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Constructing t-norms and t-conorms by using interior and closure operators on bounded lattices, respectively

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Abstract

In this paper, we propose construction methods for triangular norms (t-norms) and triangular conorms (t-conorms) on bounded lattices by using interior and closure operators, respectively. Thus, we obtain some proposed methods by Ertuğrul, Karaçal, Mesiar [15] and Çaylı [8] as results. Also, we give some illustrative examples. Finally, we show that the introduced construction methods can not be generalized by induction to a modified ordinal sum for t-norms and t-conorms on bounded lattices. This paper has further constructed the t-norms and t-conorms on bounded lattices from a mathematical viewpoint.

Keywords: Bounded lattice, t-norms, t-conorms.

1 Introduction and motivation

Aggregation operators [18] play an important role in theories of fuzzy sets and fuzzy logic. Two basic types of aggregation functions, namely t-norms and t-conorms, were introduced by Schweizer and Sklar [26], in 1963. Although the t-norms and t-conorms were strictly defined on the unit interval $[0,1]$, they were mostly studied on bounded lattices. The notion of ordinal sum of semigroups in Clifford's sense [7] was further developed by Mostert and Shields [22] and later used for introducing new t-norms and conorms on the unit interval $[0,1]$, see [20]. Note that there is a minor difference in ordinal sum construction for triangular norms (based on min operator) with those for triangular conorms (based on max operator). Since Goguen' s [17] generalization of the classical fuzzy sets (with membership values from $[0,1]$) to L-fuzzy sets (with membership values from a bounded lattice L), there is a growing interest in t-norms and t-conorms on bounded lattices, in particular in ordinal sum constructions.

In general topology [14], closure and interior operators on the powerset $P(X)$ of a nonempty set X are common tools to construct topologies on X. Actually, there is a one-to-one correspondence between the set of all closure and interior operators on $P(X)$ and that of all topologies on X. Note that closure and interior operators on $P(X)$ are essentially defined on the inherent lattice structure on $P(X)$ with set inclusion, set intersection and set union as the partial order, the meet and the join on $P(X)$, respectively.

In 1996, Drossos, Navara [11] studied a class of t-norms and t-conorms on any bounded lattice was generated by the use of interior operators and closure operators, respectively. In 2006, Saminger [25] focused on ordinal sums of t-norms acting on some particular bounded lattice which is not necessarily a chain or an ordinal sum of lattices. Also, it was provided necessary and sufficient conditions for an ordinal sum operation yielding again a t-norm on some bounded lattice whereas the operation is determined by an arbitrary selection of subintervals as carriers for arbitrary summand t-norms. In 2012, Medina [21] presented several necessary and sufficient conditions for ensuring whether an ordinal sum on a bounded lattice of arbitrary t-norms is a t-norm.

In 2015, a modification of ordinal sums of t-norms and t-conorms resulting to a t-norms and t-conorms on an arbitrary bounded lattice was shown by Ertuğrul, Karaçal, Mesiar [15]. Further modifications were proposed by Aşıcı, Mesiar [3, 4], Aşıcı [2], Çaylı [9, 8], Ouyang, Zhang, Baets [23] and Dan, Hu, Qiao [10]. In 2020, a new ordinal sum

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construction of t-norms and t-conorms on bounded lattices based on interior and closure operators was proposed by Dvořák, Holčapek [13]. Also, the proposed method generalized several known constructions and provided a simple tool to introduce new classes of t-norms and t-conorms.

In this paper, we introduce some new constructions of t-norms and t-conorms by using interior and closure operators on bounded lattices, respectively. The rest of this paper is organized as follows. In Section 2, some basic concepts and results about t-norms, t-conorms, lattices are given. In Section 3, we propose a new method for constructing t-norms on bounded lattices. Using this method, in Corollary 3.10 and 3.8, we obtain the methods proposed by Ertuğrul, Karaçal, Mesiar [15] and Çaylı [8], respectively. In Section 4, we propose a new method for constructing t-conorms on bounded lattices. Using this method, in Corollary 4.8 and 4.10, we obtain the methods proposed by Ertuğrul, Karaçal, Mesiar [15] and Çaylı [8], respectively. In Section 5, we show that the introduced construction methods can not be generalized by induction to a modified ordinal sum for t-norms and t-conorms on bounded lattices.

2 Preliminaries

In this section, we present some basic facts about lattices, t-norms and t-conorms.

A lattice [6] is a partially ordered set $(L, \leq)$ in which each two element subset $\{x, y\}$ has an infimum, denoted as $x \wedge y$, and a supremum, denoted as $x \vee y$. A bounded lattice $(L, \leq, 0, 1)$ is a lattice that has the bottom and top elements written as 0 and 1, respectively. For short, we use the notation L instead of $(L, \leq, 0, 1)$ throughout all of the paper.

Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case, we use the notation $a \parallel b$. We denote the set of elements which are incomparable with a by I_a. So $I_a = \{x \in L \mid x \parallel a\}$.

Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of L is defined as [19]

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Similarly, $[a, b) = \{x \in L \mid a \leq x < b\}$, $(a, b] = \{x \in L \mid a < x \leq b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

Definition 2.1. [20, 25] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A triangular norm T (t-norm) is a binary operation on L which is commutative, associative, increasing with respect to both variables and satisfies $T(x, 1) = x$ for all $x \in L$.*

Definition 2.2. [1, 5, 25] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A triangular conorm S (t-conorm) is a binary operation on L which is commutative, associative, increasing with respect to both variables and satisfies $S(x, 0) = x$ for all $x \in L$.*

Extremal t-norms $T_\wedge$ and T_W on a general bounded lattice L are defined, independently of L, as follows, respectively:

$$T_\wedge(x, y) = x \wedge y, \qquad T_W(x, y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x, y\}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the t-conorms $S_\vee$ and S_W on L are defined as follows, respectively:

$$S_\vee(x, y) = x \vee y, \qquad S_W(x, y) = \begin{cases} x \vee y & \text{if } 0 \in \{x, y\}, \\ 1 & \text{otherwise.} \end{cases}$$

The following definition of an ordinal sum of t-norms defined on subintervals of a bounded lattice $(L, \leq, 0, 1)$ has been extracted from [25], which generalizes the methods given in [20] on subintervals of $[0, 1]$.

Definition 2.3. [25] *Let $(L, \leq, 0, 1)$ be a bounded lattice and fix some subinterval $[a, b]$ of L. Let V be a t-norm on $[a, b]$. Then $T : L^2 \to L$ defined by*

$$T(x, y) = \begin{cases} V(x, y) & \text{if } (x, y) \in [a, b]^2, \\ x \wedge y & \text{otherwise.} \end{cases} \tag{1}$$

is an ordinal sum $(< a, b, V >)$ of V on L.

Definition 2.4. [25] *Let $(L, \leq, 0, 1)$ be a bounded lattice and fix some subinterval $[a, b]$ of L. Let W be a t-conorm on $[a, b]$. Then $S : L^2 \to L$ defined by*

$$S(x, y) = \begin{cases} W(x, y) & \text{if } (x, y) \in [a, b]^2, \\ x \vee y & \text{otherwise.} \end{cases} \tag{2}$$

is an ordinal sum $(< a, b, W >)$ of W on L.

However, the operation T (resp. S) given by Formula (1) (resp. Formula (2)) need not be a t-norm (resp. t-conorm), in general. Observe that condition ensuring that T (resp. S) given by (1) ((2)) is a t-norm (t-conorm) on L are given in [25].

Definition 2.5. [16] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A mapping $cl : L \to L$ is said to be a closure operator if for any $x, y \in L$, it satisfies the following three conditions:*
(i) $x \leq cl(x)$.
(ii) $cl(x \vee y) = cl(x) \vee cl(y)$.
(iii) $cl(cl(x)) = cl(x)$.

Definition 2.6. [16] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $b \in L$ be given. Then the mapping $cl_b : L \to L$ defined as $cl_b(x) = x \vee b$ $(\forall x \in L)$ is a closure operator.*

Definition 2.7. [23] *Let $(L, \leq, 0, 1)$ be a bounded lattice. The set of all universally comparable elements in L, denoted by $UC(L)$, be defined as*

$$UC(L) = \{b \in L \mid \forall c \in L, \text{ either } b \leq c \text{ or } c \leq b\ \}.$$

Definition 2.8. [23] *Let $(L, \leq, 0, 1)$ be a complete lattice. The mapping $\Uparrow : L \to L$ defined as, for any $x \in L$,*

$$\Uparrow (x) = \bigwedge \{b \in UC(L) \mid b \geq x\},$$

is a closure operator.

Definition 2.9. [23] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A mapping $int : L \to L$ is said to be an interior operator if for any $x, y \in L$, it satisfies the following three conditions:*
(i) $int(x) \leq x$,
(ii) $int(x \wedge y) = int(x) \wedge int(y)$,
(iii) $int(int(x)) = int(x)$.

Definition 2.10. [23] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $b \in L$ be given. Then the mapping $int_b : L \to L$ defined as*

$$int_b(x) = x \wedge b \ (\forall x \in L),$$

is an interior operator.

Definition 2.11. [23] *Let $(L, \leq, 0, 1)$ be a complete lattice. The mapping $\Downarrow : L \to L$ defined as, for any $x \in L$,*

$$\Downarrow (x) = \bigvee \{b \in UC(L) \mid b \leq x\},$$

is an interior operator.

In the following, it is proposed a method for generating t-norms and t-conorms on bounded lattices based on interior and closure operators, respectively.

Theorem 2.12. [11, 12] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $int : L \to L$ and $cl : L \to L$ be an interior and a closure operators on L, respectively. Then, the functions $T : L^2 \to L$ and $S : L^2 \to L$ are, respectively, a t-norm and a t-conorm on L, where*

$$T(x, y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x, y\}, \\ int(x) \wedge int(y) & \text{otherwise.} \end{cases} \tag{3}$$

$$S(x, y) = \begin{cases} x \vee y & \text{if } 0 \in \{x, y\}, \\ cl(x) \vee cl(y) & \text{otherwise.} \end{cases} \tag{4}$$

3 New construction method for t-norms on bounded lattices by using interior operators

In this section, we propose new construction method for t-norms on bounded lattices with the given t-norms by using interior operators. The main aim of this section is to present a rather effective method to construct t-norms by using interior operators on a bounded lattice. Using this method, in Corollary 3.8 and Corollary 3.10, we obtain the methods proposed by Çaylı [8] and Ertuğrul, Karaçal, Mesiar [15], respectively.

Theorem 3.1. *Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L$ and $int : L \to L$ be an interior operator such that for all $x \in I_a$ it holds $x \wedge a = int(x \wedge a)$. Given a t-norm V on $[a, 1]$, then the function $T : L^2 \to L$ defined as follows is a t-norm on L where*

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2, \\ y \wedge a & \text{if } (x,y) \in [a,1) \times I_a, \\ x \wedge a & \text{if } (x,y) \in I_a \times [a,1), \\ x \wedge y \wedge a & \text{if } (x,y) \in I_a \times I_a, \\ x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ int(x) \wedge int(y) & \text{otherwise }. \end{cases}$$

Proof. It is easy to see that T is commutative and has 1 as the neutral element.

i) Monotonicity: We prove that if $x \leq y$, then $T(x,z) \leq T(y,z)$ for all $z \in L$. If $z = 1$, then we have that $T(x,z) = T(x,1) = x \leq y = T(y,1) = T(y,z)$ for all $x, y \in L$. The proof can be split into all possible cases.

1. $x \in [0, a)$,

   1.1 $y \in [0, a)$,

   1.1.1. $z \in [0, a)$ or $z \in [a, 1)$ or $z \in I_a$,

$$T(x,z) = int(x) \wedge int(z) \leq int(y) \wedge int(z) = T(y,z),$$

   1.2. $y \in [a, 1)$,

   1.2.1. $z \in [0, a)$,

$$T(x,z) = int(x) \wedge int(z) \leq int(y) \wedge int(z) = T(y,z),$$

   1.2.2. $z \in [a, 1)$,

$$T(x,z) = int(x) \wedge int(z) \leq x \leq a \leq V(y,z) = T(y,z),$$

   1.2.3. $z \in I_a$,

$$T(x,z) = int(x) \wedge int(z) \leq x \wedge z \leq a \wedge z = T(y,z),$$

   1.3. $y \in I_a$,

   1.3.1. $z \in [0, a)$,

$$T(x,z) = int(x) \wedge int(z) \leq int(y) \wedge int(z) = T(y,z),$$

   1.3.2. $z \in [a, 1)$,

$$T(x,z) = int(x) \wedge int(z) \leq x \leq a \wedge y = T(y,z),$$

   1.3.3. $z \in I_a$,

$$T(x,z) = int(x) \wedge int(z) \leq x \wedge z \leq y \wedge z \wedge a = T(y,z),$$

   1.4. $y = 1$,

   1.4.1. $z \in [0, a)$ or $z \in [a, 1)$ or $z \in I_a$,

$$T(x,z) = int(x) \wedge int(z) \leq z = T(1,z),$$

2. $x \in [a, 1)$,

   2.1 $y \in [a, 1)$,

   2.1.1. $z \in [0, a)$,

$$T(x,z) = int(x) \wedge int(z) \leq int(y) \wedge int(z) = T(y,z),$$

   2.1.2. $z \in [a, 1)$,

$$T(x,z) = V(x,z) \leq V(y,z) = T(y,z),$$

   2.1.3. $z \in I_a$,

$$T(x,z) = z \wedge a = T(y,z),$$

   2.2 $y = 1$,

2.2.1. $z \in [0, a)$,

$$T(x, z) = int(x) \wedge int(z) \leq z = T(1, z),$$

2.1.2. $z \in [a, 1)$,

$$T(x, z) = V(x, z) \leq z = T(1, z),$$

2.1.3. $z \in I_a$,

$$T(x, z) = z \wedge a \leq z = T(1, z),$$

3. $x \in I_a$,

3.1. $y \in [a, 1)$,

3.1.1. $z \in [0, a)$,

$$T(x, z) = int(x) \wedge int(z) \leq int(y) \wedge int(z) = T(y, z),$$

3.1.2. $z \in [a, 1)$,

$$T(x, z) = x \wedge a \leq a \leq V(y, z) = T(y, z),$$

3.1.3. $z \in I_a$,

$$T(x, z) = x \wedge z \wedge a \leq z \wedge a = T(y, z),$$

3.2. $y = 1$,

3.2.1. $z \in [0, a)$,

$$T(x, z) = int(x) \wedge int(z) \leq z = T(1, z),$$

3.2.2. $z \in [a, 1)$,

$$T(x, z) = x \wedge a \leq a \leq z = T(1, z),$$

3.2.3. $z \in I_a$,

$$T(x, z) = x \wedge z \wedge a \leq z = T(1, z),$$

4. $x = 1$,
Then, it must be $y = 1$. Clearly, monotonicity holds.

ii) Associativity: We need to prove that $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in L$. If at least one of x, y, z in L is 1, then it is obvious. So, the proof is split into all possible cases.

1. $x \in [0, a)$,

1.1 $y \in [0, a)$,

1.1.1. $z \in [0, a)$ or $z \in [a, 1)$ or $z \in I_a$,

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z) = T(int(x) \wedge int(z), z) = T(T(x, y), z),$$

1.2. $y \in [a, 1)$,

1.2.1. $z \in [0, a)$,

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z) = T(int(x) \wedge int(z), z) = T(T(x, y), z),$$

1.2.2. $z \in [a, 1)$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, V(y, z)) = int(x) \wedge int(V(y, z)) \\ &= int(x), = int(x) \wedge int(y) \wedge int(z) \\ &= T(int(x) \wedge int(y), z) = T(T(x, y), z), \end{aligned}$$

1.2.3. $z \in I_a$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, z \wedge a) = int(x) \wedge int(z \wedge a) \\ &= int(x \wedge z) = int(x) \wedge int(y) \wedge int(z) \\ &= T(int(x) \wedge int(y), z) = T(T(x, y), z), \end{aligned}$$

1.3. $y \in I_a$,

1.3.1. $z \in [0, a)$,

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z) = T(int(x) \wedge int(y), z) = T(T(x, y), z),$$

1.3.2. $z \in [a, 1)$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge a) = int(x) \wedge int(y \wedge a) \\ &= int(x \wedge y) = int(x) \wedge int(y) \wedge int(z) \\ &= T(int(x) \wedge int(y), z) = T(int(x) \wedge int(y), z) \\ &= T(T(x, y), z), \end{aligned}$$

1.3.3. $z \in I_a$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge z \wedge a) = int(x) \wedge int(y \wedge z \wedge a) \\ &= int(x \wedge y \wedge z \wedge a) = int(x \wedge y \wedge z) \\ &= int(int(x) \wedge int(y)) \wedge int(z) = T(int(x) \wedge int(y), z) \\ &= T(T(x, y), z), \end{aligned}$$

2. $x \in [a, 1)$,

2.1 $y \in [0, a)$,

2.1.1. $z \in [0, a)$ or $z \in [a, 1)$ or $z \in I_a$,

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z) = T(int(x) \wedge int(y), z) = T(T(x, y), z),$$

2.2. $y \in [a, 1)$,

2.2.1. $z \in [0, a)$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z) \\ &= int(z) = int(V(x, y)) \wedge int(z) \\ &= T(V(x, y), z) = T(T(x, y), z), \end{aligned}$$

2.2.2. $z \in [a, 1)$,

$$T(x, T(y, z)) = T(x, V(y, z)) = V(x, V(y, z)) = V(V(x, y), z) = T(V(x, y), z) = T(T(x, y), z),$$

2.2.3. $z \in I_a$,

$$T(x, T(y, z)) = T(x, z \wedge a) = int(z \wedge a) = z \wedge a = T(V(x, y), z) = T(T(x, y), z),$$

2.3. $y \in I_a$,

2.3.1. $z \in [0, a)$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z) \\ &= int(y \wedge z) = int(y \wedge a) \wedge int(z) \\ &= T(y \wedge a, z) = T(T(x, y), z), \end{aligned}$$

2.3.2. $z \in [a, 1)$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge a) = int(x) \wedge int(y \wedge a) \\ &= int(y \wedge a) = int(y \wedge a) \wedge int(z) \\ &= T(y \wedge a, z) = T(T(x, y), z), \end{aligned}$$

2.3.3. $z \in I_a$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge z \wedge a) = int(x) \wedge int(y \wedge z \wedge a) \\ &= int(y \wedge z \wedge a) = int(y \wedge a) \wedge int(z) \\ &= T(y \wedge a, z) = T(T(x, y), z), \end{aligned}$$

3. $x \in I_a$,

3.1 $y \in [0, a)$,

3.1.1. $z \in [0, a)$ or $z \in [a, 1)$ or $z \in I_a$,

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z) = T(int(x) \wedge int(y), z) = T(T(x, y), z),$$

3.2. $y \in [a, 1)$,

3.2.1. $z \in [0, a)$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z) \\ &= int(x \wedge z) = int(x \wedge a) \wedge int(z) \\ &= T(x \wedge a, z) = T(T(x, y), z), \end{aligned}$$

3.2.2. $z \in [a, 1)$,

$$T(x, T(y, z)) = T(x, V(y, z)) = x \wedge a = int(x \wedge a) = int(x \wedge a) \wedge int(z) = T(x \wedge a, z) = T(T(x, y), z),$$

3.2.3. $z \in I_a$,

$$T(x, T(y, z)) = T(x, z \wedge a) = int(x) \wedge int(z \wedge a) = int(x \wedge a) \wedge int(z) = T(x \wedge a, z) = T(T(x, y), z),$$

3.3. $y \in I_a$,

3.3.1. $z \in [0, a)$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z) \\ &= int(x \wedge y \wedge a) \wedge int(z) = T(x \wedge y \wedge a, z) \\ &= T(T(x, y), z), \end{aligned}$$

3.3.2. $z \in [a, 1)$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge a) = int(x) \wedge int(y \wedge a) \\ &= int(x \wedge y \wedge a) = int(x \wedge y \wedge a) \wedge int(z) \\ &= T(x \wedge y \wedge a, z) = T(T(x, y), z), \end{aligned}$$

3.3.3. $z \in I_a$,

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge z \wedge a) = int(x) \wedge int(y \wedge z \wedge a) \\ &= int(x \wedge y \wedge z \wedge a) = int(x \wedge y \wedge a) \wedge int(z) \\ &= T(x \wedge y \wedge a, z) = T(T(x, y), z), \end{aligned}$$

So, we have the fact that T is a t-norm on L. □

Remark 3.2. *Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L$. In Theorem 3.1, observe that the condition for all $x \in I_a$ it holds $x \wedge a = int(x \wedge a)$ can not be omitted, in general. The following example illustrates this fact that the function $T : L^2 \to L$ defined by Theorem 3.1 is not a t-norm.*

Example 3.3. *Consider the lattice $(L_1 = \{0_{L_1}, b, c, d, a, k, m, 1_{L_1}\}, \leq, 0_{L_1}, 1_{L_1})$ in Figure 1. And we take the t-norm $V(x, y) = x \wedge y$ on $[a, 1_{L_1}]$. So, we obtain $int(0_{L_1}) = 0_{L_1}$, $int(b) = int(c) = int(d) = int(a) = int(k) = b$, $int(m) = m$ and $int(1_{L_1}) = 1_{L_1}$. For all $x \in I_a$ it does not hold $x \wedge a = int(x \wedge a)$. Because, $k \wedge a = c \neq b = int(c) = int(k \wedge a)$. Then, the function T on L_1 defined by Table 1 is not a t-norm. Indeed, it does not satisfy the associativity. Because $T(k, T(m, m)) = T(k, m) = c \neq b = T(c, m) = T(T(k, m), m)$.*

Corollary 3.4. *Let $(L, \leq, 0, 1)$ be a bounded lattice with $a, b \in L$ such that for all $x \in I_a$ it holds $x \wedge a = x \wedge a \wedge b$ and V be a t-norm on $[a, 1]$. Then, the function $T : L^2 \to L$ defined by*

$$T(x, y) = \begin{cases} V(x, y) & \text{if } (x, y) \in [a, 1)^2, \\ y \wedge a & \text{if } (x, y) \in [a, 1) \times I_a, \\ x \wedge a & \text{if } (x, y) \in I_a \times [a, 1), \\ x \wedge y \wedge a & \text{if } (x, y) \in I_a \times I_a, \\ x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ x \wedge y \wedge b & \text{otherwise }. \end{cases}$$

is a t-norm on L.

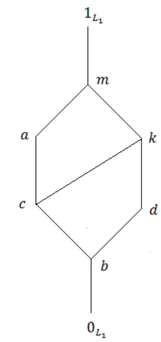

Figure 1: The lattice L_1

Table 1: The function T on L_1

T	0_{L_1}	b	c	d	a	k	m	1_{L_1}
0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}
b	0_{L_1}	b	b	b	b	b	b	b
c	0_{L_1}	b	b	b	b	b	b	c
d	0_{L_1}	b	b	b	b	b	b	d
a	0_{L_1}	b	b	b	a	c	a	a
k	0_{L_1}	b	b	b	c	c	c	k
m	0_{L_1}	b	b	b	a	c	m	m
1_{L_1}	0_{L_1}	b	c	d	a	k	m	1_{L_1}

We give next construction methods for t-norms on complete lattices from Definition 2.9 and Definition 2.11.

Corollary 3.5. *Let $(L, \leq, 0, 1)$ be a complete lattice with $a \in L$, $\Downarrow : L \to L$ be defined in Definition 2.9 such that for all $x \in I_a$ it holds $x \wedge a = \Downarrow (x \wedge a)$ and V be a t-norm on $[a, 1]$. Then, the binary operation $T : L^2 \to L$ defined by*

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2, \\ y \wedge a & \text{if } (x,y) \in [a,1) \times I_a, \\ x \wedge a & \text{if } (x,y) \in I_a \times [a,1), \\ x \wedge y \wedge a & \text{if } (x,y) \in I_a \times I_a, \\ x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ \Downarrow (x) \wedge \Downarrow (y) & \text{otherwise }. \end{cases}$$

is a t-norm on L.

We can give an example to illustrate Corollary 3.5.

Example 3.6. *Consider the complete lattice $(L_2 = \{0_{L_2}, t, p, q, a, s, n, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})$ in Figure 2. And we take the t-norm $V(x, y) = x \wedge y$ on $[a, 1_{L_2}]$. It is clear that $UC(L_2) = \{0_{L_2}, t, n, 1_{L_2}\}$. So, we obtain $\Downarrow (0_{L_2}) = 0_{L_2}$, $\Downarrow (t) = \Downarrow (p) = \Downarrow (q) = \Downarrow (a) = \Downarrow (s) = t$, $\Downarrow (n) = n$ and $\Downarrow (1_{L_2}) = 1_{L_2}$. Since for all $x \in I_a$ it holds $x \wedge a = \Downarrow (x \wedge a)$, L_2 satisfies the constraint of Corollary 3.5. That is, $q \wedge a = t = \Downarrow (t) = \Downarrow (q \wedge a)$ and $s \wedge a = t = \Downarrow (t) = \Downarrow (s \wedge a)$. Then the t-norm $T : L_2^2 \to L_2$ constructed via Corollary 3.5 is given by Table 2.*

Remark 3.7. *If we take $b = 0$ in Corollary 3.4, then it must be $x \wedge a = 0$ for all $x \in I_a$. So, we obtain corresponding t-norm as follows constructed by Çaylı* [8].

Corollary 3.8. [8] *Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L \setminus \{0, 1\}$ and V be a t-norm on $[a, 1]$. Then the function $T_1 : L^2 \to L$ is a t-norm on L, where*

$$T_1(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2, \\ x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise }. \end{cases}$$

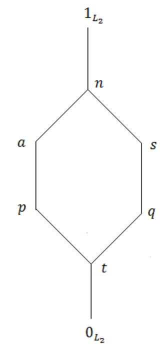

Figure 2: The lattice L_2

Table 2: The t-norm T on L_2

T	0_{L_2}	t	p	q	a	s	n	1_{L_2}
0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}
t	0_{L_2}	t	t	t	t	t	t	t
p	0_{L_2}	t	t	t	t	t	t	p
q	0_{L_2}	t	t	t	t	t	t	q
a	0_{L_2}	t	t	t	a	t	a	a
s	0_{L_2}	t	t	t	t	t	t	s
n	0_{L_2}	t	t	t	a	t	n	n
1_{L_2}	0_{L_2}	t	p	q	a	s	n	1_{L_2}

Remark 3.9. *If we take $b = 1$ in Corollary 3.4, then we obtain corresponding t-norm as follows constructed by Ertuğrul, Karaçal and Mesiar* [15].

Corollary 3.10. [15] *Let $(L, \leq, 0, 1)$ be a bounded lattice and V be a t-norm on $[a, 1]$. Then the function $T_2 : L^2 \to L$ is a t-norm on L, where*

$$T_2(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2, \\ x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ x \wedge y \wedge a & \text{otherwise} . \end{cases}$$

Remark 3.11. *It should be noted that the t-norms T_1 and T_2 in Corollary 3.8 and Corollary 3.10, respectively are different from the t-norm T in Theorem 3.1. To show that this claim, we shall consider the bounded lattice $(L_2 = \{0_{L_2}, t, p, q, a, s, n, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})$ described in Figure 2., we take the t-norm $V(x, y) = x \wedge y$ on $[a, 1_{L_2}]$ and $int(0_{L_2}) = 0_{L_2}$, $int(t) = int(p) = int(q) = int(a) = int(s) = t$, $int(n) = n$ and $int(1_{L_2}) = 1_{L_2}$. According to the Table 2, Table 3 and Table 4, it is clear that the t-norms T, T_1 and T_2 different from each other.*

Table 3: The t-norm T_1 on L_2

T_1	0_{L_2}	t	p	q	a	s	n	1_{L_2}
0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}
t	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	t
p	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	p
q	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	q
a	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	a	0_{L_2}	a	a
s	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	s
n	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	a	0_{L_2}	n	n
1_{L_2}	0_{L_2}	t	p	q	a	s	n	1_{L_2}

Table 4: The t-norm T_2 on L_2

T_2	0_{L_2}	t	p	q	a	s	n	1_{L_2}
0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}
t	0_{L_2}	t	t	t	t	t	t	t
p	0_{L_2}	t	p	t	p	t	p	p
q	0_{L_2}	t	t	t	t	t	t	q
a	0_{L_2}	t	p	t	a	t	a	a
s	0_{L_2}	t	t	t	t	t	t	s
n	0_{L_2}	t	p	t	a	t	n	n
1_{L_2}	0_{L_2}	t	p	q	a	s	n	1_{L_2}

4 New construction method for t-conorms on bounded lattices by using closure operators

In this section, we propose new construction method for t-conorms on bounded lattices with the given t-conorms by using closure operators. The main aim of this section is to present a rather effective method to construct t-conorms by using closure operators on a bounded lattice. Using this method, in Corollary 4.8 and Corollary 4.10, we obtain the methods proposed by Ertuğrul, Karaçal, Mesiar [15] and Çaylı [8], respectively.

Theorem 4.1. *Let $(L,\leq,0,1)$ be a bounded lattice with $a\in L$ such that for all $x\in I_a$ it holds $x\vee a=cl(x\vee a)$ and $cl:L\to L$ be a closure operator. Given a t-conorm W on $[0,a]$, then the function $S:L^2\to L$ defined as follows is a t-conorm on L where*

$$S(x,y)=\begin{cases} W(x,y) & if\ (x,y)\in(0,a]^2\ , \\ y\vee a & \text{if } (x,y)\in(0,a]\times I_a, \\ x\vee a & \text{if } (x,y)\in I_a\times(0,a], \\ x\vee y\vee a & \text{if } (x,y)\in I_a\times I_a, \\ x\vee y & \text{if } x=0 \text{ or } y=0, \\ cl(x)\vee cl(y) & \text{otherwise}\ . \end{cases}$$

Remark 4.2. *Let $(L,\leq,0,1)$ be a bounded lattice with $a\in L$. In Theorem 4.1, observe that the condition for all $x\in I_a$ it holds $x\vee a=cl(x\vee a)$ can not be omitted, in general. The following example illustrates this fact that the function $S:L^2\to L$ defined by Theorem 4.1 is not a t-conorm.*

Example 4.3. *Consider the lattice $(L_3=\{0_{L_3},t,a,n,p,s,q,1_{L_3}\},\leq,0_{L_3},1_{L_3})$ in Figure 3. And we take the t-conorm $W(x,y)=x\vee y$ on $[0_{L_3},a]$. So, we obtain $cl(0_{L_3})=0_{L_3}$, $cl(t)=t$, $cl(n)=cl(a)=cl(s)=cl(p)=cl(q)=q$, and $cl(1_{L_3})=1_{L_3}$. For all $x\in I_a$ it does not hold $x\vee a=cl(x\vee a)$. Because, $n\vee a=p\neq q=cl(p)=cl(n\vee a)$. Then, the function S on L_3 defined by Table 5 is not a t-conorm. Indeed, it does not satisfy the associativity. Because $S(n,S(t,t))=S(n,t)=p\neq q=S(p,t)=S(S(n,t),t)$.*

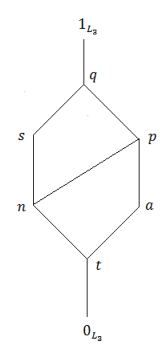

Figure 3: The lattice L_3

Table 5: The t-function S on L_3

S	0_{L_3}	t	a	n	p	s	q	1_{L_3}
0_{L_3}	0_{L_3}	t	a	n	p	s	q	1_{L_3}
t	t	t	a	p	q	q	q	1_{L_3}
a	a	a	a	p	q	q	q	1_{L_3}
n	n	p	p	p	q	q	q	1_{L_3}
p	p	q	q	q	q	q	q	1_{L_3}
s	s	q	q	q	q	q	q	1_{L_3}
q	q	q	q	q	q	q	q	1_{L_3}
1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}

Corollary 4.4. *Let $(L, \leq, 0, 1)$ be a bounded lattice with $a, b \in L$ such that for all $x \in I_a$ it holds $x \vee a = x \vee a \vee b$ and W be a t-conorm on $[0, a]$. Then, the function $S : L^2 \to L$ defined by*

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2, \\ y \vee a & \text{if } (x,y) \in (0,a] \times I_a, \\ x \vee a & \text{if } (x,y) \in I_a \times (0,a], \\ x \vee y \vee a & \text{if } (x,y) \in I_a \times I_a, \\ x \vee y & \text{if } x = 0 \text{ or } y = 0, \\ x \vee y \vee b & \text{otherwise} . \end{cases}$$

is a t-conorm on L.

We give next construction methods for t-conorms on complete lattices from Definition 2.5 and Definition 2.8.

Corollary 4.5. *Let $(L, \leq, 0, 1)$ be a complete lattice with $a \in L$, $\Uparrow: L \to L$ be defined in Definition 2.5 such that for all $x \in I_a$ it holds $x \vee a = \Uparrow (x \vee a)$ and W be a t-conorm on $[0, a]$. Then, the binary operation $S : L^2 \to L$ defined by*

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2, \\ y \vee a & \text{if } (x,y) \in (0,a] \times I_a, \\ x \vee a & \text{if } (x,y) \in I_a \times (0,a], \\ x \vee y \vee a & \text{if } (x,y) \in I_a \times I_a, \\ x \vee y & \text{if } x = 0 \text{ or } y = 0, \\ \Uparrow (x) \vee \Uparrow (y) & \text{otherwise} . \end{cases}$$

is a t-conorm on L.

We can give an example to illustrate Corollary 4.5.

Example 4.6. *Consider the complete lattice $(L_4 = \{0_{L_4}, m, r, a, k, c, d, 1_{L_4}\}, \leq, 0_{L_4}, 1_{L_4})$ in Figure 4. And we take the t-conorm $W(x, y) = x \vee y$ on $[0_{L_4}, a]$. It is clear that $UC(L_4) = \{0_{L_4}, m, d, 1_{L_4}\}$. So, we obtain $\Uparrow (0_{L_4}) = 0_{L_4}$, $\Uparrow (m) = m$, $\Uparrow (r) = \Uparrow (a) = \Uparrow (k) = \Uparrow (c) = \Uparrow (d) = d$, and $\Uparrow (1_{L_4}) = 1_{L_4}$. Since for all $x \in I_a$ it holds $x \vee a = \Uparrow (x \vee a)$, L_4 satisfies the constraint of Corollary 4.5. That is, $k \vee a = d = \Uparrow (d) = \Uparrow (k \vee a)$ and $r \vee a = d = \Uparrow (d) = \Uparrow (r \vee a)$. Then the t-conorm $S : L_4^2 \to L_4$ constructed via Corollary 4.5 is given by Table 6.*

Remark 4.7. *If we take $b = 0$ in Corollary 4.4, then we obtain corresponding t-conorm as follows constructed by Ertuğrul, Karaçal and Mesiar* [15].

Corollary 4.8. [15] *Let $(L, \leq, 0, 1)$ be a bounded lattice and W be a t-conorm on $[0, a]$. Then the function $S_1 : L^2 \to L$ is a t-conorm on L, where*

$$S_1(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2, \\ x \vee y & \text{if } x = 0 \text{ or } y = 0, \\ x \vee y \vee a & \text{otherwise} . \end{cases}$$

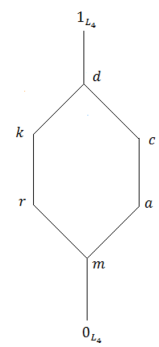

Figure 4: The lattice L_4

Table 6: The t-conorm S on L_4

S	0_{L_4}	m	r	a	k	c	d	1_{L_4}
0_{L_4}	0_{L_4}	m	r	a	k	c	d	1_{L_4}
m	m	m	d	a	d	d	d	1_{L_4}
r	r	d	d	d	d	d	d	1_{L_4}
a	a	a	d	a	d	d	d	1_{L_4}
k	k	d	d	d	d	d	d	1_{L_4}
c	c	d	d	d	d	d	d	1_{L_4}
d	d	d	d	d	d	d	d	1_{L_4}
1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}

Remark 4.9. *If we take $b = 1$ in Corollary 4.4, then it must be $x \vee a = 1$ for all $x \in I_a$. So, we obtain corresponding t-conorm as follows constructed by Çaylı* [8].

Corollary 4.10. [8] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0, 1\}$. If W be a t-conorm on $[0, a]$, then the function $S_2 : L^2 \to L$ is a t-conorm on L, where*

$$S_2(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2, \\ x \vee y & \text{if } x = 0 \text{ or } y = 0, \\ 1 & \text{otherwise .} \end{cases}$$

Remark 4.11. *It should be noted that the t-conorms S_1 and S_2 in Corollary 4.8 and Corollary 4.10, respectively are different from the t-conorm S in Theorem 4.1. To show that this claim, we consider the bounded lattice $(L_4 = \{0_{L_4}, m, r, a, k, c, d, 1_{L_4}\}, \leq, 0_{L_4}, 1_{L_4})$ in Figure 4., we take the t-conorm $W(x,y) = x \vee y$ on $[0_{L_4}, a]$ and $cl(0_{L_4}) = 0_{L_4}$, $cl(m) = m$, $cl(r) = cl(a) = cl(k) = cl(c) = cl(d) = d$ and $cl(1_{L_4}) = 1_{L_4}$. According to the Table 6, Table 7 and Table 8, it is clear that t-conorms S, S_1 and S_2 different from each other.*

Table 7: The t-conorm S_2 on L_4

S_1	0_{L_4}	m	r	a	k	c	d	1_{L_4}
0_{L_4}	0_{L_4}	m	r	a	k	c	d	1_{L_4}
m	m	m	d	a	d	c	d	1_{L_4}
r	r	d	d	d	d	d	d	1_{L_4}
a	a	a	d	a	d	c	d	1_{L_4}
k	k	d	d	d	d	d	d	1_{L_4}
c	c	c	d	c	d	c	d	1_{L_4}
d	d	d	d	d	d	d	d	1_{L_4}
1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}

Table 8: The t-conorm S_1 on L_4

S_2	0_{L_4}	m	r	a	k	c	d	1_{L_4}
0_{L_4}	0_{L_4}	m	r	a	k	c	d	1_{L_4}
m	m	m	1_{L_4}	a	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}
r	r	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}
a	a	a	1_{L_4}	a	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}
k	k	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}
c	c	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}
d	d	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}
1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}

5 Modified ordinal sum constructions of t-norms and t-conorms on bounded lattices

From [8] and [15], we know that new t-norms and t-conorms on bounded lattices can be obtained using recursion in Theorem 5.1, Theorem 5.2 and Theorem 5.5, Theorem 5.6, respectively. In this section, based on the approaches of constructing t-norms and t-conorms by using interior and closure operators, respectively, proposed in Section 3 and Section 4, we show that it can not be obtained ordinal sum constructions of t-norms and t-conorms on bounded lattice L using recursion.

Theorem 5.1. [8] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \cdots, a_n\}$ be a finite chain in L such that $1 = a_0 > a_1 > a_2 > ... > a_n = 0$. Let $V : [a_1, 1]^2 \to [a_1, 1]$ be a t-norm. Then, the function $T_n : L^2 \to L$ defined recursively as follows is a t-norm, where $V = T_1$ and for $i \in \{2, \cdots, n\}$, the function $T_i : [a_i, 1]^2 \to [a_i, 1]$ is given by*

$$T_i(x, y) = \begin{cases} T_{i-1}(x, y) & \text{if } (x, y) \in [a_{i-1}, 1)^2, \\ x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ a_i & \text{otherwise }. \end{cases} \quad (5)$$

Theorem 5.2. [15] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \cdots, a_n\}$ be a finite chain in L such that $1 = a_0 > a_1 > a_2 > ... > a_n = 0$. Let $V : [a_1, 1]^2 \to [a_1, 1]$ be a t-norm. Then, the function $T_n : L^2 \to L$ defined recursively as follows is a t-norm, where $V = T_1$ and for $i \in \{2, \cdots, n\}$,*

$$T_i(x, y) = \begin{cases} T_{i-1}(x, y) & \text{if } (x, y) \in [a_{i-1}, 1)^2, \\ x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ x \wedge y \wedge a_{i-1} & \text{otherwise }. \end{cases} \quad (6)$$

Remark 5.3. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \cdots, a_n\}$ be a finite chain in L such that $1 = a_0 > a_1 > a_2 > ... > a_n = 0$. Let $x \wedge a_i = int(x \wedge a_i)$ for all $x \in I_{a_i}$, let $V : [a_1, 1]^2 \to [a_1, 1]$ be a t-norm and $int : L \to L$ be an interior operator. It should be noted that our construction method in Theorem 3.1 can not be obtained using recursion. Because, we can not obtain the binary operation $T_i : [a_i, 1]^2 \to [a_i, 1]$ as follows, where $T_1 = V$ and for $i \in \{2, \cdots, n\}$,*

$$T_i(x, y) = \begin{cases} T_{i-1}(x, y) & \text{if } (x, y) \in [a_{i-1}, 1)^2, \\ y \wedge a_{i-1} & \text{if } (x, y) \in [a_{i-1}, 1) \times I_{a_{i-1}}, \\ x \wedge a_{i-1} & \text{if } (x, y) \in I_{a_{i-1}} \times [a_{i-1}, 1), \\ x \wedge y \wedge a_{i-1} & \text{if } (x, y) \in I_{a_{i-1}} \times I_{a_{i-1}}, \\ x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ int(x) \wedge int(y) & \text{otherwise }. \end{cases} \quad (7)$$

To illustrate this claim we shall give the following example:

Example 5.4. *Consider the lattice $(L_5 = \{0_5, a_4, b, c, a_3, a_2, a_1, 1_{L_5}\}, \leq, 0_{L_5}, 1_{L_5})$ described in Figure 5 with the finite chain $0_{L_5} < a_4 < a_3 < a_2 < a_1 < 1_{L_5}$ in L_5. Then, we obtain $int(0_{L_5}) = 0_{L_5}$, $int(a_4) = int(a_3) = int(a_2) = int(a_1) = int(c) = int(b) = a_4$, $int(1_{L_5}) = 1_{L_5}$. It is clear that $x \wedge a_i = int(x \wedge a_i)$ for all $x \in I_{a_i}$. Define the t-norm $V : [a_1, 1_{L_5}]^2 \to [a_1, 1_{L_5}]$ by $V = T_\wedge$. Since $int(a_1) \wedge int(a_2) = a_4 \notin [a_2, 1_{L_5}]$, we can not obtain the binary operation T_2 on $[a_2, 1_{L_5}]$. Since $int(a_3) \wedge int(a_1) = a_4 \notin [a_3, 1_{L_5}]$, we can not obtain the binary operation T_3 on $[a_3, 1_{L_5}]$.*

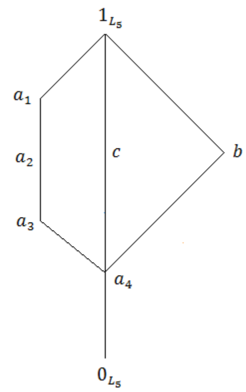

Figure 5: The lattice L_5

Theorem 5.5. [8] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \cdots, a_n\}$ be a finite chain in L such that $0 = a_0 < a_1 < a_2 < ... < a_n = 1$. Let $W : [0, a_1]^2 \to [0, a_1]$ be a t-conorm. Then, the function $S_n : L^2 \to L$ defined recursively as follows is a t-conorm, where $S_1 = W$ and for $i \in \{2, \cdots, n\}$, the binary function $S_i : [0, a_i]^2 \to [0, a_i]$ is given by*

$$S_i(x, y) = \begin{cases} S_{i-1}(x, y) & \text{if } (x, y) \in (0, a_{i-1}]^2, \\ x \vee y & \text{if } x = 0 \text{ or } y = 0, \\ a_i & \text{otherwise }. \end{cases} \tag{8}$$

Theorem 5.6. [15] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \cdots, a_n\}$ be a finite chain in L such that $0 = a_0 < a_1 < a_2 < ... < a_n = 1$. Let $W : [0, a_1]^2 \to [0, a_1]$ be a t-conorm. Then, the function $S_n : L^2 \to L$ defined recursively as follows is a t-conorm, where $S_1 = W$ and for $i \in \{2, \cdots, n\}$,*

$$S_i(x, y) = \begin{cases} S_{i-1}(x, y) & \text{if } (x, y) \in (0, a_{i-1}]^2, \\ x \vee y & \text{if } x = 0 \text{ or } y = 0, \\ x \vee y \vee a_{i-1} & \text{otherwise }. \end{cases} \tag{9}$$

Remark 5.7. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \cdots, a_n\}$ be a finite chain in L such that $0 = a_0 < a_1 < a_2 < ... < a_n = 1$. Let $x \vee a_i = cl(x \vee a_i)$ for all $x \in I_{a_i}$, let $W : [0, a_1]^2 \to [0, a_1]$ be a t-conorm and $cl : L \to L$ be a closure operator. It should be noted that our construction method in Theorem 4.1 can not be obtained using recursion. Because we can not obtain the binary operation $S_i : [0, a_i]^2 \to [0, a_i]$ as follows, where $S_1 = W$ and for $i \in \{2, \cdots, n\}$,*

$$S_i(x, y) = \begin{cases} S_{i-1}(x, y) & \text{if } (x, y) \in (0, a_{i-1}]^2, \\ y \vee a_{i-1} & \text{if } (x, y) \in (0, a_{i-1}] \times I_{a_{i-1}}, \\ x \vee a_{i-1} & \text{if } (x, y) \in I_{a_{i-1}} \times (0, a_{i-1}], \\ x \vee y \vee a_{i-1} & \text{if } (x, y) \in I_{a_{i-1}} \times I_{a_{i-1}}, \\ x \vee y & \text{if } x = 0 \text{ or } y = 0, \\ cl(x) \vee cl(y) & \text{otherwise }. \end{cases} \tag{10}$$

To illustrate this claim we shall give the following example

Example 5.8. *Consider the lattice $(L_6 = \{0_{L_6}, a_1, a_2, a_3, m, n, a_4, 1_{L_6}\}, \leq, 0_{L_6}, 1_{L_6})$ described in Figure 6 with the finite chain $0_{L_6} < a_1 < a_2 < a_3 < a_4 < 1_{L_6}$ in L_6. Then, we obtain $cl(0_{L_6}) = 0_{L_6}$, $cl(m) = cl(n) = cl(a_1) = cl(a_2) = cl(a_3) = cl(a_4) = a_4$, $cl(1_{L_6}) = 1_{L_6}$. It is clear that $x \vee a_i = cl(x \vee a_i)$ for all $x \in I_{a_i}$. Define the t-conorm $W : [0_{L_6}, a_1]^2 \to [0_{L_6}, a_1]$ by $W = S_\vee$. Since $int(a_1) \vee int(a_2) = a_4 \notin [0_{L_5}, a_2]$, we can not obtain the binary operation S_2 on $[0_{L_5}, a_2]$. Since $int(a_3) \vee int(a_1) = a_4 \notin [0_{L_5}, a_3]$, we can not obtain the binary operation S_3 on $[0_{L_5}, a_3]$.*

6 Concluding remarks

In this paper, we have proposed the constructions of t-norms and t-conorms on bounded lattices with interior and closure operators, respectively. The main aim of this paper is to present a rather effective method to construct t-norms

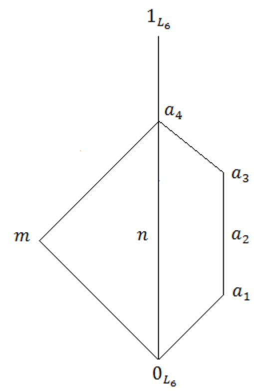

Figure 6: The lattice L_6

and t-conorms by using interior and closure operators on a bounded lattice, respectively. Also, using these methods, in Corollary 3.10 and Corollary 4.8, we obtain the methods proposed by Ertuğrul, Karaçal and Mesiar [15]. Also, in Corollary 3.8 and Corollary 4.10, we obtain the methods proposed by Çaylı [8]. Finally, we have shown that the new construction methods can not be generalized by induction to a modified ordinal sum for t-norms and t-conorms on arbitrary bounded lattice, respectively.

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