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Monte Carlo statistical test for fuzzy quality

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Abstract

Testing the capability of a productive process on the basis of the flexible fuzzy quality using Yongting's index is proposed in this paper by the Monte Carlo simulation. The theoretical approach and detailed steps of an algorithm are given to simulate the critical-value-based and also p-value-based approaches to statistical testing fuzzy quality. Also, the probability of type II error of the quality test simulated by Monte Carlo approach. Moreover, a real-world case study is provided to show the performance of the proposed algorithm for triangular and trapezoidal fuzzy qualities.

Keywords: Testing hypotheses, p-value, the probability of type II error, fuzzy quality, capability index, Monte Carlo.

1 Introduction and background

In quality control, such as other statistical problems, we may confront imprecise concepts. One practical case in quality analyses is a situation in which the specification tolerance is a fuzzy set. In such a vague environment, products are not qualified with a 0 and 1 boolean view, but to some degrees depending on the quality level of the products. This matter can be caused to a justified judgment in decision making on manufacturing processes. Yongting [9] introduced the concept of "fuzzy quality" in 1996 by substituting the indicator function $I_{\{x:x\in[LSL,USL]\}}$ with the membership function of the fuzzy set $\tilde{Q}$. Motivations and merits of applying this flexible approach instead of using the classical quality were discussed in [5]. Sadeghpour-Gildeh [7] compared capability indices C_p, C_{pk} and Yongting's index with respect to the measurement error occurrence. Amirzadeh et al. [1] proposed a new p-chart controlling the degree of nonconformity based on fuzzy quality (FQ), and they expressed that the proposed control chart is more sensitive not only to changes in the process mean, but also to changes in the variance. Another generation of process capability indices was developed by Parchami and Mashinchi [4] to measure the capability of fuzzy quality.

Testing capability is a common method to check the performance of industrial production processes. Usually, the test statistic of such test is the process capability index. Due to the complexity of the capability indices formulas, the statistical distribution of the process capability estimator may not be clear, which makes a challenge in testing process capability in applications. This challenge is also seen to test the process capability based on fuzzy quality, and we have tried to solve it by proposing a simple and practical algorithm in this paper. In the proposed algorithm, a statistical significance test is simulated for testing the capability of a fuzzy process on the basis of normal data.

This paper is organized as follows. The probabilistic capability index based on fuzzy quality is reviewed in Section 2. Moreover, a significance fuzzy quality test is investigated in Section 2. Then, the Monte Carlo testing fuzzy quality is investigated by a practical algorithm in Section 3. A case study based on two triangular and trapezoidal fuzzy qualities has been provided with real-world data in Section 4. The final section is conclusions and future works.

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2 Testing fuzzy quality

2.1 Capability index to measure fuzzy quality

Let f is the probability density/mass function of a one-dimensional quality characteristic X. Yongting (1996) introduced the process capability index

$$C_{\tilde{Q}} = \begin{cases} \int_{-\infty}^{+\infty} \tilde{Q}(x)\ f(x)\ dx, & \textit{for continuous quality characteristic,} \\ \sum_{i=1}^{n} \tilde{Q}(x_i)\ f(x_i), & \textit{for discrete quality characteristic,} \end{cases} \tag{1}$$

based on fuzzy quality for precise data in which $\tilde{Q}$ is the membership function of fuzzy quality to construct the degree of conformity with the standard fuzzy quality. Note that $\tilde{Q}(x)$ represents the degree of conformity with standard quality (or briefly, the degree of quality) when the measured quality characteristic of a product is x; see [5]. It must be mentioned that the introduced process capability index in Eq. (1) is equal to the probability of the fuzzy quality event, by considering Zadehs probabilistic definition [10], that is, $C_{\tilde{Q}} = P\left(X \in \tilde{Q}\right)$.

2.2 Significance fuzzy quality test and p-value

The main problem. For fuzzy quality analysis under the normality condition of one-dimensional quality characteristic, the main problem is testing the null hypothesis

$H_0:\ C_{\tilde{Q}} \leq c_0$ (process is not capable),

against the alternative hypothesis

$H_1:\ C_{\tilde{Q}} > c_0$ (process is capable),

based on the random sample $x_1, x_2, \ldots, x_n$ with unknown mean and unknown variance parameters, where $c_0 \in (0, 1)$ is the standard minimal criterion for Yongting's capability index. From now on, we briefly call this problem Testing Fuzzy Quality (TFQ) throughout this paper and the Monte Carlo simulation approach is investigated for TFQ in Section 3.

Now, we are going to derive a construction of a precise (and not fuzzy) test on the basis of the simulated distribution of $C_{\tilde{Q}}$, which is formally similar to recent tests. Obviously, the critical region of the proposed capability test is $\widehat{C_{\tilde{Q}}} > c$ in which c is the precise unknown critical value and

$$\widehat{C_{\tilde{Q}}} = \int_{-\infty}^{+\infty} \tilde{Q}(x)\ \widehat{\phi_{\mu,\sigma}}(x)\ dx = \int_{-\infty}^{+\infty} \tilde{Q}(x)\ \phi_{\hat{\mu},\hat{\sigma}}(x)\ dx, \tag{2}$$

is the estimator of Yongting's capability index based on the fuzzy quality $\tilde{Q}$ and random sample from normal distribution with unknown parameters μ and σ^2. Note that, the estimator of Yongting's capability index can be easily obtained by replacing unknown parameters μ and σ^2 with their estimators $\hat{\mu} = \bar{X}$ and $\widehat{\sigma^2} = S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$, respectively. Considering random sample $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N\left(\mu, \sigma^2\right)$ with unknown parameters, the probability of type I error in TFQ can be formulated by

$$\alpha = \sup_{H_0} Pr\left(\widehat{C_{\tilde{Q}}} > c\right) = Pr\left(\widehat{C_{\tilde{Q}}} > c \mid C_{\tilde{Q}} = c_0\right), \tag{3}$$

and so

$$Pr\left(\widehat{C_{\tilde{Q}}} \leq c \mid C_{\tilde{Q}} = c_0\right) = 1 - \alpha. \tag{4}$$

Therefore, the unknown critical value c is equal to the $(1-\alpha)$th quantile of $\widehat{C_{\tilde{Q}}}$ distribution under the assumption $C_{\tilde{Q}} = c_0$, and finding a precise critical value c by the Monte Carlo simulation is investigated in Section 3 as the main goal of this paper. Moreover, the p-value in TFQ is equal to

$$p\text{-}value = Pr\left(\widehat{C_{\tilde{Q}}} > \widehat{c_{\tilde{Q}}} \mid C_{\tilde{Q}} = c_0\right) = E\left[I\left(\widehat{C_{\tilde{Q}}} > \widehat{c_{\tilde{Q}}} \mid C_{\tilde{Q}} = c_0\right)\right], \tag{5}$$

where $\widehat{c_{\tilde{Q}}}$ is the observed value of Yongting's capability index based on $x_1, \ldots, x_n$ from Eq. (2) and $I(A)$ is the indicator function of event/set A. The power function of the fuzzy quality test using Yongting's index is

$$\Pi(C_{\tilde{Q}}) = Pr\left(\widehat{C_{\tilde{Q}}} > c\right), \tag{6}$$

where c is the critical value. Therefore, the probability of type II error at $c^*_{\tilde{Q}}$ is equal to

$$\beta(c^*_{\tilde{Q}}) = 1 - \Pi(c^*_{\tilde{Q}}) = Pr\left(\widehat{C_{\tilde{Q}}} \leq c \mid C_{\tilde{Q}} = c^*_{\tilde{Q}}\right) = E\left[I\left(\widehat{C_{\tilde{Q}}} \leq c \mid C_{\tilde{Q}} = c^*_{\tilde{Q}}\right)\right], \tag{7}$$

for any $c^*_{\tilde{Q}} > c_0$.

3 Monte Carlo testing fuzzy quality

An algorithm is proposed in bellow to simulate/compute the critical value, p-value and the probability of type II error for the Monte Carlo TFQ at the given significance level α.

Algorithm 1.
Step 1: Compute the observed/estimated Yongting's index $\widehat{c_{\tilde{Q}}}$ on the basis of the observed random sample $x_1, x_2, \ldots, x_n$ by Eq. (2).
Step 2: Calculate sequence $\mu_1 < \mu_2 < \ldots < \mu_k$ to cover the interquartile range $[Q_1, Q_3]$ by the following formula

$$\mu_j = Q_1 + \frac{j-1}{k-1}(Q_3 - Q_1), \quad j = 1, 2, \ldots, k, \tag{8}$$

where Q_1 and Q_3 are 25th and 75th percentiles of observations $x_1, x_2, \ldots, x_n$.
Step 3: For any $\mu_j \in \{\mu_1, \mu_2, \ldots, \mu_k\}$,

a) compute the unknown value of root σ_0 from the equation $C_{\tilde{Q}} = c_0$, which is equivalent to the equation

$$\int_{-\infty}^{+\infty} \tilde{Q}(x)\ \phi_{\mu_j,\sigma_0}(x)\ dx = c_0, \tag{9}$$

b) simulate $m = 10^3$ random samples with size n from $N\left(\mu_j, \sigma_0^2\right)$,

c) estimates $\widehat{c_{\tilde{Q}}}^{[1]}, \widehat{c_{\tilde{Q}}}^{[2]}, \ldots, \widehat{c_{\tilde{Q}}}^{[m]}$ for capability index, using Eq. (2) based on the fuzzy quality $\tilde{Q}$ for each simulated sample from part (b),

d) considering Eq. (4), the critical value for m simulated samples in part (b) is equal to the $(1-\alpha)$th quantile of $\widehat{C_{\tilde{Q}}}$ distribution, that is

$$c_j = \widehat{c_{\tilde{Q}}}^{(m(1-\alpha))}, \quad j = 1, \ldots, k, \tag{10}$$

where $\widehat{c_{\tilde{Q}}}^{(1)}, \widehat{c_{\tilde{Q}}}^{(2)}, \ldots, \widehat{c_{\tilde{Q}}}^{(m)}$ are the ordered estimated indices in part (c),

e) simulate the p-value by

$$\begin{aligned} p\text{-}value_j &= \overline{I\left(\widehat{C_{\tilde{Q}}} > \widehat{c_{\tilde{Q}}} \mid C_{\tilde{Q}} = c_0\right)} \\ &= \frac{1}{m}\sum_{r=1}^{m} I\left(\widehat{c_{\tilde{Q}}}^{[r]} > \widehat{c_{\tilde{Q}}} \mid \mu = \mu_j, \sigma = \sigma_0\right), \ j = 1, \ldots, k, \end{aligned} \tag{11}$$

in which the simulated capability indices are denoted by $\widehat{c_{\tilde{Q}}}^{[1]}, \ldots, \widehat{c_{\tilde{Q}}}^{[m]}$ and σ_0 is the computable root of the equation $C_{\tilde{Q}} = c_0$ from Part (a) by the Newton-Raphson method. Regarding to the strong low of large numbers, it must be mentioned that the Monte Carlo estimator p-value$_j$ almost surely converges to Eq. (5), for each iteration, as $m \to \infty$,

f) for any arbitrary point $c^*_{\tilde{Q}} > c_0$, the simulated probability of type II error at $c^*_{\tilde{Q}}$ is

$$\begin{aligned} \beta(c^*_{\tilde{Q}})_j &= Pr\left(\widehat{C_{\tilde{Q}}} \leq c_j \mid C_{\tilde{Q}} = c^*_{\tilde{Q}}\right) \\ &= \overline{I\left(\widehat{C_{\tilde{Q}}} \leq c_j \mid C_{\tilde{Q}} = c^*_{\tilde{Q}}\right)} \\ &= \frac{1}{m}\sum_{r=1}^{m} I\left(\widehat{c_{\tilde{Q}}}^{[r]} \leq c_j \mid \mu = \mu_j, \sigma = \sigma^*_0\right), \ j = 1, \ldots, k, \end{aligned} \tag{12}$$

in which σ^*_0 is the computable root of equation $C_{\tilde{Q}} = c^*_{\tilde{Q}}$ by Newton-Raphson method and also the simulated indices basis of μ_j and σ^*_0 are denoted by $\widehat{c_{\tilde{Q}}}^{[1]}, \ldots, \widehat{c_{\tilde{Q}}}^{[m]}$. Note that the Monte Carlo estimator $\beta(c^*_{\tilde{Q}})_j$ almost surely converges

to Eq. (7), as $m \to \infty$.
Step 4: Monte Carlo critical value in TFQ is equal to the average of k calculated critical values in Step 3 and therefore

$$c = \frac{1}{k}\sum_{j=1}^{k} c_j. \tag{13}$$

Step 5 (Decision rule): The process is capable, that is, the null hypothesis is rejected at significance level α, if $\widehat{c_{\tilde{Q}}} > c$; otherwise the process is incapable.
Step 6 (p-value): The Monte Carlo p-value in TFQ is equal to the average of k calculated p-values in iterations of Part (e), that is,

$$\hat{p}\text{-}value = \frac{1}{k}\sum_{j=1}^{k} p\text{-}value_j. \tag{14}$$

Step 7 (Probability of type *II* error): Finally, the Monte Carlo probability of type II error at $c^*_{\tilde{Q}}$, for any arbitrary point $c^*_{\tilde{Q}} > c_0$, is simulated in TFQ by the average of k calculated β in iterations of Part (f), i.e.

$$\hat{\beta}(c^*_{\tilde{Q}}) = \frac{1}{k}\sum_{j=1}^{k} \beta(c^*_{\tilde{Q}})_j. \tag{15}$$

A real-world case study is presented in the next section to show the performance of Algorithm 1.

4 Case study

Piston rings for a vehicle engine are produced in a forging process. Twenty-five samples, each of size five, are taken from the inside diameter length (in terms of a millimeter) [3]; see Figs. 1 and 2. The degree of nonconformity was defined by Amirzadeh et. al [1] based on the trapezoidal fuzzy quality, and a fuzzy p-chart was proposed. It must be emphasized that all the programs and plots in this paper have been carried out with R software, and moreover the piston rings data are accessible by "qcc" package [8].

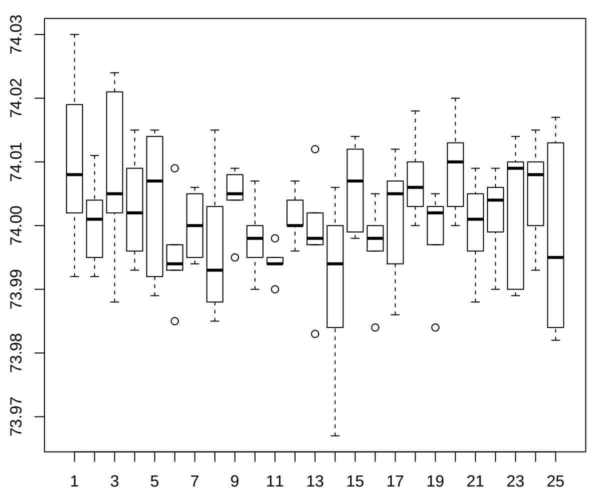

Figure 1: Box plots of 25 samples of piston rings in millimeter.

In continue, we are going to test the capability of a manufacturing process for the inside diameter measurement of the produced rings based on Yongting's index at significance level 0.01 by considering the trapezoidal fuzzy quality

$$\tilde{Q}(x) = \begin{cases} \frac{x-73.96}{0.03} & \text{if} \quad 73.96 \le x < 73.99, \\ 1 & \text{if} \quad 73.99 \le x < 74.02, \\ \frac{74.03-x}{0.01} & \text{if} \quad 74.02 \le x < 74.03, \\ 0 & \quad \text{elsewhere.} \end{cases}$$

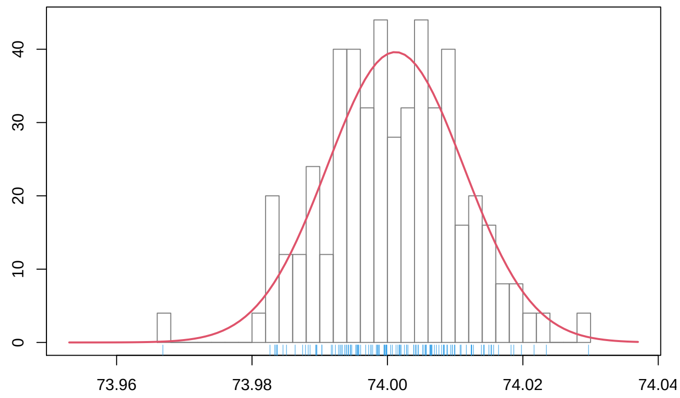

Figure 2: Histogram and kernel density estimation for 125 observed inside diameters.

The above membership function of the non-symmetric trapezoidal fuzzy quality is drawn in Fig. 3, and also the degrees of conformity and nonconformity for each observation are denoted at the left and right sides of Fig. 3, respectively.

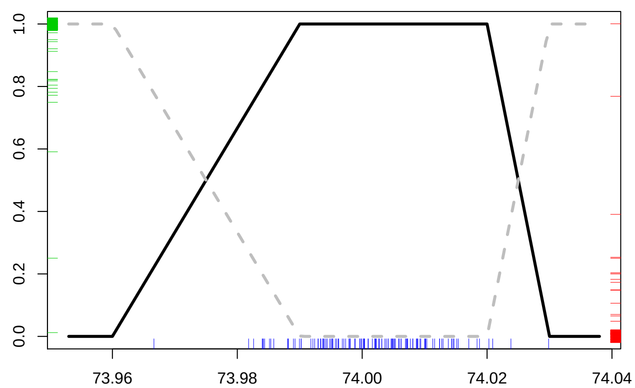

Figure 3: Membership function of trapezoidal fuzzy quality $\tilde{Q}$ and the related degree of nonconformity.

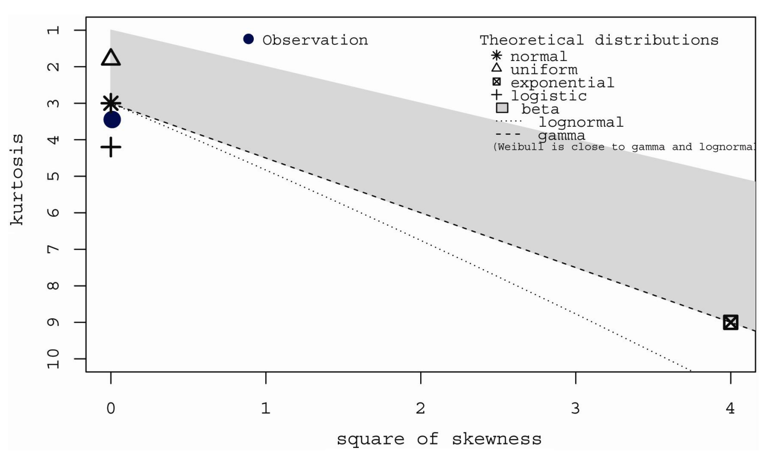

Figure 4: Cullen and Frey graph of the observed inside diameters.

Monte Carlo simulation approach is considered in this study to test $H_0: \ C_{\tilde{Q}} \leq 0.95$, against $H_1: \ C_{\tilde{Q}} > 0.95$, based on the observed random sample $x_1, \ldots, x_{125}$. Shapiro-Wilk test strongly confirms the normality assumption of the observed data with p-value $= 0.786$. Furthermore, to determine data distribution for these data set, a common method can be used namely, the Cullen and Frey graph [2]. It suggests the selection of a best fit for an unknown distribution regarding to skewness level and kurtosis. On this graph, values for common distributions are shown to help in the model selection. Fig. 4 shows the distribution of observation has a skewness of zero, which also normal, uniform, and logistic distribution have zero skewness, but the kurtosis of observation model is close to the normal distribution. Thus, by examining the skewness and kurtosis graph of the observed inside diameters of piston rings in Fig. 4, the normal distribution model is appropriate to fit the data. Therefore by Eq. (2), the estimated Yongting's index is equal to

$$\widehat{c_{\tilde{Q}}} = \int_{73.96}^{74.03} \tilde{Q}(x)\ \phi_{\hat{\mu},\hat{\sigma}}(x)\ dx = 0.966,$$

in which ϕ is the probability density function of standard normal distribution, and unknown mean and variance of normal random variable are estimable by $\hat{\mu} = \bar{x} = 74.00118$ and $\widehat{\sigma^2} = s^2 = 0.01007^2$, respectively. Hence, $0.966 < c$ is the critical region of Monte Carlo TFQ at significance level $\alpha = 0.01$, in which the critical value must be simulated by using Algorithm 1. We did a simulation to compute the Monte Carlo critical value in TFQ, where the mean changes over the following sequence: 73.99400, 73.99633, 73.99867, 74.00100, 74.00333, 74.00567 and 74.00800.

Considering the membership function of the fuzzy quality, the unknown root σ_0 in the equation $\int \tilde{Q}(x)\ \phi_{\mu_j,\sigma_0}(x)\ dx = c_0$ is computed for all seven possible cases by the Newton-Raphson method. Then, 1000 independent random samples are simulated from the normal distribution $N\left(\mu, \sigma_0^2\right)$ for each case. For each simulated sample, we estimated Yongting's index using Eq. (2). After ordering 1000 estimated capability indices, the 990th capability index is considered as the critical value for every seven possible cases. The left side graphs in Fig. 5 were shown the curve $h(\sigma_0) = C_{\tilde{Q}} - c_0$ in first three iterations of Part (a) to compute its unknown root σ_0. Moreover, corresponding histogram of the simulated Yongting's index with σ_0 is shown in the right side graphs of Fig. 5 from Parts (a) in gray color, and also, histogram of the simulated Yongting's index with σ_0^* is shown in the right side graphs based on Part (f). The total average of seven captured critical values is equal to $c = 0.973$, which is considered as the Monte Carlo critical value of TFQ in this study.

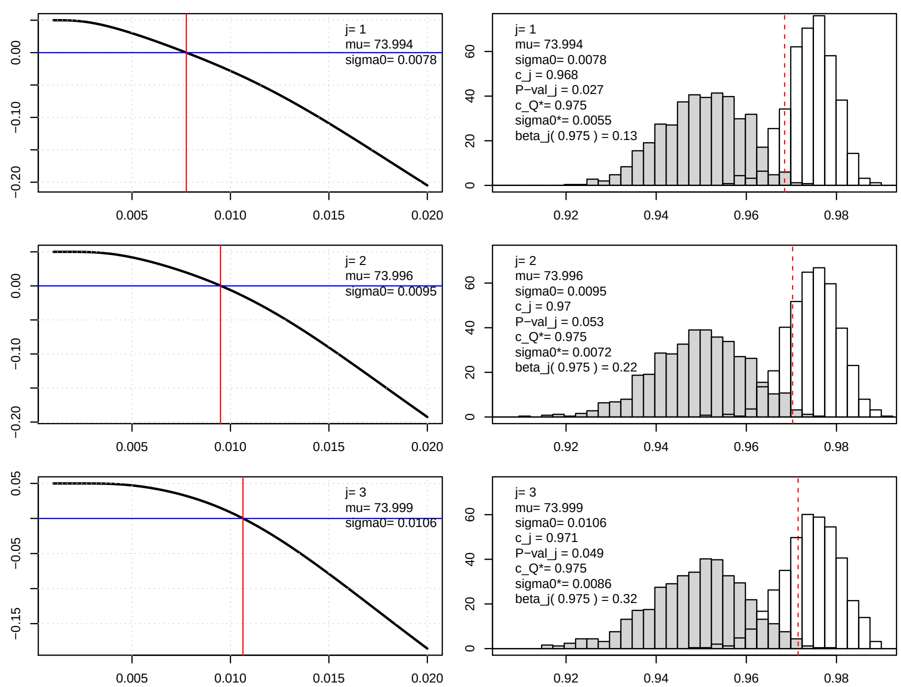

Figure 5: **Left:** Curve $h(\sigma_0) = C_{\tilde{Q}} - c_0$ to compute its unknown root for first three iterations of Part (a). **Right:** Histograms of the simulated Yongting's index with σ_0 and σ_0^* for first three iterations in parts (a) and (f), respectively.

By comparing $\widehat{c_{\tilde{Q}}} = 0.966$ and c, the null hypothesis cannot be rejected and so the process is determined as an incapable process at significance level 0.01. The results of the Monte Carlo simulation approach are summarized for every iterations in Table 1 based on the considered trapezoidal fuzzy quality in this case study.

Table 1: Results of seven iterations in the Monte Carlo simulation based on the trapezoidal fuzzy quality at different significance levels 0.01 and 0.05.

	j	μ_j	σ_{0j}	c_j	p-$value_j$	$\beta(c^*_{\tilde{Q}})_j$
$\alpha = 0.01$	1	73.99400	0.0078	0.968	0.027	0.132
$c_0 = 0.95$	2	73.99633	0.0095	0.970	0.053	0.225
$c^*_{\tilde{Q}} = 0.975$	3	73.99867	0.0106	0.971	0.049	0.320
	4	74.00100	0.0111	0.972	0.062	0.357
	5	74.00333	0.0111	0.975	0.079	0.494
	6	74.00567	0.0106	0.975	0.087	0.515
	7	74.00800	0.0097	0.975	0.097	0.531
				$c = 0.973$	$\hat{p}$-$value = 0.065$	$\hat{\beta} = 0.368$
$\alpha = 0.05$	1	73.99400	0.0086	0.956	0.002	0.008
$c_0 = 0.94$	2	73.99633	0.0103	0.958	0.004	0.008
$c^*_{\tilde{Q}} = 0.973$	3	73.99867	0.0113	0.958	0.005	0.032
	4	74.00100	0.0117	0.959	0.009	0.045
	5	74.00333	0.0117	0.961	0.016	0.079
	6	74.00567	0.0111	0.962	0.024	0.103
	7	74.00800	0.0102	0.962	0.025	0.129
				$c = 0.959$	$\hat{p}$-$value = 0.012$	$\hat{\beta} = 0.058$

Now, let us consider the following triangular fuzzy quality in TFQ (see Fig. 6)

$$\tilde{Q}_\Delta(x) = \begin{cases} \frac{x-73.96}{0.045} & \text{if} \quad 73.96 \leq x < 74.005, \\ \frac{74.03-x}{0.025} & \text{if} \quad 74.005 \leq x < 74.03, \\ 0 & \quad \text{elsewhere.} \end{cases}$$

By considering the triangular fuzzy quality instead of the trapezoidal fuzzy quality, the estimated Yongting's capability index strictly decreases to $\widehat{c_{\tilde{Q}_\Delta}} = 0.7665$, so that the simpler quality test $H_0: C_{\tilde{Q}_\Delta} \leq 0.72$, versus $H_1: C_{\tilde{Q}_\Delta} > 0.72$ by the Monte Carlo approach lead us to the critical value 0.748. Therefore, the null hypothesis rejected at significance level 0.05, by considering the triangular fuzzy quality $\tilde{Q}_\Delta$, and hence the process is capable at this level. Graphical information about first three iterations of Algorithm 1 is depicted in Fig. 7 based on triangular fuzzy quality.

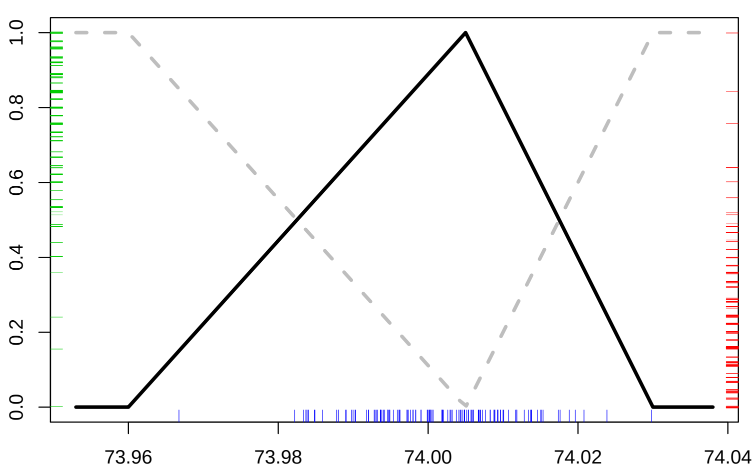

Figure 6: Membership function of the triangular fuzzy quality $\tilde{Q}_\Delta$ and the related degree of nonconformity.

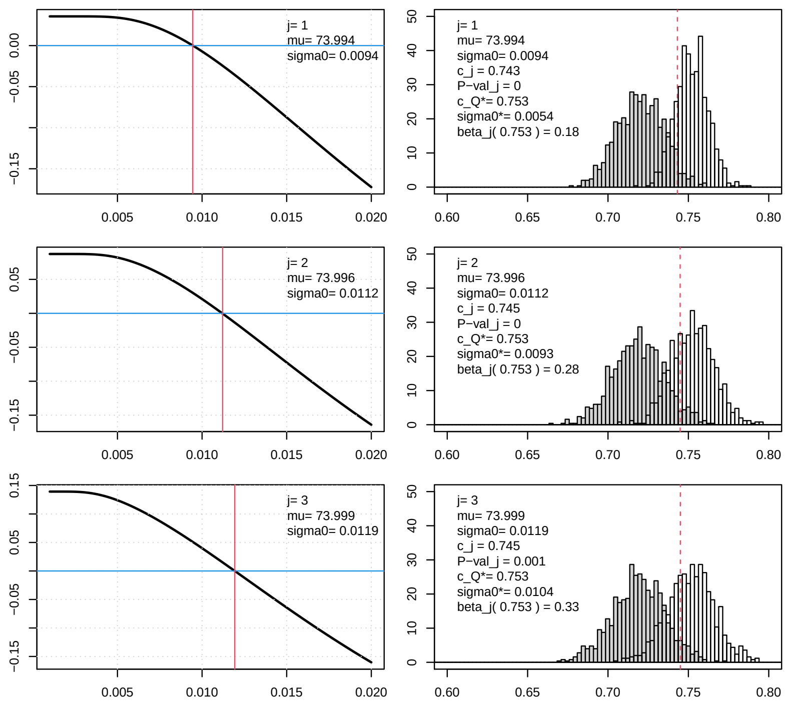

Figure 7: **Left:** Curve $h(\sigma_0) = C_{\tilde{Q}_\Delta} - c_0$ to compute its unknown root σ_0 for first three iterations of Part (a). **Right:** Histograms of the simulated Yongting's index with σ_0 and σ_0^* for first three iterations in parts (a) and (f), respectively.

5 Comparison with traditional capability indices

Traditional process capability indices $C_p = \frac{USL-LSL}{6\sigma}$ and $C_{pm} = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}}$ have many applications in industry, where LSL is the lower specification limit, USL is the upper specification limit, T is the target value and σ is the process standard deviation. In this section, we are going to test the quality of inside diameters of piston rings - by considering both crisp quality and fuzzy quality - on the basis of indices C_p, C_{pm} and the Yongting's index $C_{\tilde{Q}}$ at different significance levels.

In quality test by C_p, the critical value based on the natural estimator $\widehat{C_p} = \frac{USL-LSL}{6S}$ is equal to $c_0\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}$, in which c_0 is the standard minimal criterion and $\chi^2_{n-1,\alpha}$ is the lower α-quantile of chi-square distribution with $n-1$ degrees of freedom [6].

Also, the p-value for testing quality is equal to

$$p\text{-}value = Pr\left(\widehat{C_p} > \widehat{c_p} \mid C_p = c_0\right) = Pr\left(\chi^2_{n-1} < \frac{(n-1)c_0^2}{\widehat{c_p}^{\,2}}\right) \tag{16}$$

where $\widehat{c_p}$ is the observed value of the index C_p based on $x_1, \ldots, x_n$. Furthermore, the probability of type II error at point c_p^* is equal to

$$\beta(c_p^*) = Pr\left(\widehat{C_p} \le c \mid C_p = c_p^*\right) = 1 - Pr\left(\chi^2_{n-1} < \frac{(n-1)c_p^{*2}}{c^2}\right) \tag{17}$$

for any $c_p^* > c_0$. It must be mentioned that the critical value, the probability of type II error and the p-value for the crisp quality test by C_{pm} based on the natural estimator $\widehat{C_{pm}} = \frac{USL-LSL}{6\sqrt{S^2+(\bar{X}-T)^2}}$ were simulated by the Monte Carlo simulation approach similar to what was presented in Algorithm 1. Moreover, in testing crisp quality based on indices C_p and C_{pm}, specification $LSL = 73.96$ and $USL = 74.03$, and also the target value $T = 73.999$ were considered.

Regarding to the above disscussion, four following quality tests were considered with variuse critical values in Table 2 at different significance levels for inside diameters of piston rings:

(1) testing the crisp quality based on C_p,
(2) testing the crisp quality based on C_{pm},
(3) testing the trapezoidal fuzzy quality based on Yongting's index $C_{\tilde{Q}}$, and
(4) testing the triangular fuzzy quality based on Yongting's index $C_{\tilde{Q}_\Delta}$.

For each case, the critical value, the p-value and the result of the quality test were computed/simulated in Table 2 at significance levels 0.010, 0.025, 0.05 and 0.01. Furthermore, the Monte Carlo probability of type II error were simulated in Table 2 by Eq. (15), for each quality test at three various points $c^*_{\tilde{Q}}$.

Table 2: Results of several testing crisp quality and fuzzy quality for inside diameters of piston rings with various critical values and significance levels.

		$\alpha = 0.010$	$\alpha = 0.025$	$\alpha = 0.050$	$\alpha = 0.100$
		Quality test based on $c_0 = 1.00$, $LSL = 73.96$ and $USL = 74.03$			
Crisp Quality	Statistical critical value for C_p	1.172	1.142	1.118	1.090
	Decision based on $\hat{c}_p = 1.1586$	Not reject H_0	Reject H_0	Reject H_0	Reject H_0
	Process	Incapable	Capable	Capable	Capable
	p-value	0.015	0.015	0.015	0.015
	Probability of type II error	$\beta(1.160) = 0.546$	$\beta(1.160) = 0.386$	$\beta(1.160) = 0.262$	$\beta(1.160) = 0.150$
		$\beta(1.340) = 0.012$	$\beta(1.280) = 0.028$	$\beta(1.230) = 0.055$	$\beta(1.170) = 0.111$
		$\beta(1.344) = 0.011$	$\beta(1.289) = 0.021$	$\beta(1.239) = 0.042$	$\beta(1.179) = 0.096$
		Quality test based on $c_0 = 1.00$, $LSL = 73.96$, $T = 73.999$ and $USL = 74.03$			
Crisp Quality	Monte Carlo critical value for C_{pm}	1.156	1.132	1.109	1.082
	Decision based on $\hat{c}_{pm} = 1.1324$	Not reject H_0	Reject H_0	Reject H_0	Reject H_0
	Process	Incapable	Capable	Capable	Capable
	$\hat{p}$*-value*	0.025	0.025	0.025	0.025
	Probability of type II error	$\beta(1.164) = 0.454$	$\beta(1.164) = 0.318$	$\beta(1.164) = 0.208$	$\beta(1.164) = 0.112$
		$\beta(1.290) = 0.032$	$\beta(1.235) = 0.052$	$\beta(1.220) = 0.056$	$\beta(1.165) = 0.109$
		$\beta(1.295) = 0.030$	$\beta(1.239) = 0.046$	$\beta(1.223) = 0.051$	$\beta(1.168) = 0.101$
		Fuzzy quality test based on $c_0 = 0.95$			
Trapezoidal FQ	Monte Carlo critical value for $C_{\tilde{Q}}$	0.973	0.970	0.967	0.964
	Decision based on $c\tilde{Q} = 0.9660$	Not reject H_0	Not reject H_0	Not reject H_0	Reject H_0
	Process	Incapable	Incapable	Incapable	Capable
	$\hat{p}$*-value*	0.065	0.065	0.065	0.065
	Probability of type II error	$\hat{\beta}(0.975) = 0.368$	$\hat{\beta}(0.975) = 0.247$	$\hat{\beta}(0.975) = 0.156$	$\hat{\beta}(0.975) = 0.078$
		$\hat{\beta}(0.985) = 0.029$	$\hat{\beta}(0.982) = 0.041$	$\hat{\beta}(0.979) = 0.056$	$\hat{\beta}(0.976) = 0.061$
		$\hat{\beta}(0.987) = 0.012$	$\hat{\beta}(0.984) = 0.019$	$\hat{\beta}(0.980) = 0.042$	$\hat{\beta}(0.977) = 0.045$
		Fuzzy quality test based on $c_0 = 0.72$			
Triangular FQ	Monte Carlo critical value for $C_{\tilde{Q}_\Delta}$	0.758	0.753	0.748	0.742
	Decision based on $c\,\tilde{Q}_\Delta = 0.7665$	Reject H_0	Reject H_0	Reject H_0	Reject H_0
	Process	Capable	Capable	Capable	Capable
	$\hat{p}$*-value*	0.003	0.003	0.003	0.003
	Probability of type II error	$\hat{\beta}(0.751) = 0.680$	$\hat{\beta}(0.751) = 0.563$	$\hat{\beta}(0.751) = 0.418$	$\hat{\beta}(0.751) = 0.275$
		$\hat{\beta}(0.755) = 0.588$	$\hat{\beta}(0.754) = 0.485$	$\hat{\beta}(0.753) = 0.372$	$\hat{\beta}(0.752) = 0.255$
		$\hat{\beta}(0.750) = 0.702$	$\hat{\beta}(0.749) = 0.613$	$\hat{\beta}(0.748) = 0.502$	$\hat{\beta}(0.747) = 0.373$

6 Conclusions and future works

Evaluation and testing of the capability of manufacturing processes was investigated in this article by using the Yongting's capability index. The main goal of this investigation was to estimate the critical value and p-value as well as the probability of type II error by a Monte Carlo simulation approach for the normal quality characteristic based on flexible fuzzy quality. The practitioners can use the proposed algorithm to determine whether their process meets the preset capability requirement and make reliable decisions when the fuzzy quality is considered instead of crisp specification limits. Also, a p-value-based approach is proposed for the capability test. Finally, a case study based on two triangular and trapezoidal fuzzy qualities has been investigated with real-world data. Testing capability based on one-dimensional and multivariate capability indices are two potential subjects for further research.

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