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Multidimensional interval type 2 epistemic fuzzy arithmetic

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Abstract

The article presents a new, multidimensional arithmetic of type 2 fuzzy numbers (M-IT2-F arithmetic) in which the result is a multidimensional fuzzy set. This arithmetic increases the accuracy of calculations and the scope of problems solved in relation to the currently used interval type 2 standard fuzzy arithmetic (IT2-SF arithmetic). The proposed M-IT2-F arithmetic has mathematical properties that IT2-SF arithmetic does not have. Thanks to these properties, it provides accurate calculation results that are not over- or under-estimated in terms of uncertainty. The paper contains comparisons of both types of arithmetic in application to two problems. Fuzzy arithmetic is not a finished work and is in a phase of continuous improvement and development. M-IT2-F arithmetic is a higher form of M-IT2 (non-fuzzy) arithmetic.

Keywords: Fuzzy arithmetic, epistemic fuzzy arithmetic, multidimensional fuzzy arithmetic, fuzzy arithmetic type 2, arithmetic of interval type 2 fuzzy numbers, granular computing, soft computing, uncertainty analysis, artificial intelligence.

1 Introduction

Since the creation of the first uncertain number arithmetic (Warmus, Sunaga and Moore's interval arithmetic) scientific work is underway to improve it. Scientists are introducing increasingly better forms of uncertainty description. This article is part of this trend of research and proposes a new type of arithmetic for fuzzy interval numbers, which are a specific form of type 2 fuzzy sets. Scientists are still looking for better techniques for performing arithmetic operations on type 2 fuzzy numbers, e.g. [5, 6, 30, 32, 46].

Concept of type 2 fuzzy sets (T2FSs) as extension of type 1 fuzzy sets (T1FSs) was introduced by Zadeh [45]. Fuzzy community quickly noticed advantages of T2FSs. J. Mendel [14, 25] proposed interesting application of T2FSs to linguistic values in problems of Computing with Words. To describe these values representing evaluations of only one expert T1FSs are used. However, T1FSs are not recommended in the case when many evaluations of one and the same uncertain variable value are at disposal, because each expert can understand the considered linguistic value more or less differently, e.g. young age. If the understanding differences occur, then the linguistic value should be modeled not by one, but by a set of membership functions (MFs) of type 1. This set can be aggregated in one T2FS with uncertain MF. Apart from Computing with Words, T2FSs have proved usefulness in control problems [12], decision-making [7, 16, 42], transport and supplier selection problems [43], medicine problems [10], inventory management [28], electrical problems [33], pulp washing processes [33] etc. Generally, T2FSs can be used in the same problems in which T1FSs are used if they allow for more precise description of uncertainty. In practical examples the standard IT2FA is most often used. In this arithmetic the 2D IT2F number is considered.

The question is why T2FSs are more and more often used in practice, even though their mathematical models are more complicated than the models of T1FSs? The reason is that they provide more knowledge about the uncertain variable value than T1FSs. This allows you to increase the efficiency of real technical systems using T2FSs. The least

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knowledge of the uncertain value of the variable x is provided by the interval set (T0FS), more the usual fuzzy set T1FS, and even more T2FS.

The motivation of the article is to make progress in the sense of computation accuracy and increase the number of solvable problems to the current interval type 2 fuzzy arithmetic (IT2FA) by basing it not only on the standard 2-dimensional fuzzy arithmetic but also on the multidimensional fuzzy arithmetic.

An analysis of the literature on the applications of T2FSs shows that in practice IT2Fs are the most commonly used and that their popularity is not decreasing, but increasing (there are some reasons for this). The application examples on the one hand show that IT2FSs can be used for any uncertainty problems and on the other hand that they are badly needed. This is evidenced by such examples of applications as the application to fuzzy optimization [32], to control the motion of a robot [46], to control the motion of a vehicle [16, 42], to choose the best suppliers [43], to inventory management [28]. In practical examples, the IT2FA standard is most often used, which proves that the T2FA currently being developed should not be too complicated if we want it to be applied in practice. This arithmetic makes calculations only with borders of fuzzy sets. IT2-SF arithmetic solves some problems thoroughly, some inaccurately, and cannot solve some problems at all. The reason for this is that in this arithmetic the 2D IT2F number is considered the main and only result. However, with this assumption, the number of problems that can be solved exactly is limited. To increase this number, moving to a larger dimension is necessary. This is the M-IT2-F arithmetic option. It has much more needed mathematical properties, which will be shown in the article. The multidimensional fuzzy arithmetic type 1 and its applications, since 2012, has been presented in over 40 publications, of which only a part is quoted in References. This arithmetic is used not only by the research team from West Pomeranian University of Technology (Szczecin, Poland) but also by foreign scientists, e.g. [2, 3, 20, 21, 22, 23, 34, 35, 36]. The most important applications were in the control and modeling of uncertain dynamic systems [21, 22, 34], decision making in uncertainty conditions, e.g. [2], number theory Z [3], computing with words [40], and others. One disadvantage of M-IT2-F arithmetic over IT2-SF arithmetic is the greater complexity of calculations. However, according to the principle of non-existence of "free lunch" it is the cost you have to pay for greater accuracy and the ability to solve more problems. M-IT2-F arithmetic is a higher form of M-IT2 (non-fuzzy) arithmetic [37].

Fuzzy arithmetic is currently differentiated [18] into epistemic and ontic set arithmetic. The two arithmetic differ, as do the concepts of the two sets. This article deals with fuzzy epistemic arithmetic. The definitions of the epistemic and ontic set will be provided below to help you understand the meaning of the arithmetic presented in this article.

The difference between the epistemic set and ontic set has recently been well explained in [18].

Definition 1.1. (An epistemic set [18]) *A set representing an incomplete information about a single unknown object is called disjunctive or epistemic.*

Example of an epistemic set is interval $[1.7, 1.9]$ m informing about the height of a scientist who we are to meet at the airport. However, the height of this scientist at the time of the meeting is not an interval. At this time, only one height value will be true, e.g. $1.81 \ldots m$. The second example is the volume of water in a cylindrical tank with a slightly distorted shape. Based on the measurement of the inside and outside diameter and height of the tank, we determined that the real water volume $V[m^3]$ is in the range $[3.1, 3.9]$. This interval is an epistemic set.

Definition 1.2. (An ontic set [18]) *A set representing a population of items or a collection of elements forming a composite object is called conjunctive or ontic.*

An example of an ontic set is the interval $[24, 39]$ informing about the price of an ounce of gold over a one-year period expressed in the currency of a certain country. This interval is an ontic set, because each of the numerical values of the interval actually occurred. Another example is the water flow rate $[21, 23]m^3/h$ of a river during the day. Each of the values of the flow rate occurred, so it is true. We are dealing here not with one true value but with an infinite set of true values.

Organization of the paper is as follows: in Section 2 basic notions connected with fuzzy numbers type 2 (FNT2) and the mostly used IT2FNs standard arithmetic (IT2SF arithmetic) with its weak-points are presented. Section 3 contains short description of multidimensional T1 arithmetic and its mathematical properties. This arithmetic is used as basis of the multidimensional interval type 2 fuzzy arithmetic (M-IT2-F arithmetic). Section 4 presents concept of M-IT2-F arithmetic. In Section 5 the basic properties of the M-IT2-F arithmetic are presented. Section 6 shows examples of applications of M-IT2-F arithmetic with comparison of the achieved results with results delivered by IT2-SF arithmetic. Section 7 presents conclusions of investigations.

2 Preliminaries

2.1 Basic definitions

Definition 2.1. [44] *A T1FS A on the universe of discourse X can be characterized by its membership function* $\mu_A(x)$ *and represented as follows:*

$$A = \{(x, \mu_A(x)) | \forall x \in X, \mu_A(x) \in [0, 1]\}. \tag{1}$$

Fuzzy sets of type 2 (T2FSs) were introduced by Zadeh in 1975 [45]. Since the original definition was a little vague some scientists introduced their own interpretations and changes. It led to some confusion in T2FSs. Therefore J.M. Mendel, M.R. Rajati and P. Sussner proposed ordering of this definition [26]. The new, ordered definition of a T2FS is given on the basis of [26, 24].

In the definitions X is the universe of discourse for primary variable x and the U for the secondary variable $u \in [0, 1]$. A T2FS denoted $\tilde{A}$ is characterized by a type-2 membership function (T2MF) $\mu_{\tilde{A}}(x, u)$, where $x \in X$ and $u \in U$, can be expressed as in (2).

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | x \in X, u \in [0, 1]\}. \tag{2}$$

It should be noted that in equation (2) it does not occur the primary membership J_x, which is expressed by the formula (3).

$$J_x = \{(x, u) | u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0\}. \tag{3}$$

However, although J_x does not already appear in the definition (2), it can be used in some cases if needed [26]. The set $\tilde{A}$ can also be expressed [24] in special fuzzy set notation as in (4).

$$\tilde{A} = \int_{x \in X} \int_{u \in [0,1]} \mu_{\tilde{A}}(x, u)/(x, u). \tag{4}$$

The symbol $\int$ denotes in (4) union over all admissible values of x and u. There is also the concept of secondary MF [1].

Definition 2.2. *A secondary MF* $\mu_{\tilde{A}(x)}(u)$*, is a restriction of function* $\mu_{\tilde{A}} : X \times [0, 1] \to [0, 1]$ *to* $x \in X$*, i.e.,* $\mu_{\tilde{A}(x)} : [0, 1] \to [0, 1]$*, or in fuzzy set notation:*

$$\mu_{\tilde{A}(x)}(u) = \int_{u \in [0,1]} \mu_{\tilde{A}}(x, u)/u = \int_{u \in [0,1]} f_x(u)/u. \tag{5}$$

$\tilde{A}(x)$ is a T1FS which is also referred to as a secondary set, and as such it can be represented by its α-cut decomposition, $\{\mu_{\tilde{A}(x)}(u) | u \in U\}$ is also called a vertical slice of $\mu_{\tilde{A}}(x, u)$.

The concept of footprint of uncertainty (FOU) can also be used further, (6).

$$FOU(\tilde{A}) = \{(x, u) | x \in X \text{ and } u \in [\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]\}. \tag{6}$$

In (6) $\underline{\mu}_{\tilde{A}}(x)$ means lower membership function (LMF) and $\overline{\mu}_{\tilde{A}}(x)$ means upper membership function (UMF). The definitions of these functions according to [26] are given by (7).

$$\underline{\mu}_{\tilde{A}}(x) = \sup\{u | u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0\}, \;\; \overline{\mu}_{\tilde{A}}(x) = \inf\{u | u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0\}. \tag{7}$$

Several notions of the definition of T2FS are illustrated in Figure1a.

Because in this paper IT2 arithmetic is considered therefore definitions of interval T2FS (IT2FS) is necessary.

Definition 2.3. [25] *Interval T2FSs (IT2FSs) are special case of T2FSs where all secondary membership grades* $\mu_{\tilde{A}}(x, u) = 1$*, (8).*

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u) = 1) | \forall x \in X, \forall u \in J_x \subseteq [0, 1]\}. \tag{8}$$

Definition 2.4. [25] *An IT2FS* $\tilde{A}$ *on the universe of discourse X can be characterized by its upper membership function (UMF) and lower membership function (LMF), and can be denoted as follows:*

$$\tilde{A} = \{[\mu_{A^U}, \mu_{A^L}] | x \in X\}, \tag{9}$$

where A^U *is called the upper T1FS whose membership function* $\mu_{A^U} = \max_{x \in X} J_x$ *and* A^L *is called the lower T1FS whose membership function* $\mu_{A^L} = \min_{x \in X} J_x$*.*

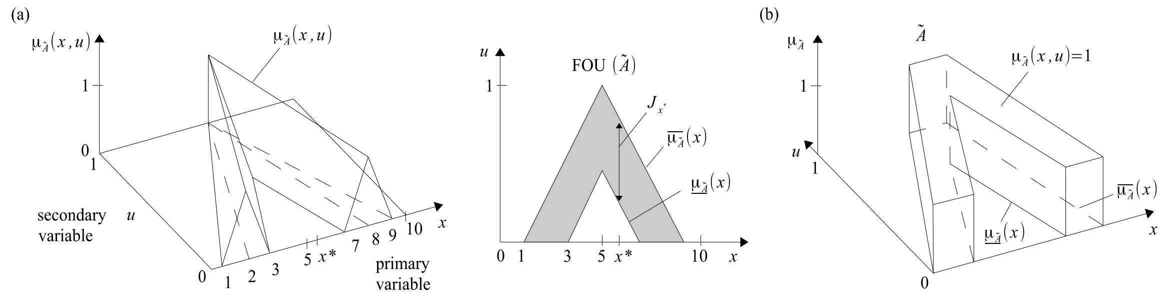

Figure 1: (a) A T2FS $\tilde{A}$ bounded by two triangular functions and its footprint of uncertainty $FOU(\tilde{A})$. J_{x*} - subinterval of uncertain primary membership, $\underline{\mu}_{\tilde{A}}(x)$, $\overline{\mu}_{\tilde{A}}(x)$ lower and upper membership functions (MFs); (b) Example of an interval type 2 fuzzy set with triangular lower MF and trapezoidal upper MF.

Visualization of the IT2FS example in 3D space is shown in Figure 1b. Figures 1a and 1b clearly explain the difference between the general T2FS and the interval T2FS, Because in IT2FSs the secondary MF $\mu_{\tilde{A}}(x,u)=1$ arithmetic and logical operations can be made with use of lower $\underline{\mu}_{\tilde{A}}(x)$ and upper $\overline{\mu}_{\tilde{A}}(x)$ T1MFs. Hence definition of the type 1 fuzzy number is necessary.

Definition 2.5. [25] *Fuzzy numbers (FNs) are special case of fuzzy sets. A FN is a fuzzy set that is bounded, convex, and its universe of discourse is the set of real numbers* $\mathbb{R}$. *Trapezoidal and triangular FNs are frequently used FNs. A triangular FN is special case of trapezoidal FNs with infinitely small core. To distinguish this type of FNs defined in terms of T1FSs from T2FSs they will be called type 1 trapezoidal FNs (trapezoidal T1FNs). A trapezoidal T1FN, denoted by* $A=(a_1,a_2,a_3,a_4,h(A))$ *is characterized by MF (10).*

$$\mu(a)=\begin{cases} \frac{a-a_1}{a_2-a_1}h(A), & a_1\le a<a_2\\ h(A), & a_2\le a<a_3\\ \frac{a_4-a}{a_4-a_3}h(A), & a_3\le a\le a_4\\ 0, & otherwise\end{cases} \tag{10}$$

where $0<h(A)\le 1$ *is the height of the trapezoidal T1FN A.*

If $h(A)=1$, then A is called a normal FN. If $h(A)<1$, then A is called a non-normal FN. If $a_1\ge 0$, then A is called a non-negative trapezoidal FN. If $a_2=a_3$ and $a_3=a_4$, then A is an interval FN. If $a_1=a_2=a_3=a_4$, then A is a crisp number. Figure 2 shows all cases of a normal trapezoidal T1FN.

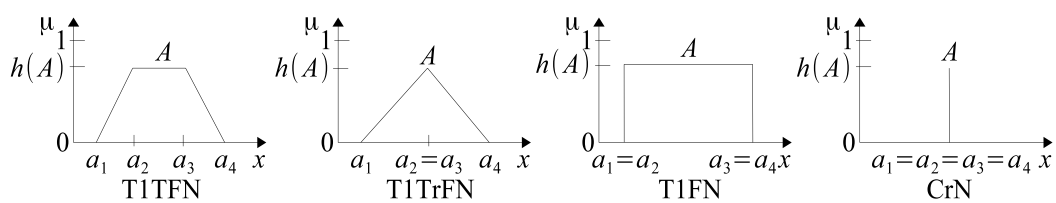

Figure 2: Trapezoidal fuzzy number type 1 and its special cases.

Trapezoidal FNs and their special cases are most often used because of the easy identification of their parameters and the ease of their mathematical modeling.

An example of how T2FNs originate is identification of a single expert opinion in the case when the expert is uncertain of parameters (a_1,a_2,a_3,a_4) of his/her evaluation. This uncertainty also is well characterized by T2FN. Apart from triangular and trapezoidal T2FNs also other types as type 2, elliptic FNs have been proposed [15].

There are several ways to identify T2FSs. Some of them are given, for example, in [29]. The most commonly used FNs type 1 and type 2 are trapezoidal ones. Standard T1 fuzzy arithmetic is usually used for trapezoidal T1FNs. It does not differentiate between epistemic and ontic sets.

Definition 2.6. [8, 17] *(Standard arithmetic of trapezoidal T1FNs) Let us assume that* $A=(a_1,a_2,a_3,a_4,h(A))$ *and* $B=(b_1,b_2,b_3,b_4,h(B))$ *are two arbitrary, non-negative, trapezoidal T1FNs, then the addition and multiplication operations for them are defined as follows:*

1. Addition: $A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, \min\{h(A), h(B)\})$
2. Multiplication: $AB = (a_1b_1, a_2b_2, a_3b_3, a_4b_4, \min\{h(A), h(B)\})$

In a similar way standard fuzzy interval arithmetic (IT2-SF arithmetic) operations are performed. Figure 3 shows denotations of trapezoidal IT2FN and Definition 2.7 defines the lower and upper membership function of this number.

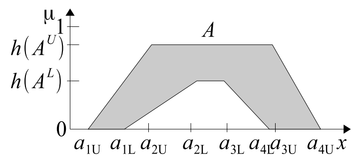

Figure 3: A trapezoidal IT2TFN $\tilde{A} = [A^U, A^L] = [a_{1U}, a_{2U}, a_{3U}, a_{4U}, h(A^U); a_{1L}, a_{2L}, a_{3L}, a_{4L}, h(A^L)]$.

Definition 2.7. [8] *(Membership functions of trapezoidal IT2FNs) Let $\tilde{A} = [A^U, A^L] = [a_{1U}, a_{2U}, a_{3U}, a_{4U}, h(A^U); a_{1L}, a_{2L}, a_{3L}, a_{4L}, h(A^L)]$ be an IT2Fs on the real set of real numbers $\mathbb{R}$. A is an interval type 2 FN if its upper MF $\mu_{A^U}(x)$ and its lower MF $\mu_{A^L}(x)$ are defined as below:*

$$\mu_{A^U}(x) = \begin{cases} h(A^U)\frac{x-a_{1U}}{a_{2U}-a_{1U}}, & a_{1U} \le x < a_{2U} \\ h(A^U), & a_{2U} \le x < a_{3U} \\ h(A^U)\frac{a_{4U}-x}{a_{4U}-a_{3U}}, & a_{3U} \le x \le a_{4U} \\ 0, & otherwise \end{cases}, \quad \mu_{A^L}(x) = \begin{cases} h(A^L)\frac{x-a_{1L}}{a_{2L}-a_{1L}}, & a_{1L} \le x < a_{2L} \\ h(A^L), & a_{2L} \le x < a_{3L} \\ h(A^L)\frac{a_{4L}-x}{a_{4L}-a_{3L}}, & a_{3L} \le x \le a_{4L} \\ 0, & otherwise \end{cases} \tag{11}$$

Definition 2.8 describes how to perform the main arithmetic operations of IT2-SF arithmetic.

Definition 2.8. [7] *(Standard arithmetic of trapezoidal IT2TFNs) Let $\tilde{A}_1 = [A_1^U, A_1^L] = [a_{11U}, a_{12U}, a_{13U}, a_{14U}, h(A_1^U); a_{11L}, a_{12L}, a_{13L}, a_{14L}, h(A_1^L)]$ and $\tilde{A}_2 = [A_2^U, A_2^L] = [a_{21U}, a_{22U}, a_{23U}, a_{24U}, h(A_2^U); a_{21L}, a_{22L}, a_{23L}, a_{24L}, h(A_2^L)]$ be two positive trapezoidal IT2TFNs, then the addition and multiplication operations on them are defined as follows:*
1. Addition: $\tilde{A}_1 + \tilde{A}_2 = [a_{11U} + a_{21U}, a_{12U} + a_{22U}, a_{13U} + a_{23U}, a_{14U} + a_{24U}, \min\{h(A_1^U), h(A_2^U)\}; a_{11L} + a_{21L}, a_{12L} + a_{22L}, a_{13L} + a_{23L}, a_{14L} + a_{24L}, \min\{h(A_1^L), h(A_2^L)\}]$.
2. Multiplication: $\tilde{A}_1\tilde{A}_2 = [a_{11U}a_{21U}, a_{12U}a_{22U}, a_{13U}a_{23U}, a_{14U}a_{24U}, \min\{h(A_1^U), h(A_2^U)\}; a_{11L}a_{21L}, a_{12L}a_{22L}, a_{13L}a_{23L}, a_{14L}a_{24L}, \min\{h(A_1^L), h(A_2^L)\}]$.

2.2 Positives and negatives of the standard IT2F-arithmetic

Authors' comment to Definition 2.8. Standard arithmetic of IT2TFNs is based on standard fuzzy arithmetic of T1FNs, and in the sense of μ-cuts approach on standard interval arithmetic (SIA). Hence the standard IT2TFNs arithmetic possess the same advantages and disadvantages as standard interval and standard fuzzy arithmetic (SFA). Its advantage is simplicity of calculations, (apparent) easiness of understanding and intuitiveness. In some problems of simpler nature the standard FA delivers correct results. However, in more complicated problems this arithmetic gives more or less incorrect and sometimes even paradoxical results. But many scientists using it are not conscious of its inaccuracy. Disadvantages of the standard fuzzy and interval arithmetic has been described in many publications, e.g. in [11, 21, 31, 39].

IT2-SF arithmetic is based on standard interval arithmetic in sense of horizontal μ-cuts and therefore it does not possess very important mathematical properties which are essential and necessary for solving more complicated problems. A few of them are mentioned below.

1. IT2-SF arithmetic does not differentiate between epistemic and ontic sets. Operations $\{+, -, \times, /\}$ are the same for both types of sets.

2. Anonymity of arithmetic operations performed. If two IT2-FNs have identical values of the parameters a_{iL} and a_{jL}, $i, j \in \{1, 2, 3, 4\}$, see Figure 3, then arithmetic operations performed on them by IT2-SF arithmetic always give one and the same result. Meanwhile, these two IT2FNs may in one case represent two different values and in the other case 2 identical variable values. Accordingly, the results of arithmetic operations should be different instead of equal. Therefore, fuzzy numbers should not be anonymous, as is the case with classical arithmetic of crisp numbers.

3. Lack of inverse element $-A$ of addition satisfying condition $A - A = 0$.

4. Lack of inverse element $(1/A)$ of multiplication satisfying $A \cdot (1/A) = 1$.

5. Not holding the sub-distributive law $A(B + C) = AB + AC$ enabling formula transformation.

6. Not holding the multiplicative cancellation law $(AC = BC) \Rightarrow A = B$ necessary for solution derivation.

7. Lack of property of algebraicity and universality of results. For example, if we solve the equation $A + X = B$, where A and B are known and X is unknown, the calculated result of X should be universal. This means that it should satisfy not only the original equation $A + X = B$ but also all other forms of this equation: $A = B - X$, $X = B - A$, $A + X - B = 0$. The requirement of result universality applies to all arithmetic operations performed by any given type of fuzzy arithmetic.

Lack of properties 1–7 causes that e.g. equations cannot be transformed from one to another form, which is necessary for problems' solving. Lack of these properties is also reason of achievement of over- or underestimated and even paradoxical, unacceptable calculation results. Instead, M-IT2-F arithmetic possess P1-P6 properties and thanks to them it delivers precise (in the sense of span), not over and not underestimated results, which also are not burdened by phenomenon of increasing entropy [11].

Disadvantages of the SFA and standard interval arithmetic (SIA) result mainly from assumption (which can be a surprise) that the direct result of arithmetic operations on intervals is also an interval, the direct result of operations on fuzzy numbers type 1 is also a fuzzy number type 1, and the direct result of operations on T2FNs also is an T2FN (the same mathematical object). This is called closure property (requirement). Unfortunately, it is not true. Direct results of operations on the above objects describing uncertain values are multidimensional granules. Intervals, T1FNs and T2FNs are only simplified, low-dimensional indicators of these granules. Hence, at most, they can be called secondary results. Further on, multidimensional RDM arithmetic of T1TFNs and T2TFNs will be presented. They are based on horizontal membership functions (HMFs) and do not possess disadvantages of standard arithmetic versions. It should be added , that apart from the IT2-SF arithmetic also other types of arithmetic of type 2 FNs have been proposed, e.g. [9, 13]. However, the mostly used and dominant type is IT2-SF arithmetic.

3 Short introduction in multidimensional arithmetic of T1FNs (M-T1TFN arithmetic)

The multidimensional approach to F-arithmetic is based on observation that problems of this arithmetic cannot be solved in low-D space. They have to be solved in multi-D space.

Information concerning the concept of this arithmetic can be found e.g. in [38, 40, 41], and examples of its application in [2, 3, 21, 22, 33, 34, 40, 41]. Information about multidimensional interval arithmetic can be found e.g. in [38, 39].

Definition 3.1. *Horizontal (inverse), trapezoidal membership function type 1 (T1HTMF) is formulated by (12). RDM means here Relative-Distance-Measure and is mostly denoted by $\alpha \in [0, 1]$. T1HTMF is mathematical model of the true and possible single value (epistemic case) of uncertain variable x in conditions where our knowledge about this values is expressed in form of T1HTMF of height $h(A)$, Figure 2.*

$$x = [a_1 + (a_2 - a_1)\mu/h(A)] + [(a_4 - a_1) - \mu/h(A)(a_4 - a_3 + a_2 - a_1)]\alpha_x,\ \mu \in [0, h(A)], \alpha_x \in [0, 1]. \tag{12}$$

If the MF is normal then $\mu \in [0, 1]$.

Definition 3.2. *(M-T1TFN arithmetic) Let $A_1 = (a_{11}, a_{12}.a_{13}, a_{14}, h(A_1))$ and $A_2 = (a_{21}, a_{22}, a_{23}, a_{24}, h(A_2))$ are two arbitrary T1TFNs. Then addition and multiplication of them are defined as follows:*
1. Addition:

$$\begin{aligned}
&Z : z \in Z,\ \ z = x_{A1} + x_{A2}\\
&x_{A1} = [a_{11} + (a_{12} - a_{11})\mu/h(A_1)] + [(a_{14} - a_{11}) - \mu/h(A_1)(a_{14} - a_{13} + a_{12} - a_{11})]\alpha_{xA1}\\
&x_{A2} = [a_{21} + (a_{22} - a_{21})\mu/h(A_2)] + [(a_{24} - a_{21}) - \mu/h(A_2)(a_{24} - a_{23} + a_{22} - a_{21})]\alpha_{xA2}\\
&\mu \in [0, \min\{h(A_1), h(A_2)\}], \alpha_{xA1}, \alpha_{xA2} \in [0, 1].
\end{aligned} \tag{13}$$

If we have additional knowledge about real values of x_{A1} and x_{A2}, e.g. $x_{A2} \geq x_{A1}$ then this knowledge can be taken into account in the arithmetic operation. If MFs μ_{A1}, μ_{A2} are normal then the condition $\mu \in [0, \min\{h(A_1), h(A_2)\}]$ takes the usual form $\mu \in [0, 1]$.
2. Multiplication:

$$Z : z \in Z,\ \ z = x_{A1} \cdot x_{A2}. \tag{14}$$

Sense of x_{A1} and x_{A2} is the same as in the addition operation. Special case of multiplication is division $z = x_{A1} \cdot (1/x_{A2})$. If the support of A_2 contains zero, then the division is of multi-granular character. This case has been described in [41].

Example 3.3. *Addition of two T1TFNs with M-T1TFN arithmetic.*

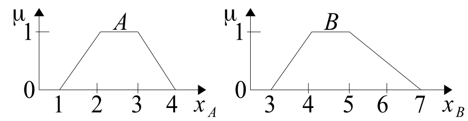

Figure 4: Membership functions $\mu_A(x_A)$ and $\mu_B(x_B)$ of uncertain values x_A and x_B added in the Example 3.3.

Let us assume that two uncertain numbers x_A and x_B should be added and that they are not known precisely. Our knowledge about them is only approximate and is expressed in form of normal T1TFNs shown in Figure 4, $x_A \in A = (1,2,3,4,1)$, $x_B \in B = (3,4,5,7,1)$.

Horizontal (inverse) MFs from Example 3.3 in terms of M-T1TFN-arithmetic are expressed by (15).

$$x_A = (1+\mu) + (3-2\mu)\alpha_{xA};\ x_B = (3+\mu) + (4-3\mu)\alpha_{xB};\ \mu, \alpha_{xA}, \alpha_{xB} \in [0,1]. \tag{15}$$

Addition result $z = x_A + x_B$ is given by (16)

$$z = x_A + x_B;\ z = (4+2\mu) + (3-2\mu)\alpha_{xA} + (4-3\mu)\alpha_{xB};\ \mu, \alpha_{xA}, \alpha_{xB} \in [0,1]. \tag{16}$$

Formula (16) is mathematical model of the true (though precisely not known) single value of the addition result. It also can be interpreted as a model of the set Z of all possible addition results z, which create multidimensional information granule existing in space $M \times A_{xA} \times A_{xB} \times Z$. This granule cannot be shown in the full 4D-space, but with certain simplification it can be shown in 3D-space $M \times A_{xA} \times A_{xB}$ in which z-values will additionally be added to particular $(\mu, \alpha_{xA}, \alpha_{xB})$-points of the space, Figure 5a.

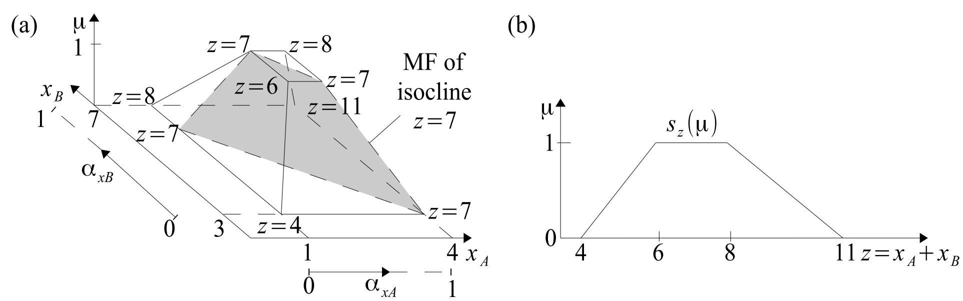

Figure 5: (a) Simplified 3D visualization of the 4D granule Z of addition results of uncertain values x_A and x_B determined by trapezoidal FNs $A = (1,2,3,4,1)$ and $B = (3,4,5,7,1)$; (b) The span function $s_Z(\mu)$ of the 4D addition result Z.

Result sets of arithmetical (mathematical) operations on uncertain numbers exist in multi-dimensional space and are difficult to imagine. Hence scientists try to use less complicated, low-dimensional indicators to "see" the results. The simplest indicator of multi-dimensional result set is its span function $s_Z = f(\mu)$, which can be calculated from (17). In most types of fuzzy arithmetic, the span of the calculation result is considered as the main "result" of the calculation.

$$s_Z(\mu) = [\min_{\alpha_{xA},\alpha_{xB}} z(\mu, \alpha_{xA}, \alpha_{xB}), \max_{\alpha_{xA},\alpha_{xB}} z(\mu, \alpha_{xA}, \alpha_{xB})];\ \alpha_{xA}, \alpha_{xB} \in [0,1], \mu \in [0, \min\{h(A), h(B)\}]. \tag{17}$$

The span should be determined for different, successive μ-levels with appropriately small step $\Delta\mu$.

In the considered case the addition result is determined by [24]. This function is monotonic in relation to α_{xA}, α_{xB}. Hence $\min z$ *occurs for $\alpha_{xA} = \alpha_{xB} = 0$ and* $\max z$ *in $\alpha_{x_A} = \alpha_{x_B} = 1$, (18).*

$$\begin{aligned} &z = x_A + x_B;\ z = (4+2\mu) + (3-2\mu)\alpha_{xA} + (4-3\mu)\alpha_{xB};\ \mu, \alpha_{xA}, \alpha_{xB} \in [0,1], \\ &\min_{\alpha_{xA},\alpha_{xB}} z = 4+2\mu,\ \max_{\alpha_{xA},\alpha_{xB}} z = 11-3\mu;\ \ s_Z = [4+2\mu, 11-3\mu]. \end{aligned} \tag{18}$$

The span function $s_Z(\mu)$ of the 4D addition result Z is shown in Figure 5b. It should be remembered that the span function is not the direct addition result but only its 2D indicator. Using $s_Z(\mu)$ as result to next arithmetic operations causes phenomenon of increasing entropy [11] of uncertain calculation. It also causes other negative phenomena [21, 39, 37]. The next simplified result indicator is cardinality distribution (histogram) $V_{card_{z_i,z_j}}$ being measure of the number *of possible results contained between isoclines $z_i = const$, $z_j = const$,where z_i, z_j are two possible values of the sum $x_A + x_B$ and $V_{zi,zj}$ is the volume contained between two isoclines z_i, z_j in multidimensional space, (19).*

$$V_{card_{z_i,z_j}} = \frac{V_{z_i,z_j}}{z_j - z_i},\ i, j \in \{0, n\}. \tag{19}$$

Due to the limitation of the volume of this article to 18 pages, readers interested in further explanations on cardinality are referred to our articles, e.g. [39].

Cardinality distribution $card(z)$ of the addition result (16) achieved in Example 3.3 is shown in Figure 6.

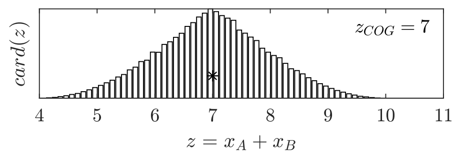

Figure 6: Cardinality distribution $card(z)$ of the addition of uncertain values x_A and x_B in Example 3.3, z_{COG}-center of gravity of the multidimensional result set $Z : z \in Z$.

Third simplified indicator of the multidimensional result set Z is its center of gravity position z_{COG}. It can be calculated with formula (20).

$$z_{COG} = \int_{\underline{z}}^{\overline{z}} z card_{z_i,z_j}(z)dz \Bigg/ \int_{\underline{z}}^{\overline{z}} card_{z_i,z_j}(z)dz. \tag{20}$$

For the considered addition position $z_{COG} = 7$.

The most frequently used indicator of the multidimensional algebraic result Z is its span $s_z(\mu)$. Therefore, it will be used in the following examples.

4 Multidimensional interval type 2 fuzzy arithmetic (M-IT2-F arithmetic)

4.1 Calculation of the main multidimensional result

An interval type 2 fuzzy number $\tilde{A} = [A^U, A^L]$ in parametric form is two ordered pairs of functions $[(\phi(\mu_U), \psi(\mu_U)), (\phi(\mu_L), \psi(\mu_L))]$. The IT2-F number in RDM notation, called interval type 2 RDM fuzzy number (IT2-RDM-FN) has an upper $x_{AU}(\mu_U, \alpha_{xAU})$ and a lower $x_{AL}(\mu_L, \alpha_{xAL})$ horizontal membership functions in the form of (21).

$$\begin{aligned} x_{AU}(\mu_U, \alpha_{xAU}) &= \phi(\mu_U) + [\psi(\mu_U) - \phi(\mu_U)]\alpha_{xAU},\ \mu_U \in [0, h(A^U)], \alpha_{xAU} \in [0, 1], \\ x_{AL}(\mu_L, \alpha_{xAL}) &= \phi(\mu_L) + [\psi(\mu_L) - \phi(\mu_L)]\alpha_{xAL},\ \mu_L \in [0, h(A^L)], \alpha_{xAL} \in [0, 1]. \end{aligned} \tag{21}$$

where $\phi(\mu_U)$ and $\phi(\mu_L)$ are bounded left continuous nondecreasing functions over $[0, h(A^U)]$ and $[0, h(A^L)]$, while $\psi(\mu_U))$ and $\psi(\mu_L)$ are bounded right continuous nonincreasing functions over $[0, h(A^U)]$ and $[0, h(A^L)]$, respectively.

Let us assume that arithmetic operation $* \in \{+, -, \cdot, /\}$ on two uncertain variable values x and y is to be realized and that possessed knowledge about these values is expressed by two trapezoidal horizontal membership functions type 2: $\mu_{T2}(x_1)$ and $\mu_{T2}(x_2)$, Figure 7.

To define arithmetic operations $* \in \{+, -, \cdot, /\}$ in terms of multidimensional approach, horizontal membership functions (HMFs) corresponding to vertical MFs from Figure 7 have to be determined. These HMFs are given by (22)–(29).

$$\begin{aligned} &x_{A1U} = [a_{11U} + (a_{12U} - a_{11U})\mu/h(A_1^U)] + [(a_{14U} - a_{11U}) - \mu/h(A_1^U)(a_{14U} - a_{13U} + a_{12U} - a_{11U})]\alpha_{xA1U}, \\ &\mu \in [0, h(A_1^U)], \alpha_{xA1U} \in [0, 1] \end{aligned} \tag{22}$$

$$x_{A1U} = (1 + 4\mu/0.9) + (12 - 8\mu/0.9)\alpha_{xA1U},\ \mu \in [0, 0.9], \alpha_{xA1U} \in [0, 1]. \tag{23}$$

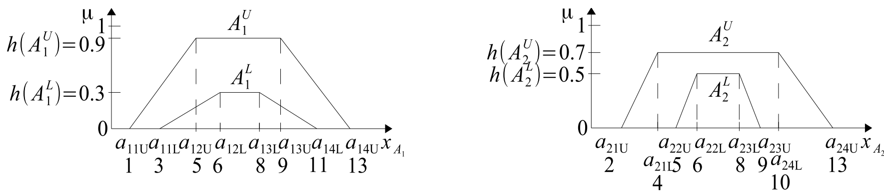

Figure 7: Interval fuzzy numbers type 2 representing knowledge about two uncertain values of variables x_{A1} and x_{A2}, taking part in arithmetic operations.

$$x_{A1L} = [a_{11L} + (a_{12L} - a_{11L})\mu/h(A_1^L)] + [(a_{14L} - a_{11L}) - \mu/h(A_1^L)(a_{14L} - a_{13L} + a_{12L} - a_{11L})]\alpha_{xA1L},\\ \mu \in [0, h(A_1^L)], \alpha_{xA1L} \in [0,1]. \tag{24}$$

$$x_{A1L} = (3 + 3\mu/0.3) + (8 - 6\mu/0.3)\alpha_{xA1L},\ \mu \in [0, 0.3], \alpha_{xA1L} \in [0,1]. \tag{25}$$

$$x_{A2U} = [a_{21U} + (a_{22U} - a_{21U})\mu/h(A_2^U)] + [(a_{24U} - a_{21U}) - \mu/h(A_2^U)(a_{24U} - a_{23U} + a_{22U} - a_{21U})]\alpha_{xA2U},\\ \mu \in [0, h(A_2^U)], \alpha_{xA2U} \in [0,1]. \tag{26}$$

$$x_{A2U} = (2 + 2\mu/0.7) + (11 - 5\mu/0.7)\alpha_{xA2U},\ \mu \in [0, 0.7], \alpha_{xA2U} \in [0,1]. \tag{27}$$

$$x_{A2L} = [a_{21L} + (a_{22L} - a_{21L})\mu/h(A_2^L)] + [(a_{24L} - a_{21L}) - \mu/h(A_2^L)(a_{24L} - a_{23L} + a_{22L} - a_{21L})]\alpha_{xA2L},\\ \mu \in [0, h(A_2^L)], \alpha_{xA2L} \in [0,1]. \tag{28}$$

$$x_{A2L} = (5 + \mu/0.5) + (4 - 2\mu/0.5)\alpha_{xA2L},\ \mu \in [0, 0.5], \alpha_{xA2L} \in [0,1]. \tag{29}$$

It can be observed in IT2-FNs that any crisp element of the lower FN is also contained in the upper FN of T2FN, see Figure 7. Hence, any result of operations on elements from two lower, input MFs will be contained in a lower result set that will be part of the upper result set. This approach is also used by IT2-SF arithmetic, see Section 2.

General formula for arithmetic operations on upper HMFs x_{A1U} and x_{A2U} gives formula (30).

$$\begin{aligned} &x_{A1U} * x_{A2U} = z^U \\ &= \{[a_{11U} + (a_{12U} - a_{11U})\mu/h(A_1^U)] + [(a_{14U} - a_{11U}) - \mu/h(A_1^U)(a_{14U} - a_{13U} + a_{12U} - a_{11U})]\alpha_{xA1U}\}, \\ &*\{[a_{21U} + (a_{22U} - a_{21U})\mu/h(A_2^U)] + [(a_{24U} - a_{21U}) - \mu/h(A_2^U)(a_{24U} - a_{23U} + a_{22U} - a_{21U})]\alpha_{xA2U}\}, \\ &\mu \in [0, \min\{h(A_1^U), h(A_2^U)\}], \alpha_{xA1U}, \alpha_{xA2U} \in [0,1]. \end{aligned} \tag{30}$$

Appropriately, for lower HMFs x_{A1L} and x_{A2L} arithmetic operations are defined by (31).

$$\begin{aligned} &x_{A1L} * x_{A2L} = z^L \\ &= \{[a_{11L} + (a_{12L} - a_{11L})\mu/h(A_1^L)] + [(a_{14L} - a_{11L}) - \mu/h(A_1^L)(a_{14L} - a_{13L} + a_{12L} - a_{11L})]\alpha_{xA1L}\}, \\ &*\{[a_{21L} + (a_{22L} - a_{21L})\mu/h(A_2^L)] + [(a_{24L} - a_{21L}) - \mu/h(A_2^L)(a_{24L} - a_{23L} + a_{22L} - a_{21L})]\alpha_{xA2L}\}, \\ &\mu \in [0, \min\{h(A_1^L), h(A_2^L)\}], \alpha_{xA1L}, \alpha_{xA2L} \in [0,1]. \end{aligned} \tag{31}$$

On the basis of (30) and (31) it can easily be observed that the upper and lower result set Z^U, Z^L exist not in 2D but 4D space, because they consist of elements $z^U = f(\mu, \alpha_{xA1U}, \alpha_{xA2U})$ and $z^L = f(\mu, \alpha_{xA1L}, \alpha_{xA2L})$. The sets Z^U, Z^L are constrained hyper-surfaces in 4D space and they can not directly be visualized. Hence, the statement that results of arithmetic operations on T2FNs are also T2FNs (are objects in 2D space) is incorrect and leads to less or more inexact or even paradoxical results. It also will be shown on examples in next chapter. In this chapter addition of two T2FNs will be shown. Two uncertain values x_{A1} and x_{A2} are to be added ($x_{A1} + x_{A2} = z$) when knowledge about these values is expressed by T2FNs shown in Figure 7.

Both T2FNs are decomposed and defined as HMFs, (23), (25), (27) and (29).

In terms of the decomposition approach first addition of the upper and then of the lower HMFs will be realized, (32), (33).

$$z^U = x_{A1U} + x_{A2U} = [(1 + 4\mu/0.9) + (12 - 8\mu/0.9)\alpha_{xA1U}] + [(2 + 2\mu/0.7) + (11 - 5\mu/0.7)\alpha_{xA2U}],\\ \mu \in [0, \min\{0.9, 0.7\}] = [0, 0.7], \alpha_{xA1U}, \alpha_{xA2U} \in [0,1]. \tag{32}$$

$$z^L = x_{A1L} + x_{A2L} = [(3 + 3\mu/0.3) + (8 - 6\mu/0.3)\alpha_{xA1L}] + [(5 + \mu/0.5) + (4 - 2\mu/0.5)\alpha_{xA2L}],\\ \mu \in [0, \min\{0.3, 0.5\}] = [0, 0.3], \alpha_{xA1L}, \alpha_{xA2L} \in [0,1]. \tag{33}$$

4.2 Calculation of the secondary result - the span of the main, multidimensional result

Result sets Z^U and Z^L of possible crisp addition results exist in 4D space ($M \times A_{xA1U} \times A_{xA2U} \times Z^U$ and $M \times A_{xA1L} \times A_{xA2L} \times Z^L$) and cannot be fully visualized. However, their simplified, 2D indicators as span function $s_Z(\mu)$, cardinality distribution $card(z)$ and position z_{COG} of the center of gravity can be determined. The span $s_Z(\mu)$ can be calculated on the basis of formula (17) that for the considered addition takes form (34).

$$s_Z^U(\mu) = [\min_{\alpha_{xA1U},\alpha_{xA2U}} z^U(\mu, \alpha_{xA1U}, \alpha_{xA2U}), \max_{\alpha_{xA1U},\alpha_{xA2U}} z^U(\mu, \alpha_{xA1U}, \alpha_{xA2U})], \quad (34)$$

where: $\mu \in [0, \min\{h(A_1^U), h(A_2^U)\}]$, $\alpha_{xA1U}, \alpha_{xA2U} \in [0, 1]$.

Because function (32) is monotonic, $\min z^U$ occurs for $\alpha_{xA1U} = \alpha_{xA2U} = 0$ and $\max z^U$ for $\alpha_{xA1U} = \alpha_{xA2U} = 1$. $\min z^U = 3+4\mu/0.9+2\mu/0.7$ and $\max z^U = 26-4\mu/0.9-3\mu/0.7$. Hence the span function $s_Z(\mu)$ is determined by (35).

$$s_Z^U(\mu) = [3 + 4\mu/0.9 + 2\mu/0.7, 26 - 4\mu/0.9 - 3\mu/0.7], \ \mu \in [0, 0.7]. \quad (35)$$

For the lower result set Z^L, formula (33), the span function $s_Z^L(\mu)$ is determined with (36).

$$s_Z^L(\mu) = [8 + 3\mu/0.3 + \mu/0.5, 20 - 3\mu/0.3 - \mu/0.5], \ \mu \in [0, 0.3]. \quad (36)$$

Both span function of the upper and lower result sets are shown in Figure 8.

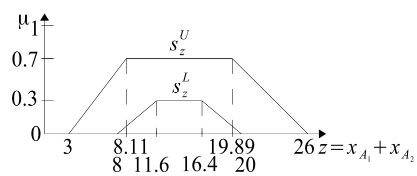

Figure 8: Span functions $s_Z^U(\mu)$ and $s_Z^L(\mu)$ of the multidimensional result set of addition of two uncertain values x_{A1} and x_{A2} in conditions when our knowledge about them is given in form of T2FNs shown in Figure 6.

The lower s_Z^L area in Figure 8 can be interpreted as area of greater confidence and the upper s_Z^U area as the general area of possible result values.

Arithmetic operations performed under M-IT2 arithmetic are not limited to trapezoidal IT2-FNs with linear borders. Borders of IT2-FNs may also be non-linear.

5 Properties of the epistemic M-IT2-F arithmetic

In the following section the basic properties of the multidimensional interval type 2 fuzzy arithmetic are presented.

Lemma 5.1. *Addition and multiplication of epistemic fuzzy numbers with M-IT2-F arithmetic are commutative and associative.*

For fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ the following equations hold:

$$\tilde{A} + \tilde{B} = \tilde{B} + \tilde{A}, \quad \tilde{A} + (\tilde{B} + \tilde{C}) = (\tilde{A} + \tilde{B}) + \tilde{C}, \quad \tilde{A}\tilde{B} = \tilde{B}\tilde{A}, \quad \tilde{A}(\tilde{B}\tilde{C}) = (\tilde{A}\tilde{B})\tilde{C}. \quad (37)$$

where $card\tilde{A} = card\tilde{B} = card\tilde{C} = 1$. Equations (37) are easy to prove with M-IT2-FA, so the proofs are left to the reader.

Lemma 5.2. *The degenerate fuzzy numbers 0 and 1 are additive and multiplicative identity elements with M-IT2-FA, respectively. Equations (38) are true, $\tilde{A}$ is any fuzzy number.*

$$\tilde{A} + 0 = 0 + \tilde{A} = \tilde{A}, \ \tilde{A} \cdot 1 = 1 \cdot \tilde{A} = \tilde{A}. \quad (38)$$

The proofs of equations (38) are left to the reader.

Lemma 5.3. *The additive inverse element of a M-IT2-FN (39)*

$$x_{AU}(\mu_U, \alpha_{xAU}) = \phi(\mu_U) + [\psi(\mu_U) - \phi(\mu_U)]\alpha_{xAU}, \ \ x_{AL}(\mu_L, \alpha_{xAL}) = \phi(\mu_L) + [\psi(\mu_L) - \phi(\mu_L)]\alpha_{xAL}, \quad (39)$$

is the M-IT2-F number (40):

$$-x_{AU}(\mu_U, \alpha_{xAU}) = -\phi(\mu_U) - [\psi(\mu_U) - \phi(\mu_U)]\alpha_{xAU}, \ \ -x_{AL}(\mu_L, \alpha_{xAL}) = -\phi(\mu_L) - [\psi(\mu_L) - \phi(\mu_L)]\alpha_{xAL}, \quad (40)$$

where $\mu_U \in [0, h(A^U)], \mu_L \in [0, h(A^L)], \alpha_{xAU}, \alpha_{xAL} \in [0, 1]$.

Proof. In M-IT2-F arithmetic we have:
$x_{AU}(\mu_U, \alpha_{xAU}) + [-x_{AU}(\mu_U, \alpha_{xAU})] = \phi(\mu_U) + [\psi(\mu_U) - \phi(\mu_U)]\alpha_{xAU} + [-\phi(\mu_U) - [\psi(\mu_U) - \phi(\mu_U)]\alpha_{xAU}] = 0$,
$x_{AL}(\mu_L, \alpha_{xAL}) + [-x_{AL}(\mu_L, \alpha_{xAL})] = \phi(\mu_L) + [\psi(\mu_L) - \phi(\mu_L)]\alpha_{xAL} + [-\phi(\mu_L) - [\psi(\mu_L) - \phi(\mu_L)]\alpha_{xAL}] = 0$.

Lemma 5.4. *In M-IT2-F arithmetic multiplicative inverse element of a fuzzy number (41)*

$$x_{AU}(\mu_U, \alpha_{xAU}) = \phi(\mu_U) + [\psi(\mu_U) - \phi(\mu_U)]\alpha_{xAU}, \;\; x_{AL}(\mu_L, \alpha_{xAL}) = \phi(\mu_L) + [\psi(\mu_L) - \phi(\mu_L)]\alpha_{xAL}, \tag{41}$$

$0 \notin x_{AU}(\mu_U, \alpha_{xAU})$ *and* $0 \notin x_{AL}(\mu_L, \alpha_{xAL})$*, is the IT2-RDM-F number (42)*

$$1/x_{AU}(\mu_U, \alpha_{xAU}) = 1/\{\phi(\mu_U) + [\psi(\mu_U) - \phi(\mu_U)]\alpha_{xAU}\}, \;\; 1/x_{AL}(\mu_L, \alpha_{xAL}) = 1/\{\phi(\mu_L) + [\psi(\mu_L) - \phi(\mu_L)]\alpha_{xAL}\}. \tag{42}$$

Proof. In M-IT2-F arithmetic we have:
$x_{AU}(\mu_U, \alpha_{xAU}) \cdot [1/x_{AU}(\mu_U, \alpha_{xAU})] = \{\phi(\mu_U) + [\psi(\mu_U) - \phi(\mu_U)]\alpha_{xAU}\} \cdot \{1/\{\phi(\mu_U) + [\psi(\mu_U) - \phi(\mu_U)]\alpha_{xAU}\}\} = 1$,
$x_{AL}(\mu_L, \alpha_{xAL}) \cdot [1/x_{AL}(\mu_L, \alpha_{xAL})] = \{\phi(\mu_L) + [\psi(\mu_L) - \phi(\mu_L)]\alpha_{xAL}\} \cdot \{1/\{\phi(\mu_L) + [\psi(\mu_L) - \phi(\mu_L)]\alpha_{xAL}\}\} = 1$.

Lemma 5.5. *The full distributive law in the form (43)*

$$\tilde{A}(\tilde{B} + \tilde{C}) = \tilde{A}\tilde{B} + \tilde{A}\tilde{C}, \tag{43}$$

in M-IT2-F arithmetic holds.

Proof. Let us consider any interval type 2 RDM fuzzy numbers $[x_{AU}(\mu_U, \alpha_{xAU}), x_{AL}(\mu_L, \alpha_{xAL})]$, $[x_{BU}(\mu_U, \alpha_{xBU}), x_{BL}(\mu_L, \alpha_{xBL})]$ and $[x_{CU}(\mu_U, \alpha_{xCU}), x_{CL}(\mu_L, \alpha_{xCL})]$.
In M-IT2-F arithmetic we have:
$x_{AU}(\mu_U, \alpha_{xAU})[x_{BU}(\mu_U, \alpha_{xBU}) + x_{CU}(\mu_U, \alpha_{xCU})] = \phi_A(\mu_U) + [\psi_A(\mu_U) - \phi_A(\mu_U)]\alpha_{xAU}[\phi_B(\mu_U) + [\psi_B(\mu_U) - \phi_B(\mu_U)]\alpha_{xBU} + \phi_C(\mu_U) + [\psi_C(\mu_U) - \phi_C(\mu_U)]\alpha_{xCU} = [\phi_A(\mu_U) + [\psi_A(\mu_U) - \phi_A(\mu_U)]\alpha_{xAU}][\phi_B(\mu_U) + [\psi_B(\mu_U) - \phi_B(\mu_U)]\alpha_{xBU}] + [\phi_A(\mu_U) + [\psi_A(\mu_U) - \phi_A(\mu_U)]\alpha_{xAU}][\phi_C(\mu_U) + [\psi_C(\mu_U) - \phi_C(\mu_U)]\alpha_{xCU}] = [x_{AU}(\mu_U, \alpha_{xAU})][x_{BU}(\mu_U, \alpha_{xBU})] + [x_{AU}(\mu_U, \alpha_{xAU})][x_{CU}(\mu_U, \alpha_{xCU})]$.
Similarly it can be proved with lower membership function of IT2-RDM-FNs.

Thanks to RDM variables dependences existing between uncertain variables can in M-IT2-F arithmetic be taken into account.

The complexity of multidimensional M-IT2-F-arithmetic level is greater than that of all 2-dimensional IT2-FA types, whose common feature is the attempt to directly compute a "2D-result" in the form of span (horizontal uncertainty of the result). M-IT2-FA does not do this directly as it is only possible with simple problems. However, in more complex problems, the direct calculation of span is inaccurate or not possible at all. Hence the higher complexity level of M-IT2-FA is the price for its higher accuracy and the ability to solve more problems. However, this higher level of complexity will no longer be a problem if software supporting the M-IT2-FA calculations is developed.

6 Comparative experiments with use of epistemic M-IT2-F arithmetic and IT2-SF arithmetic

6.1 Experiment 1 - Solving uncertain equation with one unknown

Equation and equation system solving is probably the most difficult task for any type of fuzzy arithmetic.

Let us assume, that a system operates according to dependence $a + b = c$, where a and b are physical inputs of the system and c is its output. Assume, that about true values of variables a and c we have only approximate epistemic knowledge from the system experts in form of T2FNs A and C shown in Figure 9, and that we possess no knowledge about value of the input b but we want to get this knowledge. Because value of b is unknown let us denote it by x. Hence, the task consists in solving the uncertain equation $a + x = c$ in situation when we know that $a \in A$ and $c \in C$. Let us note, that a and c are in the problem informational inputs and $x = b$ is informational output, though it is physical input of the considered system and that uncertainty of the left hand side of the equation is greater than of the right-hand side.

Solving trial with use of IT2-SF arithmetic

In terms of IT2-SF arithmetic solution X of the equation $\tilde{A} + \tilde{X} = \tilde{C}$ is to be determined, where $\tilde{A}$ and $\tilde{C}$ are T2-FNs of the form: $\tilde{A} = [A^U, A^L] = [1, 2, 5, 6, 1; 2, 3, 4, 5, 0.6]$, $\tilde{C} = [C^U, C^L] = [5, 6, 7, 8, 1; 6, 6.5, 6.5, 7, 1]$.

Solution X can be expressed in general form $X = [X^U, X^L] = [x_{1U}, x_{2U}, x_{3U}, x_{4U}, h(X^U); x_{1L}, x_{2L}, x_{3L}, x_{4L}, h(X^L)]$. Also the equation to be solved can be expressed in general form (44).

$$\tilde{A} + \tilde{X} = \tilde{C}, \;\; [A^U, A^L] + [X^U, X^L] = [C^U, C^L]. \tag{44}$$

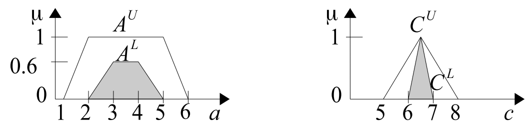

Figure 9: Type 2 fuzzy numbers representing knowledge about true values of uncertain variables a and c of a system operating according to physical law $a + b = c$, $b = x$ is the value to be determined.

To make calculations more transparent the problem can be decomposed in two sub-problems concerning upper and lower equation (45) and (46).

$$A^U + X^U = C^U, \quad [1, 2, 5, 6, 1] + [x_{1U}, x_{2U}, x_{3U}, x_{4U}, h(X^U)] = [5, 6, 7, 8, 1]. \tag{45}$$

$$A^L + X^L = C^L, \quad [2, 3, 4, 5, 0.6] + [x_{1L}, x_{2L}, x_{3L}, x_{4L}, h(X^L)] = [6, 6.5, 6.5, 7, 1]. \tag{46}$$

On the basis of (45) for upper FNs, with use of the IT2-SF arithmetic (see definitions 2.8) set of component equations (47) and their solutions is achieved.

$$\begin{aligned} &1 + x_{1U} = 5 \ : \ x_{1U} = 4, \ \ 2 + x_{2U} = 6 \ : \ x_{2U} = 4, \ \ 5 + x_{3U} = 7 \ : \ x_{3U} = 2, \ \ 6 + x_{4U} = 8 \ : \ x_{4U} = 2, \\ &\min\{1, h(X^U)\} = 1 \ : \ h(X^U) = 1. \end{aligned} \tag{47}$$

The solution (47) is logically inconsistent and can not be realized physically because it has form of interval X^U in which the lower border $\underline{x}_U = x_{1U} = x_{2U} = 4$ has greater values than the upper border $\overline{x}_U = x_{1U} = x_{2U} = 2$. According to the interval definition the true x_U-value has to satisfy inequality $\underline{x}_U \leq x_U \leq \overline{x}_U$ therefore it easily can be concluded that there exists no such x_U-value that could at the same time satisfy inequalities $x_U \geq 4$ and $x_U \leq 2$. The situation is illustrated by Fig 10.

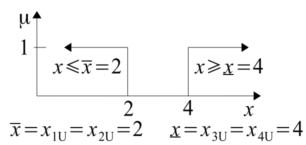

Figure 10: Illustration of inconsistency of the achieved upper solution $X^U = [4, 4, 2, 2, 1]$ of equations (47).

Now, let us determine the lower solution x_L of the considered equation. On the basis of (46) the set (48) of component equations and solutions is achieved.

$$\begin{aligned} &2 + x_{1L} = 6 \ : \ x_{1L} = 4, \ \ 3 + x_{2L} = 6.5 \ : \ x_{2L} = 3.5, \ \ 4 + x_{3L} = 6.5 \ : \ x_{3L} = 2.5, \ \ 5 + x_{4L} = 7 \ : \ x_{4L} = 2, \\ &\min\{0.6, h(X^L)\} = 1 \ : \ \text{Logically inconsistent condition} \end{aligned} \tag{48}$$

It can easily be concluded from (48) that the lower solution set X^L of the equation $\tilde{A} + \tilde{X} = \tilde{C}$ does not exists from two reasons, though one reason is sufficient. The first reason is the same as in the case of the upper solution set X^U. It is not possible that there exists such x_L-value which at the same time satisfies $x_L \geq 4$ and $x_L \leq 2$. The next reason of nonexistence of the lower solution results is caused by the fact that there exists no such membership value $\mu = h(X^L)$, which could satisfy the condition $\min[0.6, h(X^L)] = 1$. Summarizing the above: IT2-SF arithmetic is not in state to determine solutions of some, even simple equations for which, as it will be proved, solutions exist and can be determined. The main reason of this situation is the incorrect, basic assumption of IT2-SF arithmetic that result of arithmetic operations on IT2-FNs is also an IT2-FN (the same type of mathematical object).

This problem has been explained by authors, e.g. in [39].

Solution with use of M-IT2-F arithmetic:

The upper set $A^U = [1, 2, 5, 6, 1]$ is corresponded in terms of epistemic M-IT2-F arithmetic by HMF given by $a_U = (1 + \mu) + (5 - 2\mu)\alpha_{aU}$, $\mu, \alpha_{aU} \in [0, 1]$. The lower fuzzy set $A^L = [2, 3, 4, 5, 0.6]$ is corresponded by HMF $a_L = (2 + \mu/0.6) + (3 - 2\mu/0.6)\alpha_{aL}$, $\mu \in [0, 0.6], \alpha_{aL} \in [0, 1]$. The upper fuzzy set $C^U = [5, 6, 7, 8, 1]$ is corresponded by HMF $c_U = (5 + \mu) + (3 - 2\mu)\alpha_{cU}$, $\mu, \alpha_{cU} \in [0, 1]$. The lower fuzzy set $C^L = [6, 6.5, 6.5, 7, 1]$ is corresponded by HMF $c_L = (6 + 0.5\mu) + (1 - \mu)\alpha_{cL}$, $\mu, \alpha_{cL} \in [0, 1]$. In the first step the upper, multidimensional solution set $X^U : x_U \in X^U$, on the basis of (49) will be determined.

$$\begin{aligned} &a_U + x_U = c_U \ \Rightarrow \ x_U = c_U - a_U, \ \ x_U = [(5 + \mu) + (3 - 2\mu)\alpha_{cU}] - [(1 + \mu) + (5 - 2\mu)\alpha_{aU}], \\ &\mu \in [0, \min\{h(A^U), h(C^U)\}] = [0, \min\{1, 1\}] = [0, 1], \ \alpha_{aU}, \alpha_{cU} \in [0, 1]. \end{aligned} \tag{49}$$

The lower, multidimensional solution set $X^L : x_L \in X^L$ will be determined on the basis of (50).

$$a_L + x_L = c_L \;\Rightarrow\; x_L = c_L - a_L,\;\; x_L = [(6+0.5\mu) + (1-\mu)\alpha_{cL}] - [(2+\mu/0.6) + (3-2\mu/0.6)\alpha_{aL}],$$
$$\mu \in [0, \min\{h(A^L), h(C^L)\}] = [0, \min\{0.6, 1\}] = [0, 0.6],\;\; \alpha_{aL}, \alpha_{cL} \in [0, 1]. \tag{50}$$

The achieved epistemic solution $x_U = f_U(\mu, \alpha_{AU}, \alpha_{CU})$ and $x_L = f_L(\mu, \alpha_{AL}, \alpha_{CL})$ exist not in 2D but in 4D space. It can easily be checked that they are universal [21] solutions of the problem because x_U satisfies all possible mathematical forms of the problem shown in (51) and x_L satisfies all forms presented by (52).

$$a_U + x_U = c_U,\; a_U = c_U - x_U,\; x_U = c_U - a_U,\; a_U + x_U - c_U = 0. \tag{51}$$

$$a_L + x_L = c_L,\; a_L = c_L - x_L,\; x_L = c_L - a_L,\; a_L + x_L - c_L = 0. \tag{52}$$

No "solution" in form of 2D FN suggested by IT2-SF arithmetic can satisfy this condition. Solutions x_U and x_L delivered by M-IT2-F arithmetic are knowledge about possible values of variable x that we are in position to obtain on the basis of the knowledge about uncertain variables a and c expressed in form of T2-MFs shown in Figure 9. Though the solutions exists in 4D space they can in a simplified way be visualized in 2D space, e.g. for the membership level $\mu = 0$, Figure 11a, and similarly for other μ-levels. Figure 11a proves that solution set X of the considered equation $a + x = c$, $a \in \tilde{A}$, $c \in \tilde{C}$, exists in spite of the fact that IT2-SF arithmetic is not in position to determine it. The span function of the upper solution can be determined from (53).

$$s_{xU}(\mu) = [\min_{\alpha_{aU},\alpha_{cU}} x_U(\mu, \alpha_{aU}, \alpha_{cU}),\; \max_{\alpha_{aU},\alpha_{cU}} x_U(\mu, \alpha_{aU}, \alpha_{cU})]. \tag{53}$$

Value $\min x_U$ occurs for $\alpha_{aU} = 1$ and $\alpha_{cU} = 0$ and $\max x_U$ for $\alpha_{aU} = 0$ and $\alpha_{cU} = 1$. Hence, the span function has form of (54).

$$s_{xU}(\mu) = [(-1+2\mu), (7-2\mu)], \mu \in [0, 1]. \tag{54}$$

The span function s_{xL} of the lower solution set can be determined on the basis of (55).

$$s_{xL}(\mu) = [\min_{\alpha_{aL},\alpha_{cL}} x_L(\mu, \alpha_{aL}, \alpha_{cL}),\; \max_{\alpha_{aL},\alpha_{cL}} x_L(\mu, \alpha_{aL}, \alpha_{cL})]. \tag{55}$$

Here, $\min x_L$ occurs for $\alpha_{aL} = 1$ and $\alpha_{cL} = 0$ and $\max x_L$ for $\alpha_{aL} = 0$ and $\alpha_{cL} = 1$. The span function in precise form is given by (56).

$$s_{xL}(\mu) = [(1+13/6\mu), (5-13/6\mu)], \mu \in [0, 0.6]. \tag{56}$$

The upper and lower span functions are shown in Figure 11b. Span functions as shown in Figure 11b are regarded by IT2-SF arithmetic as solutions of equation of the type $\tilde{A} + \tilde{X} = \tilde{C}$. However, they (if exist) are not algebraic, universal solutions but only indicators of multidimensional solution sets.

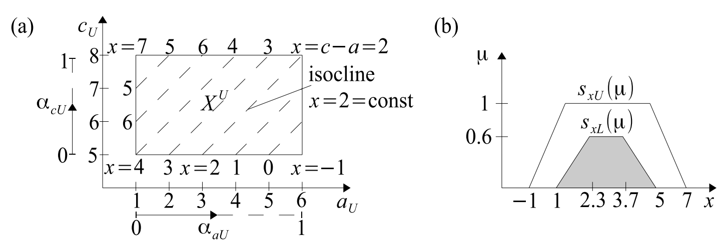

Figure 11: (a) Visualization of the upper solution set X^U of possible crisp solutions x_U of equation solved in section 6.1 for membership level $\mu = 0$. The solutions are contained here in the span $[-1, 7]$; (b) Span function $s_{xU}(\mu)$ and $s_{xL}(\mu)$ of the upper and lower solution set X^U and X^L of the equation $a + x = c$, $a \in \tilde{A}$, $c \in \tilde{C}$, solved in section 6.1.

6.2 Experiment 2 - Solving uncertain linear equation system

Linear equation system (LES) are very important for modeling of MIMO (Many Inputs, Many Outputs) systems which describe dependencies existing in e.g. economic balance systems, heat and energy balance systems, ecologic, biologic systems etc. In real systems values of variables and coefficients mostly are known not precisely but only approximately. Hence, uncertain LES, because of the great difficulty in solving it, are investigated in many scientific papers, e.g. in

[4, 27]. In section 6.2 an LES in which knowledge about uncertain coefficients is given in form of T2FNs will be considered. Formula (57) presents description of an LES in which x_1, x_2 are system inputs and y_1, y_2 are system outputs, a_1, a_2, b_1, b_2 are coefficients.

$$\begin{aligned} a_1x_1 + b_1x_2 &= y_1, \\ a_2x_1 + b_2x_2 &= y_2. \end{aligned} \tag{57}$$

The task in section 6.2 consists in determining the inputs' values x_1, x_2 when knowledge about the outputs y_1, y_2 and the coefficients a_1, a_2, b_1, b_2 is only approximate and is given in form of IT2-FNs described in terms of IT2-SF arithmetic.

$$\begin{array}{ll} a_1 \in \tilde{A}_1 = [1, 1.5, 2.5, 3, 1; 1.5, 2, 2, 2.5, 1], & a_2 \in \tilde{A}_2 = [3, 3.5, 4.5, 5, 1; 3.5, 4, 4, 4.5, 1], \\ b_1 \in \tilde{B}_1 = [4, 4.5, 5.5, 6, 1; 4.5, 5, 5, 5.5, 1], & b_2 \in \tilde{B}_2 = [-2, -1.5, -0.5, 0, 1; -1.5, -1, -1, -0.5, 1], \\ y_1 \in \tilde{Y}_1 = [17, 18, 20, 21, 1; 18, 19, 19, 20, 1], & y_2 \in \tilde{Y}_2 = [5, 6, 8, 9, 1; 6, 7, 7, 8, 1]. \end{array} \tag{58}$$

IT2-MFs corresponding to (58) are shown in Figure 12.

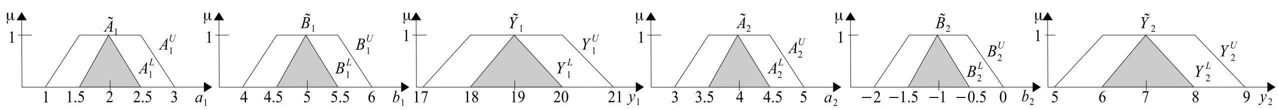

Figure 12: Membership functions type 2 of variables and coefficients occurring in the linear system equation (57).

LES (57) can be solved with generally known Cramer's rule (59).

$$x_1 = \frac{b_2y_1 - b_1y_2}{a_1b_2 - a_2b_1},\ x_2 = \frac{a_1y_2 - a_2y_1}{a_1b_2 - a_2b_1},\ 0 \notin a_1b_2 - a_2b_1. \tag{59}$$

Solution of the LES with use of IT2-SF arithmetic:

In terms of IT2-SF arithmetic solutions $\tilde{X}_1$, $\tilde{X}_2$ are fuzzy numbers type 2 and can be determined from (60).

$$\tilde{X}_1 = \frac{\tilde{B}_2\tilde{Y}_1 - \tilde{B}_1\tilde{Y}_2}{\tilde{A}_1\tilde{B}_2 - \tilde{A}_2\tilde{B}_1},\ \tilde{X}_2 = \frac{\tilde{A}_1\tilde{Y}_2 - \tilde{A}_2\tilde{Y}_1}{\tilde{A}_1\tilde{B}_2 - \tilde{A}_2\tilde{B}_1},\ 0 \notin \tilde{A}_1\tilde{B}_2 - \tilde{A}_2\tilde{B}_1. \tag{60}$$

Obtained solutions are given by (61) and are shown in Figure 13a.

$$\tilde{X}_1 = [0.56, 1.26, 4.48, 8, 1; 1.26, 2.45, 2.45, 4.48, 1],\ \ \tilde{X}_2 = [0.67, 1.51, 4.91, 8.33, 1; 1.51, 2.82, 2.82, 4.91, 1]. \tag{61}$$

It can easily be proved that "solutions" $\tilde{X}_1$, $\tilde{X}_2$ achieved with IT2-SF arithmetic are not real algebraic solutions of the LES (57) solutions. It is sufficient to insert them in the original LES and check equality of right-hand and left-hand sides of equations.

Solution of the LES obtained with use of M-IT2-F arithmetic

In the first step upper and lower MFs that present our knowledge about true values of variables and coefficients of the LES from Figure 12 should be determined in forms of HMFs. The achieved HMFs are given by (62).

$$\begin{array}{lll} a_1^U = (1+0.5\mu)+(2-\mu)\alpha_{a1U}, & a_1^L = (1.5+0.5\mu)+(1-\mu)\alpha_{a1L}, & \mu, \alpha_{a1U}, \alpha_{a1L} \in [0,1], \\ a_2^U = (3+0.5\mu)+(2-\mu)\alpha_{a2U}, & a_2^L = (3.5+0.5\mu)+(1-\mu)\alpha_{a2L}, & \mu, \alpha_{a2U} \in [0,1], \alpha_{a2L} \in [0,1], \\ b_1^U = (4+0.5\mu)+(2-\mu)\alpha_{b1U}, & b_1^L = (4.5+0.5\mu)+(1-\mu)\alpha_{b1L}, & \mu, \alpha_{b1U} \in [0,1], \alpha_{b1L} \in [0,1], \\ b_2^U = (-2+0.5\mu)+(2-\mu)\alpha_{b2U}, & b_2^L = (-1.5+0.5\mu)+(1-\mu)\alpha_{b2L}, & \mu, \alpha_{b2U} \in [0,1], \alpha_{b2L} \in [0,1], \\ y_1^U = (17+\mu)+(4-2\mu)\alpha_{y1U}, & y_1^L = (18+\mu)+(2-2\mu)\alpha_{y1L}, & \mu, \alpha_{y1U} \in [0,1], \alpha_{y1L} \in [0,1], \\ y_2^U = (5+\mu)+(4-2\mu)\alpha_{y2U}, & y_2^L = (6+\mu)+(2-2\mu)\alpha_{y2L}, & \mu, \alpha_{y2U} \in [0,1], \alpha_{y2L} \in [0,1], \end{array} \tag{62}$$

On the basis of general Cramer's rule (59), after inserting appropriate HMFs taken from (62), the multidimensional upper and lower solutions (63) are achieved. They are primary algebraic and universal solutions which, after inserting in the LES (60) satisfy it.

$$\begin{array}{l} X_1^U : x_1^U \in X_1^U,\ x_1^U = \frac{[(-2+0.5\mu)+(2-\mu)\alpha_{b2U}][(17+\mu)+(4-2\mu)\alpha_{y1U}]-[(4+0.5\mu)+(2-\mu)\alpha_{b1U}][(5+\mu)+(4-2\mu)\alpha_{y2U}]}{[(1+0.5\mu)+(2-\mu)\alpha_{a1U}][(-2+0.5\mu)+(2-\mu)\alpha_{b2U}]-[(3+0.5\mu)+(2-\mu)\alpha_{a2U}][(4+0.5\mu)+(2-\mu)\alpha_{b1U}]}, \\ X_1^L : x_1^L \in X_1^L,\ x_1^L = \frac{[(-1.5+0.5\mu)+(1-\mu)\alpha_{b2L}][(18+\mu)+(2-2\mu)\alpha_{y1L}]-[(4.5+0.5\mu)+(1-\mu)\alpha_{b1L}][(6+\mu)+(2-2\mu)\alpha_{y2L}]}{[(1.5+0.5\mu)+(1-\mu)\alpha_{a1L}][(-1.5+0.5\mu)+(1-\mu)\alpha_{b2L}]-[(3.5+0.5\mu)+(1-\mu)\alpha_{a2L}][(4.5+0.5\mu)+(1-\mu)\alpha_{b1L}]}, \\ X_2^U : x_2^U \in X_2^U,\ x_2^U = \frac{[(1+0.5\mu)+(2-\mu)\alpha_{a1U}][(5+\mu)+(4-2\mu)\alpha_{y2U}]-[(3+0.5\mu)+(2-\mu)\alpha_{a2U}][(17+\mu)+(4-2\mu)\alpha_{y1U}]}{[(1+0.5\mu)+(2-\mu)\alpha_{a1U}][(-2+0.5\mu)+(2-\mu)\alpha_{b2U}]-[(3+0.5\mu)+(2-\mu)\alpha_{a2U}][(4+0.5\mu)+(2-\mu)\alpha_{b1U}]}, \\ X_2^L : x_2^L \in X_2^L,\ x_2^L = \frac{[(1.5+0.5\mu)+(1-\mu)\alpha_{a1L}][(6+\mu)+(2-2\mu)\alpha_{y2L}]-[(3.5+0.5\mu)+(1-\mu)\alpha_{a2L}][(18+\mu)+(2-2\mu)\alpha_{y1L}]}{[(1.5+0.5\mu)+(1-\mu)\alpha_{a1L}][(-1.5+0.5\mu)+(1-\mu)\alpha_{b2L}]-[(3.5+0.5\mu)+(1-\mu)\alpha_{a2L}][(4.5+0.5\mu)+(1-\mu)\alpha_{b1L}]} \end{array} \tag{63}$$

As formulas (63) shown direct solution sets X_1^U, X_1^L, X_2^U, X_2^L are multidimensional functions, e.g. $x_{1U} = f_{1U}(\mu, \alpha_{a1U}, \alpha_{a2U}, \alpha_{b1U}, \alpha_{b2U}, \alpha_{y1U}, \alpha_{y2U})$, which cannot be visualized. Only simplified, low dimensional indicators of them can be determined and shown. The first indicator, the span function of the set X_1^U can be determined from (64).

$$s_{x1U} = x_1^U[\min(\mu, \alpha_{a1U}, \alpha_{a2U}, \alpha_{b1U}, \alpha_{b2U}, \alpha_{y1U}, \alpha_{y2U}), \max x_1^U(\mu, \alpha_{a1U}, \alpha_{a2U}, \alpha_{b1U}, \alpha_{b2U}, \alpha_{y1U}, \alpha_{y2U})]. \quad (64)$$

Examination of (63) shows that $\min X_1^U$ occurs for $\alpha_{a1U} = \alpha_{a2U} = \alpha_{b1U} = \alpha_{b2U} = 1$, $\alpha_{y1U} = 0$,$\alpha_{y2U} = 0$ and $\max X_1^U$ occurs for $\alpha_{a1U} = \alpha_{a2U} = \alpha_{b1U} = \alpha_{b2U} = 0$, $\alpha_{y1U} = \alpha_{y2U} = 1$. Hence, the precise span function is given by (65).

$$s_{x1U}(\mu) = \left[\frac{-0.5\mu(17+\mu)-(6-0.5\mu)(5+\mu)}{-0.5\mu(3-0.5\mu)-(5-0.5\mu)(6-0.5\mu)}, \frac{(-2+0.5\mu)(21-\mu)-(4+0.5\mu)(9-\mu)}{(1+0.5\mu)(-2+0.5\mu)-(3+0.5\mu)(4+0.5\mu)}\right]. \quad (65)$$

Similarly span of X_2^U can be calculated, $\min X_2^U$ corresponds to $\alpha_{a1U} = 1$, $\alpha_{a2U} = 0$, $\alpha_{b1U} = 1$, $\alpha_{b2U} = 0$, $\alpha_{y1U} = 0$,$\alpha_{y2U} = 1$ and $\max X_2^U$ corresponds to $\alpha_{a1U} = 0$, $\alpha_{a2U} = 1$, $\alpha_{b1U} = 0$, $\alpha_{b2U} = 1$, $\alpha_{y1U} = 1$, $\alpha_{y2U} = 0$. Precise span presents (66).

$$s_{x2U}(\mu) = \left[\frac{(3-0.5\mu)(9-\mu)-(3+0.5\mu)(17+\mu)}{(3-0.5\mu)(-2+0.5\mu)-(3+0.5\mu)(6-0.5\mu)}, \frac{(1+0.5\mu)(5+\mu)-(5-0.5\mu)(21-\mu)}{(1+0.5\mu)(-0.5\mu)-(5-0.5\mu)(4+0.5\mu)}\right]. \quad (66)$$

Value $\min X_1^L$ corresponds to $\alpha_{a1L} = 1$, $\alpha_{a2L} = 1$, $\alpha_{b1L} = 1$, $\alpha_{b2L} = 1$, $\alpha_{y1L} = 0$,$\alpha_{y2L} = 0$ and $\max X_1^L$ corresponds to $\alpha_{a1L} = 0$, $\alpha_{a2L} = 0$, $\alpha_{b1L} = 0$, $\alpha_{b2L} = 0$, $\alpha_{y1L} = 1$, $\alpha_{y2L} = 1$, see formula (67).

$$s_{x1L}(\mu) = \left[\frac{(-0.5-0.5\mu)(18+\mu)-(5.5-0.5\mu)(6+\mu)}{(2.5-0.5\mu)(-0.5-0.5\mu)-(4.5-0.5\mu)(5.5-0.5\mu)}, \frac{(-1.5+0.5\mu)(20-\mu)-(4.5+0.5\mu)(8-\mu)}{(1.5+0.5\mu)(-1.5+0.5\mu)-(3.5+0.5\mu)(4.5+0.5\mu)}\right]. \quad (67)$$

Value $\min X_2^L$ corresponds to $\alpha_{a1L} = 1$, $\alpha_{a2L} = 0$, $\alpha_{b1L} = 1$, $\alpha_{b2L} = 0$, $\alpha_{y1L} = 0$,$\alpha_{y2L} = 1$ and $\max X_2^L$ corresponds to $\alpha_{a1L} = 0$, $\alpha_{a2L} = 1$, $\alpha_{b1L} = 0$, $\alpha_{b2L} = 1$, $\alpha_{y1L} = 1$, $\alpha_{y2L} = 0$, see formula (68).

$$s_{x2L}(\mu) = \left[\frac{(2.5-0.5\mu)(8-\mu)-(3.5+0.5\mu)(18+\mu)}{(2.5-0.5\mu)(-1.5+0.5\mu)-(3.5+0.5\mu)(5.5-0.5\mu)}, \frac{(1.5+0.5\mu)(6+\mu)-(4.5-0.5\mu)(20-\mu)}{(1.5+0.5\mu)(-0.5-0.5\mu)-(4.5-0.5\mu)(4.5+0.5\mu)}\right]. \quad (68)$$

Figure 13b presents the spans of the primary multidimensional upper and lower solutions (63).

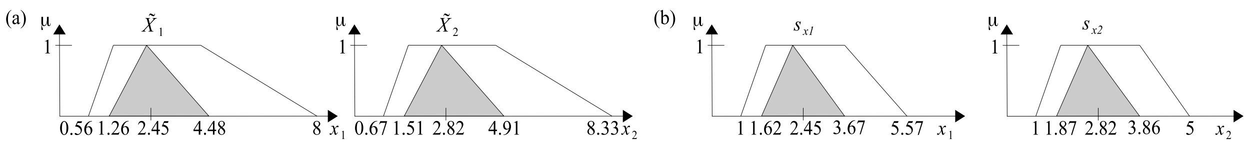

Figure 13: (a) 2D solutions of the LES (57) in form of IT2-FNs obtained with use of IT2-SF arithmetic; (b) Spans of the multidimensional solutions (63) of the LES (57) obtained with use of M-IT2-F arithmetic.

Errors of calculated spans can be understood in different ways. Many scientists believe that the precision of the fuzzy calculation result is the higher the lower the span of this result. However, this interpretation is completely wrong. The results of fuzzy calculations are accurate when they are algebraic and universal results of the problem. This means that after inserting them into the solved equation they satisfy this equation regardless of the form in which the equation will be presented, i.e. they meet the requirement of universality of the result. The span of the universal, algebraic result calculated by a given type of fuzzy arithmetic must be identical to the true (objective) span of this result. Such span can be calculated on the basis of a multivariate result obtained using multidimensional fuzzy arithmetic, or it can be determined experimentally using Monte Carlo computer simulation. As example, we can compare the supports of the solution span x_1 obtained using IT2-SF arithmetic and the span of correct solution x_1 obtained using multidimensional M-IT2-F arithmetic, Figure 14. As can be easily seen in Figure 14 the span support of the solution x_1 provided by

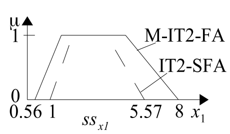

Figure 14: Comparison of span supports of the solution x_1 obtained using IT2-SF arithmetic and M-IT2-F arithmetic, ss_{x1} - span supports of both solutions.

IT2-SF arithmetic is overestimated by 63% relative to the correct span support of the solution x_1 provided by M-IT2-F arithmetic. The x_1 solution provided by IT2-SF arithmetic suggests that wrong values of $x_1 \in [0.56, 1)$ and $x_1 \in (5.57, 8]$ are possible while such values are not possible at all. This can also be checked by the Monte-Carlo method. In this way this solution misleads us partly because only the range of solutions $x_1 \in [1, 5.57]$ is possible.

6.3 Experiment 3 - Solving the type-2 fuzzy differential equation [19]

Let us consider Newton's law of cooling with equation (69)

$$\frac{d\tilde{\theta}(t)}{dt} = -k\tilde{\theta}(t) + k\tilde{\theta}_s, \tag{69}$$

where $k = 0.05$, $t \in [0, 10]$, $\tilde{\theta}(0) = [150, 180, 180, 210, 1; 170, 180, 180, 190, 1]$, $\tilde{\theta}_s = [40, 70, 70, 100, 1; 60, 70, 70, 80, 1]$.

To find a solution of FDE with of M-IT2-F arithmetic the uncertain values $l \in \tilde{\theta}(0)$ and $r \in \tilde{\theta}_s$ should be written in HMFs notation as in (70).

$$\begin{array}{l} l^U = 150 + 30\mu + 60(1-\mu)\alpha_{lU},\ l^L = 170 + 10\mu + 20(1-\mu)\alpha_{lL},\ \mu, \alpha_{lU}, \alpha_{lL} \in [0,1], \\ r^U = 40 + 30\mu + 60(1-\mu)\alpha_{rU},\ r^L = 60 + 10\mu + 20(1-\mu)\alpha_{rL},\ \mu, \alpha_{rU} \in [0,1], \alpha_{rL} \in [0,1]. \end{array} \tag{70}$$

The multidimensional upper and lower solution of equation (69) in HFN notation has a form of (71)

$$\begin{array}{l} \tilde{\theta}(t)^U : m^U(t) \in \tilde{\theta}(t)^U,\ m^U(t) = [110 + 60(1-\mu)(\alpha_{lU} - \alpha_{rU})]e^{-0.05t} + 40 + 30\mu + 60(1-\mu)\alpha_{rU}, \\ \tilde{\theta}(t)^L : m^L(t) \in \tilde{\theta}(t)^L,\ m^L(t) = [110 + 20(1-\mu)(\alpha_{lL} - \alpha_{rL})]e^{-0.05t} + 60 + 10\mu + 20(1-\mu)\alpha_{rL}. \end{array} \tag{71}$$

Figure 15 presents the spans of the full multidimensional upper and lower solutions (71).

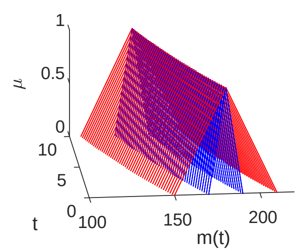

Figure 15: Spans of the multidimensional solutions (71) of the type-2 FDE (69) obtained with use of M-IT2-F arithmetic.

The results obtained with M-IT2-F arithmetic and those of example 5.1. [19] are different. It can be shown that there are crisp solutions that are in the set of full multidimensional solution (71) and do not occur as result of example 5.1 [19]. E.g. let us consider crisp differential equation obtained from FDE (69). For $k = 0.05$, $\theta_s = 40$, initial value $\theta(0) = 150$ the crisp solution of crisp DE is $\theta(t) = 110e^{-0.05t} + 40$. For $t = 10$ crisp solution equals $\theta(10) = 110e^{-0.5} + 40 \approx 106.72$. Value $\theta(10) \approx 106.72$ does not occure in the result of example 5.1 [19], all results are greater than 130, see Fig. 7 in [19]. In the full solution of type-2 FDE (71) value $\theta(10) \approx 106.72$ is obtained from m^U for $t = 10$ and $\mu = \alpha_{rU} = \alpha_{lU} = 0$.

7 Conclusions

The paper has presented two arithmetic types of interval fuzzy numbers type 2: the mostly used IT2-SF arithmetic and multidimensional epistemic M-IT2-F arithmetic, which allows to improve the calculation results on IT2-F numbers. The paper explained the calculation way with use of this new arithmetic and its most important algebraic properties. For comparison of both competitive arithmetic types they were applied to solve two problems with granular data. The examples show that IT2-SF arithmetic delivers imprecise results which are burden by increasing entropy phenomenon and is not able to solve more difficult problems. The reason is its 2-dimensional character and lack of certain but very important mathematical properties. The comparisons presented in the paper show clear benefits of using multidimensional M-IT2-F arithmetic which, though more complicated, has greater effectiveness and precision.

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