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Original paper

End-point linear functions

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Abstract

Positive homogeneity is represented as a constraint $\mathbf{0}$-homogeneity and generalized into $\mathbf{z}$-homogeneity, called also $\mathbf{z}$-end point linearity. Several special $\mathbf{z}$-homogeneous aggregation functions are studied, in particular semicopulas, quasi-copulas, copulas, overlap functions, etc.

Keywords: Aggregation function, copula, end-point linear function.

1 Introduction

Linearity of functions defined on a segment is a most applied approach in modelling of dependences between variables in physics or engineering. Recall that a function $f : D \to \mathbb{R}$ defined on a convex subset D of a vector space $(\mathbb{X}, +, \cdot)$ is linear if

$$f(a\mathbf{x} + b\mathbf{y}) = a\, f(\mathbf{x}) + b\, f(\mathbf{y}),$$

for any real constants $a, b \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in D$ such that $a\mathbf{x} + b\mathbf{y} \in D$. Equivalently, f is linear if and only if for any $\lambda \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in D$ it holds

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \lambda\, f(\mathbf{x}) + (1-\lambda)\, f(\mathbf{y}).$$

Note that in this case, $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in D$ due to the convexity of D.
A weaker form of linearity is related to a fixed point $\mathbf{z} \in D$.
We tell that f is *end-point linear in* $\mathbf{z}$ whenever for any $\mathbf{x} \in D$ and $\lambda \in [0, 1]$ it holds

$$f(\lambda\mathbf{z} + (1-\lambda)\mathbf{x}) = \lambda f(\mathbf{z}) + (1-\lambda) f(\mathbf{x}).$$

Clearly, then, for any $\mathbf{x} \in D$, f is linear on the segment $\langle \mathbf{z}, \mathbf{x} \rangle$. Also, f is linear if and only if it is $\mathbf{z}$-end point linear for any $\mathbf{z} \in D$.
In particular, if $\mathbf{0} \in D$ then f is $\mathbf{0}$-end point linear whenever

$$f(\lambda\mathbf{x}) = \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{0}),$$

for any $\mathbf{x} \in D$ and any $\lambda \in [0, 1]$.
If $\mathbf{0} \in D$, then $f : D \to \mathbb{R}$ is called positively homogeneous if

$$f(a\mathbf{x}) = a\, f(\mathbf{x}) \text{ for any } a \in [0, \infty[\text{ and any } \mathbf{x} \in D \text{ such that } a\mathbf{x} \in D.$$

Obviously, then $f(\mathbf{0}) = 0$ and, equivalently, for any $\mathbf{x} \in D$ and $\lambda \in [0, 1]$, $f(\lambda\mathbf{x}) = \lambda f(\mathbf{x})$. Observe that the positive homogeneity of the function f means that $f(\mathbf{0}) = 0$ and, for any $\mathbf{x} \in D$, the restriction of f to the segment $\langle \mathbf{0}, \mathbf{x} \rangle$ is a linear function, i.e., f is $\mathbf{0}$-end point linear. The aim of this paper is a generalization of the positive homogeneity of functions vanishing in $\mathbf{0}$ seen as linearity on $\langle \mathbf{0}, \mathbf{x} \rangle$ for any $\mathbf{x} \in D$ by $\mathbf{z}$-homogeneity, where $\mathbf{z}$ is a fixed point in D.

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Definition 1.1. *Let $f : D \to \mathbb{R}$ and $\mathbf{z} \in D$. Then f is $\mathbf{z}$-homogeneous if, for any $\mathbf{x} \in D$, f is linear on the segment $\langle \mathbf{z}, \mathbf{x} \rangle$.*

Clearly, the standard positive homogeneity implies the $\mathbf{0}$-homogeneity, but not vice-versa. Moreover, $\mathbf{z}$-homogeneity can be seen as an end-point linearity, as the $\mathbf{z}$-homogeneous f is linear on any segment in D with an end point $\mathbf{z}$. Our main aim is the study of end-point linear functions acting on the domain $[0,1]^n \subseteq \mathbb{R}^n$ (i.e., we consider the vector space $(\mathbb{R}^n, +, \cdot)$), with a special stress on the case $[0,1]^2$. Considering some additional properties, we study and characterise some particular subclasses of end-point linear functions.

The paper is organized as follows. In the next section, we discuss and exemplify some particular $\mathbf{z}$-homogeneous functions $f : [0,1]^n \to [0,1]$. In Section 3, we focus on functions $f : [0,1]^2 \to [0,1]$ and characterize end-point linear semicopulas (quasi-copulas, copulas, t-norms, t-conorms), overlap and grouping functions, quasi-arithmetic means, etc. Finally, some concluding remarks are added.

2 End-point linear fusion functions, general case

For $n \geq 2$, we will consider functions $f : [0,1]^n \to [0,1]$. Note that these functions assign to n-tuples of inputs from the unit interval $[0,1]$ an output value from the same interval $[0,1]$. To stress this property, they are also called *fusion functions*, see, e.g., [3]. Some particular subclasses of fusion functions we will deal with are:

- *semi-aggregation functions*, i.e., fusion functions satisfying two boundary conditions $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$;
- *aggregation functions*, i.e., semi-aggregation functions which are directionally increasing for any direction $\vec{r} \in [0,1]^n \setminus \{\mathbf{0}\}$;

Recall that $f : [0,1]^n \to [0,1]$ is $\vec{r}$-directionally increasing whenever $f(\mathbf{x} + c\vec{r}) \geq f(\mathbf{x})$ for any $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$. For more details we recommend [4, 8]. Note also that aggregation functions can be equivalently characterized by two boundary conditions and increasingness in each coordinate, i.e., by $\mathbf{e_i}$-directional increasingness for any unit vector $\mathbf{e_i} = (0, \dots, \underbrace{1}_{i\text{-th coordinate}}, \dots, 0)$, $i = 1, 2, \dots, n$.

As already mentioned, positively homogeneous functions are a particular case of end-point linear functions, namely, they are $\mathbf{0}$-homogeneous and vanishing in $\mathbf{0}$. As an example of $\mathbf{0}$-homogeneous fusion function which is not positively homogeneous consider, for example $f(\mathbf{x}) = \dfrac{2 - \min(\mathbf{x})}{3}$. It is not difficult to check that from any fusion function $g : [0,1]^n \to [0,1]$ one can construct a $\mathbf{0}$-homogeneous function.

Proposition 2.1. *Let $g : [0,1]^n \to [0,1]$ be a fusion function. Let $f = g_{\mathbf{0}} : [0,1]^n \to [0,1]$ be given by*

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{0}, \\ \max(\mathbf{x}) \cdot g\left(\frac{\mathbf{x}}{\max(\mathbf{x})}\right) & \text{otherwise.} \end{cases} \tag{1}$$

Then f is a $\mathbf{0}$-homogeneous fusion function. If g is also semi-aggregation function, then also f is a semi-aggregation function.

The proof is obvious and therefore omitted.

Note that we can replace max in formula (1) by any other $\mathbf{0}$-homogeneous fusion function $h : [0,1]^n \to [0,1]$, and then $f : [0,1]^n \to [0,1]$ is given by

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } h(\mathbf{x}) = 0, \\ h(\mathbf{x}) \cdot g\left(\frac{\mathbf{x}}{h(\mathbf{x})}\right) & \text{otherwise.} \end{cases} \tag{2}$$

Then again f is a $\mathbf{0}$-homogeneous fusion function. To see the correctness of (2), observe that for any $\mathbf{0}$-homogeneous fusion function h it holds, for $\mathbf{x} \neq \mathbf{0}$, $h(\mathbf{x}) = \max(\mathbf{x}) \cdot h\left(\frac{\mathbf{x}}{\max(\mathbf{x})}\right) \leq \max(\mathbf{x})$ and thus $\frac{\mathbf{x}}{h(\mathbf{x})} \in [0,1]^n$ whenever $h(\mathbf{x}) > 0$.

For constructing the $\mathbf{0}$-homogeneous aggregation function, some stronger constraints are necessary. The next result can be found in [7], see also [10].

Theorem 2.2. *Let $g : [0,1]^n \to [0,1]$ be an aggregation function such that, for any $\mathbf{x}, \mathbf{y} \in [0,1]^n, \mathbf{x} \leq \mathbf{y}$, it holds*

$$\frac{g(\mathbf{x})}{g(\mathbf{y})} \geq \min\left(\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n}\right), \tag{3}$$

with the convention $\frac{0}{0} = 1$. Then the function $f = g_{\mathbf{0}}$ given by (1) is a $\mathbf{0}$-homogeneous aggregation function.

Observe also that for any $\mathbf{0}$-homogenous aggregation function $f : [0,1]^n \to [0,1]$, necessarily $\frac{f(\mathbf{x})}{f(\mathbf{y})} \geq \min\left(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}\right)$ for any $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y} \leq \mathbf{1}$.
Recall that for any fusion function (semi-aggregation function, aggregation function) $f : [0,1]^n \to [0,1]$ its dual $f^d : [0,1]^n \to [0,1]$ given by

$$f^d(\mathbf{x}) = 1 - f(\mathbf{1} - \mathbf{x}),$$

is a fusion function (semi-aggregation function, aggregation function).

Proposition 2.3. *Let a fusion function (semi-aggregation function, aggregation function) $f : [0,1]^n \to [0,1]$ be $\mathbf{z}$-homogeneous for some $\mathbf{z} \in [0,1]^n$. Then its dual f^d is $(\mathbf{1} - \mathbf{z})$-homogeneous.*

Proof. For any $\mathbf{x} \in [0,1]$ and $\lambda \in [0,1]$, $\mathbf{y} = \lambda\mathbf{x} + (1-\lambda)(\mathbf{1} - \mathbf{z}) \in \langle \mathbf{1} - \mathbf{z}, \mathbf{x}\rangle$ and thus

$$\mathbf{1} - \mathbf{y} = \mathbf{1} - (\lambda\mathbf{x} + (1-\lambda)\mathbf{1} - (1-\lambda)\mathbf{z}) = \lambda(\mathbf{1} - \mathbf{x}) + (1-\lambda)\mathbf{z}.$$

Due to the $\mathbf{z}$-homogeneity of f it holds

$$\begin{aligned} f^d(\mathbf{y}) =& 1 - f(\mathbf{1} - \mathbf{y}) = 1 - f(\lambda(\mathbf{1} - \mathbf{x}) + (1-\lambda)\mathbf{z}) \\ =& 1 - \lambda f(\mathbf{1} - \mathbf{x}) - (1-\lambda) f(\mathbf{z}) \\ =& 1 - \lambda(1 - f^d(\mathbf{x})) - (1-\lambda)(1 - f^d(\mathbf{1} - \mathbf{z})) \\ =& \lambda f^d(\mathbf{x}) + (1-\lambda) f^d(\mathbf{1} - \mathbf{z}), \end{aligned}$$

proving the linearity of f^d on the segment $\langle \mathbf{1} - \mathbf{z}, \mathbf{x}\rangle$. Thus f^d is $(\mathbf{1} - \mathbf{z})$-homogeneous. □

Due to Proposition 2.3, one can consider $\mathbf{1}$-homogeneity of functions satisfying $f(\mathbf{1}) = 1$ as a dual positive homogeneity, and this implies also the next results.

Corollary 2.4. *Let $g : [0,1]^n \to [0,1]$ be a fusion function (semi-aggregation function). Then the function $f = g_{\mathbf{1}} : [0,1]^n \to [0,1]$ given by*

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{1}, \\ \min(\mathbf{x}) + (1 - \min(\mathbf{x})) \cdot g\left(\frac{\mathbf{x} - \min(\mathbf{x}) \cdot \mathbf{1}}{1 - \min(\mathbf{x})}\right) & \text{otherwise.} \end{cases} \tag{4}$$

is a $\mathbf{1}$-homogeneous fusion function (semi-aggregation function). The function f is an aggregation function whenever g is an aggregation function satisfying, for all $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{0} \leq \mathbf{y} \leq \mathbf{x} \leq \mathbf{1}$,

$$\frac{1 - g(\mathbf{x})}{1 - g(\mathbf{y})} \geq \min\left(\frac{1 - x_1}{1 - y_1}, \ldots, \frac{1 - x_n}{1 - y_n}\right), \tag{5}$$

with the convention $\frac{0}{0} = 1$.

Similarly as in Proposition 2.3, one can show the next result.

Proposition 2.5. *Let $f : [0,1]^n \to [0,1]$ be $\mathbf{z}$-homogeneous fusion function (semi-aggregation function, aggregation function) and let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. Then the function $f_\sigma : [0,1]^n \to [0,1]$,*

$$f_\sigma(\mathbf{x}) = f(\mathbf{x}_\sigma) = f(x_{\sigma(1)} \ldots, x_{\sigma(n)}),$$

is a $\mathbf{z}_\sigma$-homogenous fusion function (semi-aggregation function, aggregation function).

Due to Proposition 2.5, each symmetric fusion function f which is $\mathbf{z}$-homogeneous is also $\mathbf{z}_\sigma$-homogeneous for any permutation σ. Moreover, due to the symmetry of f, $f(\mathbf{z}_\sigma) = f(\mathbf{z}) = c$ for any permutation σ. Therefore, for all permutations σ and τ such that $\mathbf{z}_\sigma \neq \mathbf{z}_\tau$, the function f is constant (attaining the value c) on the segment which is the intersection of the domain $[0,1]^n$ and the straight line determined by points $\mathbf{z}_\sigma$ and $\mathbf{z}_\tau$.

Total end-point linearity can be seen as $\mathbf{z}$-homogeneity valid for any end-point $\mathbf{z} \in [0,1]^n$.

Theorem 2.6. *A fusion function $f : [0,1]^n \to [0,1]$ is total end-point linear if and only if it is linear, i.e., $f(\mathbf{x}) = a + \sum_{i=1}^{n} b_i x_i$ for some real constants $a, b_1, \ldots, b_n$ such that $a + \sum_{b_i < 0} b_i \geq 0$ and $a + \sum_{b_i > 0} b_i \leq 1$. This f is a semi-aggregation function if and only if $a = 0$ and $\sum_{i=1}^{n} b_i = 1$ and then f is also an aggregation function.*

Proof. The sufficiency of both claims is obvious.
To see the necessity, denote $a = f(\mathbf{0}), b = f(1,0,\dots,0), c = f(0,1,0,\dots,0)$, and $d = f(1,1,0,\dots,0)$. Observe that for any $x_1 \in [0,1]$, $(x_1,0,\dots,0) = (1-x_1)\,\mathbf{0} + x_1\,(1,0,\dots,0)$.

- Due to $\mathbf{0}$-homogeneity of f, it holds

$$f(x_1,0,\dots,0) = (1-x_1)\,f(\mathbf{0}) + x_1\,f(1,0,\dots,0) = (1-x_1)\,a + x_1\,b = a + (b-a)x_1.$$

  Similarly $f(0,x_2,0,\dots,0) = a + (c-a)\,x_2$. Next

- $(1,0,\dots,0)$-homogeneity ensures $f(1,x_2,0,\dots,0) = b + (d-b)\,x_2$, and
- $(0,1,0,\dots,0)$-homogeneity forces $f(x_1,1,0,\dots,0) = c + (d-c)\,x_1$.
- Also, $(0,x_2,0,\dots,0)$-homogeneity and the equality

$$(x_1,x_2,0,\dots,0) = x_1\,(1,x_2,0,\dots,0) + (1-x_1)\,(0,x_2,0\dots,0) \text{ implies,}$$

$$f(x_1,x_2,0,\dots,0) = a + (b-a)\,x_1 + (c-a)\,x_2 + (d-c-b+a)\,x_1x_2.$$

- Due to

$$\left(\frac{1}{2},\frac{1}{2},0,\dots,0\right) = \frac{1}{2}(\mathbf{0} + (1,1,0\dots,0)) = \frac{1}{2}((1,0,\dots,0) + (0,1,0,\dots,0)),$$

  it holds $f\left(\frac{1}{2},\frac{1}{2},0,\dots,0\right) = \frac{a+d}{2} = \frac{b+c}{2}$, and, finally

$$f(x_1,x_2,0,\dots,0) = a + (b-a)x_1 + (c-a)x_2.$$

  Denoting $b-a = b_1$ and $c-a = b_2$, we see that $f(x_1,x_2,0,\dots,0) = a + b_1x_1 + b_2x_2$.

In similar way, one can prove that $f(x_1,0,x_3,0,\dots,0) = a + b_1x_1 + b_3x_3$. Now, denote

$$\begin{aligned}
\alpha &= f(x_1,0,\dots,0) = a + b_1x_1 \\
\beta &= f(x_1,1,0,\dots,0) = a + b_1x_1 + b_2 \\
\gamma &= f(x_1,0,1,0,\dots,0) = a + b_1x_1 + b_3 \quad \text{and} \\
\delta &= f(x_1,1,1,0,\dots,0)
\end{aligned}$$

Similarly as in the case of $f(x_1,x_2,0,\dots,0)$, we can show that

$$f(x_1,x_2,x_3,0,\dots,0) = \alpha + (\beta-\alpha)\,x_2 + (\gamma-\alpha)\,x_3 + (\delta-\gamma-\beta+\alpha)\,x_2x_3,$$

and $f\left(x_1,\frac{1}{2},\frac{1}{2},0,\dots,0\right) = \frac{\alpha+\delta}{2} = \frac{\beta+\gamma}{2}$. Then

$$f(x_1,x_2,x_3,0,\dots,0) = a + b_1x_1 + b_2x_2 + b_3x_3.$$

By induction we get $f(x_1,\dots,x_n) = a + \sum_{i=1}^{n} b_ix_i$, i.e., f is a linear function. Its extremal values on $[0,1]^n$ are $a + \sum_{b_i<0} b_i$ and $a + \sum_{b_i>0} b_i$, and thus f is a fusion function only if $a + \sum_{b_i<0} b_i \geq 0$ and $a + \sum_{b_i>0} b_i \leq 1$.
Clearly, if f is semi-aggregation function then $f(\mathbf{0}) = a = 0$ and $f(\mathbf{1}) = a + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} b_i = 1$ and $a + \sum_{b_i<0} b_i = \sum_{b_i<0} b_i \geq 0$ ensures there is no negative b_i, i.e., each $b_i \geq 0$. Obviously, then f is an aggregation function. □

Based on Theorem 2.6 we see that the only total end-point linear (semi-) aggregation functions are just weighted arithmetic means.

Example 2.7. The functions max and min are $\mathbf{c}$-homogeneous aggregation functions for any $\mathbf{c} = (c,\dots,c)$, $c \in [0,1]$, but not $\mathbf{z}$-homogeneous whenever $\mathbf{z}$ is non-constant. Consider, e.g., $n = 2, \mathbf{z} = (1,0)$ and suppose max (min) is $\mathbf{z}$-homogeneous. Then, knowing that $\max(1,0) = \max(0,1) = 1$ ($\min(1,0) = \min(0,1) = 0$), from the $\mathbf{z}$-homogeneity it follows $\max(x,1-x) = 1$ ($\min(x,1-x) = 0$) for each $x \in [0,1]$, which is a contradiction.

Observe that $\mathbf{c}$-homogeneity, $c \in [0,1]$, holds for any Choquet integral [5, 6, 9].

Remark 2.8. *Observe that if $\mathbf{z} \in]0,1[^n$, then any $\mathbf{z}$-homogeneous fusion function f is determined by the value $f(\mathbf{z})$ and values of f on the boundary points, i.e., on $[0,1]^n \setminus]0,1[^n$. If $\mathbf{z} \notin]0,1[^n$, then even a proper subset of $[0,1]^n \setminus]0,1[^n$ is enough to be considered. For example, consider $\mathbf{z} = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$ and $f(\mathbf{z}) = \alpha$, $f(\mathbf{u}) = \beta$ for any $\mathbf{u} \in [0,1]^n \setminus]0,1[^n$, $\alpha, \beta \in [0,1]$. Then the related $\mathbf{z}$-homogeneous fusion function $f : [0,1]^n \to [0,1]$ is given by*

$$f(\mathbf{x}) = \alpha + (\beta - \alpha) \max(|2x_1 - 1|, \dots, |2x_n - 1|).$$

For $n = 2$, we continue in Remark 2.8 and show the link between the values of a $\mathbf{z}$-homogenous fusion functions and the values of f on the boundary $[0,1]^2 \setminus]0,1[^2$ and in $\mathbf{z}$. For $\mathbf{z} \in]0,1[^2$, $\mathbf{z}$-homogeneous binary functions f are fully determined by $f(\mathbf{z})$ and its four boundaries that is, $f(0,\cdot), f(\cdot,0), f(1,\cdot)$ and $f(\cdot,1)$.

Proposition 2.9. *Let $\mathbf{z} = (z_1, z_2) \in]0,1[^2$ and $f : [0,1]^2 \to [0,1]$ be $\mathbf{z}$-homogeneous. Then, if $(x_1, x_2) \neq \mathbf{z}$, f is such that, it holds*

(i) for (x_1, x_2) from the triangle $\langle \mathbf{0}, \mathbf{z}, (1,0) \rangle$,

$$f(x_1, x_2) = \frac{x_2}{z_2} f(\mathbf{z}) + \frac{z_2 - x_2}{z_2} f\left(\frac{x_1 z_2 - x_2 z_1}{z_2 - x_2}, 0\right);$$

(ii) for (x_1, x_2) from the triangle $\langle \mathbf{0}, \mathbf{z}, (0,1) \rangle$,

$$f(x_1, x_2) = \frac{x_1}{z_1} f(\mathbf{z}) + \frac{z_1 - x_1}{z_1} f\left(0, \frac{x_2 z_1 - x_1 z_2}{z_1 - x_1}\right);$$

(iii) for (x_1, x_2) from the triangle $\langle (0,1), \mathbf{z}, \mathbf{1} \rangle$,

$$f(x_1, x_2) = \frac{1 - x_2}{1 - z_2} f(\mathbf{z}) + \frac{x_2 - z_2}{1 - z_2} f\left(\frac{x_1 - z_1 - x_1 z_2 + x_2 z_1}{x_2 - z_2}, 1\right);$$

(iv) for (x_1, x_2) from the triangle $\langle (1,0), \mathbf{z}, \mathbf{1} \rangle$,

$$f(x_1, x_2) = \frac{1 - x_1}{1 - z_1} f(\mathbf{z}) + \frac{x_1 - z_1}{1 - z_1} f\left(1, \frac{x_2 - z_2 - x_2 z_1 + x_1 z_2}{x_1 - z_1}\right).$$

The proof is a matter of simple linear interpolation and therefore omitted.

3 2-dimensional end-point linear aggregation functions

Now, we focus on binary semi-aggregation functions with linear boundaries, i.e., functions $f : [0,1]^2 \to [0,1]$ which are linear on segments $\langle \mathbf{0}, (0,1) \rangle, \langle \mathbf{0}, (1,0) \rangle, \langle \mathbf{1}, (0,1) \rangle$ and $\langle \mathbf{1}, (1,0) \rangle$. Clearly, then $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$, and $f(1,0) = a, f(0,1) = b$ for some constants $a, b \in [0,1]$. Due to the linearity on boundaries, $f(x_1, 0) = a x_1, f(0, x_2) = b x_2, f(x_1, 1) = b + (1-b) x_1$ and $f(1, x_2) = a + (1-a) x_2$, see Figure 1. For the sake of brevity, f will be then called $\langle a, b \rangle$*-boundary linear function.*

As a particular $\langle a, b \rangle$-boundary linear function we recall the Choquet integral [5]. Note that the Choquet integral $\mathcal{C}_m$ with respect to a capacity m such that $m(\{1\}) = a$ and $m(\{2\}) = b$, is a function $\mathcal{C}_m : [0,1]^2 \to [0,1]$ given by

$$\mathcal{C}_m(x_1, x_2) = \begin{cases} a x_1 + (1-a) x_2 & \text{if } x_1 \geq x_2, \\ (1-b) x_1 + b x_2 & \text{otherwise.} \end{cases}$$

The next result relates the Choquet integrals and $\mathbf{0}$-homogeneous ($\mathbf{1}$-homogeneous) $\langle a, b \rangle$-boundary linear functions.

Proposition 3.1. *The semi-aggregation function $f : [0,1]^2 \to [0,1]$ is $\langle a, b \rangle$-boundary linear for some $a, b \in [0,1]$ and $\mathbf{0}$-homogeneous (or $\mathbf{1}$-homogeneous) if and only if $f = \mathcal{C}_m$ is the Choquet integral with respect to a capacity m such that $m(\{1\}) = a$ and $m(\{2\}) = b$.*

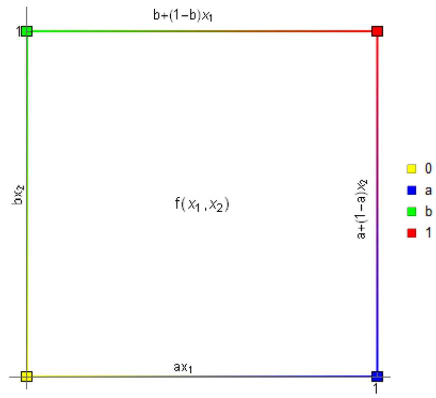

Figure 1: The $\langle a,b\rangle$-boundary linear function

Proof. Obviously, $\mathcal{C}_m$ is linear on triangles determined by points $(0,0),(1,1),(1,0)$, and $(0,0),(1,1),(0,1)$, which ensures the $\langle a,b\rangle$-boundary linearity, $\mathbf{0}$-homogeneity and $\mathbf{1}$-homogeneity of $f=\mathcal{C}_m$.
On the other hand, suppose that f is $\langle a,b\rangle$-boundary linear function which is $\mathbf{0}$-homogeneous (i.e., positively homogeneous).
Consider $(x_1,x_2)\in[0,1]^2$ such that $x_1\geq x_2>0$.
Then $(x_1,x_2)\in\left\langle(0,0),\left(1,\frac{x_2}{x_1}\right)\right\rangle$, $(x_1,x_2)=(1-x_1)(0,0)+x_1\left(1,\frac{x_2}{x_1}\right)$, and thus

$$f(x_1,x_2)=x_1 f\left(1,\frac{x_2}{x_1}\right)=x_1\left(a+(1-a)\frac{x_2}{x_1}\right)=ax_1+(1-a)x_2=\mathcal{C}_m(x_1,x_2).$$

Similarly, if $0<x_1\leq x_2$ then

$$f(x_1,x_2)=(1-b)x_1+bx_2=\mathcal{C}_m(x_1,x_2).$$

Summarizing, we have $f=\mathcal{C}_m$. □

Proposition 3.1 can be modified replacing the $\langle a,b\rangle$-boundary linearity and $\mathbf{0}$-homogeneity by $\mathbf{z}$-homogeneity for any $\mathbf{z}=(z,z)$, $z\in[0,1]$.

When considering $\mathbf{z}$-homogeneous $\langle a,b\rangle$-boundary linear functions on $[0,1]^2$, then each such f is linear on (possibly degenerated) triangles determined by point $\mathbf{z}$ and two neighbouring vertices of $[0,1]^2$ square. In general, denote $f(\mathbf{z})=c$ (clearly, $f(0,1)=b$, similarly $f(1,0)=a$.) Then f is univocally determined by $\mathbf{z}$ and parameters a,b,c but it need not be, in general, increasing and thus not aggregation function.

Theorem 3.2. *Let $f:[0,1]^2\to[0,1]$ be an $\langle a,b\rangle$-boundary linear function which is $\mathbf{z}$-homogeneous and $f(\mathbf{z})=c$. Then f is a semi-aggregation function such that*

(i) for (x_1,x_2) from the triangle $\langle\mathbf{0},\mathbf{z},(1,0)\rangle$,

$$f(x_1,x_2)=ax_1+\frac{x_2(c-az_1)}{z_2},$$

(if $z_2=0$, the degenerated triangle $\langle\mathbf{0},\mathbf{z},(1,0)\rangle$ coincides with the segment $\langle\mathbf{0},(1,0)\rangle$ and then $f(x_1,0)=ax_1$);

(ii) for (x_1,x_2) from the triangle $\langle\mathbf{0},\mathbf{z},(0,1)\rangle$,

$$f(x_1,x_2)=\frac{x_1(c-bz_2)}{z_1}+bx_2,$$

(if $z_1=0$, the degenerated triangle $\langle\mathbf{0},\mathbf{z},(1,0)\rangle\equiv\langle\mathbf{0},(0,1)\rangle$ and then $f(0,x_2)=bx_2$);

(iii) for (x_1,x_2) from the triangle $\langle(0,1),\mathbf{z},\mathbf{1}\rangle$,

$$f(x_1,x_2)=(1-b)x_1+\frac{(x_2-1)(b+(1-b)z_1-c)}{1-z_2}+b,$$

(if $z_2=1$, the degenerated triangle $\langle(0,1),\mathbf{z},\mathbf{1}\rangle\equiv\langle(0,1),\mathbf{1}\rangle$ and then $f(x_1,1)=b+(1-b)x_1$);

(*iv*) *for* (x_1, x_2) *from the triangle* $\langle (1,0), \mathbf{z}, \mathbf{1} \rangle$,

$$f(x_1, x_2) = \frac{(x_1 - 1)(a + (1-a)z_2 - c)}{1 - z_1} + (1-a)x_2 + a,$$

(if $z_1 = 1$*, the degenerated triangle* $\langle (1,0), \mathbf{z}, \mathbf{1} \rangle \equiv \langle (1,0), \mathbf{1} \rangle$ *and then* $f(1, x_2) = a + (1-a)x_2$*).*

See, Figure 2.

Proof. The function f is linear on each of 4 discussed triangles and thus to determine $f(x_1, x_2)$ it is enough to find the triangle containing (x_1, x_2). Then $f(x_1, x_2)$ is a convex combination of values of f on the related triangle vertices corresponding to the convex combination of related vertices resulting into (x_1, x_2). As all 4 vertices of the square $[0,1]^2$ have value from the set $\{0, 1, a, b\} \subset [0,1]$, f is a fusion function. More, due to $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$ we see that f is a semi-aggregation function. □

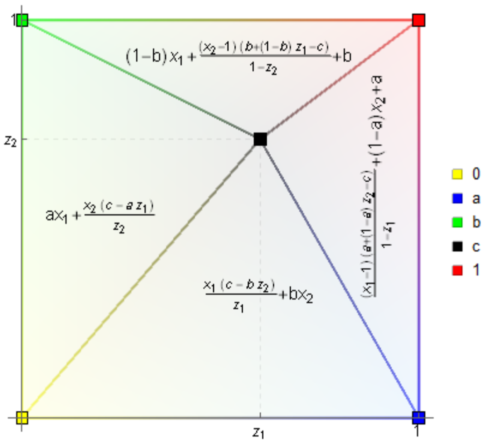

Figure 2: The semi-aggregation function

Based on Theorem 3.2, it is not difficult to see when the discussed f is an aggregation function.

Corollary 3.3. *Under the constraints of Theorem 3.2, f is an aggregation function if and only if*

$$c \geq az_1 \quad \text{and} \quad c \geq bz_2 \quad \text{and} \quad c \leq a + z_2 - az_2 \quad \text{and} \quad c \leq b + z_1 - bz_1.$$

Considering some particular subclasses of aggregation functions, note that the only boundary linear (weighted) quasi-arithmetic means are weighted arithmetic means $W_{(w,1-w)}$ given by

$$W_{(w,1-w)}(x_1, x_2) = wx_1 + (1-w)x_2, \qquad w \in [0,1],$$

which are $\langle w, 1-w \rangle$-boundary linear and $\mathbf{z}$-homogeneous for an arbitrary $\mathbf{z} \in [0,1]^2$ and $c = f(\mathbf{z}) = wz_1 + (1-w)z_2$.

Next we focus on semicopulas and their subclasses quasi-copulas, copulas and triangular norms. Recall that an aggregation function $f : [0,1]^2 \to [0,1]$ is *a semicopula* whenever 1 is its neutral element, $f(x,1) = f(1,x)$ for all $x \in [0,1]$. Obviously, semicopulas are just $\langle 0, 0 \rangle$- boundary linear aggregation functions. Next, a semicopula $f : [0,1]^2 \to [0,1]$ is *a quasi-copula* if it is 1-Lipschitz, i.e.,

$$|f(x_1, x_2) - f(y_1, y_2)| \leq |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } (x_1, x_2), (y_1, y_2) \in [0,1]^2.$$

A quasi-copula $f : [0,1]^2 \to [0,1]$ is *a copula* if it is supermodular, i.e.

$$f((x_1, x_2) \vee (y_1, y_2)) + f((x_1, x_2) \wedge (y_1, y_2)) \geq f(x_1, x_2) + f(y_1, y_2),$$

for all $(x_1, x_2), (y_1, y_2) \in [0,1]^2$. Finally, a semicopula f is *a triangular norm* if it is symmetric (commutative) and associative. For more details we recommend [7].

Proposition 3.4. *A $\langle 0,0\rangle$-boundary linear $\mathbf{z}$-homogeneous function $f:[0,1]^2\to[0,1]$, $f(\mathbf{z})=c$, is a semicopula if and only if $c\le \min(z_1,z_2)$.*

Proof. For any semicopula f and $(x_1,x_2)\in[0,1]^2$, it holds $f(x_1,x_2)\le\min(x_1,x_2)$, and thus $f(\mathbf{z})=c\le\min(z_1,z_2)$, showing the necessity. To see the sufficiency, note that due to Corollary 3.3, the above considered f is an aggregation function if and only if $c\le\min(z_1,z_2)$. Also, due to $a=0$ we have $f(1,y)=y$ for all $y\in[0,1]$. Similarly, due to $b=0$, it holds $f(x,1)=x$ for all $x\in[0,1]$, and thus 1 is a neutral element of f. Hence, f is a semicopula. □

Example 3.5. *Consider $\langle 0,0\rangle$-boundary linear (k,k)-homogeneous semicopula $f_{k,c}$, where $c\in[0,k]$. Then, for the greatest $c=k$,*

$$f_{k,k}(x_1,x_2)=\min(x_1,x_2),$$

i.e., $f_{k,k}$ is also a quasi-copula, a copula and a t-norm. On the other hand, for the smallest $c=0$, it holds

$$f_{k,0}(x_1,x_2)=\begin{cases}0 & \text{if } x_2\le\min\left(1+\frac{k-1}{k}x_1,\frac{k}{k-1}x_1-\frac{k}{k-1}\right),\\ \min(x_1,x_2)+\frac{k}{1-k}\max(x_1,x_2)-\frac{k}{1-k} & \text{otherwise.}\end{cases}$$

Then $f_{k,0}$ is a semicopula for each $k\in[0,1[$, it is quasicopula and copula for each $k\in\left[0,\frac12\right]$ and it is a t-norm only for $k\in\left\{0,\frac12\right\}$. Note that $f_{0,0}=\min$ and $f_{\frac12,0}=T_L$ is the Lukasziewicz t-norm given by

$$T_L(x_1,x_2)=\max(x_1+x_2-1,0).$$

Theorem 3.6. *For a $\langle 0,0\rangle$-boundary linear $\mathbf{z}$-homogeneous function $f:[0,1]^2\to[0,1]$, $f(\mathbf{z})=c$, the following are equivalent:*

(*i*) *f is a quasi-copula;*

(*ii*) *f is a copula;*

(*iii*) $\max(z_1+z_2-1,0)\le c\le\min(z_1,z_2)$.

Proof. Recall that f is a semicopula if and only if $c\le\min(z_1,z_2)$, see Proposition 3.4. Then f is a quasi-copula only if it is 1-Lipschitz on each of 4 triangles determined by $\mathbf{z}$ considered in Theorem 3.2. As on each of these triangles f is linear (and increasing in both coordinates), it is 1-Lipschitz if and only if the coefficients by z_1 and by z_2 are bounded from above by 1. Hence, f is a quasi-copula if and only if

$$c\le\min(z_1,z_2)\quad\text{and}\quad\frac{z_2-c}{1-z_1}\le 1,(\text{i.e., } c\ge z_1+z_2-1)\quad\text{and}\quad\frac{z_1-c}{1-z_2}\le 1,(\text{i.e., } c\ge z_1+z_2-1).$$

Summarizing, we see that (i) and (iii) are equivalent. Next, each copula is also a quasi-copula, hence (ii) implies (i) and (iii).
On the other hand suppose that (iii) holds. Recall that the supermodularity of a function $f:[0,1]^2\to[0,1]$ trivially holds if (x_1,x_2) and (y_1,y_2) are comparable, and hence we need to discuss the case when (x_1,x_2) and (y_1,y_2) are incomparable only. Suppose $x_1<y_1$ and $x_2>y_2$. Then f is supermodular only if

$$f(y_1,x_2)+f(x_1,y_2)\ge f(x_1,x_2)+f(y_1,y_2),$$

i.e., if $V_f(R)\ge 0$, where R is the rectangle $[x_1,y_2]\times[y_1,x_2]$ and $V_f(R)$ is its volume given by

$$V_f(R)=f(y_1,x_2)+f(x_1,y_2)-f(x_1,x_2)-f(y_1,y_2).$$

Note that if f is linear on some domain D, then $V_f(R)=0$ for any $R\subseteq D$. Also, if $R=R_1\cup R_2$ for some non-overlapping rectangles R_1 and R_2 (i.e., $R_1\cap R_2$ has Lebesque measure 0), then $V_f(R)=V_f(R_1)+V_f(R_2)$. These facts allow to restrict our considerations to special rectangles only, namely to $R=[x_1,y_2]\times[y_1,x_2]$ with diagonal segment $\langle(x_1,y_2),(y_1,x_2)\rangle$ which is a subset of segment $\langle\mathbf{z},\mathbf{u}\rangle$, where $\mathbf{u}\in\{(0,0),(1,0),(0,1),(1,1)\}$. It is not difficult to check the next 4 cases:

i) if $\langle(x_1,y_2),(y_1,x_2)\rangle\subseteq\langle\mathbf{z},\mathbf{0}\rangle$ then

$$V_f(R)=\frac{c(x_2-y_2)}{z_2}=\frac{c(y_1-x_1)}{z_1}\ge 0;$$

ii) if $\langle (x_1, y_2), (y_1, x_2)\rangle \subseteq \langle \mathbf{z}, (1,0)\rangle$ then

$$V_f(R) = (z_2 - c)\frac{x_2 - y_2}{z_2} \geq 0, \quad \text{(note that } c \leq z_2).$$

iii) if $\langle (x_1, y_2), (y_1, x_2)\rangle \subseteq \langle \mathbf{z}, (0,1)\rangle$ then

$$V_f(R) = (z_1 - c)\frac{y_1 - x_1}{z_1} \geq 0, \quad \text{(note that } c \leq z_1).$$

iv) if $\langle (x_1, y_2), (y_1, x_2)\rangle \subseteq \langle \mathbf{z}, \mathbf{1}\rangle$ then

$$V_f(R) = (1 - z_1 - z_2 + c)\frac{x_2 - y_2}{1 - z_2} = (1 - z_1 - z_2 + c)\frac{y_1 - x_1}{1 - z_1} \geq 0,$$

(note that $c \geq \max(z_1 + z_2 - 1, 0)$ and thus $1 - z_1 - z_2 + c \geq 0$).

Summarizing, we see that $V_f(R) \geq 0$ for any rectangle $R \subseteq [0,1]^2$, and thus f is a copula. □

Remark 3.7. *Due to Theorem 3.6, each $\langle 0, 0\rangle$-boundary linear $\mathbf{z}$-homogeneous function $f : [0,1]^2 \to [0,1]$, $f(\mathbf{z}) = c \in [T_L(z_1, z_2), \min(z_1, z_2)]$ is a copula. This copula is singular and its support with the corresponding masses is formed by 4 segments (possibly degenerated):*

- $\langle \mathbf{0}, \mathbf{z}\rangle$ *with mass c uniformly distributed over this segment;*
- $\langle \mathbf{z}, (0,1)\rangle$ *with mass $z_1 - c$ uniformly distributed over this segment;*
- $\langle \mathbf{z}, (1,0)\rangle$ *with mass $z_2 - c$ uniformly distributed over this segment;*
- $\langle \mathbf{z}, \mathbf{1}\rangle$ *with mass $1 - z_1 - z_2 + c$ uniformly distributed over this segment.*

As we have seen, the class of $\langle a, b\rangle$-boundary linear $\mathbf{z}$-homogeneous copulas is quite rich. This is not the case of triangular norms as shown in the next theorem.

Theorem 3.8. *An $\langle a, b\rangle$-boundary linear function f which is $\mathbf{z}$-homogeneous for some $\mathbf{z} \in [0,1]^2$ is a t-norm if and only if $f \in \{T_L, \min\}$.*

Proof. The sufficiency is obvious. Indeed, the Łukasiewicz t-norm T_L is $\langle 0, 0\rangle$-boundary linear and $\mathbf{z} = (z, 1 - z)$-homogeneous for any $z \in [0,1]$ and $f(\mathbf{z}) = 0$. Similarly, the strongest t-norm, min, is $\langle 0, 0\rangle$-boundary linear and $\mathbf{z} = (z, z)$-homogeneous with $c = z, z \in [0,1]$.
The necessity is more tricky. Obviously, each t-norm f is $\langle 0, 0\rangle$-boundary linear. Suppose it is $\mathbf{z}$-homogeneous and $f(\mathbf{z}) = c$. Clearly, $c \in [0, \min(z_1, z_2)]$. Suppose first $z_1 \leq z_2$.

If $z_2 = 1$ then necessarily $f(\mathbf{z}) = z_1 = c$ and f is symmetric and t-norm only if $z_1 = c = 1$ (then $f = \min$).

If $z_2 < 1$, we will consider x sufficiently large, close to 1. Then

$$x^{(2)} = f(x, x) = u = x\,\frac{1 - z_1 + z_2 - c}{1 - z_1} - \frac{z_2 - c}{1 - z_1},$$

and

$$x^{(4)} = u^{(2)} = f(u, u) = u\,\frac{1 - z_1 + z_2 - c}{1 - z_1} - \frac{z_2 - c}{1 - z_1} = x\left(\frac{1 - z_1 + z_2 - c}{1 - z_1}\right)^2 - \left(\frac{z_2 - c}{1 - z_1}\right)\left(2 + \frac{z_2 - c}{1 - z_1}\right).$$

On the other hand,

$$x^{(3)} = f(x, x^{(2)}) = f(x, u) = u + \frac{(x - 1)(z_2 - c)}{1 - z_1} = x\left(1 + 2\,\frac{z_2 - c}{1 - z_1}\right) - 2\,\frac{z_2 - c}{1 - z_1}$$

and, due to the associativity of t-norms,

$$x^{(4)} = f(x, x^{(3)}) = x^{(3)} + \frac{(x - 1)(z_2 - c)}{1 - z_1} = x\left(1 + 3\,\frac{z_2 - c}{1 - z_1}\right) - 3\,\frac{z_2 - c}{1 - z_1}.$$

This two expressions for $x^{(4)}$ result into the next two equalities:

$$\left(\frac{1-z_1+z_2-c}{1-z_1}\right)^2 = 1+3\,\frac{z_2-c}{1-z_1} \quad \text{and} \quad 3\,\frac{z_2-c}{1-z_1} = 2\,\frac{z_2-c}{1-z_1} = \left(\frac{z_2-c}{1-z_1}\right)^2.$$

Hence $\left(\frac{z_2-c}{1-z_1}\right)^2 = \frac{z_2-c}{1-z_1}$ which implies either $c = z_2$ (and then necessarily $z_1 = z_2 = c$) or $c = z_1 + z_2 - 1$.

On the other hand, if $z_1 = 0$, then necessarily $c = 0$ and $f = \min$ if $z_2 = 0$. If $z_2 > 0$ in this case, then f is not symmetric and thus not a t-norm.

Suppose $z_1 > 0$. For any $x \in]0, z_1]$ it holds

$$x^{(2)} = u = \frac{xc}{z_2},\ x^{(4)} = u^{(2)} = \frac{uc}{z_2} = x\left(\frac{c}{z_2}\right)^2.$$

More,

$$x^{(3)} = f(x,u) = \frac{uc}{z_2} = x\left(\frac{c}{z_2}\right)^2 \quad \text{and} \quad x^{(4)} = f(x, x^(3)) = \frac{x^{(3)}c}{z_2} = x\left(\frac{c}{z_2}\right)^3.$$

Thus $\left(\frac{c}{z_2}\right)^2 = \left(\frac{c}{z_2}\right)^3$, which implies either $c = 0$ or $c = z_2$ (and hence $z_1 = z_2 = c$).
Summarizing, we see that

$$c = z_1 = z_2 \quad \text{or} \quad c = z_1 + z_2 - 1, \quad \text{and} \quad c = 0 \quad \text{or} \quad c = z_2 = z_1.$$

Then either $c = z_1 = z_2$ and the resulting $f = \min$, or $c = 0 = z_1 + z_2 - 1$, and then $\mathbf{z} = (z_1, 1 - z_1)$, $z_1 \in \left]0, \frac{1}{2}\right]$, $f(\mathbf{z}) = 0$ and $f = T_L$.

Similar results are obtained if we suppose $z_1 \geq z_2$ (then we have the case $\mathbf{z} = (z_1, 1 - z_1)$ for $z_1 \in \left]\frac{1}{2}, 1\right]$). In all possible situations, the only possibilities are $f = \min$ or $f = T_L$. □

Remark 3.9. *Recall that if f is a t-norm then its dual f^d is a t-conorm. Due to Theorem 3.8 we see that an $\langle a, b\rangle$-boundary linear $\mathbf{z}$-homogeneous function $f : [0,1]^2 \to [0,1]$ is a t-conorm if and only if $f \in \{S_L, \max\}$, (S_L is the Lukasziewicz t-conorm given by $S_L(x_1, x_2) = \min(x_1 + x_2, 1)$). Note that the t-conorm* max *is $\langle 1, 1\rangle$-boundary linear and (z, z)-homogeneous with $c = z, z \in [0,1]$. Similarly, S_L is $\langle 1, 1\rangle$-boundary linear and $(z, 1 - z)$-homogeneous with $c = 1, z \in [0,1]$.*

Recently, overlap and grouping functions were introduced, especially for applications in image processing [2].

Definition 3.10. *A fusion function $f : [0,1]^2 \to [0,1]$ is called an overlap function whenever*

(*i*) *f is continuous and symmetric;*

(*ii*) *f is increasing in both coordinates;*

(*iii*) *$f(x, y) = 0$ if and only if $x = 0$ or $y = 0$;*

(*iv*) *$f(x, y) = 1$ if and only if $x = y = 1$.*

Dual function $f^d : [0,1]^2 \to [0,1]$ to an overlap function f is called a grouping function.

Obviously, for any overlap function f it holds $f(1, 0) = f(0, 1) = 0$, and thus the particular subclass of overlap functions belong to the class of $\langle 0, 0\rangle$-boundary linear aggregation functions. Clearly, these overlap functions are symmetric semicopulas, too. Thus, based on Proposition 3.4, we have the next result.

Corollary 3.11. *A $\langle 0, 0\rangle$-boundary linear $\mathbf{z}$-homogeneous function $f : [0,1]^2 \to [0,1]$, is an overlap function if and only if $\mathbf{z} = (z, z), z \in [0,1]$ and $c = z$ if $z \in \{0, 1\}$, and $c \in]0, z[$ if $z \in]0, 1[$. Then $f = f_z$, where*

$$f_z(x,y) = \begin{cases} \min(x,y) & \text{if } c = z \\ \max\left(\frac{c}{z}\cdot\min(x,y), \min(x,y) + \frac{z-c}{1-z}\cdot\max(y-1, x-1)\right) & \text{else,} \end{cases}$$

see Figure 3.

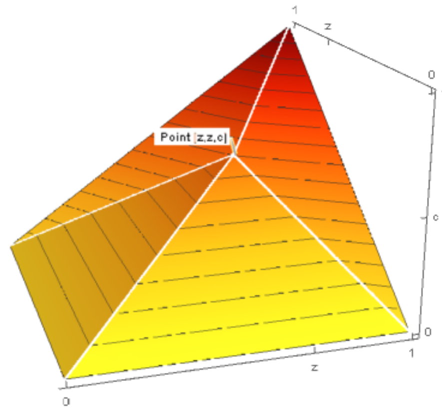

Figure 3: The **z**-homogeneous overlap function

Proof. The result that $f = \min$ in the case when $\mathbf{z} = (z, z)$ and $c = z$ is trivial. Suppose that f is (z_1, z_2)-homogeneous, $f(\mathbf{z}) = c$. Due to the symmetry, f is also (z_2, z_1)-homogeneous, and $f(z_2, z_1) = c$. Due to properties of overlap functions, $\mathbf{z} \in]0, 1[^2$ or $\mathbf{z} \in \{\mathbf{0}, \mathbf{1}\}$. Consider $\mathbf{z} \in]0, 1[^2$. Clearly, $f(\mathbf{z}) = c > 0$.
As our f is $\langle 0, 0\rangle$-boundary linear semicopula, we have 4 possibilities how to evaluate $f(z_2, z_1)$, depending on the triangle with vertex $\mathbf{z}$ considered in Theorem 3.2 where (z_2, z_1) belongs. Thus either

i) $\dfrac{z_2\, c}{z_1} = c$ implying $z_1 = z_2 = z$;

ii) $\dfrac{z_1\, c}{z_2} = c$ implying $z_1 = z_2$;

iii) $z_2 + \dfrac{(z_1 - 1)(z_1 - c)}{1 - z_2} = c$ implying $z_1 = z_2$ or $c = z_1 + z_2 - 1$, and thus also $z_1 + z_2 > 1$ (else the axiom (iii) from Definition 3.10 of overlap functions will be violated);

iv) $z_1 + \dfrac{(z_2 - 1)(z_2 - c)}{1 - z_1} = c$, leading to the same conclusions as in the case iii).

Suppose $z_1 + z_2 > 1$ and $f(z_1, z_2) = f(z_2, z_1) = c = z_1 + z_2 - 1 > 0$. Due to **z**-homogeneity of f, $f\left(\dfrac{z_1}{2}, \dfrac{z_2}{2}\right) = \dfrac{c}{2}$. On the other hand, f is also (z_2, z_1)-homogeneous and then $f\left(\dfrac{z_1}{2}, \dfrac{z_2}{2}\right) = c\,\dfrac{z_1}{2z_2}$, implying $z_1 = z_2$. Summarizing necessity $\mathbf{z} = (z, z)$ for $z \in [0, 1]$ and $c = 0$ if $z = 0$ (then $f = \min$), $c = 1$ if $z = 1$ (then also $f = \min$) and $c \in]0, z]$ if $z \in]0, 1[$. □

Corollary 3.11 has introduced a 2-parametric family (f_z, c) of overlap functions, where $(z, c) \in \{(0, 0), (1, 1), (u, v) | u \in]0, 1[, v \in]0, u]\}$. Due to duality of overlap and grouping functions, one can characterize all $\langle 1, 1\rangle$-boundary linear **z**-homogenous grouping functions, with $\mathbf{z} = (z, z), z \in [0, 1]$, where $z = c$ if $z \in \{0, 1\}$, and else $c \in [z, 1[$.

4 Concluding remarks

We have introduced and discussed **z**-homogeneous fusion functions, in particular aggregation functions and semi-aggregation functions. Our approach generalizes the positive homogeneity of functions, which in our terminology, for functions vanishing in $\mathbf{0}$ (i.e., $f(\mathbf{0}) = 0$), is just the $\mathbf{0}$-homogeneity. We expect applications of our results in several engineering domains and physics where the end-point linearity related to one fixed end-point **z** is considered. Then the **z**-homogeneity, possibly with some other given properties, allows to build consistent models of real world dependencies, requiring few accurate measurements only. This fact was exemplified in the case of binary (semi-)aggregation functions $f : [0, 1]^2 \to [0, 1]$ supposing the boundary linearity. We have completely characterized several particular classes of binary aggregation functions which are boundary linear and **z**-homogenous, such as semicopulas, quasi-copulas, copulas, triangular norms, overlap functions. Then also dual aggregation functions are completely characterized what was exemplified on t-conorms and grouping functions.

Our approach can be helpful also when constructing some other types of fusion functions, such as fuzzy implications, co-implications, restricted dissimilarity functions, etc. For example, consider boundary linear $\mathbf{z}$-homogenous fuzzy implication functions $I : [0,1]^2 \to [0,1]$ (for more details see [1]). Recall that I is a fuzzy implication if it extends the classical Boolean implication, i.e., $I(0,0) = I(0,1) = I(1,1) = 0$ and $I(1,0) = 0$, and it is decreasing in the first coordinate and increasing in the second coordinate. Clearly, if I is boundary linear, then $I(0,y) = I(x,1) = 1, I(x,0) = 1-x$ and $I(1,y) = y$ for all $x, y \in [0,1]$. Then the function $f : [0,1]^2 \to [0,1]$ given by

$$f(x,y) = 1 - I(x, 1-y),$$

is a semicopula. If I is also $\mathbf{z}$-homogeneous, $\mathbf{z} = (z_1, z_2), I(\mathbf{z}) = c$, then f is $(z_1, 1-z_2)$-homogeneous and $f(z_1, 1-z_2) = 1-c$. Based on Proposition 3.4, $1-c \le \min(z_1, 1-z_2)$, i.e., $c \ge \max(1-z_1, z_2)$.
For example, consider $z_1 = z_2 = c = \frac{1}{2}$. Then the corresponding fuzzy implication I which is $\mathbf{z}$-homogeneous and $I(\mathbf{z}) = \frac{1}{2}$ is just the Kleene-Dienes implication I_{KD} given by $I_{KD}(x,y) = \max(1-x, y)$.
Similarly, for any $\mathbf{z} = (z,z)$, $z \in [0,1]$, and $I(\mathbf{z}) = 1$, the corresponding $\mathbf{z}$-homogeneous implication I_L is given by $I_L(x,y) = \min(1, 1-x+y)$.

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