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Topological structures induced by L-fuzzifying approximation operators

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Abstract

This paper further studies topological structures induced by L-fuzzifying approximation operators, where L denotes a completely distributive De Morgan algebra. Firstly, the Alexandrov topologies induced by L-fuzzy relations are investigated with respect to L-fuzzifying approximation operators. Especially, the relationships among those Alexandrov topologies are discussed. Secondly, pseudo-similarity sets of L-fuzzy relations are proposed based on the Alexandrov topologies induced by upper and lower L-fuzzifying approximation operators. Meanwhile, the properties of pseudo-similarity sets are discussed, where some examples are presented to show the differences between similarity set of fuzzy relations and pseudo-similarity set of L-fuzzy relations.

Keywords: L-fuzzy relation, topology, L-fuzzifying approximation operator, pseudo-similarity set.

1 Introduction

Pawlak [20, 21] proposed rough set theory, which is an effective mathematical tool to model and process vagueness, imprecision and uncertainty in information systems. The knowledge hidden in information systems can be expressed in the form of decision rule(s) by the lower and upper rough approximation operators. The equivalence relations in Pawlak's rough sets were replaced as binary relations [23, 32], covers [2, 16], or neighborhood [26, 30]. Moreover, the topological structures of rough sets were investigated in [6, 8, 15, 33] with respect to different binary relations.

Dubois and Prade [4] proposed rough fuzzy sets and fuzzy rough sets, which were further studied in [3, 14, 17, 18, 24, 27, 28, 29, 31]. There were three kinds of combinations of fuzzy sets and rough sets to dates. Firstly, crisp relations and fuzzy sets were applied to proposed rough fuzzy sets in [4, 29]. Secondly, the combination of rough sets and fuzzy sets in [31] was defined with fuzzy relations and crisp subsets. Moreover, Pang et al. [19] proposed L-fuzzifying rough sets with L-fuzzy relations and crisp subsets, where L denotes a completely distributive De Morgan algebra. Thirdly, fuzzy rough sets were studied with fuzzy relations and fuzzy sets, such as fuzzy rough sets based on t-(co)norms [18, 27, 29], fuzzy rough sets based on t-norms and fuzzy implications [3, 24, 28], fuzzy covering rough sets [14] and so on. Especially, there are some differences between L-fuzzifying rough sets and the other fuzzy rough sets. For example, a transitive fuzzy relation in fuzzy rough sets based on t-(co)norms or fuzzy rough sets based on t-norms and fuzzy implications is often characterized by the relationship between upper (resp. lower) fuzzy rough approximation operators and the composition of upper (resp. lower) fuzzy rough approximation operators. However, the composition of upper or lower L-fuzzifying approximation operators does not necessarily make sense in L-fuzzifying rough sets, because the approximated subsets are required to be crisp. Therefore, L-fuzzifying rough sets should be studied more thoroughly.

In recent years, the topological structures of fuzzy rough sets were investigated in many published works [11, 12, 13, 22, 25, 27]. Li and Cui [12] studied topological structures of fuzzy rough sets [4], where similarity sets of fuzzy relations were investigated. The topological properties of fuzzy rough sets based on t-(co)norms were discussed in [13, 27]. Moreover, Wang [25] studied fuzzy topologies induced by the lower fuzzy rough approximation operators determined

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by fuzzy implications. Considering the particularity of L-fuzzifying rough sets mentioned above, this paper intends to investigate the topological structures of L-fuzzifying rough sets.

The purpose of this paper is to provide a theoretical background for the topological properties of L-fuzzifying rough sets. Firstly, we study the Alexandrov topologies induced by upper and lower L-fuzzifying approximation operators with respect to an L-fuzzifying approximation space (U, R), where R is an L-fuzzy relation on U. We also point out that the upper L-fuzzifying approximation operator and the lower L-fuzzifying approximation operator are not necessarily a closure operator and an interior operator with respect to the Alexandrov topology $\mathscr{L}(R)$, respectively, even if R is an L-fuzzy preorder. Secondly, we propose pseudo-similarity sets of L-fuzzy relations based on the Alexandrov topologies induced by upper and lower L-fuzzifying approximation operators. We also give some examples to show the differences between similarity sets of fuzzy relations and pseudo-similarity sets of L-fuzzy relations.

The content of this paper is organized as follows. In Section 2, we recall some fundamental concepts and related properties of L-fuzzifying rough sets. Section 3 studies the Alexandrov topologies induced by upper and lower L-fuzzifying approximation operators. In Section 4, we investigate pseudo-similarity sets of L-fuzzy relations. Moreover, we provide some examples to show the differences between similarity sets of fuzzy relations and pseudo-similarity sets of L-fuzzy relations.

2 Preliminaries

In this section, we present some basic concepts and terminologies used throughout the paper.

A *completely distributive De Morgan algebra* is an algebra $(L, \wedge, \vee, ', 0, 1)$ such that $(L, \wedge, \vee, 0, 1)$ is a completely distributive lattice with an order-reversing involution "$'$". In the sequel, L always denotes a completely distributive De Morgan algebra for short, if not otherwise specified.

Let U and W denote two nonempty sets, which are called universes. The powerset of W is denoted by $\mathcal{P}(W)$. For each $A \in \mathcal{P}(W)$, the complement of A is denoted by A^c and the characteristic function of A is denoted by 1_A. Assume that U is a universe, then a mapping $A : U \to L$ is called an *L-fuzzy set* [5] on U. The family of all L-fuzzy sets on U is denoted as $\mathcal{F}_L(U)$.

The operations on L-fuzzy sets are defined as follows: for all $A, B \in \mathcal{F}_L(U)$, $\{A_i\}_{i\in I} \subseteq \mathcal{F}_L(U)$ and $x \in U$,

$$(A \wedge B)(x) = A(x) \wedge B(x), \quad (A \vee B)(x) = A(x) \vee B(x),$$
$$\Big(\bigwedge_{i\in I} A_i\Big)(x) = \bigwedge_{i\in I} A_i(x), \quad \Big(\bigvee_{i\in I} A_i\Big)(x) = \bigvee_{i\in I} A_i(x).$$

The order relation on L-fuzzy sets is defined as $A \leqslant B$ if and only if $A(x) \leqslant B(x)$ for all $x \in U$.

An L-fuzzy set $R \in \mathcal{F}_L(U \times W)$ is called an L-fuzzy relation from U to W. If $\bigvee_{y\in W} R(x, y) = 1$ for all $x \in U$, then R is said to be *serial* [1]. When there exists $y \in W$ such that $R(x, y) = 1$ for all $x \in U$, R is said to be *strongly serial*. If $U = W$, then R is an L-fuzzy relation on U. For every L-fuzzy relation R on U, a fuzzy relation R^{-1} is defined as $R^{-1}(x, y) = R(y, x)$ for all $x, y \in U$.

Definition 2.1. [34] *Consider R be an L-fuzzy relation on U. Then R is said to be*

(1) *reflexive if $R(x, x) = 1$ for all $x \in U$;*

(2) *symmetric if $R(x, y) = R(y, x)$ for all $x, y \in U$;*

(3) *transitive if $R(x, y) \wedge R(y, z) \leqslant R(x, z)$ for all $x, y, z \in U$.*

An L-fuzzy relation R is called an L-fuzzy tolerance, if it is reflexive and symmetric, and an L-fuzzy preorder, if it is reflexive and transitive. Moreover, two L-fuzzy relations R^0 and R^r [12, 13] *induced by R are defined as*

$$R^0(x, y) = \begin{cases} 0, & x = y; \\ R(x, y), & x \neq y; \end{cases} \text{ and } R^r(x, y) = \begin{cases} 1, & x = y; \\ R(x, y), & x \neq y; \end{cases} \text{ respectively.}$$

Definition 2.2. [10] *A subset $\omega \subseteq \mathcal{P}(U)$ is called a topology, if it satisfies:*

(1) $\emptyset, U \in \omega$,

(2) $A \cap B \in \omega$ *for all* $A, B \in \omega$,

(3) $\bigcup_{i\in I} A_i \in \omega$, *for all* $\{A_i\}_{i\in I} \subseteq \omega$.

A topology $\omega \subseteq \mathcal{P}(U)$ is said to be Alexandrov, if item (2) is replaced as
(2) $\bigcap_{i\in I} A_i \in \omega$, *for all* $\{A_i\}_{i\in I} \subseteq \omega$.

Remark 2.3. *Pang et al.* [19] *studied topological structures of L-fuzzifying rough sets with L-fuzzifying topologies [7]. A mapping $\tau : \mathcal{P}(U) \to L$ is called an L-fuzzifying topology on U, if it satisfies*

(1) $\tau(\emptyset) = \tau(U) = 1$,

(2) $\tau(A \cap B) \geqslant \tau(A) \wedge \tau(B)$, *for all* $A, B \in \mathcal{P}(U)$,

(3) $\tau\big(\bigcup_{i\in I} A_i\big) \geqslant \bigwedge_{i\in I} \tau(A_i)$, *for all* $\{A_i\}_{i\in I} \subseteq \mathcal{P}(U)$.

In fact, a topology ω is a special case of L-fuzzifying topology, that is,

$$\tau_\omega(A) = \begin{cases} 1, & \text{if } A \in \omega; \\ 0, & \text{if } A \notin \omega. \end{cases}$$

Hence, we obtain that τ_ω is an L-fuzzifying topology. Thus, our conclusions are much stricter than the conclusions obtained in [19].

Definition 2.4. [12] *Consider $\omega \subseteq \mathcal{P}(U)$ be a topology. Then the closure operator of $A \in \mathcal{P}(U)$ with respect to the topology ω denoted as $cl_\omega(A)$ is defined as*

$$cl_\omega(A) = \bigcap \big\{B \in \omega^c | A \subseteq B\big\},$$

where ω^c is defined as $\omega^c = \{D^c | D \in \omega\}$. The interior operator of $A \in \mathcal{P}(U)$ with respect to the topology ω denoted as $int_\omega(A)$ is defined as

$$int_\omega(A) = \bigcup \big\{B \in \omega | B \subseteq A\big\}.$$

Consider R be an L-fuzzy relation from U to W. Then the triple (U, W, R) is called an *L-fuzzifying approximation space* [19]. If $U = W$ and R is an L-fuzzy relation on U, then we also call (U, R) an L-fuzzifying approximation space.

Definition 2.5. [19] *Let (U, W, R) be an L-fuzzifying approximation space. Then the following mappings $\overline{R}, \underline{R} : \mathcal{P}(W) \to \mathcal{F}_L(U)$ are defined as follows: for all $A \in \mathcal{P}(W)$ and $x \in U$,*

$$\overline{R}(A)(x) = \bigvee_{y\in A} R(x, y) \text{ and } \underline{R}(A)(x) = \bigwedge_{y\notin A} R(x, y)'.$$

The mappings $\underline{R}$ and $\overline{R}$ are called the lower and upper L-fuzzifying approximation operators, respectively. The pair $\big(\underline{R}(A), \overline{R}(A)\big)$ is called an L-fuzzifying rough set with respect to (U, W, R). Especially, consider R be an L-fuzzy relation on U. Then a crisp set $A \in \mathcal{P}(U)$ is said to be

(1) *upper definable with respect to an L-fuzzy relation R, if $\overline{R}(A) = 1_A$; the family of all upper definable set(s) with respect to R is denoted as $\mathscr{D}_U(R)$, that is, $\mathscr{D}_U(R) = \big\{A \in \mathcal{P}(U) | \overline{R}(A) = 1_A\big\}$.*

(2) *lower definable with respect to an L-fuzzy relation R, if $\underline{R}(A) = 1_A$; the family of all lower definable set(s) with respect to R is denoted as $\mathscr{D}_L(R)$, that is, $\mathscr{D}_L(R) = \big\{A \in \mathcal{P}(U) | \underline{R}(A) = 1_A\big\}$.*

The properties of L-fuzzifying rough sets are collected as follows.

Theorem 2.6. [19] *Let (U, W, R) be an L-fuzzifying approximation space. Then the following hold.*

(1) $\overline{R}(A) = \underline{R}(A^c)'$ *and* $\underline{R}(A) = \overline{R}(A^c)'$ *for all* $A \in \mathcal{P}(W)$.

(2) $\overline{R}\big(\bigcup_{i\in I} A_i\big) = \bigvee_{i\in I} \overline{R}(A_i)$ *and* $\underline{R}\big(\bigcap_{i\in I} A_i\big) = \bigwedge_{i\in I} \underline{R}(A_i)$ *for all* $\{A_i\}_{i\in I} \subseteq \mathcal{P}(W)$.

(3) *If R is serial, then* $\overline{R}(W) = 1_U$ *and* $\underline{R}(\emptyset) = 1_\emptyset$.

(4) *If $U = W$ and R is a reflexive L-fuzzy relation, then* $\underline{R}(A) \leqslant 1_A \leqslant \overline{R}(A)$ *for all* $A \in \mathcal{P}(U)$.

3 Topological structures induced by *L*-fuzzifying approximation operators

In this section, we study topologies induced by L-fuzzy relations and discuss their relationships.

Theorem 3.1. *Let (U, W, R) be an L-fuzzifying approximation space and R be strongly serial. Then $\mathscr{D}(R) = \{A \in \mathcal{P}(W) | \overline{R}(A) = \underline{R}(A)\}$ is an Alexandrov topology.*

Proof. We verify the items in Definition 2.2 as follows.

(1) By Definition 2.5, we have for all $x \in U$,

$$\overline{R}(\emptyset)(x) = \bigvee_{y \in \emptyset} R(x, y) = \bigvee \emptyset = 0,$$
$$\underline{R}(W)(x) = \bigwedge_{y \notin W} R(x, y)' = \bigwedge \emptyset = 1.$$

Hence, it follows from Theorem 2.6(3) that $\emptyset, W \in \mathscr{D}(R)$.

(2) Consider $x_0 \in U$. Then there is a $y_0 \in W$ such that $R(x_0, y_0) = 1$, because R is strongly serial. Let $B_{x_0} = \{y \in W | R(x_0, y) = 1\}$. We verify $\underline{R}(A) \leqslant \overline{R}(A)$ for all $A \in \mathcal{P}(W)$ in the following two cases.

Case 1: $B_{x_0} \cap A \neq \emptyset$:

By Definition 2.5, we have

$$\overline{R}(A)(x_0) = \bigvee_{y \in A} R(x_0, y) \geqslant \bigvee_{y \in B_{x_0} \cap A} R(x_0, y) = 1 \geqslant \underline{R}(A)(x_0).$$

Case 2: $B_{x_0} \cap A = \emptyset$:

Hence we have $B_{x_0} \subseteq A^c$. It follows from Definition 2.5 that the following holds

$$\underline{R}(A)(x_0) = \bigwedge_{y \notin A} R(x_0, y)' = \bigwedge_{y \in A^c} R(x_0, y)' \leqslant \bigwedge_{y \in B_{x_0}} R(x_0, y)' = 0 \leqslant \overline{R}(A)(x_0).$$

Thus it follows from Theorem 2.6(2) that the following holds for all $\{A_i\}_{i \in I} \subseteq \mathscr{D}(R)$,

$$\underline{R}\Big(\bigcap_{i \in I} A_i\Big) \leqslant \overline{R}\Big(\bigcap_{i \in I} A_i\Big) \leqslant \bigwedge_{i \in I} \overline{R}(A_i) = \bigwedge_{i \in I} \underline{R}(A_i) = \underline{R}\Big(\bigcap_{i \in I} A_i\Big).$$

Therefore, $\bigcap_{i \in I} A_i \in \mathscr{D}(R)$ for all $\{A_i\}_{i \in I} \subseteq \mathscr{D}(R)$.

(3) Similarly, by Theorem 2.6(2) and $\underline{R}(A) \leqslant \overline{R}(A)$ for all $A \in \mathcal{P}(W)$, we obtain for all $\{A_i\}_{i \in I} \subseteq \mathscr{D}(R)$,

$$\overline{R}\Big(\bigcup_{i \in I} A_i\Big) = \bigvee_{i \in I} \overline{R}(A_i) = \bigvee_{i \in I} \underline{R}(A_i) \leqslant \underline{R}\Big(\bigcup_{i \in I} A_i\Big) \leqslant \overline{R}\Big(\bigcup_{i \in I} A_i\Big).$$

Thus we have $\bigcup_{i \in I} A_i \in \mathscr{D}(R)$ for all $\{A_i\}_{i \in I} \subseteq \mathscr{D}(R)$. □

Theorem 3.2. *Let (U, R) be an L-fuzzifying approximation space and R be reflexive. Then $\mathscr{D}_U(R)$ and $\mathscr{D}_L(R)$ are Alexandrov topologies.*

Proof. By Theorem 2.6(1), we obtain for all $A \in \mathcal{P}(U)$

$$A \in \mathscr{D}_U(R) \Leftrightarrow A^c \in \mathscr{D}_L(R) \text{ and } A \in \mathscr{D}_L(R) \Leftrightarrow A^c \in \mathscr{D}_U(R).$$

Thus we only need to verify that $\mathscr{D}_U(R)$ is an Alexandrov topology.

As a reflexive L-fuzzy relation R must be strongly serial, it follows from Theorems 3.1 and 2.6(2) that $\emptyset, W \in \mathscr{D}_U(R)$ and $\bigcup_{i \in I} A_i \in \mathscr{D}_U(R)$ for all $\{A_i\}_{i \in I} \subseteq \mathscr{D}_U(R)$, respectively. Consider $\{A_i\}_{i \in I} \subseteq \mathscr{D}_U(R)$. Then it follows from Theorem 2.6(2) and (4) that the following holds,

$$\bigwedge_{i \in I} 1_{A_i} \leqslant \overline{R}\Big(\bigcap_{i \in I} A_i\Big) \leqslant \bigwedge_{i \in I} \overline{R}(A_i) = \bigwedge_{i \in I} 1_{A_i}.$$

Thus we obtain $\bigcap_{i \in I} A_i \in \mathscr{D}_U(R)$ for all $\{A_i\}_{i \in I} \subseteq \mathscr{D}_U(R)$. Therefore, $\mathscr{D}_U(R)$ is an Alexandrov topology. □

The relationship among Alexandrov topologies mentioned above is studied as follows:

Theorem 3.3. *Let (U, R) be an L-fuzzifying approximation space and R be reflexive. Then $\mathscr{D}(R) = \mathscr{D}_U(R) \cap \mathscr{D}_L(R)$.*

Proof. Let $A \in \mathscr{D}_U(R) \cap \mathscr{D}_L(R)$. Then it is obvious that $\underline{R}(A) = 1_A = \overline{R}(A)$ holds. Thus we have $A \in \mathscr{D}(R)$. Conversely, consider $A \in \mathscr{D}(R)$. Then it follows from Theorem 2.6(4) that $\underline{R}(A) = 1_A = \overline{R}(A)$. Thus we obtain $A \in \mathscr{D}_U(R) \cap \mathscr{D}_L(R)$. Therefore, $\mathscr{D}(R) = \mathscr{D}_U(R) \cap \mathscr{D}_L(R)$ holds. □

Considering a special L-fuzzy relation, we obtain the following conclusion.

Theorem 3.4. *Let (U, R) be an L-fuzzifying approximation space and R be an L-fuzzy tolerance. Then $\mathscr{D}_U(R) = \mathscr{D}(R) = \mathscr{D}_L(R)$.*

Proof. Because R is an L-fuzzy tolerance, it follows from Theorems 3.2 and 3.3 that $\mathscr{D}(R)$, $\mathscr{D}_U(R)$ and $\mathscr{D}_L(R)$ are Alexandrov topologies and $\emptyset, U \in \mathscr{D}(R) = \mathscr{D}_U(R) \cap \mathscr{D}_L(R)$. Thus it is obvious that we only need show when $A \in \mathcal{P}(U)$ and A is neither $\emptyset$ nor U, $A \in \mathscr{D}_U(R)$ if and only if $A \in \mathscr{D}_L(R)$. We verify that proposition in the following two cases.

Case 1: $A \in \mathscr{D}_U(R)$:

If $x_0 \notin A$, then it follows from Theorem 2.6(4) that $\underline{R}(A)(x_0) \leqslant 1_A(x_0) = 0$. Meanwhile, as $A \in \mathscr{D}_U(R)$ and $x_0 \notin A$ hold, we have

$$\overline{R}(A)(x_0) = \bigvee_{y \in A} R(x_0, y) = 1_A(x_0) = 0.$$

Hence, we obtain $R(x_0, y) = 0$, when $x_0 \notin A$ and $y \in A$.

Conversely, let $y_0 \in A$. Then it follows from Definition 2.5 that $\underline{R}(A)(y_0) = \bigwedge_{x \notin A} R(y_0, x)'$ holds. As R is symmetric, we have $R(y_0, x) = 0$, when $x \notin A$ and $y_0 \in A$. Thus $\underline{R}(A)(y_0) = 1$ holds. On the basis of discussion mentioned above, we obtain that $A \in \mathscr{D}_U(R)$ implies $A \in \mathscr{D}_L(R)$.

Case 2: $A \in \mathscr{D}_L(R)$:

If $x_0 \in A$, then we have $\overline{R}(A)(x_0) \geqslant 1_A(x_0) = 1$ by Theorem 2.6(4). Moreover, by $A \in \mathscr{D}_L(R)$ and $x_0 \in A$, the following holds,

$$\underline{R}(A)(x_0) = \bigwedge_{y \notin A} R(x_0, y)' = 1_A(x_0) = 1.$$

Thus we have $R(x_0, y) = 0$, when $x_0 \in A$ and $y \notin A$.

Conversely, consider $y_0 \notin A$. Then it follows from Definition 2.5 that $\overline{R}(A)(y_0) = \bigvee_{x \in A} R(y_0, x)$ holds. Because R is an L-fuzzy tolerance, we have $R(y_0, x) = 0$, when $x \in A$ and $y_0 \notin A$. Thus $\overline{R}(A)(y_0) = 0$ holds.

Based on discussion mentioned above, we have that $A \in \mathscr{D}_L(R)$ implies $A \in \mathscr{D}_U(R)$. □

Meanwhile, there is a natural problem whether those Alexandrov topologies $\mathscr{D}(R)$, $\mathscr{D}_U(R)$ and $\mathscr{D}_L(R)$ are still equal to each other without the symmetry of L-fuzzy relation R. In fact, the Alexandrov topologies $\mathscr{D}(R)$, $\mathscr{D}_U(R)$ and $\mathscr{D}_L(R)$ are not equal to each other, even if R is an L-fuzzy preorder.

Example 3.5. *Consider the complete lattice $\mathscr{L} = \{0, a, 1\}$. Then the order relation is defined as $0 < a < 1$. The order-reversing involution $'$ is defined as $0' = 1$, $a' = a$ and $1' = 0$. Let $U = \{x_1, x_2, x_3\}$ and L-fuzzy relation R on U as*

$$R = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then R is an L-fuzzy preorder.

By Definition 2.5, we have

$$\overline{R}(\{x_1\}) = \frac{1}{x_1} + \frac{0}{x_2} + \frac{0}{x_3}, \qquad \underline{R}(\{x_1\}) = \frac{a}{x_1} + \frac{0}{x_2} + \frac{0}{x_3},$$
$$\overline{R}(\{x_2\}) = \frac{a}{x_1} + \frac{1}{x_2} + \frac{0}{x_3}, \qquad \underline{R}(\{x_2\}) = \frac{0}{x_1} + \frac{1}{x_2} + \frac{0}{x_3},$$
$$\overline{R}(\{x_3\}) = \frac{0}{x_1} + \frac{0}{x_2} + \frac{1}{x_3}, \qquad \underline{R}(\{x_3\}) = \frac{0}{x_1} + \frac{0}{x_2} + \frac{1}{x_3},$$

$$\overline{R}(\{x_1,x_2\})=\frac{1}{x_1}+\frac{1}{x_2}+\frac{0}{x_3}, \qquad \underline{R}(\{x_1,x_2\})=\frac{1}{x_1}+\frac{1}{x_2}+\frac{0}{x_3},$$
$$\overline{R}(\{x_1,x_3\})=\frac{1}{x_1}+\frac{0}{x_2}+\frac{1}{x_3}, \qquad \underline{R}(\{x_1,x_3\})=\frac{a}{x_1}+\frac{0}{x_2}+\frac{1}{x_3},$$
$$\overline{R}(\{x_2,x_3\})=\frac{a}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}, \qquad \underline{R}(\{x_2,x_3\})=\frac{0}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}.$$

Hence, we obtain

$$\mathscr{D}_U(R)=\{\emptyset,\{x_1\},\{x_3\},\{x_1,x_2\},\{x_1,x_3\},\{x_1,x_2,x_3\}\},$$
$$\mathscr{D}_L(R)=\{\emptyset,\{x_2\},\{x_3\},\{x_1,x_2\},\{x_2,x_3\},\{x_1,x_2,x_3\}\},$$
$$\mathscr{D}(R)=\{\emptyset,\{x_3\},\{x_1,x_2\},\{x_1,x_2,x_3\}\}.$$

Therefore, Theorem 3.4 does not necessarily hold without the symmetry of L-fuzzy relation R.

Theorem 3.6. *Let (U,R) be an L-fuzzifying approximation space. Then the following hold for all $A\in\mathcal{P}(U)$,*

(1) $\overline{R^r}(A)=1_A \Leftrightarrow \overline{R}(A)\leqslant 1_A \Leftrightarrow \overline{R^0}(A)\leqslant 1_A$.

(2) $\underline{R^r}(A)=1_A \Leftrightarrow 1_A\leqslant \underline{R}(A) \Leftrightarrow 1_A\leqslant \underline{R^0}(A)$.

Proof. (1) Consider $\overline{R^r}(A)=1_A$. Then we verify $\overline{R}(A)\leqslant 1_A$ in the following two cases.

Case 1: $x\in A$

We obtain

$$\overline{R}(A)(x)=R(x,x)\vee\Big\{\bigvee_{y\in A,y\neq x}R(x,y)\Big\}\leqslant R^r(x,x)\vee\Big\{\bigvee_{y\in A,y\neq x}R^r(x,y)\Big\}=\overline{R^r}(A)(x).$$

By $\overline{R^r}(A)=1_A$, we have $\overline{R}(A)(x)\leqslant 1_A(x)$, if $x\in A$.

Case 2: $x\notin A$

Similarly, the following holds

$$\overline{R}(A)(x)=\bigvee_{y\in A,y\neq x}R(x,y)=\bigvee_{y\in A,y\neq x}R^r(x,y)=\overline{R^r}(A)(x)=1_A(x).$$

Thus we obtain $\overline{R}(A)\leqslant 1_A$.

On the other hand, consider $\overline{R}(A)\leqslant 1_A$. Then we prove $\overline{R^r}(A)=1_A$ in the following two cases.

Case 1: $x\in A$

$$\overline{R^r}(A)(x)=\bigvee_{y\in A}R^r(x,y)=R^r(x,x)\vee\Big\{\bigvee_{y\in A,y\neq x}R^r(x,y)\Big\}=1=1_A(x).$$

Case 2: $x\notin A$

We have

$$\overline{R^r}(A)(x)=\bigvee_{y\in A,y\neq x}R^r(x,y)=\bigvee_{y\in A,y\neq x}R(x,y)=\overline{R}(A)(x)\leqslant 1_A(x)=0.$$

Thus we obtain $\overline{R^r}(A)=1_A$. Similarly, we can verify the equivalence between $\overline{R}(A)\leqslant 1_A$ and $\overline{R^0}(A)\leqslant 1_A$.

(2) By Theorem 2.6(1), the equivalent expression of $\underline{R^r}(A)=1_A\Leftrightarrow 1_A\leqslant\underline{R}(A)\Leftrightarrow 1_A\leqslant\underline{R^0}(A)$ is

$$\overline{R^r}(A^c)=1_{A^c}\Leftrightarrow\overline{R}(A^c)\leqslant 1_{A^c}\Leftrightarrow\overline{R^0}(A^c)\leqslant 1_{A^c}.$$

Thus it follows immediately from item (1). □

Let (U,R) be an L-fuzzifying approximation space and denote

$$\mathscr{U}(R)=\{A\in\mathcal{P}(U)|\overline{R}(A)\leqslant 1_A\} \text{ and } \mathscr{L}(R)=\{A\in\mathcal{P}(U)|1_A\leqslant\underline{R}(A)\}.$$

Then we have the following conclusions for a general L-fuzzy relation.

Theorem 3.7. *Let (U, R) be an L-fuzzifying approximation space. Then*

(1) *$\mathscr{U}(R)$ and $\mathscr{L}(R)$ are Alexandrov topologies.*

(2) *$\mathscr{U}(R) = \mathscr{U}(R^0) = \mathscr{U}(R^r) = \mathscr{D}_U(R^r)$ and $\mathscr{L}(R) = \mathscr{L}(R^0) = \mathscr{L}(R^r) = \mathscr{D}_L(R^r)$.*

(3) *$\overline{R}(A) \leqslant 1_{cl_{\mathscr{L}(R)}(A)}$ and $\underline{R}(A) \geqslant 1_{int_{\mathscr{L}(R)}(A)}$ for all $A \in \mathcal{P}(U)$.*

Proof. (1) By Definition 2.5, it is obvious that $\overline{R}(\emptyset) = 1_\emptyset$ and $\overline{R}(U) \leqslant 1_U$ hold. Consider $\{A_i\}_{i\in I} \subseteq \mathscr{U}(R)$. Then it follows from Theorem 2.6(2) that the following hold

$$\overline{R}\Big(\bigcup_{i\in I} A_i\Big) = \bigvee_{i\in I} \overline{R}(A_i) \leqslant \bigvee_{i\in I} 1_{A_i} \text{ and } \overline{R}\Big(\bigcap_{i\in I} A_i\Big) \leqslant \bigwedge_{i\in I} \overline{R}(A_i) \leqslant \bigwedge_{i\in I} 1_{A_i}.$$

Thus $\mathscr{U}(R)$ is an Alexandrov topology. Moreover, as $A \in \mathscr{L}(R)$ if and only if $A^c \in \mathscr{U}(R)$, it is obvious that $\mathscr{L}(R)$ is also an Alexandrov topology.
(2) It follows immediately from Theorems 2.6(4) and 3.6.
(3) It follows from Definition 2.4 that the following holds for all $A \in \mathcal{P}(U)$,

$$cl_{\mathscr{L}(R)}(A) = \bigcap \big\{B \in \mathscr{L}(R)^c | A \subseteq B\big\} = \bigcap \big\{B \in \mathscr{U}(R) | A \subseteq B\big\} \supseteq \bigcap \big\{B \in \mathscr{U}(R) | \overline{R}(A) \leqslant \overline{R}(B)\big\}.$$

Thus we have $\overline{R}(A) \leqslant 1_{cl_{\mathscr{L}(R)}(A)}$ for all $A \in \mathcal{P}(U)$. Similarly, we can obtain $\underline{R}(A) \geqslant 1_{int_{\mathscr{L}(R)}(A)}$ for all $A \in \mathcal{P}(U)$. □

Remark 3.8. *In fact, $\overline{R}$ and $\underline{R}$ are not necessarily a closure operator and an interior operator with respect to the Alexandrov topology $\mathscr{L}(R)$, respectively, even if R is an L-fuzzy preorder. For example, considering L-fuzzifying rough sets in Example 3.5, it follows from Definition 2.4 that the following hold*

$$cl_{\mathscr{L}(R)}\big(\{x_2\}\big) = \{x_1, x_2\} \text{ and } int_{\mathscr{L}(R)}\big(\{x_1\}\big) = \emptyset.$$

In fact, Example 3.5 shows that

$$\overline{R}\big(\{x_2\}\big) = \frac{a}{x_1} + \frac{1}{x_2} + \frac{0}{x_3} \text{ and } \underline{R}\big(\{x_1\}\big) = \frac{a}{x_1} + \frac{0}{x_2} + \frac{0}{x_3}.$$

However, Pang et al. [19] *pointed out that $\overline{R}$ and $\underline{R}$ are an L-fuzzifying closure operator and an L-fuzzifying interior operator, respectively, where R is an L-fuzzy preorder. There is a big difference between topology and L-fuzzifying topology, which further shows that our conclusions are much stricter than the conclusions in* [19].

Theorem 3.9. *Consider R_i be an L-fuzzy relation on U for all $i \in I$. Then the following hold*

$$\mathscr{U}\Big(\bigvee_{i\in I} R_i\Big) = \bigcap_{i\in I} \mathscr{U}(R_i) \text{ and } \mathscr{L}\Big(\bigvee_{i\in I} R_i\Big) = \bigcap_{i\in I} \mathscr{L}(R_i).$$

Proof. By Definition 2.5, the following hold for all $A \in \mathcal{P}(U)$ and $x \in U$,

$$\Big(\overline{\bigvee_{i\in I} R_i}\Big)(A)(x) = \bigvee_{y\in A} \Big(\bigvee_{i\in I} R_i(x,y)\Big) = \bigvee_{i\in I} \bigvee_{y\in A} R_i(x,y) = \bigvee_{i\in I} \overline{R_i}(A)(x),$$

$$\Big(\underline{\bigvee_{i\in I} R_i}\Big)(A)(x) = \bigwedge_{y\notin A} \Big(\bigvee_{i\in I} R_i(x,y)\Big)' = \bigwedge_{i\in I} \bigwedge_{y\notin A} R_i(x,y)' = \bigwedge_{i\in I} \underline{R_i}(A)(x).$$

Hence, we obtain for all $A \in \mathcal{P}(U)$,

$$\begin{aligned} A \in \mathscr{U}\Big(\bigvee_{i\in I} R_i\Big) &\Leftrightarrow \Big(\overline{\bigvee_{i\in I} R_i}\Big)(A) \leqslant 1_A \\ &\Leftrightarrow \bigvee_{i\in I} \overline{R_i}(A) \leqslant 1_A \\ &\Leftrightarrow \overline{R_i}(A) \leqslant 1_A \text{ for all } i \in I \\ &\Leftrightarrow A \in \mathscr{U}(R_i) \text{ for all } i \in I \\ &\Leftrightarrow A \in \bigcap_{i\in I} \mathscr{U}(R_i). \end{aligned}$$

Similarly, the following holds for all $A \in \mathcal{P}(U)$,

$$\begin{aligned} A \in \mathscr{L}\Big(\bigvee_{i\in I} R_i\Big) &\Leftrightarrow 1_A \leqslant \underline{\Big(\bigvee_{i\in I} R_i\Big)}(A) \\ &\Leftrightarrow 1_A \leqslant \bigwedge_{i\in I} \underline{R_i}(A) \\ &\Leftrightarrow 1_A \leqslant \underline{R_i}(A) \text{ for all } i \in I \\ &\Leftrightarrow A \in \mathscr{L}(R_i) \text{ for all } i \in I \\ &\Leftrightarrow A \in \bigcap_{i\in I} \mathscr{L}(R_i). \end{aligned}$$

Thus we obtain

$$\mathscr{U}\Big(\bigvee_{i\in I} R_i\Big) = \bigcap_{i\in I} \mathscr{U}(R_i) \text{ and } \mathscr{L}\Big(\bigvee_{i\in I} R_i\Big) = \bigcap_{i\in I} \mathscr{L}(R_i). \qquad \square$$

Corollary 3.10. *Consider R, S be two L-fuzzy relations on U and $R \leqslant S$. Then the following hold,*

$$\mathscr{U}(S) \subseteq \mathscr{U}(R) \text{ and } \mathscr{L}(S) \subseteq \mathscr{L}(R).$$

Proof. By Theorem 3.9, we have

$$\mathscr{U}(S) = \mathscr{U}(R \vee S) = \mathscr{U}(R) \cap \mathscr{U}(S) \text{ and } \mathscr{L}(S) = \mathscr{L}(R \vee S) = \mathscr{L}(R) \cap \mathscr{L}(S).$$

Hence, we obtain $\mathscr{U}(S) \subseteq \mathscr{U}(R)$ and $\mathscr{L}(S) \subseteq \mathscr{L}(R)$. $\square$

Consider R and S be two L-fuzzy relations on U. Then the composition $R \circ S$ of R and S is defined for all $x, y \in U$ as

$$(R \circ S)(x, y) = \bigvee_{z\in U} \big\{R(x, z) \wedge S(z, y)\big\}.$$

Meanwhile, we denote

$$R^n \triangleq \underbrace{R \circ R \circ \cdots \circ R}_{n} \text{ for all } n \in \mathbb{N}.$$

We call the smallest transitive L-fuzzy relation containing L-fuzzy relation R as the *transitive closure* of R and denoted as $t(R)$. Kaufmann [9] pointed out that the transitive closure of L-fuzzy relation R on U is the L-fuzzy relation

$$t(R) = \bigvee_{n=1}^{\infty} R^n.$$

Theorem 3.11. *Let R, S_1 and S_2 be L-fuzzy relations on U. Then the following hold.*

(1) *If $\mathscr{U}(R) = \mathscr{U}(S_1) = \mathscr{U}(S_2)$, then $\mathscr{U}(R) \subseteq \mathscr{U}(S_1 \circ S_2)$.*

(2) *If $\mathscr{L}(R) = \mathscr{L}(S_1) = \mathscr{L}(S_2)$, then $\mathscr{L}(R) \subseteq \mathscr{L}(S_1 \circ S_2)$.*

Proof. (1) Consider $A \in \mathscr{U}(R)$. Then we verify $\mathscr{U}(R) \subseteq \mathscr{U}(S_1 \circ S_2)$ in the following two cases.

Case 1: $x_0 \in A$

It is obvious that the following holds

$$\big(\overline{S_1 \circ S_2}\big)(A)(x_0) \leqslant 1 = 1_A(x_0).$$

Case 2: $x_0 \notin A$

It follows from $\mathscr{U}(R) = \mathscr{U}(S_1) = \mathscr{U}(S_2)$ that the following hold,

$$\begin{aligned} \overline{S_1}(A)(x_0) &= \bigvee_{y\in A} S_1(x_0, y) \leqslant 1_A(x_0) = 0, \\ \overline{S_2}(A)(x_0) &= \bigvee_{y\in A} S_2(x_0, y) \leqslant 1_A(x_0) = 0. \end{aligned}$$

Hence, we obtain if $x_0 \notin A$ and $y \in A$, then $S_1(x_0, y) = S_2(x_0, y) = 0$.

Assume that $(S_1 \circ S_2)(x_0, y) \neq 0$ and $y \in A$ hold, then there is $z_0 \in U$ such that $S_1(x_0, z_0) \wedge S_2(z_0, y) \neq 0$. Hence it is obvious that $S_1(x_0, z_0) \neq 0$ and $S_2(z_0, y) \neq 0$. By $S_1(x_0, z_0) \neq 0$, we have $z_0 \notin A$. Otherwise, we have $S_1(x_0, z_0) = 0$. However, when $z_0 \notin A$ and $y \in A$, we obtain $S_2(z_0, y) = 0$, which implies a contradiction. Thus, when $x_0 \notin A$ and $y \in A$, we have

$$(S_1 \circ S_2)(x_0, y) = 0.$$

It follows from Definition 2.5 that the following holds

$$\big(\overline{S_1 \circ S_2}\big)(A)(x_0) = \bigvee_{y \in A} (S_1 \circ S_2)(x_0, y) = 0.$$

Hence, we obtain $\big(\overline{S_1 \circ S_2}\big)(A)(x_0) \leqslant 1_A(x_0)$, if $x_0 \notin A$.

Therefore, $\mathscr{U}(R) \subseteq \mathscr{U}(S_1 \circ S_2)$ holds.

(2) It can be proven in a similar way as for item (1). □

4 Pseudo-similarity sets of *L*-fuzzy relations

On the basis of the Alexandrov topologies $\mathscr{U}(R)$ and $\mathscr{L}(R)$ induced by a general L-fuzzy relation R, we study the relationship between two L-fuzzy relations R and S on U, when $\mathscr{U}(R) = \mathscr{U}(S)$ and $\mathscr{L}(R) = \mathscr{L}(S)$ hold.

Definition 4.1. *Let R and S be two L-fuzzy relations on U and $\mathscr{U}(R) = \mathscr{U}(S)$. Then L-fuzzy relation S is said to be pseudo-similar to L-fuzzy relation R. The family of all pseudo-similar L-fuzzy relations of L-fuzzy relation R is called pseudo-similarity set of L-fuzzy relation R and denoted as*

$$\mathscr{S}(R) = \big\{S \in \mathcal{F}_L(U \times U) | \mathscr{U}(R) = \mathscr{U}(S)\big\}.$$

Remark 4.2. *We can also define two L-fuzzy relations R and S on U are pseudo-similar with respect to $\mathscr{L}(R) = \mathscr{L}(S)$. The reason is explained as follows:*

By Theorem 2.6(1), the following hold for all $A \in \mathcal{P}(U)$,

$$A \in \mathscr{L}(R) \Leftrightarrow A^c \in \mathscr{U}(R) \text{ and } A \in \mathscr{U}(R) \Leftrightarrow A^c \in \mathscr{L}(R).$$

Thus we obtain $\mathscr{L}(R) = \mathscr{L}(S)$ if and only if $\mathscr{U}(R) = \mathscr{U}(S)$.

We collect the properties of pseudo-similarity set of L-fuzzy relation R in the following conclusion.

Corollary 4.3. *Let (U, R) be an L-fuzzifying approximation space. Then the following hold.*

(1) $R^0, R^r \in \mathscr{S}(R)$.

(2) *If $S_i \in \mathscr{S}(R)$ for all $i \in I$, then $\bigvee_{i \in I} S_i \in \mathscr{S}(R)$.*

(3) *If $S_1, S_3 \in \mathscr{S}(R)$ and $S_1 \leqslant S_2 \leqslant S_3$, then $S_2 \in \mathscr{S}(R)$.*

Proof. It follows immediately from Theorems 3.7(2), 3.9 and Corollary 3.10. □

Theorem 4.4. *Let (U, R) be an L-fuzzifying approximation space. Then the following hold for all $S_1, S_2 \in \mathscr{S}(R)$,*

$$S_1 \circ (S_2)^r, (S_1)^r \circ S_2, (S_1)^r \circ (S_2)^r \in \mathscr{S}(R).$$

Proof. We only need to verify $S_1 \circ (S_2)^r$. The others can be proven in a similar way.

By Corollary 4.3(1), we have $(S_2)^r \in \mathscr{S}(R)$. Moreover, it follows from Theorem 3.11(1) that $\mathscr{U}(R) \subseteq \mathscr{U}\big(S_1 \circ (S_2)^r\big)$. Meanwhile, it is obvious that $S_1 \leqslant S_1 \circ (S_2)^r$ holds. By Corollary 3.10, we have

$$\mathscr{U}\big(S_1 \circ (S_2)^r\big) \subseteq \mathscr{U}(S_1) = \mathscr{U}(R).$$

Thus we have $\mathscr{U}\big(S_1 \circ (S_2)^r\big) = \mathscr{U}(R)$, which implies $S_1 \circ (S_2)^r \in \mathscr{S}(R)$. □

Corollary 4.5. *Let (U, R) be an L-fuzzifying approximation space. Then $t\big(R^r\big) \in \mathscr{S}(R)$.*

Proof. It follows immediately from Corollary 4.3(1), (2) and Theorem 4.4. □

Remark 4.6. *L-fuzzy preorder $t(R^r)$ is not the unique L-fuzzy preorder in the pseudo-similarity set $\mathscr{S}(R)$, although Li and Cui* [12, 13] *had pointed out that $t(S^r)$ is the unique fuzzy preorder in the similarity set of fuzzy relation S. The reason is that the pseudo-similarity set and similarity set depend on crisp topology and fuzzy topology, respectively. We give the counterexample as follows.*

Example 4.7. *Consider the complete lattice $L = \{0, a, b, c, d, 1\}$. Then the order relation is defined as $0 < a < b, c < d < 1$, where b and c are incomparable. Moreover, the order-reversing involution $'$ is defined as $0' = 1$, $a' = d$ and $b' = c$. Let $U = \{x_1, x_2, x_3\}$, two L-fuzzy relations R and S as*

$$R = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ a & c & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ a & b & 1 \end{bmatrix}, \text{ respectively.}$$

Then both R and S are L-fuzzy preorders.

By Definition 2.5, we have

$$\begin{aligned}
\overline{R}(\{x_1\}) &= \frac{1}{x_1} + \frac{b}{x_2} + \frac{a}{x_3}, & \overline{S}(\{x_1\}) &= \frac{1}{x_1} + \frac{c}{x_2} + \frac{a}{x_3};\\
\overline{R}(\{x_2\}) &= \frac{0}{x_1} + \frac{1}{x_2} + \frac{c}{x_3}, & \overline{S}(\{x_2\}) &= \frac{0}{x_1} + \frac{1}{x_2} + \frac{b}{x_3};\\
\overline{R}(\{x_3\}) &= \frac{0}{x_1} + \frac{0}{x_2} + \frac{1}{x_3}, & \overline{S}(\{x_3\}) &= \frac{0}{x_1} + \frac{0}{x_2} + \frac{1}{x_3};\\
\overline{R}(\{x_1, x_2\}) &= \frac{1}{x_1} + \frac{1}{x_2} + \frac{c}{x_3}, & \overline{S}(\{x_1, x_2\}) &= \frac{1}{x_1} + \frac{1}{x_2} + \frac{b}{x_3};\\
\overline{R}(\{x_1, x_3\}) &= \frac{1}{x_1} + \frac{b}{x_2} + \frac{1}{x_3}, & \overline{S}(\{x_1, x_3\}) &= \frac{1}{x_1} + \frac{c}{x_2} + \frac{1}{x_3};\\
\overline{R}(\{x_2, x_3\}) &= \frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, & \overline{S}(\{x_2, x_3\}) &= \frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}.
\end{aligned}$$

Hence, we obtain

$$\mathscr{U}(R) = \mathscr{U}(S) = \big\{\emptyset, \{x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\big\}.$$

Thus $R, S \in \mathscr{S}(R)$ holds, where both R and S are L-fuzzy preorders. However, $R \neq S$ holds. Similarly, we have the following counterexample, when the completely distributive De morgan algebra degenerates into the closed unit interval.

Consider $L = [0, 1]$ and the order-reversing involution $'$ is defined as $a' = 1 - a$ for all $a \in [0, 1]$. Let $U = \{x_1, x_2, x_3\}$, two fuzzy relation R_1 and S_1 as

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 1 & 0 \\ 0.1 & 0.2 & 1 \end{bmatrix} \text{ and } S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.1 & 0.1 & 1 \end{bmatrix}, \text{ respectively.}$$

Then both R_1 and S_1 are fuzzy preorders. Moreover, we have $R_1, S_1 \in \mathscr{S}(R_1)$. However, we obtain $R_1 \neq S_1$.

In the sequel, we study the algebraic structures of pseudo-similarity set.

Let (U, R) be an L-fuzzifying approximation space and denote

$$\mathscr{S}^0(R) = \big\{S^0 | S \in \mathscr{S}(R)\big\} \text{ and } \mathscr{S}^r(R) = \big\{S^r | S \in \mathscr{S}(R)\big\}.$$

Definition 4.8. *Let U be a universe and a binary operation $\diamond : U \times U \to U$. Then the algebra $(U, \diamond)$ is called a semigroup, if $(x \diamond y) \diamond z = x \diamond (y \diamond z)$ for all $x, y, z \in U$. Meanwhile, the semigroup $(U, \diamond)$ is said to be commutative, that is, $x \diamond y = y \diamond x$ for all $x, y \in U$. Let $(U, \diamond)$ be a semigroup and $W \subseteq U$. Then $(W, \diamond)$ is called a sub-semigroup of $(U, \diamond)$, if $\diamond$ is closed on W.*

Theorem 4.9. *Let (U, R) be an L-fuzzifying approximation space. Then*

(1) $\big(\mathscr{S}(R), \cup\big)$ *is a commutative semigroup.*

(2) $\big(\mathscr{S}^0(R),\cup\big)$ *and* $\big(\mathscr{S}^r(R),\cup\big)$ *are two sub-semigroups of* $\big(\mathscr{S}(R),\cup\big)$.

Proof. It follows immediately from Corollary 4.3(1) and (2). □

Theorem 4.10. *Let* (U,R) *be an L-fuzzifying approximation space. Then* $\big(\mathscr{S}^r(R),\circ\big)$ *is a commutative semigroup.*

Proof. It follows immediately from Theorem 4.4. □

Remark 4.11. *Although Li and Cui* [12, 13] *had pointed out that the transitive subset of similarity set of fuzzy relation is a complete distributive lattice with respect to intersections and unions, the transitive subset of pseudo-similarity set of L-fuzzy relation is not a complete lattice with respect to intersections and unions.*

Example 4.12. *Consider the complete lattice* $L=\{0,a,b,1\}$. *Then the order relation is defined as* $0<a,b<1$, *where* a *and* b *are incomparable. Moreover, the order-reversing involution* $'$ *is defined as* $0'=1$ *and* $a'=b$. *Let* $U=\{x_1,x_2,x_3\}$, *two L-fuzzy relations* R *and* S *as*

$$R=\begin{bmatrix}1&0&0\\a&1&0\\a&0&1\end{bmatrix}\ \textit{and}\ S=\begin{bmatrix}1&0&0\\b&1&0\\b&0&1\end{bmatrix}\ \textit{respectively.}$$

Then both R *and* S *are L-fuzzy preorders.*

By Definition 2.5, we have

$$\begin{aligned}
\overline{R}\big(\{x_1\}\big)&=\frac{1}{x_1}+\frac{a}{x_2}+\frac{a}{x_3}, & \overline{S}\big(\{x_1\}\big)&=\frac{1}{x_1}+\frac{b}{x_2}+\frac{b}{x_3};\\
\overline{R}\big(\{x_2\}\big)&=\frac{0}{x_1}+\frac{1}{x_2}+\frac{0}{x_3}, & \overline{S}\big(\{x_2\}\big)&=\frac{0}{x_1}+\frac{1}{x_2}+\frac{0}{x_3};\\
\overline{R}\big(\{x_3\}\big)&=\frac{0}{x_1}+\frac{0}{x_2}+\frac{1}{x_3}, & \overline{S}\big(\{x_3\}\big)&=\frac{0}{x_1}+\frac{0}{x_2}+\frac{1}{x_3};\\
\overline{R}\big(\{x_1,x_2\}\big)&=\frac{1}{x_1}+\frac{1}{x_2}+\frac{a}{x_3}, & \overline{S}\big(\{x_1,x_2\}\big)&=\frac{1}{x_1}+\frac{1}{x_2}+\frac{b}{x_3};\\
\overline{R}\big(\{x_1,x_3\}\big)&=\frac{1}{x_1}+\frac{a}{x_2}+\frac{1}{x_3}, & \overline{S}\big(\{x_1,x_3\}\big)&=\frac{1}{x_1}+\frac{b}{x_2}+\frac{1}{x_3};\\
\overline{R}\big(\{x_2,x_3\}\big)&=\frac{0}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}, & \overline{S}\big(\{x_2,x_3\}\big)&=\frac{0}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}.
\end{aligned}$$

Hence, we obtain

$$\mathscr{U}(R)=\mathscr{U}(S)=\big\{\emptyset,\{x_2\},\{x_3\},\{x_2,x_3\},\{x_1,x_2,x_3\}\big\}.$$

Meanwhile, by Corollary 4.3(2), we have $R\vee S\in\mathscr{S}(R)$ *and* $R\vee S$ *is an L-fuzzy preorder. However, the intersection of L-fuzzy relations* R *and* S *is*

$$R\wedge S=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Hence, we obtain $\mathscr{U}(R\wedge S)=\mathcal{P}(U)$ *and* $R\wedge S$ *is also an L-fuzzy preorder. Thus* $R\wedge S\notin\mathscr{S}(R)$ *holds. Therefore, the transitive subset of pseudo-similarity set of L-fuzzy relation is not a complete lattice with respect to intersections and unions.*

Consider $L=[0,1]$, $U=\{x_1,x_2,x_3\}$ *and the order-reversing involution* $'$ *is defined as* $a'=1-a$ *for all* $a\in[0,1]$. *Then fuzzy relation* R_i *is*

$$R_i=\begin{bmatrix}1&0&0\\\frac{1}{i}&1&0\\\frac{1}{i}&\frac{1}{i}&1\end{bmatrix}\ \textit{for all}\ i\in\mathbb{N}.$$

It is obvious that R_i *is a fuzzy preorder for all* $i\in\mathbb{N}$.

It follows from Definition 2.5 that the following hold for all $i \in \mathbb{N}$,

$$\begin{aligned}
\overline{R_i}\big(\{x_1\}\big) &= \frac{1}{x_1} + \frac{\frac{1}{i}}{x_2} + \frac{\frac{1}{i}}{x_3},\\
\overline{R_i}\big(\{x_2\}\big) &= \frac{0}{x_1} + \frac{1}{x_2} + \frac{\frac{1}{i}}{x_3},\\
\overline{R_i}\big(\{x_3\}\big) &= \frac{0}{x_1} + \frac{0}{x_2} + \frac{1}{x_3},\\
\overline{R_i}\big(\{x_1,x_2\}\big) &= \frac{1}{x_1} + \frac{1}{x_2} + \frac{\frac{1}{i}}{x_3},\\
\overline{R_i}\big(\{x_1,x_3\}\big) &= \frac{1}{x_1} + \frac{\frac{1}{i}}{x_2} + \frac{1}{x_3},\\
\overline{R_i}\big(\{x_2,x_3\}\big) &= \frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}.
\end{aligned}$$

Hence, we have for all $i \in \mathbb{N}$,

$$\mathscr{U}(R_i) = \big\{\emptyset, \{x_3\}, \{x_2,x_3\}, \{x_1,x_2,x_3\}\big\}.$$

By Corollary 4.3(2), we have $\bigvee_{i\in\mathbb{N}} R_i = R_1 \in \mathscr{S}(R_1)$ and $\bigvee_{i\in\mathbb{N}} R_i$ is a fuzzy preorder. However, the intersection of $\{R_i\}_{i\in\mathbb{N}}$ is

$$\bigwedge_{i\in\mathbb{N}} R_i = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, we obtain $\mathscr{U}\big(\bigwedge_{i\in\mathbb{N}} R_i\big) = \mathcal{P}(U)$ and $\bigwedge_{i\in\mathbb{N}} R_i$ is a fuzzy preorder, which implies $\bigwedge_{i\in\mathbb{N}} R_i \notin \mathscr{S}(R_1)$. Therefore, the transitive subset of pseudo-similarity set of fuzzy relation is not a complete lattice with respect to intersections and unions.

Remark 4.13. *The condition that L is a completely distributive De Morgan algebra is indispensable in* [19] *(see Theorems 3.10, 3.13, 4.8, 4.10 and 4.13 in* [19]*), but the completely distributive law of L is redundant in this paper. In fact, this paper only requires that L is a completely De Morgan algebra.*

5 Conclusions

In this paper, we further investigate topological structures induced by L-fuzzifying approximation operators. We discuss the Alexandrov topologies induced by L-fuzzy relations. Meanwhile, we study the relationships among those Alexandrov topologies. Especially, we point out that the upper (resp. lower) L-fuzzifying approximation operator is not necessarily a closure operator (resp. an interior operator) with respect to the Alexandrov topology $\mathscr{L}(R)$, even if R is an L-fuzzy preorder. On the basis of the Alexandrov topologies induced by a general L-fuzzy relation, we propose pseudo-similarity set. We point out that $t\big(R^r\big)$ is not the unique L-fuzzy preorder in the pseudo-similarity set of L-fuzzy relation R with counterexamples. Moreover, we point out that the transitive subset of pseudo-similarity set of L-fuzzy relation is not a complete lattice with respect to intersections and unions.

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