implicational principle for FMP: The conclusion B^* to the FMP problem (2) is the smallest fuzzy set in $\langle F(Y), \leq_F \rangle$ such that (8) gets its maximum value.

Definition 3.1. Let $B \in F(Y)$, $A, A^* \in F(X)$. If B^* in $F(Y)$ is a fuzzy set such that (8) obtains its maximum value, then B^* is called the FMP-symmetric quintuple implicational solution.

Suppose that the maximum of (8) for FMP at each point (x, y) is $M(x, y)$ ($x \in X, y \in Y$). Let \mathbb{E} be the set of all FMP-symmetric quintuple implicational solution.

Notation 3.2. Assume that $\rightarrow_1, \rightarrow_2$ satisfy (C1). For (8), if there exists a FMP-symmetric quintuple implicational solution B^* , then it is easy to know that each fuzzy set D such that $B^* \leq_F D$ ($D \in F(Y)$) shall be a FMP-symmetric quintuple implicational solution. Consequently, many FMP-symmetric quintuple implicational solutions exist, which include $B_0^*(y) \equiv 1$ ($y \in Y$). It is emphasized that B_0 is a specific solution, for which (8) gets its maximum value no matter what $A \rightarrow_1 B$ and A^* are employed. As a result, if the ideal FMP-symmetric quintuple implicational solution exists, it should be the minimum of all FMP-symmetric quintuple implicational solutions.

Theorem 3.3. If $\rightarrow_1, \rightarrow_2$ are the R-implications and \otimes_1, \otimes_2 the left-continuous t-norms in adjunction pairs $(\otimes_1, \rightarrow_1), (\otimes_2, \rightarrow_2)$, then the minimum of all symmetric quintuple implicational solution B^* is as follows:

$$B^*(y) = \vee_{x \in X} \{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A(x)) \otimes_2 (A(x) \rightarrow_1 B(y)))\}, \quad y \in Y. \quad (15)$$

Proof. It is easy to find the maximum of (8) is 1 when $\rightarrow_1, \rightarrow_2$ are the R-implications.

a) To begin with, we verify $B^* \in \mathbb{E}$, that is $(x \in X, y \in Y)$:

$$(A(x) \rightarrow_1 B(y)) \rightarrow_2 ((A^*(x) \rightarrow_1 A(x)) \rightarrow_2 (A^*(x) \rightarrow_1 B^*(y))) = 1. \quad (16)$$

In fact, according to (15), we know $(x \in X, y \in Y)$:

$$A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A(x)) \otimes_2 (A(x) \rightarrow_1 B(y))) \leq B^*(y).$$

Since $(\otimes_1, \rightarrow_1)$ is an adjunction pair, we have $(x \in X, y \in Y)$:

$$(A^*(x) \rightarrow_1 A(x)) \otimes_2 (A(x) \rightarrow_1 B(y)) \leq A^*(x) \rightarrow_1 B^*(y).$$

Note that $(\otimes_2, \rightarrow_2)$ is an adjunction pair, then $(x \in X, y \in Y)$:

$$(A(x) \rightarrow_1 B(y)) \leq (A^*(x) \rightarrow_1 A(x)) \rightarrow_2 (A^*(x) \rightarrow_1 B^*(y)).$$

Because \rightarrow_2 satisfies (C4) (from Lemma 2.6), it can be found that (16) holds. That is, B^* expressed as (15) is an FMP-symmetric quintuple implicational solution.

b) Furthermore, we prove that B^* is the minimum of \mathbb{E} .

Suppose that $C \in \langle F(Y), \leq_F \rangle$ and that $(x \in X, y \in Y)$

$$(A(x) \rightarrow_1 B(y)) \rightarrow_2 ((A^*(x) \rightarrow_1 A(x)) \rightarrow_2 (A^*(x) \rightarrow_1 C(y))) = 1.$$

Note that \rightarrow_2 satisfies (C4) (from Lemma 2.6), one has $(x \in X, y \in Y)$

$$A(x) \rightarrow_1 B(y) \leq (A^*(x) \rightarrow_1 A(x)) \rightarrow_2 (A^*(x) \rightarrow_1 C(y)).$$

Due to $(\otimes_1, \rightarrow_1)$ and $(\otimes_2, \rightarrow_2)$ are two adjunction pairs, we get $(x \in X, y \in Y)$:

$$(A^*(x) \rightarrow_1 A(x)) \otimes_2 (A(x) \rightarrow_1 B(y)) \leq A^*(x) \rightarrow_1 C(y),$$

and thus $(x \in X, y \in Y)$

$$A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A(x)) \otimes_2 (A(x) \rightarrow_1 B(y))) \leq C(y).$$

Then $C(y)$ is an upper bound of the set

$$\{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A(x)) \otimes_2 (A(x) \rightarrow_1 B(y))) \mid x \in X\}, \quad y \in Y,$$

then $B^*(y) \leq C(y)$ ($y \in Y$). So B^* is the minimum of \mathbb{E} .

We can find B^* expressed as (15) is the minimum of all symmetric quintuple implicational solution. \square

Definition 3.4. [33] *As for a fuzzy reasoning algorithm for solving FMP problem, if $A^* = A$ implies $B^* = B$ whenever condition (P) is satisfied, then this method is called P-reversible.*

Theorem 3.5. *Let $\rightarrow_1, \rightarrow_2$ be the R-implications, then the symmetric quintuple implicational method for FMP is P-reversible, where P means that A is a normal fuzzy set.*

Proof. Since \rightarrow_1 is an R-implication, we get \rightarrow_1 satisfies (C4) and (C5) according to Lemma 2.6.

When $A^* = A$, the minimum of all symmetric quintuple implicational solution can be obtained from Theorem 3.3 ($y \in Y$):

$$B^*(y) = \vee_{x \in X} \{A(x) \otimes_1 ((A(x) \rightarrow_1 A(x)) \otimes_2 (A(x) \rightarrow_1 B(y)))\} = \vee_{x \in X} \{A(x) \otimes_1 (A(x) \rightarrow_1 B(y))\}.$$

Here $(\otimes_1, \rightarrow_1)$ and $(\otimes_2, \rightarrow_2)$ are two adjunction pairs, while \rightarrow_1 satisfies (C4) (from Lemma 2.6).

To begin with, we prove $B \leq_F B^*$. Note that A is the normal fuzzy set in $\langle F(X), \leq_F \rangle$, then there is x_0 such that $A(x_0) = 1$. Noting that \rightarrow_1 satisfies (C5) (from Lemma 2.6), then for any $y \in Y$, there is $B^*(y) \geq A(x_0) \otimes_1 (A(x_0) \rightarrow_1 B(y)) = 1 \otimes_1 (1 \rightarrow_1 B(y)) = B(y)$.

Furthermore, we prove $B^* \leq_F B$. Note that $A(x) \rightarrow_1 B(y) \leq A(x) \rightarrow_1 B(y)$ ($x \in X, y \in Y$). According to the adjunction condition (11), we get $A(x) \otimes_1 (A(x) \rightarrow_1 B(y)) \leq B(y)$ ($x \in X, y \in Y$), and thus $B^* \leq_F B$ is obtained.

In summary, $B^*(y) = B(y)$ holds for any $y \in Y$. Therefore, the SQI method for FMP is P-reversible. \square

Example 3.6. *These fuzzy implications are R-implications, which are respectively Lukasiewicz implication \rightarrow_{LK} , Gödel implication \rightarrow_{GD} , Goguen implication \rightarrow_{GG} , Fodor implication \rightarrow_{FD} , \rightarrow_{EP} , and \rightarrow_{YG} .*

$$\begin{aligned} x \rightarrow_{LK} y &= \begin{cases} 1, & x \leq y \\ 1 - x + y, & x > y \end{cases}, & x \rightarrow_{GD} y &= \begin{cases} 1, & x \leq y \\ y, & x > y \end{cases}, \\ x \rightarrow_{GG} y &= \begin{cases} 1, & x \leq y \\ y/x, & x > 0 \end{cases}, & x \rightarrow_{FD} y &= \begin{cases} 1, & x \leq y \\ (1-x) \vee y, & x > y \end{cases}, \\ x \rightarrow_{EP} y &= \begin{cases} 1, & x \leq y \\ (2y - xy)/(x + y - xy), & x > y \end{cases}, & x \rightarrow_{YG} y &= \begin{cases} 1, & x \leq y \\ 1 - (\sqrt{1-x} - \sqrt{1-y})^2, & x > y \end{cases}. \end{aligned}$$

The operations associated with $\rightarrow_{LK}, \rightarrow_{GD}, \rightarrow_{GG}, \rightarrow_{FD}, \rightarrow_{EP}, \rightarrow_{YG}$ are:

$$\begin{aligned} x \otimes_{LK} y &= \begin{cases} x + y - 1, & x + y > 1 \\ 0, & x + y \leq 1 \end{cases}, & x \otimes_{GD} y &= x \wedge y, & x \otimes_{GG} y &= x \times y; \\ x \otimes_{FD} y &= \begin{cases} x \wedge y, & x + y > 1 \\ 0, & x + y \leq 1 \end{cases}, & x \otimes_{EP} y &= xy/(2 - x - y + xy); \\ x \otimes_{YG} y &= \begin{cases} 1 - (k(x, y))^2, & k(x, y) \leq 1 \\ 0, & k(x, y) > 1 \end{cases}, & k(x, y) &= \sqrt{1-x} + \sqrt{1-y}. \end{aligned}$$

Here \otimes_{EP} is the t-norm of Einstein product, and \otimes_{YG} is the t-norm of Yager defined as $\otimes_{YG-\omega}(x, y) = 1 - \min[1, ((1-x)^\omega + (1-y)^\omega)^{1/\omega}]$, where ω is equal to 0.5.

From Theorem 3.3, we can find that if $\rightarrow_1, \rightarrow_2 \in \{\rightarrow_{LK}, \rightarrow_{GD}, \rightarrow_{GG}, \rightarrow_{FD}, \rightarrow_{EP}, \rightarrow_{YG}\}$, then the minimum of all symmetric quintuple implicational solution are expressed as (15).

4 The symmetric quintuple implicational method for FMT

For the FMT problem (3), from the perspective of the SQI method, we can obtain the following principle:

Symmetric quintuple implicational principle for FMT: The conclusion A^* of FMT problem (3) is the smallest fuzzy set in $\langle F(X), \leq_F \rangle$ such that (9) obtains its maximum value.

Definition 4.1. Let $A \in F(X), B, B^* \in F(Y)$. If A^* (in $F(X)$) maximizes (9), then A^* is called the FMT-symmetric quintuple implicational solution.

Suppose that the maximum of (9) for FMT at each point (x, y) is $N(x, y)$ ($x \in X, y \in Y$). Let \mathbb{F} be the set of all FMT-symmetric quintuple implicational solution.

Notation 4.2. Assume that (C1) holds for $\rightarrow_1, \rightarrow_2$. For (9), if there exists a FMT-symmetric quintuple implicational solution A^* , then it is obvious to know that each fuzzy set C such that $A^* \leq_F C$ ($C \in F(X)$) shall be a FMT-symmetric quintuple implicational solution. Therefore, many FMT-symmetric quintuple implicational solutions exist, which incorporate $A_0^*(x) \equiv 1$ ($x \in X$). It is noticed that A_0 is a specific solution, for which (9) gets its maximum value no matter what $A \rightarrow_1 B$ and B^* are employed. Consequently, if the ideal FMT-symmetric quintuple implicational solution exists, it should be the minimum of \mathbb{F} .

Theorem 4.3. If $\rightarrow_1, \rightarrow_2$ are the R-implications and \otimes_1, \otimes_2 the left-continuous t -norms in adjunction pairs $(\otimes_1, \rightarrow_1)$ and $(\otimes_2, \rightarrow_2)$, then the minimum of all symmetric quintuple implicational solution A^* is as follows:

$$A^*(x) = \bigvee_{y \in Y} \{A(x) \otimes_1 ((B(y) \rightarrow_1 B^*(y)) \otimes_2 (A(x) \rightarrow_1 B(y)))\}, \quad x \in X. \quad (17)$$

Proof. It is easy to find $N(x, y) = 1$ ($x \in X, y \in Y$) when $\rightarrow_1, \rightarrow_2$ are the R-implications.

a) First of all, we prove $A^* \in \mathbb{F}$, that is ($x \in X, y \in Y$),

$$(A(x) \rightarrow_1 B(y)) \rightarrow_2 ((B(y) \rightarrow_1 B^*(y)) \rightarrow_2 (A(x) \rightarrow_1 A^*(x))) = 1. \quad (18)$$

Actually, it follows from (17) that we get ($x \in X, y \in Y$):

$$A(x) \otimes_1 ((B(y) \rightarrow_1 B^*(y)) \otimes_2 (A(x) \rightarrow_1 B(y))) \leq A^*(x).$$

Noting that $(\otimes_1, \rightarrow_1)$ is an adjunction pair, we have ($x \in X, y \in Y$),

$$(B(y) \rightarrow_1 B^*(y)) \otimes_2 (A(x) \rightarrow_1 B(y)) \leq A(x) \rightarrow_1 A^*(x).$$

Because $(\otimes_2, \rightarrow_2)$ is an adjunction pair, one has ($x \in X, y \in Y$),

$$A(x) \rightarrow_1 B(y) \leq (B(y) \rightarrow_1 B^*(y)) \rightarrow_2 (A(x) \rightarrow_1 A^*(x)).$$

Since \rightarrow_2 satisfies (C4) (from Lemma 2.6), it can be known that (18) holds. That is, A^* denoted as (17) is an FMT-symmetric quintuple implicational solution.

b) Moreover, we verify that A^* is the minimum of \mathbb{F} .

Suppose that $C \in \langle F(X), \leq_F \rangle$ and that ($x \in X, y \in Y$),

$$(A(x) \rightarrow_1 B(y)) \rightarrow_2 ((B(y) \rightarrow_1 B^*(y)) \rightarrow_2 (A(x) \rightarrow_1 C(x))) = 1.$$

Since \rightarrow_2 satisfies (C4) (from Lemma 2.6), we get ($x \in X, y \in Y$),

$$A(x) \rightarrow_1 B(y) \leq (B(y) \rightarrow_1 B^*(y)) \rightarrow_2 (A(x) \rightarrow_1 C(x)).$$

Since $(\otimes_1, \rightarrow_1)$ and $(\otimes_2, \rightarrow_2)$ are two adjunction pairs, we get ($x \in X, y \in Y$):

$$(B(y) \rightarrow_1 B^*(y)) \otimes_2 (A(x) \rightarrow_1 B(y)) \leq A(x) \rightarrow_1 C(x),$$

and thus ($x \in X, y \in Y$),

$$A(x) \otimes_1 ((B(y) \rightarrow_1 B^*(y)) \otimes_2 (A(x) \rightarrow_1 B(y))) \leq C(x).$$

Then $C(x)$ is an upper bound of the set

$$\{A(x) \otimes_1 ((B(y) \rightarrow_1 B^*(y)) \otimes_2 (A(x) \rightarrow_1 B(y))) \mid y \in Y\}, \quad x \in X,$$

then $A^*(x) \leq C(x)$ ($x \in X$). So A^* is the minimum of \mathbb{F} .

We can find A^* expressed as (17) is the minimum of all symmetric quintuple implicational solution. □

Definition 4.4. [33] *As for a fuzzy reasoning algorithm for solving FMT problem, if $B^* = B$ implies $A^* = A$ whenever condition (P) is satisfied, then this method is called P-reversible.*

Theorem 4.5. *Let $\rightarrow_1, \rightarrow_2$ be the R-implications, then the symmetric quintuple implicational method for FMT is P-reversible, where P means that B is a normal fuzzy set.*

Proof. When $B^* = B$, the minimum of all symmetric quintuple implicational solution A^* can be obtained from Theorem 4.3 ($x \in X$):

$$A^*(x) = \bigvee_{y \in Y} \{A(x) \otimes_1 ((B(y) \rightarrow_1 B(y)) \otimes_2 (A(x) \rightarrow_1 B(y)))\} = \bigvee_{y \in Y} \{A(x) \otimes_1 (A(x) \rightarrow_1 B(y))\}.$$

Here \otimes_1, \otimes_2 are the left-continuous t-norms in adjunction pairs $(\otimes_1, \rightarrow_1)$ and $(\otimes_2, \rightarrow_2)$, and \rightarrow_1 satisfies (C4) (from Lemma 2.6).

Firstly, we verify $A \leq_F A^*$. Considering that B is the normal fuzzy set in $\langle F(Y), \leq_F \rangle$, there is y_0 such that $B(y_0) = 1$. Noting that \rightarrow_1 satisfies (C4) (from Lemma 2.6), then for any $x \in X$, one has $A^*(x) \geq A(x) \otimes_1 (A(x) \rightarrow_1 B(y_0)) = A(x) \otimes_1 (A(x) \rightarrow_1 1) = A(x) \otimes_1 1 = A(x)$.

Secondly, we prove $A^* \leq_F A$. Note that $A(x) \otimes_1 (A(x) \rightarrow_1 B(y)) \leq A(x) \otimes_1 1 = A(x)$ holds ($x \in X, y \in Y$), and thus $A^* \leq_F A$ is obtained. To sum up, $A^*(x) = A(x)$ holds for any $x \in X$. Consequently, the SQI method for FMT is P-reversible. \square

5 Symmetrical quintuple implication method for multiple rules

In practice, one is often faced with multiple rules. That is, FMP (2) is changed into a more complex form:

$$\text{FMP: Given } n \text{ rules } A_i \longrightarrow B_i \text{ and input } A^*, \text{ calculate the output } B^*. \quad (19)$$

Here $A_i, A^* \in F(X)$ and $B_i, B^* \in F(Y)$ ($i = 1, 2, \dots, n$).

In this section, an FMP-symmetric quintuple implicational solution aiming at (19) is also simply called a symmetric quintuple implicational solution.

Suppose that S is a system of n rules $A_i \longrightarrow B_i$, in which $A_i \longrightarrow B_i$ is expressed as the following fuzzy relation:

$$R_i(x, y) = A_i(x) \rightarrow_1 B_i(y). \quad (20)$$

Here \rightarrow_1 is a fuzzy implication.

For multiple rules of system S , there are two general approaches to carry on inference [2], i.e., FITA (First-Infer-Then-Aggregate) and FATI (First-Aggregate-Then-Infer). For the former, FITA gets reasoning result B'_i from R_i and A^* , and then aggregate all B'_i into the formal solution B^* . FATI aggregates R_i into a total relation R in which $R(x, y) = \bigwedge_{i=1}^n R_i(x, y)$ is normally used, and then achieve the formal solution B^* from R and A^* .

CRI employs FATI to carry through inferences. Let R be a fuzzy relation from (20) in S and A^* is an input. The inference result of CRI with FATI is as follows:

$$B^*(y) = \bigvee_{x \in X} \{A^*(x) \otimes_1 R(x, y)\}, \quad y \in Y. \quad (21)$$

Here \otimes_1 is the left-continuous t-norms in an adjunction pair $(\otimes_1, \rightarrow_1)$.

In this section, we put emphasis on the SQI method for FMP with regard to the system S of multiple fuzzy rules. Assume that R is a fuzzy relation from the rules in S and A^* is an input.

Theorem 5.1. *If $\rightarrow_1, \rightarrow_2$ are the R-implications and \otimes_1, \otimes_2 the left-continuous t-norms in adjunction pairs $(\otimes_1, \rightarrow_1)$, $(\otimes_2, \rightarrow_2)$, then the minimum of all symmetric quintuple implicational solution B^* with FITA is as follows:*

$$B^*(y) = \bigvee_{i=1}^n \bigvee_{x \in X} \{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y)))\}, \quad y \in Y. \quad (22)$$

Proof. For $i = 1, 2, \dots, n$, it follows from Theorem 3.3 that the minimum of all symmetric quintuple implicational solutions B_i^* (for A^*, A_i, B_i) is

$$B_i^*(y) = \bigvee_{x \in X} \{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y)))\} \quad y \in Y.$$

Then we show B^* expressed by (22) is what we demand.

a) According to (22), we know $(x \in X, y \in Y, i = 1, 2, \dots, n)$:

$$A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y))) \leq B^*(y).$$

Since $(\otimes_1, \rightarrow_1)$ is an adjunction pair, we have $(x \in X, y \in Y, i = 1, 2, \dots, n)$:

$$(A^*(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y)) \leq A^*(x) \rightarrow_1 B^*(y).$$

Note that $(\otimes_2, \rightarrow_2)$ is an adjunction pair, then $(x \in X, y \in Y, i = 1, 2, \dots, n)$:

$$A_i(x) \rightarrow_1 B_i(y) \leq (A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 B^*(y)).$$

Because \rightarrow_2 satisfies (C4) (from Lemma 2.6), we have $(x \in X, y \in Y, i = 1, 2, \dots, n)$:

$$(A_i(x) \rightarrow_1 B_i(y)) \rightarrow_2 ((A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 B^*(y))) = 1.$$

That is, B^* expressed as (22) is an FMP-symmetric quintuple implicational solution (for all A^*, A_i, B_i).

b) Furthermore, we prove that B^* is the minimum of FMP-symmetric quintuple implicational solutions (for all A^*, A_i, B_i). Suppose that $C \in \langle F(Y), \leq_F \rangle$ and that C lets

$$(A_i(x) \rightarrow_1 B_i(y)) \rightarrow_2 ((A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 C(y))) = 1.$$

hold for any $x \in X, y \in Y, i = 1, 2, \dots, n$. Note that \rightarrow_2 satisfies (C4) (from Lemma 2.6), one has $(x \in X, y \in Y, i = 1, 2, \dots, n)$

$$A_i(x) \rightarrow_1 B_i(y) \leq (A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 C(y)).$$

Due to $(\otimes_1, \rightarrow_1)$ and $(\otimes_2, \rightarrow_2)$ are two adjunction pairs, we get $(x \in X, y \in Y, i = 1, 2, \dots, n)$:

$$(A^*(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y)) \leq A^*(x) \rightarrow_1 C(y),$$

and thus $(x \in X, y \in Y, i = 1, 2, \dots, n)$

$$A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y))) \leq C(y).$$

Then $C(y)$ is an upper bound of the set

$$\{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y))) \mid x \in X, i = 1, 2, \dots, n\}, \quad y \in Y,$$

then $B^*(y) \leq C(y)$ ($y \in Y$).

To sum up, B^* expressed as (22) is the minimum of all symmetric quintuple implicational solutions with FITA. \square

Theorem 5.2. *If $\rightarrow_1, \rightarrow_2$ are the R -implications and \otimes_1, \otimes_2 the left-continuous t -norms in adjunction pairs $(\otimes_1, \rightarrow_1), (\otimes_2, \rightarrow_2)$, then the minimum of all symmetric quintuple implicational solution B^* with FATI is as follows:*

$$B^*(y) = \bigvee_{x \in X} \bigvee_{i=1}^n \{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 R(x, y))\}, \quad y \in Y. \quad (23)$$

Proof. a) According to (23), we know $(x \in X, y \in Y, i = 1, 2, \dots, n)$:

$$A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 R(x, y)) \leq B^*(y).$$

Since $(\otimes_1, \rightarrow_1)$ is an adjunction pair, we have $(x \in X, y \in Y, i = 1, 2, \dots, n)$:

$$(A^*(x) \rightarrow_1 A_i(x)) \otimes_2 R(x, y) \leq A^*(x) \rightarrow_1 B^*(y).$$

Note that $(\otimes_2, \rightarrow_2)$ is an adjunction pair, then $(x \in X, y \in Y, i = 1, 2, \dots, n)$:

$$R(x, y) \leq (A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 B^*(y)).$$

Because \rightarrow_2 satisfies (C4) (from Lemma 2.6), we have $(x \in X, y \in Y, i = 1, 2, \dots, n)$:

$$R(x, y) \rightarrow_2 ((A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 B^*(y))) = 1.$$

That is, B^* expressed as (23) is an FMP-symmetric quintuple implicational solution (for all $A^*, A_i, B_i, R(x, y)$).

b) Furthermore, we prove that B^* is the minimum of FMP-symmetric quintuple implicational solutions (for all $A^*, A_i, B_i, R(x, y)$). Suppose that $C \in \langle F(Y), \leq_F \rangle$ and that C lets

$$R(x, y) \rightarrow_2 ((A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 C(y))) = 1.$$

hold for any $x \in X, y \in Y, i = 1, 2, \dots, n$. Note that \rightarrow_2 satisfies (C4) (from Lemma 2.6), one has ($x \in X, y \in Y, i = 1, 2, \dots, n$):

$$R(x, y) \leq (A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 C(y)).$$

Due to $(\otimes_1, \rightarrow_1)$ and $(\otimes_2, \rightarrow_2)$ are two adjunction pairs, we get ($x \in X, y \in Y, i = 1, 2, \dots, n$):

$$(A^*(x) \rightarrow_1 A_i(x)) \otimes_2 R(x, y) \leq A^*(x) \rightarrow_1 C(y),$$

and thus ($x \in X, y \in Y, i = 1, 2, \dots, n$)

$$A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 R(x, y)) \leq C(y).$$

Then $C(y)$ is an upper bound of the set

$$\{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 R(x, y)) \mid x \in X, i = 1, 2, \dots, n\}, \quad y \in Y,$$

then $B^*(y) \leq C(y)$ ($y \in Y$).

To sum up, B^* expressed as (23) is the minimum of all symmetric quintuple implicational solutions with FATI. \square

We respectively use $\varphi(A^*), \eta(A^*), \rho(A^*)$ to denote these solutions defined in (21), (22) and (23). Here φ, ρ, η are three mappings from $F(X)$ to $F(Y)$, which characterize the structure of fuzzy reasoning and are instances of fuzzy adjoint functions in [27]. η is also called the SQI-FITA operator while ρ the SQI-FATI operator.

Definition 5.3. [27] Suppose that S is the system of rules denoted by (20). A fuzzy adjoint function $\delta : F(X) \rightarrow F(Y)$ is said to be continuous if

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow \delta(A)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)),$$

holds for any $i = 1, \dots, n$ and $A \in F(X)$; and δ is said to be interpolative if $\delta(A_i) = B_i$ ($i = 1, \dots, n$).

In Definition 5.3, $\bigwedge_{y \in Y} (B_i(y) \leftrightarrow \delta(A)(y))$ embodies the similarity degree between B_i and $\delta(A)$, meanwhile $\bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x))$ is the one between A_i and A . The continuity assures that the similarity degree of the outputs should be not less than the one of the inputs.

Lemma 5.4. [24, 27] Suppose that S is the system of fuzzy rules denoted by (20). Then the CRI operator φ (shown in (21)) is continuous if and only if φ is interpolative.

Lemma 5.5. Suppose that S is the system of rules denoted by (20) and that a fuzzy adjoint function δ is continuous and that \rightarrow is an R-implication. Then δ is interpolative.

Proof. Since δ is continuous, then it follows from Definition 5.3 that

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow \delta(A)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)),$$

holds for any $i = 1, 2, \dots, n$ and $A \in F(X)$.

For $A = A_i$ ($i \in \{1, 2, \dots, n\}$), we gain from (C10) of Lemma 2.6 that $A_i(x) \leftrightarrow A_i(x) = 1$ ($x \in X$) and that

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow \delta(A_i)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A_i(x)) = \bigwedge_{x \in X} 1 = 1.$$

Hence

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow \delta(A_i)(y)) = 1.$$

So it follows that

$$B_i(y) \leftrightarrow \delta(A_i)(y) = 1 \quad (y \in Y),$$

that is

$$(B_i(y) \rightarrow \delta(A_i)(y)) \wedge (\delta(A_i)(y) \rightarrow B_i(y)) = 1.$$

One has $B_i(y) \rightarrow \delta(A_i)(y) = 1$ and $\delta(A_i)(y) \rightarrow B_i(y) = 1$ ($y \in Y$). Consequently, it follows from (C4) of Lemma 2.6 that $B_i(y) \leq \delta(A_i)(y)$ and $\delta(A_i)(y) \leq B_i(y)$ ($y \in Y$). Thus $B_i = \delta(A_i)$ holds for any $i \in \{1, 2, \dots, n\}$. As a result, δ is interpolative. \square

Proposition 5.6. *The minimum of all symmetric quintuple implicational solutions with FITA (i.e., $\eta(A^*)$) is the smallest fuzzy set B^* such that $(A_i(x) \rightarrow_1 B_i(y)) \rightarrow_2 ((A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 B^*(y))) = 1$ holds for any $i \in \{1, \dots, n\}$, $x \in X$, $y \in Y$.*

Proof. From Definition 3.1 and the case of (19), for any $B_0 \in F(Y)$, if B_0 makes

$$(A_i(x) \rightarrow_1 B_i(y)) \rightarrow_2 ((A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 B_0(y))) = 1,$$

holds for any $i \in \{1, \dots, n\}$, $x \in X$, $y \in Y$, then B_0 is an FMP-symmetric quintuple implicational solution aiming at (19) with FITA, i.e., a symmetric quintuple implicational solution. Moreover, the minimum of all symmetric quintuple implicational solutions is the smallest fuzzy set B^* such that

$$(A_i(x) \rightarrow_1 B_i(y)) \rightarrow_2 ((A^*(x) \rightarrow_1 A_i(x)) \rightarrow_2 (A^*(x) \rightarrow_1 B^*(y))) = 1,$$

holds for any $i \in \{1, \dots, n\}$, $x \in X$, $y \in Y$. □

As for the SQI-FITA operator η , the following conclusion is obtained.

Theorem 5.7. *Suppose that S is the system of rules denoted by (20) and \leftrightarrow is related to \rightarrow_1 . Then*

i) the SQI-FITA operator η is interpolative if η is continuous;

ii) the SQI-FITA operator η is continuous if η is interpolative and the following property holds:

$$(C17) \quad a \otimes_2 b \geq a \otimes_1 b \quad (a, b \in [0, 1]).$$

Proof. From Lemma 5.5, we can find that if the SQI-FITA operator η is continuous then η is interpolative.

We only demand to verify ii). Assume that η is interpolative and η satisfies (C17). Then $\eta(A_i) = B_i$ for any $i = 1, \dots, n$. Thus, for any $y \in Y, x' \in X$, one has

$$\begin{aligned} B_i(y) \leftrightarrow \eta(A^*)(y) &= \eta(A_i)(y) \leftrightarrow \eta(A^*)(y) && \text{(because } \eta(A_i) = B_i) \\ &= \{\bigvee_{j=1}^n \bigvee_{x \in X} [A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y)))]\} \\ &\leftrightarrow \{\bigvee_{j=1}^n \bigvee_{x \in X} [A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y)))]\} \\ &\geq \bigvee_{j=1}^n \{\bigvee_{x \in X} [(A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y)))]\} \\ &\leftrightarrow \{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y)))]\} && \text{(from (C14))} \\ &\geq \bigvee_{x \in X} \{[A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y)))]\} \\ &\leftrightarrow \{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y)))]\} && \text{(let } j = i) \\ &\geq \{A_i(x') \otimes_1 ((A_i(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y)))]\} \\ &\leftrightarrow \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y)))]\} && \text{(let } x = x') \\ &\geq \{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ &\leftrightarrow \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y)))]\}. && \text{(from (C4))} \end{aligned}$$

Here it can be divided into three phrases.

a) Suppose that $A_i(x') \geq A^*(x')$. Then we can get $A_i(x') \leftrightarrow A^*(x') = A_i(x') \rightarrow_1 A^*(x')$ and $A^*(x') \rightarrow_1 A_i(x') = 1$. We get

$$\begin{aligned} &\{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ &\leftrightarrow \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y)))]\} \\ = &\{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ &\leftrightarrow \{A^*(x') \otimes_1 (1 \otimes_2 (A_i(x') \rightarrow_1 B_i(y)))]\} \\ = &\{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ &\leftrightarrow \{A^*(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ \geq &\{A_i(x') \leftrightarrow A^*(x')\} \\ &\otimes_1 \{(A_i(x') \rightarrow_1 B_i(y)) \leftrightarrow (A_i(x') \rightarrow_1 B_i(y))\} && \text{(from (C15))} \\ = &A_i(x') \leftrightarrow A^*(x'). && \text{(from (C10))} \end{aligned}$$

b) Suppose that $A_i(x') < A^*(x')$ and that

$$A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y)) < A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y))).$$

Then one has $A_i(x') \leftrightarrow A^*(x') = A^*(x') \rightarrow_1 A_i(x')$.

According to $(A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y)) \leq A_i(x') \rightarrow_1 B_i(y)$, we get

$$A^*(x') \otimes_1 [(A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y))] \leq A^*(x') \otimes_1 [A_i(x') \rightarrow_1 B_i(y)].$$

Meanwhile \rightarrow_1 is decreasing with respect to the first variable, then we obtain

$$\begin{aligned} & \{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ & \leftrightarrow \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y)))\} \\ = & \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y)))\} \\ & \rightarrow_1 \{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ \geq & \{A^*(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ & \rightarrow_1 \{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ \geq & \{A^*(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ & \leftrightarrow \{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ \geq & \{A^*(x') \leftrightarrow A_i(x')\} \\ & \otimes_1 \{(A_i(x') \rightarrow_1 B_i(y)) \leftrightarrow (A_i(x') \rightarrow_1 B_i(y))\} && \text{(from (C15))} \\ = & A_i(x') \leftrightarrow A^*(x'). && \text{(from (C10))} \end{aligned}$$

c) Suppose that $A_i(x') < A^*(x')$ and that $A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y)) \geq A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y)))$.

Note that $a \otimes_2 b \geq a \otimes_1 b$ ($a, b \in [0, 1]$), we get

$$A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y))) \geq A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_1 (A_i(x') \rightarrow_1 B_i(y))).$$

Then we obtain

$$\begin{aligned} & \{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ & \leftrightarrow \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y)))\} \\ = & \{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ & \rightarrow_1 \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y)))\} \\ \geq & \{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ & \rightarrow_1 \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_1 (A_i(x') \rightarrow_1 B_i(y)))\} \\ \geq & \{A_i(x') \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ & \leftrightarrow \{[A^*(x') \otimes_1 (A^*(x') \rightarrow_1 A_i(x'))] \otimes_1 (A_i(x') \rightarrow_1 B_i(y))\} \\ \geq & \{A_i(x') \leftrightarrow [A^*(x') \otimes_1 (A^*(x') \rightarrow_1 A_i(x'))]\} \\ & \otimes_1 \{[A_i(x') \rightarrow_1 B_i(y)] \leftrightarrow [A_i(x') \rightarrow_1 B_i(y)]\} \\ \geq & [A_i(x') \leftrightarrow A^*(x')] \otimes_1 1 && \text{(from (C10), (C11))} \\ = & A_i(x') \leftrightarrow A^*(x'). \end{aligned}$$

Since x', y are arbitrary in X, Y respectively, one has $\bigwedge_{y \in Y} (B_i(y) \leftrightarrow \eta(A^*)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A^*(x))$ holds for any $x \in X, y \in Y$. That is, η is continuous. \square

The next proposition shows the equivalent condition for the case that η is interpolative.

Proposition 5.8. *Suppose that S is the system of rules denoted by (20) and that all A_i are normal ($i = 1, \dots, n$). Then η is interpolative if and only if*

$$(A_i(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y)) \leq A_i(x) \rightarrow_1 B_i(y), \quad (24)$$

holds for any $x \in X, y \in Y$ and $i, j \in \{1, \dots, n\}$.

Proof. a) Suppose that η is interpolative. Then $\eta(A_i) = B_i$, i.e., for any $y \in Y$ we have

$$\bigvee_{j=1}^n \bigvee_{x \in X} \{A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y)))\} = B_i(y).$$

For any $i, j \in \{1, \dots, n\}, x \in X, y \in Y$, one has $A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y))) \leq B_i(y)$.

Since $(\otimes_1, \rightarrow_1)$ is an adjunction pair, we get from adjunction condition that $(A_i(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y)) \leq A_i(x) \rightarrow_1 B_i(y)$. That is, (24) holds.

b) Suppose that (24) holds for any $x \in X, y \in Y$ and $i, j \in \{1, \dots, n\}$. Because $(\otimes_1, \rightarrow_1)$ is an adjunction pair, it follows from adjunction condition that $A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y))) \leq B_i(y)$.

It implies that $\bigvee_{j=1}^n \bigvee_{x \in X} \{A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y)))\} \leq B_i(y)$ holds for any $y \in Y$. Thus $\eta(A_i)(y) \leq B_i(y)$ ($y \in Y$).

Moreover, let $x' \in X$ be the normal point of A_i (i.e., $A_i(x') = 1$), one has ($y \in Y$)

$$\begin{aligned} \eta(A_i)(y) &= \bigvee_{j=1}^n \bigvee_{x \in X} \{A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_j(x)) \otimes_2 (A_j(x) \rightarrow_1 B_j(y)))\} \\ &\geq \bigvee_{x \in X} \{A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_i(x)) \otimes_2 (A_i(x) \rightarrow_1 B_i(y)))\} && \text{(let } j = i\text{)} \\ &\geq A_i(x') \otimes_1 ((A_i(x') \rightarrow_1 A_i(x')) \otimes_2 (A_i(x') \rightarrow_1 B_i(y))) && \text{(let } x = x'\text{)} \\ &= 1 \otimes_1 (1 \rightarrow_1 B_i(y)) \\ &= B_i(y). && \text{(from (C5))} \end{aligned}$$

Summarizing the above findings, we have $\eta(A_i) = B_i$ ($i \in \{1, \dots, n\}$). Thus η is interpolative. \square

From Proposition 5.8 and Theorem 5.7, it is easy to establish Proposition 5.9.

Proposition 5.9. *Suppose that S is the system of rules denoted by (20) and that all A_i are normal ($i = 1, \dots, n$). Then η is continuous if (C17) and (24) hold for any $x \in X, y \in Y, i, j \in \{1, \dots, n\}$.*

Theorem 5.10. *Suppose that S is the system of rules denoted by (20) and \leftrightarrow is related to \rightarrow_1 . Then*

- i) the SQI-FATI operator ρ is interpolative if ρ is continuous.
- ii) the SQI-FATI operator ρ is continuous if ρ is interpolative and (C17) holds.

Proof. From Lemma 5.5, we can know that if the SQI-FATI operator ρ is continuous then ρ is interpolative.

We only need to prove ii). Suppose that ρ is interpolative and (C17) holds. Then $\rho(A_i) = B_i$ for any $i = 1, \dots, n$. Thus, for any $y \in Y, x' \in X$, one has

$$\begin{aligned} B_i(y) \leftrightarrow \rho(A^*)(y) &= \rho(A_i)(y) \leftrightarrow \rho(A^*)(y) && \text{(because } \rho(A_i) = B_i\text{)} \\ &= \{\bigvee_{x \in X} \bigvee_{j=1}^n [A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_j(x)) \otimes_2 R(x, y))]\} \\ &\leftrightarrow \{\bigvee_{x \in X} \bigvee_{j=1}^n [A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_j(x)) \otimes_2 R(x, y))]\} \\ &\geq \bigvee_{j=1}^n \{\bigvee_{x \in X} [(A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_j(x)) \otimes_2 R(x, y)))]\} \\ &\leftrightarrow (A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_j(x)) \otimes_2 R(x, y))) && \text{(from (C14))} \\ &\geq \bigvee_{x \in X} \{[A_i(x) \otimes_1 ((A_i(x) \rightarrow_1 A_i(x)) \otimes_2 R(x, y))]\} \\ &\leftrightarrow [A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 R(x, y))] && \text{(let } j = i\text{)} \\ &\geq \{A_i(x') \otimes_1 ((A_i(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y))\} \\ &\leftrightarrow \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y))\} && \text{(let } x = x'\text{)} \\ &= \{A_i(x') \otimes_1 R(x', y)\} \\ &\leftrightarrow \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y))\}. && \text{(from (C4))} \end{aligned}$$

Here it can be divided into three categories.

a) Suppose that $A_i(x') \geq A^*(x')$. Then we get $A_i(x') \leftrightarrow A^*(x') = A_i(x') \rightarrow_1 A^*(x')$ and $A^*(x') \rightarrow_1 A_i(x') = 1$. We have

$$\begin{aligned}
& \{A_i(x') \otimes_1 R(x', y)\} \\
& \leftrightarrow \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y))\} \\
& = \{A_i(x') \otimes_1 R(x', y)\} \\
& \leftrightarrow \{A^*(x') \otimes_1 [1 \otimes_2 R(x', y)]\} \\
& = \{A_i(x') \otimes_1 R(x', y)\} \\
& \leftrightarrow \{A^*(x') \otimes_1 R(x', y)\} \\
& \geq \{A_i(x') \leftrightarrow A^*(x')\} \\
& \quad \otimes_1 \{R(x', y) \leftrightarrow R(x', y)\} \quad (\text{from (C15)}) \\
& = A_i(x') \leftrightarrow A^*(x'). \quad (\text{from (C10)}).
\end{aligned}$$

b) Suppose that $A_i(x') < A^*(x')$ and that $A_i(x') \otimes_1 R(x', y) < A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y))$, then we can get $A_i(x') \leftrightarrow A^*(x') = A^*(x') \rightarrow_1 A_i(x')$. According to $R(x', y) \geq (A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y)$, we can get

$$A^*(x') \otimes_1 [(A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y)] \leq A^*(x') \otimes_1 R(x', y).$$

And \rightarrow_1 is decreasing about the first variable, then we can get

$$\begin{aligned}
& \{A_i(x') \otimes_1 R(x', y)\} \\
& \leftrightarrow \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y))\} \\
& = \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y))\} \\
& \rightarrow_1 \{A_i(x') \otimes_1 R(x', y)\} \\
& \geq \{A^*(x') \otimes_1 R(x', y)\} \\
& \rightarrow_1 \{A_i(x') \otimes_1 R(x', y)\} \\
& = \{A^*(x') \otimes_1 R(x', y)\} \\
& \leftrightarrow \{A_i(x') \otimes_1 R(x', y)\} \\
& \geq \{A_i(x') \leftrightarrow A^*(x')\} \\
& \quad \otimes_1 \{R(x', y) \leftrightarrow R(x', y)\} \quad (\text{from (C15)}) \\
& = A_i(x') \leftrightarrow A^*(x'). \quad (\text{from (C10)}).
\end{aligned}$$

c) Suppose that $A_i(x') < A^*(x')$ and that $A_i(x') \otimes_1 R(x', y) \geq A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y))$, then we can get $A_i(x') \leftrightarrow A^*(x') = A^*(x') \rightarrow_1 A_i(x')$.

Note that $a \otimes_2 b \geq a \otimes_1 b$ ($a, b \in [0, 1]$) and that \rightarrow_1 is increasing about the second variable, we get

$$\begin{aligned}
& \{A_i(x') \otimes_1 R(x', y)\} \\
& \leftrightarrow \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y))\} \\
& = \{A_i(x') \otimes_1 R(x', y)\} \\
& \rightarrow_1 \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_2 R(x', y))\} \\
& \geq \{A_i(x') \otimes_1 R(x', y)\} \\
& \rightarrow_1 \{A^*(x') \otimes_1 ((A^*(x') \rightarrow_1 A_i(x')) \otimes_1 R(x', y))\} \\
& = \{A_i(x') \otimes_1 R(x', y)\} \\
& \leftrightarrow \{A^*(x') \otimes_1 (A^*(x') \rightarrow_1 A_i(x')) \otimes_1 R(x', y)\} \\
& \geq \{A_i(x') \leftrightarrow (A^*(x') \otimes_1 (A^*(x') \rightarrow_1 A_i(x')))\} \\
& \quad \otimes_1 \{R(x', y) \leftrightarrow R(x', y)\} \quad (\text{from (C15)}) \\
& \geq [A_i(x') \leftrightarrow A^*(x')] \otimes_1 1 \quad (\text{from (C10), (C11)}) \\
& = A_i(x') \leftrightarrow A^*(x').
\end{aligned}$$

Note that x', y are arbitrary elements in X, Y respectively, then we get

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow \rho(A^*)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A^*(x))$$

holds for any $x \in X, y \in Y$. Consequently ρ is continuous. \square

6 Example

Affective computing [15], [17] is currently one of the most active research fields. The purpose of affective computing is to build a harmonious human-computer environment by giving computers the ability to recognize, understand, express and adapt to human emotions. As an important part of affective computing, emotion deduction research how to determine proper values of other emotions derived from some basic emotions, which is very useful for building massive emotional corpus and analyzing affect state transitions and so on. Here we give one example of emotion deduction.

Here we carry on a thorough emotion deduction using the symmetric quintuple implicational method with FATI. It is acknowledged that there are eight basic emotions (including expectation, anxiety, sorrow, angry, hate, surprise, joy, love). In these 8 emotions, it is found that the former five emotions possess a close relationship with vigilance (which is a new emotion needed to be explored). To get the value of this new emotion, we establish the emotion deduction system from five basic emotions to vigilance (see Example 6.1 in what follows).

Example 6.1. Suppose that $X = \{x_1, x_2, \dots, x_5\}$ where $x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1.0$, and that $Y = \{y_1\}$ in which $\{y_1 = 0.5\}$. Five rules $A_i \rightarrow B_i$ and input A^* are as follows:

$$\begin{aligned} A_1 &= \{0.9/x_1, 0.3/x_2, 0.3/x_3, 0.1/x_4, 0.7/x_5\}, B_1 = \{0.2/y_1\}, \\ A_2 &= \{0.7/x_1, 0.5/x_2, 0.4/x_3, 0.2/x_4, 0.9/x_5\}, B_2 = \{0.2/y_1\}, \\ A_3 &= \{0.5/x_1, 0.6/x_2, 0.5/x_3, 0.8/x_4, 0.6/x_5\}, B_3 = \{0.4/y_1\}, \\ A_4 &= \{0.4/x_1, 0.9/x_2, 0.8/x_3, 0.5/x_4, 0.3/x_5\}, B_4 = \{0.7/y_1\}, \\ A_5 &= \{0.1/x_1, 1.0/x_2, 0.9/x_3, 0.6/x_4, 0.4/x_5\}, B_5 = \{0.7/y_1\}, \\ A^* &= \{0.3/x_1, 0.2/x_2, 1.0/x_3, 0.4/x_4, 0.5/x_5\}. \end{aligned}$$

This is an example of fuzzy classification. Here 3 classes are $B(y_1) = 0.2, B(y_1) = 0.4, B(y_1) = 0.7$, where the degree of vigilance corresponding to the three cases are weak, medium, and strong, respectively. Let $\rightarrow_1 = \rightarrow_{GD}, \rightarrow_2 = \rightarrow_{LK}$. $(\otimes_1, \rightarrow_1)$ and $(\otimes_2, \rightarrow_2)$ are two adjunction pairs.

We now compute the minimum of all symmetric quintuple implicational solution with FATI.

Note that $R(x, y) = \bigwedge_{k=1}^n (A_k(x) \rightarrow_1 B_k(y))$, we have

$$\begin{aligned} R(x_1, y_1) &= \bigwedge_{k=1}^5 (A_k(x_1) \rightarrow_1 B_k(y_1)) \\ &= (0.9 \rightarrow_1 0.2) \wedge (0.7 \rightarrow_1 0.2) \wedge (0.5 \rightarrow_1 0.4) \wedge (0.4 \rightarrow_1 0.7) \wedge (0.1 \rightarrow_1 0.7) \\ &= 0.2 \wedge 0.2 \wedge 0.4 \wedge 1.0 \wedge 1.0 = 0.2. \end{aligned}$$

In the same way, we get:

$$R(x_2, y_1) = 0.2, R(x_3, y_1) = 0.2, R(x_4, y_1) = 0.4, R(x_5, y_1) = 0.2.$$

Furthermore, we obtain

$$\begin{aligned} &\bigvee_{i=1}^5 \{A^*(x_1) \otimes_1 ((A^*(x_1) \rightarrow_1 A_i(x_1)) \otimes_2 R(x_1, y_1))\} \\ &= \{0.3 \otimes_1 [(0.3 \rightarrow_1 0.9) \otimes_2 0.2]\} \vee \{0.3 \otimes_1 [(0.3 \rightarrow_1 0.7) \otimes_2 0.2]\} \vee \{0.3 \otimes_1 [(0.3 \rightarrow_1 0.5) \otimes_2 0.2]\} \\ &\quad \vee \{0.3 \otimes_1 [(0.3 \rightarrow_1 0.4) \otimes_2 0.2]\} \vee \{0.3 \otimes_1 [(0.3 \rightarrow_1 0.1) \otimes_2 0.2]\} \\ &= 0.2 \vee 0.2 \vee 0.2 \vee 0.2 \vee 0 = 0.2. \end{aligned}$$

Similarly we have

$$\begin{aligned} \bigvee_{i=1}^5 \{A^*(x_2) \otimes_1 ((A^*(x_2) \rightarrow_1 A_i(x_2)) \otimes_2 R(x_2, y_1))\} &= 0.2, \\ \bigvee_{i=1}^5 \{A^*(x_3) \otimes_1 ((A^*(x_3) \rightarrow_1 A_i(x_3)) \otimes_2 R(x_3, y_1))\} &= 0.1, \\ \bigvee_{i=1}^5 \{A^*(x_4) \otimes_1 ((A^*(x_4) \rightarrow_1 A_i(x_4)) \otimes_2 R(x_4, y_1))\} &= 0.4, \\ \bigvee_{i=1}^5 \{A^*(x_5) \otimes_1 ((A^*(x_5) \rightarrow_1 A_i(x_5)) \otimes_2 R(x_5, y_1))\} &= 0.2. \end{aligned}$$

From Theorem 5.2, the minimum of all symmetric quintuple implicational solution B^* with FATI is:

$$B^*(y_1) = \bigvee_{x \in X} \bigvee_{i=1}^5 \{A^*(x) \otimes_1 ((A^*(x) \rightarrow_1 A_i(x)) \otimes_2 R(x, y_1))\} = 0.2 \vee 0.2 \vee 0.1 \vee 0.4 \vee 0.2 = 0.4.$$

We show the main difference between this paper and the QIP method in [39].

Firstly, one fuzzy implication is used in the QIP method with five times. However, two fuzzy implications with symmetric structure (viz., (1, 2, 1, 2, 1) layout) are employed in the SQI method, and such mode considers the connotation of both the logic system and the reasoning model.

Secondly, when there are q fuzzy implications in the scope of discussion, the QIP method can get q kinds of particular inference forms. But, the symmetric quintuple implicational method can generate q^2 kinds, which include the q kinds from the QIP method. For instance, 6 specific fuzzy implications are used in this paper. Then the QIP method can get 6 inference forms, meanwhile the symmetric quintuple implicational method can generate $6 \times 6 = 36$ forms. Consequently, the symmetric quintuple implicational method is able to provide more forms of fuzzy reasoning than the QIP method.

Thirdly, the internal operating mechanism of the SQI method is more complex than the QIP method. Here we only show one example (while many examples can be found). For the QIP method, the QIP-FITA operator is continuous whenever it is interpolative (noting that the proving process of 1 page can verify the conclusion, see Theorem 4 in [39]). But in this paper, Theorem 5.7 pointed out the fact that the SQI-FITA operator is continuous, if it is interpolative and (C17) (i.e., $a \otimes_2 b \geq a \otimes_1 b$, $a, b \in [0, 1]$) demands to be satisfied (noting that we need the proving process of 3 pages to validate the result).

Lastly, the SQI method has larger research space than the QIP method in the future. For example, in this paper, two fuzzy implications are all R-implications. But in fact, we can analyze the status that one is an R-implication and another one is an S-implication. In this way, the research will get more interesting and complicated. Such point will be our future work.

7 Conclusions

The symmetric quintuple implicational method is put forward and investigated, which combines the symmetric implicational idea with the quintuple implication principle. The main contributions and resulting conclusions are outlined as follows.

- i) The symmetric quintuple implicational method for FMP and FMT are discovered, including the following four parts:
 - a) The symmetric quintuple implicational principles are presented, which are distinct from the previous ones.
 - b) Uniform forms of optimal solutions of the symmetric quintuple implicational method are achieved for FMP and FMT, in which $\rightarrow_1, \rightarrow_2$ employ R-implications.
 - c) The reversible properties of the symmetric quintuple implicational method are verified.
- ii) For the case of multiple rules, the optimal solutions of the symmetric quintuple implicational method are established, which includes two general approaches, i.e., FITA and FATI. Equivalence relation of continuity and interpolation is analyzed for both FITA and FATI under the environment of the symmetric quintuple implicational method.
- iii) One specific computing examples in the field of affective computing is provided for the symmetric quintuple implicational method with FATI. By analyses, the symmetric quintuple implicational method preserves the same properties as the QIP method.

For the next work, it is valuable to study the symmetric quintuple implicational method from the perspective of granular computing (see [20], [21], [25]). What is more, the fuzzy controllers via the symmetric quintuple implicational method could be constructed and analyzed in future studies.

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