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Original paper

On homogeneity of the smallest semicopula-based universal integral in the class of pseudo-continuous semicopulas

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Abstract

In this paper, we introduce the concept of pseudo-continuous semicopula. We show its relationship with continuity in the second variable and provide a characterization of all semicopulas S such that the smallest semicopula-based universal integral is S-homogeneous. This completely solves Open problem 2.29 proposed by J. Borzová-Molnárová et al. in the paper [2].

Keywords: Semicopula, the smallest semicopula-based universal integral, homogeneity.

1 Introduction

It is well known that in general, Sugeno integral is not linear and in particular it is linear only for small classes of measures [5]. However, in [2] J. Borzová-Molnárová et al. proved that Sugeno integral with respect to a maxitive measure is idempotent linear from Theorem 2.26 in [2]. Furthermore, they also presented a characterization of two semicopula-based integrals as follows:
Proposition 2.24 in [2] says that the smallest semicopula-based universal integral with respect to S is Π-homogeneous if and only if $\mathsf{S} = \Pi$.
Proposition 2.25 in [2] states that the smallest semicopula-based universal integral with respect to S is M-homogeneous if and only if $\mathsf{S} = \mathsf{M}$.
These results motivated the authors to formulate Open problem 2.29 and they also conjectured that the class of semicopulas solving the open problem will only contain two semicopulas M and Π.

Next, in [1] M. Boczek and M. Kaluszka showed that this conjecture is false, that is, any associative semicopula with continuous selections which satisfy some mild conditions satisfies Open problem 2.29. Inspired by the above, we study another approach which explicitly specifies all solutions of the problem. This approach is based on a new concept which is called "pseudo-continuity" of a semicopula. From the above with the obtained result of Theorem 2.1 in [1], we incidentally discovered one more interesting result, which is the equivalence between the two concepts the pseudo-continuity and continuity in the second variable of a semicopula S that seem to be completely different from each other. By applying this equivalence we deduce another characterization of a semicopula S satisfying Open problem 2.29 in [2]. This characterization is an improvement of Theorem 2.1 and 2.2 in [1].

The layout of our work is organized as follows: In Section 2, we recall the background of the smallest semicopula-based universal integral and the related results leading to the open problem we solve. In Section 3, a new approach solving the problem is presented. Finally, a conclusion and the appendix are given in Section 4 and 5.

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2 Preliminaries

In this section, we recall some necessary concepts and present several results of the smallest semicopula-based universal integral leading to Open problem 2.29 in [2].

Let (X, Σ) be a measurable space, where Σ is a σ-algebra of subsets of a nonempty set X.

Definition 2.1. [9] *Let $\mu : \Sigma \to [0, \infty]$ be a non-negative, extended real-valued set function. We say that μ is a monotone measure if it satisfies*
(1) $\mu(\emptyset) = 0$;
(2) $A, B \in \Sigma$ and $A \subseteq B$ imply $\mu(A) \leqslant \mu(B)$ (monotonicity).
Then the triplet (X, Σ, μ) is called a monotone measure space.

For $f : X \to [0, \infty]$, we denote by $f_\alpha = \{x \in X | \, f(x) \geqslant \alpha\}$ the α-level set of f for $\alpha > 0$.

Definition 2.2. [4] *Let $\mathscr{S}$ denote the class of all measurable spaces (X, Σ).*
1) *We denote by $\mathscr{F}^{[0,a]}_{(X,\Sigma)}$ the set of all Σ-measurable functions $f : X \to [0, a]$ for some $a \in (0, \infty]$. In the case $a = \infty$, we denote by $\mathscr{F}_{(X,\Sigma)} := \mathscr{F}^{[0,\infty]}_{(X,\Sigma)}$.*
2) *For each $b \in (0, \infty)$, we denote by $\mathscr{M}^b_{(X,\Sigma)}$ the set of all monotone measures satisfying $\mu(X) = b$.*

Definition 2.3. *A binary operation $\mathsf{S} : [0, 1] \times [0, 1] \to [0, 1]$ is a semicopula if it satisfies*

1. *S is nondecreasing, i.e., $\mathsf{S}(a, b) \leqslant \mathsf{S}(c, d)$ for all $a, b, c, d \in [0, 1]$ with $a \leqslant c$ and $b \leqslant d$.*
2. *$\mathsf{S}(1, x) = \mathsf{S}(x, 1) = x$ for all $x \in [0, 1]$.*

Moreover, we say that a semicopula S is associative if $\mathsf{S}(\mathsf{S}(x, y), z) = \mathsf{S}(x, \mathsf{S}(y, z))$ for all $x, y, z \in [0, 1]$.

Example 2.4. *The following operations are semicopulas:*

$$\begin{aligned}
&\mathsf{M}(x, y) = x \wedge y,\\
&\Pi(x, y) = x \cdot y,\\
&\mathsf{W}(x, y) = (x + y - 1) \vee 0,\\
&\mathsf{D}(x, y) = \begin{cases} x \wedge y, & \text{if } x \vee y = 1,\\ 0, & \text{otherwise,}\end{cases}\\
&\mathsf{F}_s(x, y) = \log_s \left(1 + \frac{(s^x - 1) \cdot (s^y - 1)}{s - 1}\right), \text{ where } s \in (0, \infty)/\{1\},\\
&\mathsf{FP}(x, y) = \begin{cases} x \wedge y, & \text{if } x + y > 1,\\ 0, & \text{otherwise,}\end{cases}\\
&\Delta_1(x, y) = \begin{cases} 0, & \text{if } x = y = 0,\\ \frac{x \cdot y}{x + y - x \cdot y}, & \text{otherwise,}\end{cases}\\
&\Delta_2(x, y) = \frac{x \cdot y}{2 - (x + y - x \cdot y)},\\
&\Delta_3(x, y) = x \cdot y \cdot (x \vee y).
\end{aligned}$$

Remark 2.5. *Let S be a semicopula. Then the following assertions hold:*

1. *$\mathsf{S}(x, y) \leqslant x \wedge y$ for all $x, y \in [0, 1]$.*
2. *$\mathsf{S}(x, 0) = \mathsf{S}(0, x) = 0$ for all $x \in [0, 1]$.*
3. *If $\mathsf{S}(x, x) = x$ for all $x \in (0, 1)$ then $\mathsf{S} = \mathsf{M}$.*

All the semicopulas from Example 2.4 are associative except Δ_3.

Throughout this paper, the main object of our interest is the class of the smallest semicopula-based universal integrals with respect to a semicopula S given by:

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \sup_{0<\alpha} \mathsf{S}(\alpha, \mu(f_\alpha)),$$

where $(X,\Sigma) \in \mathscr{S}$ and $(m,f) \in \mathscr{M}^1_{(X,\Sigma)} \times \mathscr{F}_{(X,\Sigma)}$.
Note that:
• This integral is also called a S-semicopula integral or a seminormed fuzzy integral (see in [3, 1, 6]).
• In particular, for $\mathsf{S} = \mathsf{M}$ we recover the original definition of the Sugeno integral (see in [8]). For $\mathsf{S} = \Pi$ the integral $\mathbf{I}_\Pi$ is the Shilkret integral (see in [7]).
• Also, for some $A \in \Sigma$ we get

$$\mathbf{I}_{\mathsf{S},A}(\mu, f) := \mathbf{I}_{\mathsf{S}}(\mu, f \cdot \chi_A) = \sup_{0<\alpha} \mathsf{S}\left(\alpha, \mu\left((f \cdot \chi_A)_\alpha\right)\right) = \sup_{0<\alpha} \mathsf{S}\left(\alpha, \mu\left(A \cap f_\alpha\right)\right).$$

The smallest semicopula-based universal integral can be represented in another form as follows:

Theorem 2.6. (Representation theorem in [2]) *Let* S *be a semicopula. Then for all* $(X,\Sigma) \in \mathscr{S}$ *and* $(m,f) \in \mathscr{M}^1_{(X,\Sigma)} \times \mathscr{F}^{[0,1]}_{(X,\Sigma)}$ *we have*

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \sup_{\emptyset \neq A \in \Sigma} \mathsf{S}\left(\inf_{x \in A} f(x), \mu(A)\right).$$

Definition 2.7. *Let* S *be a semicopula and* $\otimes : [0,1] \times [0,1] \to [0,1]$ *be a binary operation. We say that the smallest semicopula-based universal integral with respect to* S *is* $\otimes$*-homogeneous if it satisfies* $\mathbf{I}_{\mathsf{S}}(\mu, k \otimes f) = k \otimes \mathbf{I}_{\mathsf{S}}(\mu, f)$ *for any constant* $k \in [0,1], (X,\Sigma) \in \mathscr{S}$ *and* $(\mu, f) \in \mathscr{M}^1_{(X,\Sigma)} \times \mathscr{F}^{[0,1]}_{(X,\Sigma)}$.

For the homogeneity, we obtain the characterizations of two semicopulas Π and M given by:

Proposition 2.8. (Proposition 2.24 in [2]) *The smallest semicopula-based universal integral with respect to* S *is* Π*-homogeneous if and only if* $\mathsf{S} = \Pi$.

Proposition 2.9. (Proposition 2.25 in [2]) *The smallest semicopula-based universal integral with respect to* S *is* M*-homogeneous if and only if* $\mathsf{S} = \mathsf{M}$.

From the result of Proposition 2.8 and 2.9 the authors in [2] proposed the following Open problem:
"Characterize the class of semicopulas S for which the property $\mathbf{I}_{\mathsf{S}}(\mu, \mathsf{S}(\alpha, f)) = \mathsf{S}(\alpha, \mathbf{I}_{\mathsf{S}}(\mu, f))$ holds for all $\alpha \in [0,1], (X,\Sigma) \in \mathscr{S}$ and $(\mu, f) \in \mathscr{M}^1_{(X,\Sigma)} \times \mathscr{F}^{[0,1]}_{(X,\Sigma)}$."

There was a conjecture that semicopulas Π and M are the only solutions of the open problem. However, in the paper [1], M. Boczek and M. Kaluszka pointed out a necessary condition for a solution of this open problem given by the following theorem:

Theorem 2.10. (Theorem 2.1 in [1]) *If a semicopula* S *satisfies the open problem then the semicopula* S *is associative and continuous in the second variable.*

Furthermore, they also stated a sufficient condition for a solution of the open problem.

Theorem 2.11. (Theorem 2.2 in [1]) *If a semicopula* S *is associative and continuous in the second variable and* $x \mapsto \mathsf{S}(a,x)$ *is increasing on some countable number of intervals for every* $a \in (0,1)$ *then* S *satisfies the open problem.*

Motivated by the above results, we introduce another approach to solve the open problem without using Theorem 2.10 and 2.11 which is presented in Section 3.

3 Another approach for solving the open problem

To present a full characterization of semicopulas solving the open problem, first of all, we need to introduce some new concepts and related results.

Definition 3.1. *We say that a semicopula* S *is continuous in the second variable if the function* $[0,1] \ni x \mapsto \mathsf{S}(a,x)$ *is continuous for every fixed* $a \in [0,1]$.

Analogous definitions for the left and right continuity in the second variable is omitted.

Remark 3.2. *It is easy to see that the class of all semicopulas which are continuous in the second variable is very broad. It includes the following semicopulas:* $\mathsf{M}, \Pi, \mathsf{W}, \mathsf{F}_s, \Delta_1, \Delta_2, \Delta_3$.

Definition 3.3. *We say that a semicopula* S *is pseudo-continuous if it has the following properties*
1) $\inf\limits_{x\in A} \mathsf{S}(k, f(x)) = \mathsf{S}\left(k, \inf\limits_{x\in A} f(x)\right)$ *for all* $k \in [0,1], (X,\Sigma) \in \mathscr{S}$, $A \in \Sigma$ *and* $f \in \mathscr{F}^{[0,1]}_{(X,\Sigma)}$.
2) $\sup\limits_{A\in\Sigma} \mathsf{S}\left(k, \mathsf{S}\left(\inf\limits_{x\in A} f(x), \mu(A)\right)\right) = \mathsf{S}\left(k, \sup\limits_{A\in\Sigma}\left\{\mathsf{S}\left(\inf\limits_{x\in A} f(x), \mu(A)\right)\right\}\right)$ *for all* $k \in [0,1]$, $(X,\Sigma) \in \mathscr{S}$ *and* $(\mu, f) \in \mathscr{M}^1_{(X,\Sigma)} \times \mathscr{F}^{[0,1]}_{(X,\Sigma)}$.

First, we study the following two negative examples.

Example 3.4. *Semicopula* FP *is not pseudo-continuous. Indeed, by taking* $X = [0,1]$, $f(x) = x$ *for every* $x \in X$, $k = \frac{1}{2}$, $\Sigma = \mathcal{P}(X)$ *and* $A = (k,1]$. *It is easy to see that*

$$\inf_{x\in A} \mathsf{FP}(k, f(x)) = \inf_{x\in A}\{k \wedge f(x)\} = k = \frac{1}{2},$$

and

$$\mathsf{FP}\left(k, \inf_{x\in A} f(x)\right) = \mathsf{FP}(k,k) = \mathsf{FP}\left(\frac{1}{2}, \frac{1}{2}\right) = 0.$$

So, the property 1 of Definition 3.3 doesn't hold. This means that FP *is not pseudo-continuous.*

Example 3.5. *Semicopula* D *is not pseudo-continuous. Indeed, take* $X = [0,1)$, $f(x) = x$ *for every* $x \in X$, $k \in (0,1)$ *and* $\Sigma = \mathcal{P}(X)$. *Put:* $\mu : \Sigma \to \mathbb{R}_+$ *defined by* $\mu(E) = 1$ *for all* $E \in \Sigma \setminus \{\emptyset\}$ *and* $\mu(\emptyset) = 0$. *Then* μ *is a monotone measure on* (X,Σ) *and* $\mathsf{D}\left(\inf\limits_{x\in A} f(x), \mu(A)\right) = \inf\limits_{x\in A} f(x) < 1$ *for any* $A \in \Sigma \setminus \{\emptyset\}$. *This implies that*

$$\sup_{A\in\Sigma} \mathsf{D}\left(\inf_{x\in A} f(x), \mu(A)\right) \geqslant \sup_{n\in\mathbb{N}} \mathsf{D}\left(\inf_{x\in A_n} f(x), 1\right) = \sup_{n\in\mathbb{N}} \inf_{x\in A_n} f(x) = \sup_{n\in\mathbb{N}}\left\{1 - \frac{1}{n}\right\} = 1,$$

where $A_n = \left[1 - \frac{1}{n}, 1\right)$. *We deduce that*

$$\mathsf{D}\left(k, \sup_{A\in\Sigma} \mathsf{D}\left(\inf_{x\in A} f(x), \mu(A)\right)\right) = \mathsf{D}(k,1) = k.$$

On the other hand,

$$\sup_{A\in\Sigma} \mathsf{D}\left(k, \mathsf{D}\left(\inf_{x\in A} f(x), \mu(A)\right)\right) = \sup_{\emptyset\neq A\in\Sigma} \mathsf{D}\left(k, \inf_{x\in A} f(x)\right) = 0.$$

So, the property 2 of Definition 3.3 doesn't hold. This means that D *is not pseudo-continuous.*

Now we present a full solution of the open problem by another approach without using the results of Theorem 2.10 and 2.11.

Theorem 3.6. *Let* S *be a semicopula. Then the following conditions are equivalent:*

1. *The property:* $\mathbf{I}_{\mathsf{S}}(\mu, \mathsf{S}(\alpha, f)) = \mathsf{S}(\alpha, \mathbf{I}_{\mathsf{S}}(\mu, f))$ *holds for every* $\alpha \in [0,1]$, *every* $(X,\Sigma) \in \mathscr{S}$ *and every* $(\mu, f) \in \mathscr{M}^1_{(X,\Sigma)} \times \mathscr{F}^{[0,1]}_{(X,\Sigma)}$;
2. S *is associative and pseudo-continuous.*

Proof. $2 \Rightarrow 1$: By applying Theorem 2.6 for $\mathbf{I}_{\mathsf{S}}$ and Definition 3.3, one has

$$\begin{aligned}\mathbf{I}_{\mathsf{S}}(\mu, \mathsf{S}(k,f)) &= \sup_{\emptyset\neq A\in\Sigma} \mathsf{S}\left(\inf_{x\in A} \mathsf{S}(k, f(x)), \mu(A)\right) = \sup_{\emptyset\neq A\in\Sigma} \mathsf{S}\left(\mathsf{S}\left(k, \inf_{x\in A} f(x)\right), \mu(A)\right)\\ &= \sup_{\emptyset\neq A\in\Sigma} \mathsf{S}\left(k, \mathsf{S}\left(\inf_{x\in A} f(x), \mu(A)\right)\right) = \mathsf{S}\left(k, \sup_{\emptyset\neq A\in\Sigma} \mathsf{S}\left(\inf_{x\in A} f(x), \mu(A)\right)\right)\\ &= \mathsf{S}(k, \mathbf{I}_{\mathsf{S}}(\mu, f)).\end{aligned}$$

$1 \Rightarrow 2$: There are three steps:

Step 1. Associativity of S: Consider any $k, a, b \in [0,1]$, μ be the Lebesgue measure and $f(x) = a \cdot \chi_{[0,b]}(x)$ for $x \in X = [0,1]$. Then

$$\mathbf{I}_{\mathsf{S}}(\mu, f) = \sup_{0<\alpha\leqslant a} \mathsf{S}(\alpha, \mu(f_\alpha)) \vee \sup_{a<\alpha\leqslant 1} \mathsf{S}(\alpha, \mu(f_\alpha)) = \sup_{0<\alpha\leqslant a} \mathsf{S}(\alpha, b) = \mathsf{S}(a, b).$$

On the other hand, we have

$$\mathbf{I}_{\mathsf{S}}(\mu, \mathsf{S}(k, f)) = \mathbf{I}_{\mathsf{S}}\left(\mu, \mathsf{S}\left(k, a \cdot \chi_{[0,b]}\right)\right) = \mathbf{I}_{\mathsf{S}}\left(\mu, \mathsf{S}(k, a) \cdot \chi_{[0,b]}\right) = \mathsf{S}(\mathsf{S}(k, a), b).$$

Since S is a solution of the open problem, it follows that $\mathsf{S}(\mathsf{S}(k,a), b) = \mathsf{S}(k, \mathsf{S}(a,b))$. So, S is associative.

Step 2. The property 1 of Definition 3.3: For any $(X, \Sigma) \in \mathscr{S}$, $A \in \Sigma$, put: $m_A(E) = 0$ if $E \not\supseteq A$ and $m_A(E) = 1$ if $E \supseteq A$ for any $E \in \Sigma$. Then m is a monotone measure on (X, Σ). Therefore, for any $k \in [0,1]$ and $f \in \mathscr{F}^{[0,1]}_{(X,\Sigma)}$ it follows from the assumption that

$$\mathbf{I}_{\mathsf{S}}(m_A, \mathsf{S}(k, g)) = \mathsf{S}(k, \mathbf{I}_{\mathsf{S}}(m_A, g)), \text{ where } g = f \cdot \chi_A.$$

By applying Theorem 2.6, we get that

$$\begin{aligned}\mathbf{I}_{\mathsf{S},A}(m_A, f) = \mathbf{I}_{\mathsf{S}}(m_A, g) &= \sup_{\emptyset\neq E\in\Sigma} \mathsf{S}\left(\inf_{x\in E} g(x), m_A(E)\right)\\ &= \sup_{A\subseteq E\in\Sigma} \mathsf{S}\left(\inf_{x\in E} f(x) \cdot \chi_A(x), 1\right) = \inf_{x\in A} f(x).\end{aligned}$$

On the other hand, from $\mathsf{S}(k, g(x)) = \mathsf{S}(k, f(x)) \cdot \chi_A(x)$ and applying the above result it follows that

$$\mathbf{I}_{\mathsf{S}}(m_A, \mathsf{S}(k, g)) = \inf_{x\in A} \mathsf{S}(k, f(x)).$$

So,

$$\inf_{x\in A} \mathsf{S}(k, f(x)) = \mathsf{S}\left(k, \inf_{x\in A} f(x)\right).$$

Step 3. The property 2 of Definition 3.3: For all $k \in [0,1]$, $(X, \Sigma) \in \mathscr{S}$ and $(\mu, f) \in \mathscr{M}^1_{(X,\Sigma)} \times \mathscr{F}^{[0,1]}_{(X,\Sigma)}$, by applying the obtained results of steps 1, 2 and Representation theorem, we get that

$$\begin{aligned}\mathbf{I}_{\mathsf{S}}(\mu, \mathsf{S}(k, f)) &= \sup_{\emptyset\neq A\in\Sigma} \mathsf{S}\left(\inf_{x\in A} \mathsf{S}(k, f(x)), \mu(A)\right)\\ &= \sup_{A\in\Sigma} \mathsf{S}\left(\mathsf{S}\left(k, \inf_{x\in A} f(x)\right), \mu(A)\right)\\ &= \sup_{A\in\Sigma} \mathsf{S}\left(k, \mathsf{S}\left(\inf_{x\in A} f(x), \mu(A)\right)\right).\end{aligned}$$

On the other hand, it follows from the assumption and Representation theorem that

$$\mathbf{I}_{\mathsf{S}}(\mu, \mathsf{S}(k, f)) = \mathsf{S}(k, \mathbf{I}_{\mathsf{S}}(\mu, f)) = \mathsf{S}\left(k, \sup_{A\in\Sigma} \mathsf{S}\left(\inf_{x\in A} f(x), \mu(A)\right)\right).$$

By comparing the above results, we deduce that the property 2 of Definition 3.3 holds. The proof of Theorem 3.6 is completed. □

A surprising and interesting result is given by the following theorem.

Theorem 3.7. *Let* S *be a semicopula. Then*

S *is pseudo-continuous if and only if* S *is continuous in the second variable.*

Proof. The proof is given in the appendix. □

Remark 3.8. *1) From Remark 2.5, 3.2, Theorem 3.6 and 3.7, we can conclude that the semicopulas* M, Π, W, F_s, Δ_1, Δ_2 *are solutions of the open problem.*
2) From Remark 2.5, Example 3.4, 3.5 and Theorem 3.6, we can conclude that the semicopulas D, FP *and* Δ_3 *are not solutions of the open problem.*

Example 3.9. *Let* $X = [0,1]$, $\Sigma = \mathcal{P}(X)$ *and* $\mu : \Sigma \to \mathbb{R}_+$ *defined by* $\mu(E) = 1$ *if* $E \neq \emptyset$ *and* $\mu(\emptyset) = 0$. *Consider* $k \in (0,1)$ *and* $f(x) = x$ *on* X. *Then*

$$(\mathsf{D}(k,f)(x)) = \mathsf{D}(k, f(x)) = 0 \quad \text{for all } x \in X,$$

and

$$\mathbf{I}_{\mathsf{D}}(\mu, f) = \sup_{\alpha\in(0,1]} \mathsf{D}(\alpha, \mu(f_\alpha)) = \sup_{\alpha\in(0,1]} \alpha = 1.$$

These imply that

$$\mathbf{I}_{\mathsf{D}}(\mu, \mathsf{D}(k,f)) = 0 \ \text{ and } \ \mathsf{D}(k, \mathbf{I}_{\mathsf{D}}(\mu,f)) = k > 0$$

showing that $\mathbf{I}_{\mathsf{D}}$ *is not* D*-homogeneous, i.e,* D *is not a solution of the open problem.*

Example 3.10. *Consider* $X = [0,1]$ *and* λ *be the Lebesgue measure on* $\mathfrak{B}(X)$. *For the function* $f(x) = x$ *on* X, *we have* $\mathbf{I}_{\mathsf{W}}(\lambda, f) = 0$, *therefore,* $\mathsf{W}(k, \mathbf{I}_{\mathsf{W}}(\lambda, f)) = 0$ *for every* $k \in [0,1]$. *On the other hand, it is not difficult to check that* $(\mathsf{W}(k,f))_\alpha = [1+\alpha-k, 1]$ *for every* $\alpha \in (0,1]$. *This implies that* $\lambda((\mathsf{W}(k,f))_\alpha) = k - \alpha$. *This gives*

$$\mathbf{I}_{\mathsf{W}}(\lambda, \mathsf{W}(k,f)) = \sup_{0<\alpha\leqslant k} \mathsf{W}(\alpha, k-\alpha) = \sup_{0<\alpha\leqslant k} \{(k-1)\vee 0\} = 0.$$

This result is fully compatible with the assertion of Remark 3.8. Therefore, there is some confusion in Example 2.28 presented in the paper [2].

From Theorems 3.6 and 3.7, we obtain the following result which improves the results of Theorems 2.10 and 2.11.

Theorem 3.11. *A semicopula is a solution of the open problem if and only if it is associative and continuous in the second variable.*

4 Conclusion

Another approach finding out all solutions of the open problem without using Theorem 2.10 and 2.11 is proposed. Further, a full characterization of semicopulas S satisfying the property $\mathbf{I}_{\mathsf{S}}(\mu, \mathsf{S}(\alpha, f)) = \mathsf{S}(\alpha, \mathbf{I}_{\mathsf{S}}(\mu, f))$ for every $\alpha \in [0,1]$, $(X,\Sigma) \in \mathscr{S}$ and $(\mu, f) \in \mathscr{M}^1_{(X,\Sigma)} \times \mathscr{F}^{[0,1]}_{(X,\Sigma)}$ is stated in Theorem 3.11. This result refines Theorem 2.10 and 2.11. Finally, we have studied an extremely interesting property that is the equivalence between the pseudo-continuity and the continuity in the second variable of a semicopula S.

5 Appendix

This section is devoted to proving Theorem 3.7. First of all, we need the following auxiliary results.

Lemma 5.1. *Let* S *be a semicopula. Then*
S *is right-continuous in the second variable if and only if*

$$\inf_{x\in A} \mathsf{S}(k, f(x)) = \mathsf{S}\left(k, \inf_{x\in A} f(x)\right),$$

for all $k \in [0,1]$, *for all nonempty set* A *and* $f : A \to [0,1]$.

Proof. 1) The forward part: It follows from monotonicity of S that

$$\mathsf{S}\left(k, \inf_{x\in A} f(x)\right) \leqslant \inf_{x\in A} \mathsf{S}(k, f(x)).$$

On the other hand, for every $\varepsilon > 0$ there exists $x_\varepsilon \in A$ such that

$$\mathsf{S}\left(k, \left(\inf_{x\in A} f(x)\right) + \varepsilon\right) \geqslant \mathsf{S}(k, f(x_\varepsilon)) \geqslant \inf_{x\in A} \mathsf{S}(k, f(x)).$$

By applying the right continuity of S in the second variable, we get that

$$\mathsf{S}\left(k, \inf_{x\in A} f(x)\right) \geqslant \inf_{x\in A} \mathsf{S}(k, f(x)).$$

Therefore, the proof of the forward part is finished.
2) The reserve part: For any $b \in [0,1]$ fixed. Consider a nonincreasing sequence $\{b_n\} \subset [0,1]$ such that $b_n \searrow b$, put $f : A \equiv [0,1] \to [0,1]$ defined by $f(x) = b_n$ if $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right)$ for $n \in \mathbb{N}$, and $f(x) = 1$ if $x \in \{0,1\}$. Then

$$\inf_{x\in A} f(x) = b \text{ and } \inf_{x\in A} \mathsf{S}(k, f(x)) = \lim_{n\to\infty} \mathsf{S}(k, b_n) \text{ for each } k \in [0,1].$$

In view of the assumption of the reverse part, we get

$$\lim_{n\to\infty} \mathsf{S}(k, b_n) = \mathsf{S}(k, b).$$

The proof is finished. □

Lemma 5.2. *Let* S *be a semicopula. Then* S *is left-continuous in the second variable if and only if*

$$\sup_{x\in A} \mathsf{S}(k, f(x)) = \mathsf{S}\left(k, \sup_{x\in A} f(x)\right),$$

for all $k \in [0,1]$, *for all nonempty set* A *and* $f : A \to [0,1]$.

Proof. The proof of Lemma 5.2 is a dual of Lemma 5.1. Therefore, it is omitted here. □

Lemma 5.3. *Let* S *be a semicopula. Then*
S *is left-continuous in the second variable if and only if* S *satisfies the property 2 of Definition 3.3.*

Proof. 1) The forward part: It is a dual of the forward part in Lemma 5.1. So, it is omitted here.
2) The reverse part: For any nonempty set A and $k \in [0,1]$ fixed. Consider σ-algebra $\Sigma = \mathcal{P}(A)$, $\mu : \Sigma \to [0,1]$ defined by $\mu(E) = 1$ if $E \in \mathcal{P}(A) \setminus \{\emptyset\}$ and $\mu(E) = 0$ if $E = \emptyset$. In view of the property 2 of Definition 3.3, one has

$$\sup_{\emptyset\neq E\subset A} \mathsf{S}\left(k, \mathsf{S}\left(\inf_{x\in E} f(x), \mu(E)\right)\right) = \mathsf{S}\left(k, \sup_{\emptyset\neq E\subset A} \left\{\mathsf{S}\left(\inf_{x\in E} f(x), \mu(E)\right)\right\}\right).$$

This implies that

$$\sup_{\emptyset\neq E\subset A} \mathsf{S}\left(k, \inf_{x\in E} f(x)\right) = \mathsf{S}\left(k, \sup_{\emptyset\neq E\subset A} \inf_{x\in E} f(x)\right). \tag{1}$$

Now, we claim that

$$\sup_{\emptyset\neq E\subset A} \inf f(E) = \sup_{x\in A} f(x), \tag{2}$$

and

$$\sup_{\emptyset\neq E\subset A} \mathsf{S}(k, \inf f(E)) = \sup_{x\in A} \mathsf{S}(k, f(x)). \tag{3}$$

Indeed, it is obvious that

$$\sup_{x\in A} f(x) = \sup_{x\in A} \inf f(\{x\}) \leqslant \sup_{\emptyset\neq E\subset A} \inf f(E).$$

On the other hand, there exists $x_E \in E$ such that

$$\inf f(E) \leqslant f(x_E) \leqslant \sup_{x\in A} f(x).$$

Therefore,

$$\sup_{\emptyset\neq E\subset A} \inf f(E) \leqslant \sup_{x\in A} f(x).$$

So, the claim (2) holds. Next, by the same technique, the claim (3) holds, too. Combining the results (1)-(3), we obtain that

$$\sup_{x\in A} \mathsf{S}(k, f(x)) = \mathsf{S}\left(k, \sup_{x\in A} f(x)\right).$$

for all $k \in [0,1]$, for all nonempty set A and $f : A \to [0,1]$.
By applying Lemma 5.2, we conclude that S is left-continuous in the second variable.
The proof is finished. □

Now, we come back to proving Theorem 3.7. The conclusion of Theorem 3.7 immediately follows from Definition 3.3 and applying Lemmas 5.1 and 5.3.

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