Iranian Journal of Fuzzy Systems

Volume 18, Number 5, (2021), pp. 155-172

Original paper

A novel defuzzification approach of type-2 fuzzy variable to solving matrix games: An application to plastic ban problem

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Abstract

The novelty of the type-2 fuzzy variable (T2FV) is its fuzzy secondary possibility distribution. Here, we aim to formulate a new defuzzification model of triangular type-2 fuzzy variables (TT2FVs) based on type reduction. We also show its superiority compared to other existing models by providing some numerical examples. Later, we develop a matrix game problem in the type-2 fuzzy environment to show the applicability of the proposed defuzzification method in a real-world situation. Based on this defuzzification method, two crisp linear programming models are derived, which are subsequently solved by the simplex method using LINGO 17.0 software. Also, the expected pay-off for the maximizing player is calculated as TT2FV, which is desirable. Finally, to show the validity and applicability, the proposed methodology is illustrated with the plastic ban problem.

Keywords: Matrix game, triangular type-2 fuzzy variable, type reduction, ranking function, plastic ban problem.

1 Introduction

In many real-life situations, due to lack of concentration or time or both, we face to the irregularities of input data. Also, finding the actual membership degree of that data become very difficult at that time. In this situation, to formulate the problem by avoiding all such complexity type-2 fuzzy variables (T2FVs) plays a significant role in decision-making problems. Type-2 fuzzy variable(T2FV) was defined by Liu and Liu [29] as a function from the fuzzy possibility space to the real space. Qin et al. [31] extended the concept of T2FV and gave the definitions of three different kinds of T2FV: Triangular type 2 fuzzy variable(TT2FV), Normal type-2 fuzzy variable(NT2FV) and Gamma type 2 fuzzy variable(GT2FV). Based on the theory of fuzzy integral, Qin et al. [31] defined three types of critical value (CV) reduction formulae of a TT2FV, namely, optimistic CV reduction, pessimistic CV reduction and CV reduction. They also gave a concept of the generalized credibility of the reduced T2FV.

Moreover, several extensions and applictaions of T2FV can be found in the literature in different directions. Chen and Zang [10] introduced the trapezoidal type-2 fuzzy variable(TrT2FV), another extension of T2FV. Das et al. [12] proposed the defuzzification method of a TrT2FV utilizing the nearest interval approximation of a fuzzy number based on the defuzzification procedure proposed by Grzegorzewski [16] and applied it to solve a green solid transportation problem. Kundu et al. [22, 23] used T2FV in the fixed charge transportation problem and a multi-criteria group decision-making problem. Biswas and De [5] proposed a unified method to defuzzify the type-2 fuzzy set and applied it to a multi-objective decision-making problem. Debnath et al. [14] also designed a sustainable fuzzy economic production quantity inventory model with type-2 fuzzy demand. Das et al. [13] discussed solid transportation problems with T2FVs. Castro et al. [9] parameterized the general type-2 fuzzy set. Castillo et al. [8] studied the application of type-2 fuzzy sets in the area of image processing. Roy and Bhaumik [34] discuss intelligent water management through a triangular type-2 intuitionistic game-theoretic approach. Roy and Maiti [35] performed a reduction procedure of T2FV and applied it to discuss stackelberg game.

Corresponding Author: M. R. Seikh
Received: May 2020; Revised: February 2021; Accepted: May 2021.

Game theory is an efficient tool To express any chaotic phenomenon of confliction. To capture all the impreciseness and vagueness in data researchers started to discuss the matrix game problems in fuzzy environment. Campos et al. [7] utilized the ranking function approach to solve the matrix games with fuzzy pay-offs. Bector and Chandra [2] implemented fuzzy linear programming duality to study fuzzy matrix games. Li [24, 25] introduced different methodologies to solve matrix games with fuzzy pay-offs and obtained fuzzy values of the games. Seikh et al. [40] solved the fuzzy matrix games through an $\alpha-$cut based approach. In recent past, Qiu et al. [32] discussed the solution model of fuzzy matrix games by using the lower limit of the possibility degree of the pay-offs. Jana and Roy [17] studied the matrix game with pay-offs of trapezoidal fuzzy numbers. Qiu et al. [33] solved a fuzzy matrix game utilizing a ranking value function method. Khalifa [21] rendered a solution procedure of fuzzy matrix games via linear programming approach. Xia [47] discussed about the solution procedure of a matrix game with cross-evaluated pay-offs. Verma and Kumar [45] studied the fuzzy solution concept of the non-cooperative games.

In fuzzy matrix games, the decision makers or players only consider the possible acceptance degrees of players toward the supplied information. In this scenario, the non-acceptance of the players are neglected. So, aiming to portray the conflicting situation by considering both the acceptance and non-acceptance of the players towards the information, researchers discussed the matrix game theory under intuitionistic fuzzy [1] environment. Li [26] introduced different methodologies to solve matrix games with intuitionistic fuzzy pay-offs. Seikh et al. [39, 41, 42, 43] introduced different approaches to solve games with pay-offs of intuitionistic fuzzy number (IFN). Bhaumik and Roy [3] proposed a robust ranking method of intuitionistic fuzzy set and applied it to solve matrix games. Brikaa et al. [6] solved a multi-criteria matrix games whose pay-offs are considered as IFSs. Applying Ambika's method, Verma and Kumar [44] resolved matrix games with IFSs pay-offs. Xing and Qiu [48] used the notion of accuracy function method to solve intuitionistic fuzzy matrix games. Roy and Mondal [36] studied interval valued fuzzy matrix games. Aiming to consider the hesitancy of players' mind game theory has been extended in hesitant fuzzy environment. Recently, based on the similarity measure of dual hesitant fuzzy sets, Jana and Roy [18] explored the solution procedure of fuzzy matrix games with dual hesitant fuzzy pay-offs. Seikh et al. [38] implemented Lexicographic method to solve matrix games under hesitant fuzzy payoffs. Yang and Song [49] solved matrix games having intuitionistic hesitant fuzzy pay-offs. Bhaumik et al. [4] discussed the Prisoners's dilemma in the hesitant interval-valued intuitionistic fuzzy environment and applied it to describe a case study on the human-trafficking problem. Seikh et al. [37] studied matrix game in dense fuzzy environment. Karmakar et al. [19] studied fuzzy matrix games with type-2 intuitionistic fuzzy payoffs based on normalized minkowski distance measure and applied it to resolve biogas-plant implementation problem to prevent the air pollution.

Based on our sincere review of literature, we observe that the matrix game theory experienced its illustration in various extensions of the T1FS. But, due to different types of complex situations such as huge set of data, different assumption of conception provided by decision-makers(here the players of the game), T1FS fails to describe the situation properly. On the other hand, triangular or trapezoidal fuzzy sets and their extensions are two dimensional. However, in reality, some uncertain situation arises, which are failing to describe by such kind of two-dimensional fuzzy set. Therefore, T1FSs become insufficient to describe some complex situation. So, in that case, we take type-2 fuzzy variables which have three dimensions (element, primary membership degree, and secondary membership degree) to discuss a matrix game. For example, suppose two different companies are going to launch the same electronic gadget in a targeted market. Both of the companies recruit a team of experts to study the market and to make a rough sketch about the future profit. While conducting a market study, the expert teams observe that the dealers are interested to display and sell those gadgets which have up-to-date technology, modern features, and budget-friendly. So, based on these demands, initially, the team of experts suggests the corresponding company arrange their business strategy. To find the maximum profit, this conflicting scenario can be discussed through a fuzzy matrix game. But, later it is observed that the demand of the dealers also depends on some secondary issues (like the taste of the buyers, the economic condition of the buyers, etc.). A type-1 fuzzy set is not enough to explain such a scenario with secondary issues rather a T2FV is ideal to portray such kind of problems. This issue is more clearly explained in Subsection 4.1

In this paper, we have explored a two-person matrix game with TT2FV pay-offs. It is a challenging task to handle such type of matrix game. To make the solution, here we introduce a new defuzzification method, viz., average approximation method of TT2FVs. A brief comparison of this defuzzification procedure with the other existing methods is also given here. Based on this defuzzification formula two crisp equivalent linear programming problems are formulated which are solved by using LINGO 17.0 software to find the optimal strategy for each player.

The major contributions of this article are:

- A new defuzzification method, viz., average approximation method of TT2FV is presented here. We compare the proposed defuzzification procedure with the other existing methods.
- There are so many works on the game theory with the pay-offs of T1FS but to the best of our knowledge this is the first contribution to develop a matrix game with TT2FV pay-offs.

- We use this methodology to solve the plastic ban problem in India to check the applicability and validity of the proposed method.

Apart from this introductory portion the rest of the paper is organized in the following manner. In Section 2, the basic definitions associated with T1FV, T2FV, and the type reduction formulae of T2FV are revisited. We propose a new defuzzification method of TT2FV in Section 3. With a brief description, we compare the proposed defuzzification method with the other existing methods in this section with some suitable examples. Section 4 develops the concept of a matrix game with TT2FV pay-offs and provides the solution procedure of such a problem as an application of the proposed defuzzification model. The algorithm of the solution procedure is also sketched here. To validate our discussion on the game in the type-2 fuzzy environment, we illustrate a case study on the plastic ban problem in India in this section. The conclusion is finally drawn and written in Section 5.

2 Preliminary discussion

In this section, we discuss some important definitions and preliminaries which are used in the subsequent sections.

2.1 T1FV

Let $\mathbf{U}$ be the universe of discourse. According to Wang [46], S is said to be an ample field on $\mathbf{U}$, if A is defined as the class of the subsets of $\mathbf{U}$, which is closed under the arbitrary intersections, unions and complements in $\mathbf{U}$. Let us consider a function $\wp : S \to [0,1]$ and $\wp$ satisfies the following conditions:

1. $\wp(\phi) = 0,\ \wp(\mathbf{U}) = 1,$
2. for any subclass $\{A_i : i \in I\}$ of S (finite, countable or uncountable whatever it is) $\wp(\cup_{i\in I} A_i) = \sup_{i\in I} \wp(A_i)$.

Then $\wp$ is called the possibility measure and the triplet $(\mathbf{U}, S, \wp)$ is termed as a possibility space [29] in which the credibility measure is defined as, $Cr(A) = (1 + \wp(A) - \wp(A^c))/2,\ A \in S$.

Definition 2.1. (**T1FV:**) [30] *$\tilde{\kappa}$ is said to be a T1FV, if it can be defined as a function from the possibility space $(\boldsymbol{U}, S, \wp)$ to the set of the real numbers $\Re$.*

2.2 T2FV

In the previous subsection, we have already discussed about the T1FV. It is defined as a function from a possibility space to the set of real numbers. Similarly, the T2FV is defined on the fuzzy possibility space.

Definition 2.2. (**T2FV:**) [29] *Consider the triplet $(\boldsymbol{U}, S, \tilde{\wp})$ as a fuzzy possibility space. Then a mapping $\tilde{\tilde{\kappa}} = (\tilde{\tilde{\kappa}}_1, \tilde{\tilde{\kappa}}_2, \dots, \tilde{\tilde{\kappa}}_m)$ from $\boldsymbol{U}$ to $\Re^m$ is called an m-ary type-2 fuzzy vector if it is a measurable map from $\boldsymbol{U}$ to $\Re^m$ in the sense that for every $t = (t_1, t_2, \dots, t_m) \in \Re^m$, the set $\{\gamma \in \boldsymbol{U} : \tilde{\tilde{\kappa}}(\gamma) \le t\} = \{\gamma \in \boldsymbol{U} : \tilde{\tilde{\kappa}}_1 \le t_1, \tilde{\tilde{\kappa}}_2 \le t_2, \dots, \tilde{\tilde{\kappa}}_m \le t_m\}$ is an element of S. If, $m = 1$, then the map $\tilde{\tilde{\kappa}} : \boldsymbol{U} \to \Re^m$ is termed as type-2 fuzzy variable.*

Definition 2.3. (**Secondary possibility distribution(SPD):**) [29] *$\tilde{\mu}_{\tilde{\tilde{\kappa}}}(x)$ is said to be the secondary possibility distribution of the T2FV $\tilde{\tilde{\kappa}}$ if it is defined as a mapping from the set of real numbers $\Re$ to $[0,1]$, such that $\tilde{\mu}_{\tilde{\tilde{\kappa}}}(x) = \tilde{\wp}\{\gamma \in \boldsymbol{U} : \tilde{\tilde{\kappa}}(\gamma) = x\}$ for $x \in \Re$.*

Definition 2.4. (**TT2FV:**) [31] *A T2FV $\tilde{\tilde{\kappa}} = (a^s, a^m, a^u)$ is said to be a TT2FV with the uncertainty parameters $\tau^l, \tau^r \in [0,1]$, if it has the following secondary possibility distribution function,*

$$\tilde{\mu}_{\tilde{\tilde{\kappa}}}(x) = \begin{cases} \left(\frac{x-a^s}{a^m-a^s} - \tau^l \min\left\{\frac{x-a^s}{a^m-a^s}, \frac{a^m-x}{a^m-a^s}\right\}, \frac{x-a^s}{a^m-a^s}, \frac{x-a^s}{a^m-a^s} + \tau^r \min\left\{\frac{x-a^s}{a^m-a^s}, \frac{a^m-x}{a^m-a^s}\right\}\right), & \text{for any } x \in [a^s, a^m] \\ \left(\frac{a^u-x}{a^u-a^m} - \tau^l \min\left\{\frac{a^u-x}{a^u-a^m}, \frac{x-a^m}{a^u-a^m}\right\}, \frac{a^u-x}{a^u-a^m}, \frac{a^u-x}{a^u-a^m} + \tau^r \min\left\{\frac{a^u-x}{a^u-a^m}, \frac{x-a^m}{a^u-a^m}\right\}\right), & \text{for any } x \in [a^m, a^u]. \end{cases} \tag{1}$$

where x takes the value from $\boldsymbol{U}$. More conveniently, the TT2FV can be denoted as, $\tilde{\tilde{\kappa}} = (a^s, a^m, a^u; \tau^l, \tau^r)$.

Example 2.5. $\tilde{\tilde{\kappa}} = (a^s, a^m, a^u; \tau^l, \tau^r) = (120, 125, 130; 0.5, 0.6)$ *be a TT2FV. Then according to Equation (1), its secondary possibility distribution is calculated as,*

$$\tilde{\mu}_{\tilde{\tilde{\kappa}}}(x) = \begin{cases} \Big(0.1(x-120), 0.2(x-120), 0.32(x-120)\Big), & \text{for } x \in [120, 122.5] \\ \Big(0.3(x-121.67), 0.2(x-120), 0.08(x-112.5)\Big), & \text{for } x \in (122.5, 125] \\ \Big(0.3(128.33-x), 0.2(130-x), 0.08(137.5-x)\Big), & \text{for } x \in (125, 127.5] \\ \Big(0.1(130-x), 0.2(130-x), 0.32(130-x)\Big), & \text{for } x \in (127.5, 130] \end{cases}$$

Remark 2.6. *If the uncertainty parameters of the TT2FV,* $\tilde{\tilde{\kappa}}$ *are vanished, i.e., if* $\tau^l = \tau^r = 0$, *then the TT2FV reduces to a triangular type-1 fuzzy variable (TT1FV),* $\tilde{\kappa} = (a^s, a^m, a^u)$.

Definition 2.7. (**Aggregation operation on TT2FV:**) [28] *Let* $\tilde{\tilde{\kappa}}_1, \tilde{\tilde{\kappa}}_2, \ldots, \tilde{\tilde{\kappa}}_m$ *be m−number of TT2FVs, having the form* $\tilde{\tilde{\kappa}}_i = (a_i^s, a_i^m, a_i^u; \tau_i^l, \tau_i^r)$, *for* $i = 1, 2, \ldots, m$. *Also assume that,* c_i, $(i = 1, 2, \ldots, m)$ *are some real scalars, positive or negative, then the TT2FVs can be aggregated as,*

$$\sum_{i=1}^{m} c_i \tilde{\tilde{\kappa}}_i = \Big(\sum_{i=1}^{m}(c_i^+ a_i^s - c_i^- a_i^u), \sum_{i=1}^{m}(c_i^+ a_i^m - c_i^- a_i^m), \sum_{i=1}^{m}(c_i^+ a_i^u - c_i^- a_i^s); \max_{1\le i\le m} \tau_i^l, \min_{1\le i\le m} \tau_i^r\Big). \tag{2}$$

Where, $c_i^+ = \max\{c_i, 0\}$, *and* $c_i^- = \max\{-c_i, 0\}$.

Example 2.8. *Suppose* $\tilde{\tilde{\kappa}}_1 = (a_1^s, a_1^m, a_1^u; \tau_1^l, \tau_1^r) = (10, 12, 15; 0.5, 0.6)$, *and* $\tilde{\tilde{\kappa}}_2 = (a_2^s, a_2^m, a_2^u; \tau_2^l, \tau_2^r) = (15, 17, 20; 0.35, 0.4)$ *be two TT2FVs. Also, consider two scalars* $c_1 = 0.4$, *and* $c_2 = 0.6$, *then according to Definition 2.7,* $c_1^+ = 0.4$, $c_1^- = 0$, $c_2^+ = 0.6$, *and* $c_2^- = 0$. *And thus,*

$$c_1\tilde{\tilde{\kappa}}_1 + c_2\tilde{\tilde{\kappa}}_2 = \Big(c_1 a_1^s + c_2 a_2^s, c_1 a_1^m + c_2 a_2^m, c_1 a_1^u + c_2 a_2^u; \max(\tau_1^l, \tau_2^l), \min(\tau_1^r, \tau_2^r)\Big) = (13, 15, 18; 0.5, 0.4).$$

2.3 Type reduction of TT2FV

In a practical situation, it is very difficult to handle the fuzzy membership function, but the problem becomes easier when the T2FV is converted into the T1FV, i.e. reducing the type is very essential to solve such problems. So many researchers made notable works on the type reduction of T2FV [20, 27, 31]. In this article, we use the type reduction, based on the critical value(CV) reduction, proposed by Qin et al. [31], to reduce the type of TT2FV.

Qin et al. [31] introduced three types of CVs of a TT2FV. According to him, if $\tilde{\tilde{\kappa}}$ be a TT2FV, then,

(i) The optimistic CV of $\tilde{\tilde{\kappa}}$, denoted by, $CV^*[\tilde{\kappa}]$ and defined as, $CV^*[\tilde{\kappa}] = \sup_{\alpha\in[0,1]}[\alpha \wedge Pos(\mu_{\tilde{\tilde{\kappa}}} \geq \alpha)]$.

(ii) The pessimistic CV of $\tilde{\tilde{\kappa}}$, denoted by, $CV_*[\tilde{\kappa}]$ and defined as, $CV_*[\tilde{\kappa}] = \sup_{\alpha\in[0,1]}[\alpha \wedge Nec(\mu_{\tilde{\tilde{\kappa}}} \geq \alpha)]$.

(iii) The CV of $\tilde{\tilde{\kappa}}$, denoted by, $CV[\tilde{\kappa}]$ and defined as, $CV[\tilde{\kappa}] = \sup_{\alpha\in[0,1]}[\alpha \wedge Cr(\mu_{\tilde{\tilde{\kappa}}} \geq \alpha)]$.

Where the possibility, necessity and credibility functions of $\tilde{\tilde{\kappa}}$ are defined as, $Pos(\mu_{\tilde{\tilde{\kappa}}} \geq \alpha) = \sup_{x\geq\alpha} \mu_{\tilde{\tilde{\kappa}}}(x)$, $Nec(\mu_{\tilde{\tilde{\kappa}}} \geq \alpha) = 1 - \sup_{x<\alpha} \mu_{\tilde{\tilde{\kappa}}}(x)$, and $Cr(\mu_{\tilde{\tilde{\kappa}}} \geq \alpha) = 1 + \sup_{x\geq\alpha} \mu_{\tilde{\tilde{\kappa}}}(x) - \sup_{x<\alpha} \mu_{\tilde{\tilde{\kappa}}}(x)$.

Example 2.9. *Let* $\tilde{\tilde{\kappa}}$ *be a TT2FV, with* $a^s = 10, a^m = 12, a^u = 15$, *with the primary membership function* $\mu_1(10) = \{0.2, 0.4, 0.6\}$, $\mu_1(12) = \{0.1, 0.3, 0.5\}$, *and* $\mu_1(15) = \{0.2, 0.5, 0.8\}$. *Also, the secondary membership functions are given as,* $\mu_{\tilde{\tilde{\kappa}}}(10, \mu_1(10)) = (0.6, 0.8, 1)$, $\mu_{\tilde{\tilde{\kappa}}}(12, \mu_1(12)) = (0.5, 1, 0.8)$, *and* $\mu_{\tilde{\tilde{\kappa}}}(15, \mu_1(15)) = (0.4, 0.6, 0.5)$. *Then,*

$$Pos(\mu_{\tilde{\tilde{\kappa}}}(10) \geq \alpha) = \begin{cases} 0.6 & \text{for } \alpha \in [0, 0.2] \\ 0.8 & \text{for } \alpha \in (0.2, 0.4] \\ 1 & \text{for } \alpha \in (0.4, 1] \end{cases}, \quad Nec(\mu_{\tilde{\tilde{\kappa}}}(10) \geq \alpha) = \begin{cases} 0.4 & \text{for } \alpha \in [0, 0.2] \\ 0.2 & \text{for } \alpha \in (0.2, 0.4] \\ 0 & \text{for } \alpha \in (0.4, 1] \end{cases}, \text{ and}$$

$$Cr(\mu_{\tilde{\tilde{\kappa}}}(10) \geq \alpha) = \begin{cases} 0.5 & \text{for } \alpha \in [0, 0.2] \\ 0.5 & \text{for } \alpha \in (0.2, 0.4] \\ 0.5 & \text{for } \alpha \in (0.4, 1] \end{cases}$$

. So that, optimistic CV reduction, pessimistic CV reduction and CV reduction of the TT2FV $\tilde{\tilde{\kappa}}$ *for* $a^s = 10$ *are,*

$$CV^*[\mu_{\tilde{\tilde{\kappa}}}(10)] = \sup_{\alpha\in[0,1]}[\alpha \wedge Pos(\mu_{\tilde{\tilde{\kappa}}}(10) \geq \alpha)] = \sup_{\alpha\in[0,0.2]}[\alpha \wedge 0.6] \vee \sup_{\alpha\in(0.2,0.4]}[\alpha \wedge 0.8] \vee \sup_{\alpha\in(0.4,1]}[\alpha \wedge 1] = 0 \vee 0.4 \vee 1 = 1,$$

$$CV_*[\mu_{\tilde{\tilde{\kappa}}}(10)] = \sup_{\alpha\in[0,1]}[\alpha \wedge Nec(\mu_{\tilde{\tilde{\kappa}}}(10) \geq \alpha)] = \sup_{\alpha\in[0,0.2]}[\alpha \wedge 0.4] \vee \sup_{\alpha\in(0.2,0.4]}[\alpha \wedge 0.2] \vee \sup_{\alpha\in(0.4,1]}[\alpha \wedge 0] = 0 \vee 0.2 \vee 0 = 0.2,$$

$$\text{and} \quad CV[\mu_{\tilde{\tilde{\kappa}}}(10)] = \sup_{\alpha\in[0,1]}[\alpha \wedge Cr(\mu_{\tilde{\tilde{\kappa}}}(10) \geq \alpha)] = \sup_{\alpha\in[0,0.2]}[\alpha \wedge 0.5] \vee \sup_{\alpha\in(0.2,0.4]}[\alpha \wedge 0.5] \vee \sup_{\alpha\in(0.4,1]}[\alpha \wedge 0.5] = 0 \vee 0.4 \vee 0.5 = 0.5,$$

respectively. Similarly, for a^m, we obtain, $CV^*[\mu_{\tilde{\tilde{\kappa}}}(12)] = 0.8$, $CV_*[\mu_{\tilde{\tilde{\kappa}}}(12)] = 0.2$, $CV[\mu_{\tilde{\tilde{\kappa}}}(12)] = 0.5$, and for a^u, we obtain $CV^*[\mu_{\tilde{\tilde{\kappa}}}(15)] = 0.5$, $CV_*[\mu_{\tilde{\tilde{\kappa}}}(15)] = 0.5$, $CV[\mu_{\tilde{\tilde{\kappa}}}(15)] = 0.5$. Thus the TT1FV, expressed in the form of $\tilde{\kappa} = \Big((a^s, \mu_{\tilde{\kappa}}(a^s)), (a^m, \mu_{\tilde{\kappa}}(a^m)), (a^u, \mu_{\tilde{\kappa}}(a^u))\Big)$, obtained by applying optimistic CV reduction, pessimistic CV reduction and CV reduction are, $\Big((10, 1), (12, 0.8), (15, 0.5)\Big)$, $\Big((10, 0.2), (12, 0.2), (15, 0.5)\Big)$, and $\Big((10, 0.5), (12, 0.5), (15, 0.5)\Big)$, respectively.

Theorem 2.10. [31] *Let $\tilde{\tilde{\kappa}}$ be a TT2FV, where $\tilde{\tilde{\kappa}} = (a^s, a^m, a^u; \tau^l, \tau^r)$ has the secondary possibility distribution defined in Equation (1). Then,*

(i) The type reduction of $\tilde{\tilde{\kappa}}$, obtained by using optimistic CV reduction, has the following possibility distribution:

$$\mu_{\tilde{\kappa}_o}(x) = \begin{cases} \dfrac{(1+\tau^r)(x-a^s)}{a^m - a^s + \tau^r(x-a^s)}, & x \in \left[a^s, \dfrac{a^s+a^m}{2}\right] \\ \dfrac{(1-\tau^r)x + \tau^r a^m - a^s}{a^m - a^s + \tau^r(a^m - x)}, & x \in \left(\dfrac{a^s+a^m}{2}, a^m\right] \\ \dfrac{(-1+\tau^r)x - \tau^r a^m + a^u}{a^u - a^m + \tau^r(a^u - x)}, & x \in \left(a^m, \dfrac{a^m+a^u}{2}\right] \\ \dfrac{(1+\tau^r)(a^u - x)}{a^u - a^m + \tau^r(a^u - x)}, & x \in \left(\dfrac{a^m+a^u}{2}, a^u\right] \end{cases} \tag{3}$$

(ii) The type reduction of $\tilde{\tilde{\kappa}}$, obtained by using pessimistic CV reduction, has the following possibility distribution:

$$\mu_{\tilde{\kappa}_p}(x) = \begin{cases} \dfrac{x-a^s}{a^m - a^s + \tau^l(x-a^s)}, & x \in \left[a^s, \frac{a^s+a^m}{2}\right] \\ \dfrac{x-a^s}{a^m - a^s + \tau^l(a^m - x)}, & x \in \left(\frac{a^s+a^m}{2}, a^m\right] \\ \dfrac{a^u - x}{a^u - a^m + \tau^l(x - a^m)}, & x \in \left(a^m, \frac{a^m+a^u}{2}\right] \\ \dfrac{a^u - x}{a^u - a^m + \tau^l(a^u - x)}, & x \in \left(\frac{a^m+a^u}{2}, a^u\right] \end{cases} \tag{4}$$

(iii) The type reduction of $\tilde{\tilde{\kappa}}$, obtained by using CV reduction, has the following possibility distribution:

$$\mu_{\tilde{\kappa}_r}(x) = \begin{cases} \dfrac{(1+\tau^r)(x-a^s)}{a^m - a^s + 2\tau^r(x-a^s)}, & x \in \left[a^s, \frac{a^s+a^m}{2}\right] \\ \dfrac{(1-\tau^l)x + \tau^l a^m - a^s}{a^m - a^s + 2\tau^l(a^m - x)}, & x \in \left(\frac{a^s+a^m}{2}, a^m\right] \\ \dfrac{(-1+\tau^l)x - \tau^l a^m + a^u}{a^u - a^m + 2\tau^l(x - a^m)}, & x \in \left(a^m, \frac{a^m+a^u}{2}\right] \\ \dfrac{(1+\tau^r)(a^u - x)}{a^u - a^m + 2\tau^r(a^u - x)}, & x \in \left(\frac{a^m+a^u}{2}, a^u\right] \end{cases} \tag{5}$$

3 Defuzzification of TT2FV

We can not assign the order or the rank of a fuzzy number or a fuzzy set without defuzzyfying it, and thus defuzzification of a fuzzy number has vast importance for the real-world applications. The specialty of a TT2FV is its secondary possibility distribution. Actually, a TT2FV is a mapping from the fuzzy possibility space to the real space, which is discussed in Section 2. For that particular reason, the computational complexity becomes very high in the type 2 fuzzy environment. So many researchers employed their skill and knowledge to defuzzify the TT2FV [5, 12, 22, 23, 31]. Karnik and Mendel [20] defined the centroid and the generalized centroid of a type-2 fuzzy set and also discussed the computational procedure of these. Coupland and John [11] converted the T2FS into a geometric T2FS and applied the geometric defuzzification method to defuzzify this. Greenfield et al. [15] made an extension of the centroid method to defuzzify. To reduce the computational complexity and to obtain a more accurate value of a TT2FV, a new defuzzification procedure, namely, the average approximation method is proposed here with a suitable example.

3.1 Average approximation method

In this proposed defuzzification procedure, we first reduce the type of the TT2FV $\tilde{\tilde{\kappa}}$, i.e., we convert the TT2FV into a type-1 fuzzy variable by using the optimistic CV-reduction, pessimistic CV-reduction, and CV-reduction as described in Theorem 2.10 proposed by Qin et al. [31]. Then we calculate the $\alpha-$cut set of them.

Now, the $\alpha-$cut of this reduced set has the form $[\tilde{\kappa}_o^L(\alpha), \tilde{\kappa}_o^R(\alpha)]$ by optimistic CV-reduction, where,

$$\tilde{\kappa}_o^L(\alpha) = \begin{cases} \dfrac{\alpha(a^m - a^s - \tau^r a^s) + (1+\tau^r)a^s}{1+(1-\alpha)\tau^r}, & \text{for } \alpha \in [0, 0.5] \\[2ex] \dfrac{\alpha(a^m - a^s + \tau^r a^m) + a^s - \tau^r a^m}{1-(1-\alpha)\tau^r}, & \text{for } \alpha \in [0.5, 1] \end{cases} \quad \text{and} \quad \tilde{\kappa}_o^R(\alpha) = \begin{cases} \dfrac{\alpha(a^u - a^m - \tau^r a^u) + \tau^r a^m - a^u}{-1+(1+\alpha)\tau^r}, & \text{for } \alpha \in [0, 0.5] \\[2ex] \dfrac{\alpha(a^u - a^m + \tau^r a^u) - (1+\tau^r)a^u}{-1-(1-\alpha)\tau^r} & \text{for } \alpha \in [0.5, 1] \end{cases} \tag{6}$$

Similarly, the $\alpha-$cut of this reduced set has the form $[\tilde{\kappa}_p^L(\alpha), \tilde{\kappa}_p^R(\alpha)]$ by pessimistic CV-reduction, where,

$$\tilde{\kappa}_p^L(\alpha) = \begin{cases} \dfrac{\alpha(a^m - a^s - \tau^l a^s) + a^s}{1-\alpha\tau^l}, & \text{for } \alpha \in [0, 0.5] \\[2ex] \dfrac{\alpha(a^m - a^s + \tau^l a^m) + a^s}{1+\alpha\tau^l}, & \text{for } \alpha \in [0.5, 1] \end{cases} \quad \text{and} \quad \tilde{\kappa}_p^R(\alpha) = \begin{cases} \dfrac{\alpha(a^u - a^m - \tau^l a^m) - a^u}{-(1+\alpha\tau^l)}, & \text{for } \alpha \in [0, 0.5] \\[2ex] \dfrac{\alpha(a^u - a^m + \tau^l a^u) - a^u}{-(1-\alpha\tau^l)}, & \text{for } \alpha \in [0.5, 1] \end{cases} \tag{7}$$

The $\alpha-$cut of this reduced set has the form $[\tilde{\kappa}_r^L(\alpha), \tilde{\kappa}_r^R(\alpha)]$ by CV-reduction, where,

$$\tilde{\kappa}_r^L(\alpha) = \begin{cases} \dfrac{\alpha(a^m - a^s - 2\tau^r a^s) + (1+\tau^r)a^s}{1+(1-2\alpha)\tau^r}, & \text{for } \alpha \in [0, 0.5] \\[2ex] \dfrac{\alpha(a^m - a^s + 2\tau^l a^m) + a^s - \tau^l a^m}{1-(1-2\alpha)\tau^l}, & \text{for } \alpha \in [0.5, 1] \end{cases} \quad \text{and} \quad \tilde{\kappa}_r^R(\alpha) = \begin{cases} \dfrac{\alpha(a^u - a^m - 2\tau^l a^m) + \tau^l a^m - a^u}{-1+(1-2\alpha)\tau^l}, & \text{for } \alpha \in [0, 0.5] \\[2ex] \dfrac{\alpha(a^u - a^m + 2\tau^r a^u) - (1+\tau^r)a^u}{(2\alpha-1)\tau^r - 1}, & \text{for } \alpha \in [0.5, 1] \end{cases} \tag{8}$$

Hence the average approximation of the TT2FV for optimistic CV-reduction is then obtained as,

$$\begin{aligned} A_o(\tilde{\tilde{\kappa}}) &= \frac{1}{2}\Big[\int_0^1 \{\tilde{\kappa}_o^L(\alpha) + \tilde{\kappa}_o^R(\alpha)\}\Big]d\alpha \\ &= \frac{1}{2}\Big[\Big(\int_0^{0.5} + \int_{0.5}^1\Big)\tilde{\kappa}_o^L(\alpha)d\alpha + \Big(\int_0^{0.5} + \int_{0.5}^1\Big)\tilde{\kappa}_o^R d\alpha\Big] \\ &= \frac{1}{2}\Big[\frac{a^s + a^m}{2} - \frac{a^m - a^s}{(\tau^r)^2}\Big\{(1+\tau^r)\ln\Big(\frac{1+0.5\tau^r}{1+\tau^r}\Big) - \ln(1-0.5\tau^r)\Big\} + \frac{a^u + a^m}{2} - \frac{a^u - a^m}{\tau^r} + \\ &\qquad \frac{a^u - a^m}{(\tau^r)^2}\Big\{(1+\tau^r)\ln(1+0.5\tau^r) - \ln\Big(\frac{1-0.5\tau^r}{1-\tau^r}\Big)\Big\}\Big] \text{ (by using Equation (6))} \\ &= \frac{1}{2}\Big[\frac{a^s + 2a^m + a^u}{2} - \frac{a^u - a^m}{\tau^r} - \frac{a^m - a^s}{(\tau^r)^2}\Big\{(1+\tau^r)\ln\Big(\frac{1+0.5\tau^r}{1+\tau^r}\Big) - \ln(1-0.5\tau^r)\Big\} + \\ &\qquad \frac{a^u - a^m}{(\tau^r)^2}\Big\{(1+\tau^r)\ln(1+0.5\tau^r) - \ln\Big(\frac{1-0.5\tau^r}{1-\tau^r}\Big)\Big\}\Big]. \end{aligned} \tag{9}$$

Similarly, the average approximation of the TT2FV for pessimistic CV-reduction can be obtained as,

$$\begin{aligned} A_p(\tilde{\tilde{\kappa}}) &= \frac{1}{2}\Big[\int_0^1 \{\tilde{\kappa}_p^L(\alpha) + \tilde{\kappa}_p^R(\alpha)\}\Big]d\alpha \\ &= \frac{1}{2}\Big[\Big(\int_0^{0.5} + \int_{0.5}^1\Big)\tilde{\kappa}_p^L(\alpha)d\alpha + \Big(\int_0^{0.5} + \int_{0.5}^1\Big)\tilde{\kappa}_p^R d\alpha\Big] \\ &= \frac{1}{2}\Big[\frac{a^s + a^m}{2} - \frac{a^m - a^s}{(\tau^l)^2}\Big\{(1+\tau^l)\ln\Big(\frac{1+\tau^l}{1+0.5\tau^l}\Big) + \ln(1-0.5\tau^l)\Big\} + \frac{a^u + a^m}{2} - \frac{a^u - a^m}{\tau^l} + \\ &\qquad \frac{a^u - a^m}{(\tau^l)^2}\Big\{(1+\tau^l)\ln(1+0.5\tau^l) + \ln\Big(\frac{1-\tau^l}{1-0.5\tau^l}\Big)\Big\}\Big] \text{ (by using Equation (7))} \\ &= \frac{1}{2}\Big[\frac{a^s + 2a^m + a^u}{2} - \frac{a^u - a^m}{\tau^l} - \frac{a^m - a^s}{(\tau^l)^2}\Big\{(1+\tau^l)\ln\Big(\frac{1+\tau^l}{1+0.5\tau^l}\Big) + \ln(1-0.5\tau^l)\Big\} + \\ &\qquad \frac{a^u - a^m}{(\tau^l)^2}\Big\{(1+\tau^l)\ln(1+0.5\tau^l) + \ln\Big(\frac{1-\tau^l}{1-0.5\tau^l}\Big)\Big\}\Big], \end{aligned} \tag{10}$$

and the average approximation of the TT2FV for CV-reduction can be obtained as,

$$
\begin{aligned}
A_r(\tilde{\kappa}) &= \frac{1}{2}\Big[\int_0^1 \{\tilde{\kappa}_r^L(\alpha)+\tilde{\kappa}_r^R(\alpha)\}\Big]d\alpha \\
&= \frac{1}{2}\Big[\Big(\int_0^{0.5}+\int_{0.5}^1\Big)\tilde{\kappa}_r^L(\alpha)d\alpha+\Big(\int_0^{0.5}+\int_{0.5}^1\Big)\tilde{\kappa}_r^R d\alpha\Big] \\
&= \frac{1}{2}\Big[\frac{(2\tau^r+1)a^s-a^m}{4\tau^r}+\frac{(2\tau^l+1)a^m-a^s}{4\tau^l}+\frac{(a^m-a^s)(1+\tau^r)}{4(\tau^r)^2}\ln(1+\tau^r) \\
&\quad -\frac{(a^m-a^s)(1+\tau^l)}{4(\tau^l)^2}\ln(1+\tau^l)+\frac{(2\tau^l+1)a^m-a^u}{4\tau^l}+\frac{(2\tau^r+1)a^u-a^m}{4\tau^r}+ \\
&\quad \frac{(a^u-a^m)(1+\tau^r)}{4(\tau^r)^2}\ln(1-\tau^r)-\frac{(a^u-a^m)(1+\tau^l)}{4(\tau^l)^2}\ln(1-\tau^l)\Big] \text{ (by using Equation (8))} \\
&= \frac{1}{2}\Big[\frac{(2\tau^r+1)(a^s+a^u)-2a^m}{4\tau^r}-\frac{a^s-2(2\tau^l+1)a^m+a^u}{4\tau^l}+\frac{1+\tau^r}{4(\tau^r)^2}\Big\{(a^m-a^s)\ln(1+\tau^r)+ \\
&\quad (a^u-a^m)\ln(1-\tau^r)\Big\}-\frac{1+\tau^l}{4(\tau^l)^2}\Big\{(a^m-a^s)\ln(1+\tau^l)+(a^u-a^m)\ln(1-\tau^l)\Big\}\Big]. \qquad (11)
\end{aligned}
$$

It can be noted that the average approximation of a TT2FV is a crisp value, which is desirable.

Example 3.1. *Let us take a TT2FV,* $\tilde{\tilde{150}} = (145, 150, 155; 0.35, 0.38)$*. Then according to Equation (6)* $\alpha-$*cut set of* $\tilde{\tilde{150}}$ *by optimistic CV-reduction would be* $[\tilde{150}_o^L, \tilde{150}_o^R]$*, where,*

$$
\tilde{150}_o^L(\alpha)=\begin{cases}\dfrac{-50.1\alpha+200.1}{1.38-0.38\alpha}, & \text{for } \alpha\in[0,0.5]\\[2ex] \dfrac{62\alpha+88}{0.62+0.38\alpha}, & \text{for } \alpha\in[0.5,1]\end{cases} \quad \text{and} \quad \tilde{150}_o^R(\alpha)=\begin{cases}\dfrac{-53.9\alpha-98}{-0.62+0.38\alpha}, & \text{for } \alpha\in[0,0.5]\\[2ex] \dfrac{63.9\alpha-213.9}{-1.38+0.38\alpha}, & \text{for } \alpha\in[0.5,1]\end{cases}
$$

Similarly, according to Equation (7) $\alpha-$*cut set of* $\tilde{\tilde{150}}$ *by pessimistic CV-reduction would be* $[\tilde{150}_p^L, \tilde{150}_p^R]$*, where,*

$$
\tilde{150}_p^L(\alpha)=\begin{cases}\dfrac{-45.75\alpha+145}{1-0.35\alpha}, & \text{for } \alpha\in[0,0.5]\\[2ex] \dfrac{55.75\alpha+145}{1+0.35\alpha}, & \text{for } \alpha\in[0.5,1]\end{cases} \quad \text{and} \quad \tilde{150}_p^R(\alpha)=\begin{cases}\dfrac{-47.5\alpha-155}{-1-0.35\alpha}, & \text{for } \alpha\in[0,0.5]\\[2ex] \dfrac{57.5\alpha-155}{-1+0.35\alpha}, & \text{for } \alpha\in[0.5,1]\end{cases}
$$

And, according to Equation (8) $\alpha-$*cut set of* $\tilde{\tilde{150}}$ *by CV-reduction would be* $[\tilde{150}_r^L, \tilde{150}_r^R]$*, where,*

$$
\tilde{150}_r^L(\alpha)=\begin{cases}\dfrac{-105.2\alpha+200.1}{1.38-0.76\alpha}, & \text{for } \alpha\in[0,0.5]\\[2ex] \dfrac{110\alpha+92.6}{0.65-0.70\alpha}, & \text{for } \alpha\in[0.5,1]\end{cases} \quad \text{and} \quad \tilde{150}_r^R(\alpha)=\begin{cases}\dfrac{-100\alpha-102.5}{-0.65-0.70\alpha}, & \text{for } \alpha\in[0,0.5]\\[2ex] \dfrac{122.8\alpha-213.9}{0.76\alpha-1.38}, & \text{for } \alpha\in[0.5,1]\end{cases}
$$

Then by following Equations (9), (10) and (11) the defuzzified value of $\tilde{\tilde{150}}$ *by optimistic average approximation is* $A_o(\tilde{\tilde{150}}) = 152.096$*, by pessimistic average approximation is* $A_p(\tilde{\tilde{150}}) = 142.536$ *and by average approximation is* $A_r(\tilde{\tilde{150}}) = 149.969$*.*

3.2 Comparison among the CV based defuzzification method based on general credibility [31], Nearest interval approximation method [12] and proposed Average approximation method

This subsection provides a brief introduction of the above-mentioned defuzzification procedures of TT2FV. Then a comparison of the above mentioned defuzzification procedures are discussed with few examples.

CV based defuzzification method based on general credibility [31]: Qin et al. [31] developed the defuzzification process of the reduced set by applying the generalized credibility of a fuzzy number.

Theorem 3.2. [31] *Let $\tilde{\kappa}_{ij}$ be the reduction of TT2FV $\tilde{\tilde{\kappa}}_{ij} = (a^s_{ij}, a^m_{ij}, a^u_{ij}; \tau^l_{ij}, \tau^r_{ij})$, obtained by CV reduction method for $i = 1, 2, ..., m$ and $j = 1, 2, ...n$ and all the $\tilde{\kappa}_{ij}$'s are mutually independent, then,*

1. *if the generalized credibility level $\theta \in (0, 0.25]$, then the credibility $Cr\{\sum_{j=1}^n \tilde{\kappa}_{ij} y_j \leq t\} \geq \theta$ is equivalent to,* $\sum\limits_{j=1}^{n} \frac{(1-2\theta+(1-4\theta)\tau^r_{ij})a^s_{ij}+2\theta a^m_{ij}}{1+(1-4\theta)\tau^r_{ij}} y_j \leq t.$
2. *for the generalized credibility level $\theta \in (0.25, 0.5]$, the credibility $Cr\{\sum_{j=1}^n \tilde{\kappa}_{ij} y_j \leq t\} \geq \theta$ is equivalent to,* $\sum\limits_{j=1}^{n} \frac{(1-2\theta)a^s_{ij}+(2\theta+(4\theta-1)\tau^l_{ij})a^m_{ij}}{1+(4\theta-1)\tau^l_{ij}} y_j \leq t.$
3. *for the generalized credibility level $\theta \in (0.5, 0.75]$, the credibility $Cr\{\sum_{j=1}^n \tilde{\kappa}_{ij} y_j \leq t\} \geq \theta$ is equivalent to,* $\sum\limits_{j=1}^{n} \frac{(2\theta-1)a^u_{ij}+(2(1-\theta)+(3-4\theta)\tau^l_{ij})a^m_{ij}}{1+(3-4\theta)\tau^l_{ij}} y_j \leq t.$
4. *if the generalized credibility level $\theta \in (0.75, 1]$, then the credibility $Cr\{\sum_{j=1}^n \tilde{\kappa}_{ij} y_j \leq t\} \geq \theta$ is equivalent to,* $\sum\limits_{j=1}^{n} \frac{(2\theta-1+(4\theta-3)\tau^r_{ij})a^u_{ij}+2(1-\theta)a^m_{ij}}{1+(4\theta-3)\tau^r_{ij}} y_j \leq t.$

Nearest interval approximation method [12]: Das et al. [12] introduced a new defuzzification method of a T2FV in the form of a closed interval as $[\kappa^L, \kappa^R]$,. Where κ^L, and κ^R are obtained as respectively,

$$\kappa^L = \frac{(1+\tau^r)a^s}{\tau^r} \ln\Big(\frac{1+\tau^r}{1+0.5\tau^r}\Big) - \frac{a^s - a^m\tau^r}{\tau^r} \ln(1-0.5\tau^r) - \frac{a^m-(1+\tau^r)a^s}{(\tau^r)^2}\Big[0.5\tau^r - (1+\tau^r)\ln\Big(\frac{1+\tau^r}{1+0.5\tau^r}\Big)\Big] + \frac{(1+\tau^r)a^m - a^s}{(\tau^r)^2}[0.5\tau^r + (1-\tau^r)\ln(1-0.5\tau^r)]$$

and

$$\kappa^R = \frac{(1+\tau^r)a^u}{\tau^r} \ln\Big(\frac{1+\tau^r}{1+0.5\tau^r}\Big) - \frac{a^u - a^m\tau^r}{\tau^r} \ln(1-0.5\tau^r) + \frac{(1-\tau^r)a^u - a^m}{(\tau^r)^2}\Big[0.5\tau^r - (1+\tau^r)\ln\Big(\frac{1+\tau^r}{1+0.5\tau^r}\Big)\Big] - \frac{a^u-(1-\tau^r)a^m}{(\tau^r)^2}[0.5\tau^r + (1-\tau^r)\ln(1-0.5\tau^r)]$$

Thus in CV based defuzzification method [31] based on generalized credibility, some parametric equations are made with two or more than two variables. These parametric equations require more associative conditions to solve the parametric programming problem. On the other hand, the nearest interval approximation method [12] transforms a TT2FV into an interval but not a crisp value. We enlist the defuzzified values of some TT2FVs obtained by the nearest interval approximation method [12] and by proposed average approximation method in Table 1.

Table 1: Defuzzified values of some TT2FVs obtained by Nearest interval approximation and Average approximation

TT2FV($\tilde{\tilde{\kappa}}$)	Defuzzified values			
	Nearest interval [12]	Proposed method		
	($[\kappa^L, \kappa^R]$)	Optimistic ($A_o(\tilde{\kappa})$)	pessimistic ($A_p(\tilde{\kappa})$)	CV reduced ($A_r(\tilde{\kappa})$)
(8,10,12;0.47,0.65)	[8.8528,11.1472]	7.9324	7.6690	9.8792
(3,6,9;0.5,0.5)	[4.3296,7.6704]	2.4903	2.6607	6.0000
(4,6,8;0.3,0.6)	[4.8642,7.1358]	3.8770	2.5630	5.8447
(7,9,10;0.5,0.6)	[7.8642,9.5679]	7.6545	7.6653	8.7165

So, in the light of comparison of CV based defuzzification method, nearest interval approximation method, and the proposed average approximation method, the following observations are made.

- In CV based defuzzification method, to calculate the crisp value of a TT2FV we formulate a set of non-linear equations, which is made based on the concept of the generalized credibility of a fuzzy number. And, in the Nearest interval approximation method, an interval is generated by the help of the $\alpha-$cut of the type reduced set. But, in the proposed methodology, viz. the average approximation method central value of the left and right $\alpha-$cut of the reduced set is calculated to find the crisp value of the given TT2FV.
- The CV based defuzzification and the average approximation calculate a particular crisp value against the given TT2FV, but the Nearest interval approximation provides an interval in which the crisp value of the TT2FV lies.
- In CV based defuzzification, the average of the optimistic and pessimistic CV is calculated. So, this is an efficient tool to defuzzify a TT2FV. And, in the proposed method, after computing the $\alpha-$cuts of the type reduced set, the average of the left and the right $\alpha-$cuts is calculated. So, it is more efficient than the previous method, as it removes the maximum computational error. But, in the Nearest interval approximation method, there is no procedure to choose α for that reason this method has less efficiency.

4 An application of the average approximation of TT2FV to matrix game in Type-2 fuzzy environment

4.1 Implication of TT2FV in game theory

To capture the uncertainty of the conflicting situation, the notion of the fuzzy set was implemented to game theory instead of the crisp set. Later, it has been seen that the ordinary fuzzy set (i.e., the T1FS) fails to describe all the complexities properly. Due to the large volume of information, multiple sources of information, different assumptions of the players, and linguistic information with the different degrees of players fail to suggest their pay-offs by type-1 fuzzy set. In this circumstance, players are bound to describe pay-offs in the type-2 fuzzy environment to show the complexity of that situation and to obtain a fruitful result. The following example illustrates the situation better.

Suppose a laptop manufacturing company is going to launch its product with some modern features in a targeted market. They chose three suitable channels to supply information about the product to increase the interest of consumers. Also, they recruit an agency to make a rough sketch of their future sale amount by studying the market. By surveying the demand of the customers of a targeted market, the agency concludes that if the company launches the new laptop with a special offer, then the sale amount will be around 80 units. They also say that the sale amount margin will be between a minimum of 75 units to a maximum of 85 units. Initially, the agency estimated the amount of sale by a fuzzy variable $(75, 80, 85)$. But, later, they noticed that the demand for the laptop in the market is very much dependent on the advertisements given on the specific three channels. Depending on these advertisements, the gross amount of sale may be changed with a certain percentage, for which they may have a secondary possibility distribution of the profit. Such a scenario can not be explained by a T1FV, and for better estimation T2FV is needed. For example, for the advertisement, if the minimum range of sale amount is reduced by 10% and the maximum range is extended by 15%, then the sale amount can be estimated by a TT2FV as $(75, 80, 85; 0.10, 0.15)$.

4.2 Model formulation

In this subsection, we formulate a matrix game with pay-offs of type-2 fuzzy variables and discuss the solution procedure.

Let us consider a matrix game with pay-offs of TT2FV, which is denoted by $\tilde{\tilde{G}} = (\tilde{\tilde{\kappa}}_{ij})_{m\times n}$, where each $\tilde{\tilde{\kappa}}_{ij}$ represents a TT2FV having the form $\tilde{\tilde{\kappa}}_{ij} = (a^s_{ij}, a^m_{ij}, a^u_{ij}; \tau^l_{ij}, \tau^r_{ij})$. Consider, $\mathbf{S}_1 = \{\mathbf{x} = (x_1, x_2, ..., x_m) | x_i \in \Re^{m+} \text{ and } \sum_{i=1}^m x_i = 1\}$ and $\mathbf{S}_2 = \{\mathbf{y} = (y_1, y_2, ..., y_n) | y_j \in \Re^{n+} \text{ and } \sum_{j=1}^n y_j = 1\}$, where $\Re^{m+}$ and $\Re^{n+}$ are the non-negative orthant of the Euclidean space $\Re$. Let us choose, $\mathbf{S}_1$ and $\mathbf{S}_2$ as the strategy spaces of Player A_1 and Player A_2, respectively. Thus we can represent the game as a triplet, $\tilde{\tilde{M}} = \langle \tilde{\tilde{G}}, \mathbf{S}_1, \mathbf{S}_2 \rangle$.

Suppose that, $\gamma = \{\gamma_1, \gamma_2, ..., \gamma_m\}$ and $\delta = \{\delta_1, \delta_2, ..., \delta_n\}$ are the set of pure strategies chosen by Player A_1 and Player A_2, respectively. Without loss of generality, we assume that Player A_1 is the maximizing player whereas Player A_2 is the minimizing player. If Player A_1 chooses the pure strategy γ_i for i=1,2,...,m and Player A_2 chooses the the pure strategy δ_j for j=1,2,...,n, then the outcome will be, $(\gamma_i, \delta_j) = \tilde{\tilde{\kappa}}_{ij}$, the ij^{th} entry of the pay-off matrix $\tilde{\tilde{G}}$.

Definition 4.1. Expected value of the game: *If Player A_1 takes $\boldsymbol{x} \in \boldsymbol{S}_1$ and Player A_2 chooses $\boldsymbol{y} \in \boldsymbol{S}_2$, then the fuzzy expected value of the game for Player A_1 is calculated as,*

$$\tilde{\tilde{\boldsymbol{E}}}_{A_1} = \boldsymbol{x}^T \tilde{\tilde{G}} \boldsymbol{y} = \sum_{i=1}^m \sum_{j=1}^n \tilde{\tilde{\kappa}}_{ij} x_i y_j = \sum_{i=1}^m \sum_{j=1}^n \{(a^s_{ij}, a^m_{ij}, a^u_{ij}; \tau^l_{ij}, \tau^r_{ij})\} x_i y_j.$$

Using Equation (2), the expected pay-off can be calculated as,

$$\begin{aligned}\tilde{\tilde{\boldsymbol{E}}}_{A_1} = & \Big(\sum_{i=1}^m \sum_{j=1}^n \Big((x_i y_j)^+ a^s_{ij} - (x_i y_j)^- a^u_{ij}\Big), \sum_{i=1}^m \sum_{j=1}^n \Big((x_i y_j)^+ a^m_{ij} - (x_i y_j)^- a^m_{ij}\Big), \sum_{i=1}^m \sum_{j=1}^n \Big((x_i y_j)^+ a^u_{ij} - (x_i y_j)^+ a^s_{ij}\Big); \\ & \max_{1\le i\le m} \max_{1\le j\le n} \tau^l_{ij}, \min_{1\le i\le m} \min_{1\le j\le n} \tau^r_{ij}\Big),\end{aligned}$$

where $(x_i y_j)^+ = \max\{x_i y_j, 0\}$, and $(x_i y_j)^- = \max\{-x_i y_j, 0\}$.
Since $x_i \geq 0,\ \forall i$ and $y_j \geq 0,\ \forall j$, so, $(x_i y_j)^+ = x_i y_j$, and $(x_i y_j)^- = 0$ and thus,

$$\tilde{\tilde{\boldsymbol{E}}}_{A_1} = \Big(\sum_{i=1}^m \sum_{j=1}^n x_i y_j a^s_{ij}, \sum_{i=1}^m \sum_{j=1}^n x_i y_j a^m_{ij}, \sum_{i=1}^m \sum_{j=1}^n x_i y_j a^u_{ij}; \max_{1\le i\le m} \max_{1\le j\le n} \tau^l_{ij}, \min_{1\le i\le m} \min_{1\le j\le n} \tau^r_{ij}\Big). \quad (12)$$

Definition 4.2. Optimal Strategies: *Suppose $\boldsymbol{x}^* \in \boldsymbol{S}_1$ and $\boldsymbol{y}^* \in \boldsymbol{S}_2$, then $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are said to be the optimal strategies for Player A_1, and Player A_2, respectively, if it satisfy the following inequations:*

$$\boldsymbol{x}A(\tilde{\tilde{\kappa}})\boldsymbol{y}^{*T} \leq \boldsymbol{x}^* A(\tilde{\tilde{\kappa}})\boldsymbol{y}^{*T} \text{ and } \boldsymbol{x}^* A(\tilde{\tilde{\kappa}})\boldsymbol{y}^{*T} \leq \boldsymbol{x}^* A(\tilde{\tilde{\kappa}})\boldsymbol{y}^{T}. \tag{13}$$

Where $A(\tilde{\tilde{\kappa}})$ is the average approximation of the fuzzy variable $\tilde{\tilde{\kappa}}$.

Substituting the value of $A(\tilde{\tilde{\kappa}})$ in (13), we obtain,

$$\begin{aligned}
\mathbf{x}\frac{\left(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\right)d\alpha}{2}\mathbf{y}^{*T} &\leq \mathbf{x}^*\frac{\left(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\right)d\alpha}{2}\mathbf{y}^{*T},\\
\text{and } \mathbf{x}^*\frac{\left(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\right)d\alpha}{2}\mathbf{y}^{*T} &\leq \mathbf{x}^*\frac{\left(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\right)d\alpha}{2}\mathbf{y}^{T}.
\end{aligned} \tag{14}$$

Theorem 4.3. *The points $\boldsymbol{x}^* \in \boldsymbol{S}_1$ and $\boldsymbol{y}^* \in \boldsymbol{S}_2$ are called optimal strategies for Player A_1 and PlayerA_2 iff*

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{n}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* \leq \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*,\\
\textit{and}\quad &\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* \leq \frac{1}{2}\sum_{i=1}^{m} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big].
\end{aligned} \tag{15}$$

Proof. See Appendix A. □

From the above theorem, we say that, either

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* > \max_i \frac{1}{2}\sum_{j=1}^{n}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*,\\
\text{or}\quad &\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* = \max_i \frac{1}{2}\sum_{j=1}^{n}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*.
\end{aligned}$$

But, if the left hand expression be bigger than the right hand expression, then we must have,

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* > \frac{1}{2}\sum_{j=1}^{n}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*,$$

which implies that,

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* > \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*,$$

which is not possible. Thus, we always have,

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* = \max_i \frac{1}{2}\sum_{j=1}^{n}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*, \tag{16}$$

and similarly,

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* = \min_j \frac{1}{2}\sum_{i=1}^{m} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]. \tag{17}$$

Now, if we denote the expression $\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*$ as w, which is considered as the value of the game, then from the above theorem, we can say that, the points $\mathbf{x}^* \in \mathbf{S}_1$ and $\mathbf{y}^* \in \mathbf{S}_2$ are called optimal strategies for Player A_1, and Player A_2 respectively, if

$$\sum_{j=1}^{n}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* \leq w \text{ and } w \leq \sum_{i=1}^{m} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha) + \int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]. \tag{18}$$

To reach the goal, i.e., to maximize the profit, Player A_1 has to maximize the quantity w and to minimize the loss, Player A_2 has to minimize the quantity w. Thus to evaluate the value of the game and the optimal strategies for player A_1 and player A_2 we have to solve the equivalent crisp linear programming models:

$$\begin{aligned} &\max w \\ \text{s.t.}\quad &\frac{1}{2}\sum_{i=1}^{m}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]x_i \geq w, \\ &\qquad j=1,2,...n \\ &\sum_i x_i = 1 \\ &x_i \geq 0, i=1,2,...,m \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\min w \\ \text{s.t.}\quad &\frac{1}{2}\sum_{j=1}^{n}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j \leq w, \\ &\qquad i=1,2,...m \\ &\sum_j y_j = 1 \\ &y_j \geq 0, j=1,2,...,n \end{aligned} \tag{20}$$

Solving these two LP models, we obtain the optimal strategies $\mathbf{x}^*$ and $\mathbf{y}^*$ for Player A_1, and A_2 and the value of the game w.

Finally, the expected pay-off for Player A_2 is calculated by using Equation (12) as shown in Definition 4.1.

4.3 Algorithm

The algorithm of the solution procedure of a matrix game with pay-offs of TT2FV is described in the following.

Step I: Consider a matrix game with pay-offs of TT2FV, $\tilde{\tilde{G}} = \left(\tilde{\tilde{\kappa}}_{ij}\right)_{m\times n} = \left((a_{ij}^s, a_{ij}^m, a_{ij}^u; \tau_{ij}^l, \tau_{ij}^r)\right)_{m\times n}$.

Step II: Compute the possibility distribution of the type reduced variable $\tilde{\kappa}_{ij}$ of each TT2FV $\tilde{\tilde{\kappa}}_{ij}$ by using the optimistic CV reduction, pessimistic CV reduction and CV reduction which are depicted in Equations (3), (4) and (5) respectively.

Step III: Find the $\alpha-$cuts of these type reduced variables $\tilde{\kappa}_{ijs}$ as $[\tilde{\kappa}_{ijs}^L(\alpha), \tilde{\kappa}_{ijs}^R(\alpha)]$, for $s = o, p, r$, where $\tilde{\kappa}_{ijs}^L(\alpha)$ and $\tilde{\kappa}_{ijs}^R(\alpha)$ are calculated.

Step IV: Calculate the average approximation of the entries of the pay-off matrix by using the formula,

$$A_s(\tilde{\tilde{\kappa}}_{ij}) = \frac{1}{2}\Big[\int_0^1 \{\tilde{\kappa}_{ijs}^L(\alpha)+\tilde{\kappa}_{ijs}^R(\alpha)\}\Big]d\alpha., \text{ for } s=o,p,r \tag{21}$$

Step V: Substitute these average approximations of the entries in the linear programming models (19) and (20).

Step VI: Solve these two LP models and evaluate the optimal strategies $\mathbf{x}^* = (x_1, x_2, ..., x_m)$ and $\mathbf{y}^* = (y_1, y_2, ..., y_n)$ for Player A_1, and Player A_2 and the value of the game w.

Step VII: Utilizing the optimal strategies $\mathbf{x}^*$ and $\mathbf{y}^*$, the expected pay-off of Player A_1 is calculated as, $\tilde{\tilde{\mathbf{E}}}_{A_1} = \mathbf{x}^{*T}\tilde{\tilde{G}}\mathbf{y}^*$.

4.4 Numerical example

This section provides a real-life problem which is described in the scenario of the type-2 fuzzy environment.

4.4.1 Single-use plastic ban problem in India

'Plastic' is the most common and popular word in this era, which is usually a gathering of synthetic or semi-synthetic organic compounds. It becomes a useful commodity in our daily life. It makes our daily practices easier. It has great importance in industrial uses and medical grounds. Also, it is used to reduce soil erosion and wind erosion. Plastic is corrosion resistant and unbreakable, so it can be used after recycling.

But unfortunately, due to the presence of various complex polymers, plastic can not be degraded by any kind of bio-molecules and it remains unchanged for years after years and pollute our earth. Toxic chemicals, released from plastic bags or packets has the power to infertile soil and to pollute the surrounding groundwater. Toxic fumes emitted from burnt plastic are responsible for heart diseases, asthma, emphysema, which even can damage the nervous system and disrupt the re-productivity of animals including human beings. Also, it makes the water more acidic which is very harmful to aquatic plants and animals.

According to a survey (2017-2018) of CPCB (Central Pollution Control Board) India generates about 2600 truckloads (i.e., 26000 tonnes) plastic waste per day, among which 15600 tonnes are carried for recycling and the remaining are either landfills or ends up by polluting groundwater resources or throw into the ocean. But, recycling is not a permanent solution, because, after every recycling, the condition of plastic materials deteriorates due to thermal pressure. So, to take control to generate plastic waste, PWM Rules (Plastic Waste Management),2016 declare a list of dos and don'ts on this type of wastage.

However, it is noteworthy that if the government ban the use and production of such plastics without any alternative measures, it will have a significant impact on the Indian economy. According to AIPMA (All India Plastic Manufacturers Association), over 4 million peoples are employed in the plastic-processing industry which has over 30000 units and an annual turnover of it is around 2.25 lakh crore. If this industry is completely shut down a large number of people will be deprived of their livelihood. Moreover, there will be an economic recession in the small industries associated with it. So, according to AIMPA, as long as the government can stop the economic decline by taking any alternative measure, they cannot stop this production.

Out of the governmental policy and the agendas of AIPMA, as a responsible citizen, we have to maintain some practices to make our planet healthy. Plastic has indeed become an important item in our daily life; on the other hand, it is harmful to the environment. So, we must take some measures to maintain the health of the earth. For these,

- We should not throw plastic waste here and there and gather it for recycling.
- We should not carry plastic carries for grocery and prefer jute bags instead of it.

The government and other non-governmental NGOs took some strategies to implement their rules with the aforesaid must-to-do-measures. Also, AIPMA took some strategies to fulfill its requirements with the aforesaid must-to-do-measures.

The strategies taken by the alliance of Government and other non-governmental NGOs are:

A1: Effective policy formulation.

A2: Campaign to make India free from single-use plastic.

A3: Announcing penalty for using it.

Also, The strategies taken by AIMPA are:

B1: Collecting plastic and recycling it.

B2: More employment to the recycling industries.

B3: Employ advanced technology to delay the deterioration of the plastic materials due to thermal pressure.

These situations of confliction can be considered as a two-person matrix games. Here the alliance of government and the non-governmental NGOs is considered as Player A_1, and on the other side, AIPMA is considered as Player A_2. Here, it has been seen that both the Players exert some strategies to settle their views and the strategies are dependent upon some secondary must-to-do-measures. So the pay-offs of this game may be considered as fuzzy variables with some secondary possibility distribution. For this reason, we are bound to express the pay-offs in three-dimensional fuzzy sets. So, the pay-offs are assessed in the form of TT2FVs. Let the pay-off matrix $\tilde{\tilde{G}}$ of Player A_1 can be expressed as follows:

$$\tilde{\tilde{G}} = \begin{array}{c} \\ A1 \\ A2 \\ A3 \end{array} \begin{array}{c} \begin{array}{ccc} B1 & B2 & B3 \end{array} \\ \begin{pmatrix} \tilde{\tilde{120}} & \tilde{\tilde{110}} & \tilde{\tilde{75}} \\ \tilde{\tilde{180}} & \tilde{\tilde{120}} & \tilde{\tilde{72}} \\ \tilde{\tilde{143}} & \tilde{\tilde{143}} & \tilde{\tilde{150}} \end{pmatrix} \end{array}.$$

The expression of the entries of the pay-off matrix are given in Table 2.

Here, $\tilde{\tilde{120}} = (115, 120, 125; 0.15, 0.17)$ indicates that if the government and non-governmental NGOs took the strategy $A1$ and AIPMA took the strategy $B1$, then after policy formulation, the amount of producing single-use plastic may be reduced by 120 units, with a minimum of 115 units and maximum of 125 units. Also, the minimum amount is reducible to 15%,, and the maximum amount is extensible to 17%. The other entries of the pay-off matrix can be described similarly.

Table 2: Expressions of the TT2FV entries

TT2FV entries	Expressions	TT2FV entries	Expressions
$\tilde{\tilde{120}}$	(115,120,125;0.15,0.17)	$\tilde{\tilde{72}}$	(70,72,75;0.25,0.35)
$\tilde{\tilde{110}}$	(108,110,112;0.25,0.47)	$\tilde{\tilde{143}}$	(140,143,146;0.5,0.6)
$\tilde{\tilde{75}}$	(73,75,77;0.5,0.5)	$\tilde{\tilde{150}}$	(145,150,155;0.35,0.38)
$\tilde{\tilde{180}}$	(175,180,185;0.4,0.9)		

Table 3: Defuzzified value of the TT2FV entries

TT2FV entries	defuzzified values		
	optimistic CV-reduction	pessimistic CV-reduction	CV-reduction
$\tilde{\tilde{120}}$	121.934	103.224	119.985
$\tilde{\tilde{110}}$	110.885	105.919	109.907
$\tilde{\tilde{75}}$	75.904	72.774	75.000
$\tilde{\tilde{180}}$	184.403	173.358	178.519
$\tilde{\tilde{72}}$	73.509	66.113	72.196
$\tilde{\tilde{143}}$	144.479	139.661	142.904
$\tilde{\tilde{150}}$	152.096	142.536	149.969

According to the solution algorithm, and applying the proposed defuzzification method as described in Subsection 3.1, we compute the crisp values of the entries of the pay-off matrix $\tilde{\tilde{G}}$, utilizing Equation (9), Equation (10), and Equation (11) which are listed in Table 3.

Now, substituting the optimistic CV reduced entries in Problem (19) and Problem (20) we have the following two LP problems.

$$\begin{aligned} &\max w \\ \text{s.t.}\quad &121.934x_1 + 184.403x_2 + 144.479x_3 \geq w \\ &110.885x_1 + 121.934x_2 + 144.479x_3 \geq w \\ &75.904x_1 + 73.509x_2 + 152.096x_3 \geq w \\ &x_1 + x_2 + x_3 = 1 \\ &x_i \geq 0,\ i = 1,2,3 \end{aligned} \tag{22}$$

$$\begin{aligned} &\min w \\ \text{s.t.}\quad &121.934y_1 + 110.885y_2 + 75.904y_3 \leq w \\ &184.403y_1 + 121.934y_2 + 73.509y_3 \leq w \\ &144.479y_1 + 144.479y_2 + 152.096y_3 \leq w \\ &y_1 + y_2 + y_3 = 1 \\ &y_j \geq 0,\ j = 1,2,3 \end{aligned} \tag{23}$$

By solving Problem (22) and Problem (23), we obtain the optimal strategy $\mathbf{x}^* = (0, 0, 1)$ for Player A_1, i.e., the alliance of the government and non-governmental NGOs, optimal strategy $\mathbf{y}^* = (0.3360, 0.6640, 0)$ for Player A_2, i.e., AIPMA, and the optimistic value of the game, $w = 144.479$.

Similarly, to find the pessimistic value of the game we substitute the pessimistic CV reduced entries in the problems (19) and (20) as follows,

$$\begin{aligned} &\max w \\ \text{s.t.}\quad &103.224x_1 + 173.358x_2 + 139.661x_3 \geq w \\ &105.919x_1 + 103.224x_2 + 139.661x_3 \geq w \\ &72.774x_1 + 66.113x_2 + 142.536x_3 \geq w \\ &x_1 + x_2 + x_3 = 1 \\ &x_i \geq 0,\ i = 1,2,3 \end{aligned} \tag{24}$$

$$\begin{aligned} &\min w \\ \text{s.t.}\quad &103.224y_1 + 105.919y_2 + 72.774y_3 \leq w \\ &173.358y_1 + 103.224y_2 + 66.113y_3 \leq w \\ &139.661y_1 + 139.661y_2 + 142.536y_3 \leq w \\ &y_1 + y_2 + y_3 = 1 \\ &y_j \geq 0,\ j = 1,2,3 \end{aligned} \tag{25}$$

And, to find the value of the game we substitute the CV reduced entries in the problems (19) and (20) as follows,

$$\begin{aligned}&\max w\\ \text{s.t.}\quad &119.985x_1+178.519x_2+142.904x_3\geq w\\ &109.907x_1+119.985x_2+142.904x_3\geq w\\ &75x_1+72.196x_2+149.969x_3\geq w\\ &x_1+x_2+x_3=1\\ &x_i\geq 0,\ i=1,2,3\end{aligned}\tag{26}$$

$$\begin{aligned}&\min w\\ \text{s.t.}\quad &119.985y_1+109.907y_2+75y_3\leq w\\ &178.519y_1+119.985y_2+72.196y_3\leq w\\ &142.904y_1+142.904y_2+149.969y_3\leq w\\ &y_1+y_2+y_3=1\\ &y_j\geq 0,\ j=1,2,3\end{aligned}\tag{27}$$

The results are enlisted in Table 4 in Subsection 4.4.2.

4.4.2 Result and discussion

In this subsection, we summarize the results, obtained by solving Problem (22), Problem (23), Problem (24), Problem (25), Problem (26), and Problem (27) in Table 4. Substituting the optimal strategies $\mathbf{x}^*$ for Player A_1, and $\mathbf{y}^*$ for Player A_2 in Equation (12), we calculate the expected pay-off $\tilde{\tilde{\mathbf{E}}}_I$ for Player A_1, which is also enlisted in Table 4

Table 4: optimal strategies and value of the game

Type reduction used	optimal strategies $\mathbf{x}^*$	optimal strategies $\mathbf{y}^*$	Value of the game	Expected pay-off ($\tilde{\tilde{\mathbf{E}}}_I$)
Optimistic CV-reduction			144.479	
Pessimistic CV-reduction	$(0,0,1)$	$(0.3360,0.6640,0)$	139.661	$(140,143,146;0.5,0.17)$
CV-reduction			142.904	

From Table 4, following observations are made.

(i) It can be easily seen from Table 4 that the value of the game is $w=144.479$, and the optimal strategies are $\mathbf{x}^*=(0,0,1)$, and $\mathbf{y}^*=(0.3360,0.6640,0)$, when we solve the matrix game with optimistic CV reduced entries. Also if we solve the game with pessimistic CV-reduced entries then the value of the game will be $w=139.661$, and if we use CV reduced entries to solve the game, then value of the game will be $w=142.904$.

(ii) Substituting the optimal strategies $\mathbf{x}^*=(0,0,1)$ and $\mathbf{y}^*=(0.3360,0.6640,0)$, in Equation (12), the expected pay-off for Player A_1 is calculated as $\tilde{\tilde{\mathbf{E}}}_I=(140,143,146;0.50,0.17)$, which is a TT2FV. This indicates that if the alliance of government and non-governmental NGOs fully uses strategy $A3$, i.e., "Announcing penalty" and the AIPMA takes the strategy $B1$, i.e., "Collecting plastic and using it" with 33.60%, $B2$, i.e., "More emplyment to the recycling industries" with 66.40%, then the reduction of the single-used plastic will be possible 143 units upto a maximum 146 units and a minimum 140 units. Also, the uncertainty parameters $\tau^l=0.50$, and $\tau^r=0.17$ imply that the minimum unit of reduction of the single-used plastic may be reduced to 50%, and the maximum unit of reduction may be extended to 17%.

4.4.3 Comparison study

To investigate the cogency of the obtained result by imposing the proposed methodology, in this subsection, we check the resolved optimal strategies and the value of the game in the crisp and type-1 fuzzy environment. If the uncertainty parameters of the entries of the pay-off matrix have been vanished, then according to Remark 2.6, the entries are transformed into TT1FV. Using the algorithm of Campos [7], we solve the matrix game with TT1FV pay-offs. The results, obtained in three different scenarios, are summarized in Table 5.

From Table 5, following observations are made.

(i) The resolved optimal strategies remain unchanged in three different scenarios. Which shows that our proposed methodology is valid.

(ii) There is no change in the value of the game, obtained in crisp and fuzzy environment. But, a variation in the value of the game has been observed, when the matrix game is promoted in type-2 fuzzy environment.

Table 5: optimal strategies and value of the game in different environment

	Environment	optimal strategies $\mathbf{x}^*$	$\mathbf{y}^*$	Value of the game
Crisp	-	$(0, 0, 1)$	$(0.3360, 0.6640, 0)$	143
T1FV	-	$(0, 0, 1)$	$(0.3360, 0.6640, 0)$	143
	Pessimistic CV-reduction			139.661
TT2FV	CV-reduction	$(0, 0, 1)$	$(0.3360, 0.6640, 0)$	142.904
	Optimistic CV-reduction			144.479

(iii) From Table 5, we have observed that the value of the game of the matrix games depicted in the type-2 fuzzy environment depend entirely on what method is used to defuzzify it. From the comparative discussion, we can conclude that if we use the optimistic CV reduction methods for type reduction of TT2FVs, then we obtain the best result. It even gives better value than the results obtained in crisp or type-1 fuzzy environments.

5 Conclusion

Aiming to consider the secondary possibility distribution of the information, We have constructed a matrix game with TT2FV pay-offs. This is the first attempt to apply TT2FV in matrix game theory. A new defuzzification method of TT2FV, namely, average approximation, is proposed here, which consists of two parts. First, we reduce the type of the TT2FVs, and then the reduced variable is defuzzified by utilizing a formula given in our article. To solve the game, we convert the pay-offs into the crisp by using optimistic CV reduction, pessimistic CV reduction, and CV reduction. Then the problems are solved by the usual methods. We have illustrated the applicability of this methodology by considering a single-use plastic ban problem in India. Here we have discussed the situation of confliction on the ban of single-use plastic between the government and the non-governmental NGOs with the AIPMA. To check the validation of our proposed methodology, we compare the obtained result with the same problem, defined in the crisp and the fuzzy environment.

In this article, we have concentrated mainly on two works. One of them is to incarnate a matrix game with TT2FV entries, and the other is to introduce a suitable defuzzification procedure of TT2FVs. Here we have observed that the results of a matrix game problem depicted in the type-2 fuzzy environment depend entirely on what method is used to defuzzify it. If we use the optimistic CV reduction methods for type reduction of TT2FVs, then we obtain the best result. It even gives better value than the results obtained in crisp or type-1 fuzzy environments. These are the main contribution of our paper.

However, there are some drawbacks to our proposed article. In this work, the defuzzification procedure is formulated based on type reduction. Using this defuzzification rule, we find an equivalent defuzzified value of the matrix game with TT2FV pay-offs. For which we loss a lot of fuzzy information. Therefore, it needs further investigation to introduce a general methodology to solve a matrix game with TT2FV pay-offs in a direct way. We can leave this as an open problem for our future study.

In the future, an attempt could be made to design a methodology to study bi-matrix games under the type-2 fuzzy framework. Here, the proposed methodology has been illustrated by a single used plastic-ban problem. It also can be extended to different directions of decision-making problems such as market share problems, military science, medical diagnosis, etc.

Appendix A

Proof of Theorem 4.3

Necessary part of this proof is obvious.

To prove the converse part, we first take the first inequality relation of the equation (15),

$$\frac{1}{2}\sum_{j=1}^{n}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* \leq \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*.$$

We may write, $$x_i\sum_{j=1}^{n}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* \leq x_i\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*$$

Therefore, $$\sum_{i=1}^{m}x_i\sum_{j=1}^{n}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* \leq \sum_{i=1}^{m}x_i\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*.$$

Which gives, $$\sum_{i=1}^{m}\sum_{j=1}^{n}x_i\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* \leq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^*,$$

as $\sum_{i=1}^{m}x_i=1$ and $x_i\geq 0$ for $i=1,2,\ldots,m$.

which implies, $$\mathbf{x}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]\mathbf{y}^{*T} \leq \mathbf{x}^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]\mathbf{y}^{*T}.$$

i.e., $$\frac{1}{2}\mathbf{x}\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]\mathbf{y}^{*T} \leq \frac{1}{2}\mathbf{x}^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]\mathbf{y}^{*T}.$$

Similarly, we can show that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]y_j^* \leq \sum_{i=1}^{m} x_i^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]$$

which proves that, $$\frac{1}{2}\mathbf{x}^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]\mathbf{y}^{*T} \leq \frac{1}{2}\mathbf{x}^*\Big[\Big(\int_0^1 \tilde{\kappa}_{ij}^L(\alpha)+\int_0^1 \tilde{\kappa}_{ij}^R(\alpha)\Big)d\alpha\Big]\mathbf{y}^{T}.$$

Therefore, we can write $\mathbf{x}^*$ and $\mathbf{y}^*$ are the optimal strategies for Player A_1, and Player A_2. □

Acknowledgement

We would like to acknowledge the anonymous reviewers for their constructive comments, worthy suggestions, and earnestly supports, which have been helpful for us to construct a better version of our present work.

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