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A new definition of fuzzy k-pseudo metric and its induced fuzzifying structures

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Abstract

In this paper, a new definition of a fuzzy k-pseudo metric is introduced and its induced fuzzifying structures are constructed, such as a fuzzifying neighborhood system, a fuzzifying topology, a fuzzifying closure operator, a fuzzifying uniformity. Besides, it is shown that there is a one-to-one correspondence between fuzzy k-pseudo metrics and nests of crisp k-pseudo metrics.

Keywords: Nest of k-pseudo metric, fuzzy k-pseudo metric, fuzzifying topology, fuzzifying uniformity.

1 Introduction

Metric spaces play an important role in the research and applications of mathematics. A pseudo metric on a set X is a mapping $d : X \times X \to [0, \infty)$ satisfying: (d1) $d(x, x) = 0$; (d2) $d(x, y) = d(y, x)$; (d3) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality). If it satisfies a stronger version of the first axiom (d1)* $d(x, y) = 0 \Leftrightarrow x = y$, then d is called a metric. The pair (X, d) is called a metric space.

Inspired by the notion of probabilistic metric space which was presented by Menger [14] in 1942, Kramosil and Michalek [11] introduced the concept of KM-fuzzy (pseudo) metric in 1975, a mapping $M : X \times X \times [0, \infty) \to [0, 1]$ satisfying some conditions. In 1988, Grabiec [6] added the conditions $M(x, y, 0) = 0$ and $lim_{t\to 0} M(x, y, t) = 1$, so as to make the definition of KM-fuzzy (pseudo) metric more complete and reasonable. He showed that the KM-fuzzy metric is the semantic generalization of the crisp metric.

In order to get a Hausdorff topology from a fuzzy metric space, George and Veeramani [5] slightly modified the definition of KM-fuzzy (pseudo)metric and proposed a new notion of fuzzy metric, called GV-fuzzy (pseudo)metric in 1994. Subsequently, GV-fuzzy metric was widely concerned. As pointed out in [7], most results about topological properties do not depend on the modified conditions, and can be obtained under weaker conditions. Mardones Pérez and Prada Vicente [12] had shown that a Hausdorff topology can still be obtained from the KM-fuzzy metric.

As a generalization of metric spaces, the notion of a metric-type space was introduced by Bakhtin [1] in 1989, and later was independently rediscovered by Czerwik [4] under the name of b-metric space in 1993. In order to emphasize the special role of the constant k in the definition of a b-metric space, in case when k is fixed, A. šostak calls such spaces k-metric [19]. The concept of a k-metric generalize the notion of a metric by replacing the triangle inequality to a more general axiom: $d(x, z) \leq k(d(x, y) + d(y, z))$, where $k \geq 1$ is a fixed constant.

Recently, Hussain [10] et al. and Nădăban [16] introduced the concept of a fuzzy b-metric space and discussed the corresponding fixed point theorem. Based on the idea of GV-fuzzy metric [5], This class of spaces was independetly found by Šostak and studied in [19] under the name of fuzzy k-(pseudo) metric spaces. In this paper, he focused on studying two crisp structures induced by a fuzzy k-pseudo metric: topology and supratopology, and discussed their convergence, completeness and compactness in crisp topological spaces.

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The disadvantages of GV-fuzzy metric make the definition of Šostak's fuzzy k-pseudo metric a little defective. Firstly, it limits the definition to $(0,\infty)$, missing the conditions added by Grabiec [6]. Secondly, it changed the left continuity [11] into continuity, which makes fuzzy k-metric only a formal generalization, not a semantic generalization of crisp k-metric. Namely, when the value is limited to $\{0,1\}$, it cannot be reduced to a crisp k-metric. Thirdly, the crisp topology induced by a fuzzy k-pseudo metric is generated by the crisp open ball $B_M(x,r,t) = \{y \in X \mid M(x,y,t) > 1-r\}$. Thus in this way a fuzzy k-(pseudo)-metric induces a crisp topological-type structures and not fuzzy ones (see the similar situation with topology induced by GV-fuzzy mnetrics in [5], [7].). Also, he did not consider any fuzzy structures induced by a fuzzy k-pseudo metric.

The main aims of this paper are to introduce a new definition of fuzzy k-pseudo metric and discuss its induced fuzzy structures. Moreover, we shall show that there is a one-to-one correspondence between fuzzy k-pseudo metrics and nests of crisp k-pseudo metrics.

This paper is organized as follows. In Section 2, some necessary concepts of k-pseudo metric spaces and some conclusions of fuzzifying structures are recalled. In Section 3, a new definition of fuzzy k-pseudo metric and its relevant examples are introduced. In Section 4, fuzzifying structures induced by a fuzzy k-pseudo metric are presented, including a fuzzifying topology, a fuzzifying neighborhood system, a fuzzifying uniformity and a fuzzifying closure operator. In Section 5, the relationships between crisp k-pseudo metrics and fuzzy k-pseudo metrics are discussed. It is shown that there is a one-to-one correspondence between fuzzy k-pseudo metrics and nests of crisp k-pseudo metrics.

2 Preliminaries

Let X be a non-empty set. The concept of a k-metric is introduced as follows.

Definition 2.1. [1, 4, 19] *Let $k \geq 1$ be a fixed constant and let $d : X \times X \to [0,\infty)$ be a mapping satisfying $\forall x,y,z \in X$,*
$(D1)$ $d(x,x) = 0$;
$(D2)$ $d(x,y) = d(y,x)$;
$(D3)$ $d(x,z) \leq k(d(x,y) + d(y,z))$.
Then d is called a k-pseudo metric and the pair (X,d) is called a k-pseudo metric space. If the condition (D1) is replaced by a stronger axiom:
$(D1)^*$ $d(x,y) = 0 \Leftrightarrow x = y$;
then d is called a k-metric and the pair (X,d) is called a k-metric space.

If $k = 1$, then d is exactly the definition of pseudo metric. If $k < 1$, then d makes no sense.

Example 2.2. *Let $\mathbb{R}$ be the set of real numbers and let $d : X \times X \to [0,\infty)$ be a mapping defined by $\forall x,y \in \mathbb{R}$, $d(x,y) = |x-y|^2$. Then d is a 2-metric and d is not a metric. In fact, $|x-z|^2 \leq (|x-y|+|y-z|)^2 \leq 2(|x-y|^2+|y-z|^2)$. Similarly, let $(X, \| \ \|)$ be a normed space. There also exists 2-metric on X defined by $\forall x,y \in X$, $d(x,y) = \|x-y\|^2$.*

Example 2.3. *A series of k-pseudo metrics can be obtained from a crisp pseudo metric by the following construction. Let $k \geq 1$ be a fixed constant and let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous increasing mapping such that $\varphi(0) = 0$ and $\varphi(a+b) \leq k(\varphi(a) + \varphi(b))$ for any $a,b \in [0,\infty)$. And, let $\rho : X \times X \to [0,\infty)$ be a pseudo metric. Then the mapping $d_{\rho\varphi} : X \times X \to [0,\infty)$ defined by $\forall x,y \in X$,*

$$d_{\rho\varphi}(x,y) = (\varphi \circ \rho)(x,y),$$

is a k-pseudo metric. Indeed, conditions (D1) and (D2) for $d_{\rho\varphi}$ are trivial. It suffices to check (D3). For all $x,y,z \in X$, we know $d_{\rho\varphi}(x,z) = \varphi(\rho(x,z)) \leq \varphi(\rho(x,y) + \rho(y,z)) \leq k(\varphi(\rho(x,y)) + \varphi(\rho(y,z))) = k(d_{\rho\varphi}(x,y) + d_{\rho\varphi}(y,z))$.

Now, we will illustrate this construction by several specific examples.

(1) Let $\varphi(a) = a^2$. Obviously $(a+b)^2 \leq 2(a^2+b^2)$. In particular, $d_{\rho\varphi}(f,g) = \left(\int_a^b |f(x)-g(x)|dx\right)^2$ on the set of Lebesgue measurable functions on $[a,b]$.

(2) Let $\varphi(a) = a^{\frac{3}{2}}$. It is easy to see that $(a+b)^{\frac{3}{2}} \leq \sqrt{2}(a^{\frac{3}{2}} + b^{\frac{3}{2}})$. By defining $d_{\rho\varphi}(x,y) = \rho(x,y)^{\frac{3}{2}}$, we can obtain a $\sqrt{2}$-pseudo metric.

Example 2.4. *Let X be the set of Lebesgue measurable functions on $[a,b]$ such that $\int_a^b |f(x)|^2 dx < \infty$. Define $d : X \times X \to [0,\infty)$ by $\forall f,g \in X$, $d(f,g) = \int_a^b |f(x)-g(x)|^2 dx$. Then d is a 2-metric on X. But this 2-metric is not the one described in Example 2.3.*

In what follows, the notions of a fuzzifying neighborhood system, a fuzzifying topology, a fuzzifying uniformity are recalled.

Definition 2.5. [8, 21] *A fuzzifying topology on X is a mapping $\mathcal{T}: 2^X \to [0,1]$ satisfying the following conditions:*
$(FT1)$ $\mathcal{T}(\emptyset) = \mathcal{T}(X) = 1$;
$(FT2)$ $\forall A_1, A_2 \in 2^X$, $\mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \leq \mathcal{T}(A_1 \cap A_2)$;
$(FT3)$ $\forall \{A_j : j \in J\} \subseteq 2^X$, $\bigwedge_{j \in J} \mathcal{T}(A_j) \leq \mathcal{T}\left(\bigcup_{j \in J} A_j\right)$.

A continuous mapping from a fuzzifying topological space $(X, \mathcal{T}^X)$ to a fuzzifying topological space $(Y, \mathcal{T}^Y)$ is a mapping $f : X \to Y$ such that $\forall B \in 2^Y$, $\mathcal{T}^Y(B) \leq \mathcal{T}^X(f^{\leftarrow}(B))$, where $f^{\leftarrow}(B) = \{x \in X \mid f(x) \in B\}$. The category of fuzzifying topological spaces and their continuous mappings is denoted by FY-**Top**.

Definition 2.6. [21, 24] *A fuzzifying neighborhood system (or a generalized neighborhood system) on X is defined to be a set $\mathcal{N} = \{\mathcal{N}_x \mid x \in X\}$ of maps $\mathcal{N}_x : 2^X \to [0,1]$ satisfying $\forall x \in X$,*
$(FN1)$ $\mathcal{N}_x(\emptyset) = 0$, $\mathcal{N}_x(X) = 1$;
$(FN2)$ $\forall x \notin A$, $\mathcal{N}_x(A) = 0$;
$(FN3)$ $\mathcal{N}_x(A_1 \cap A_2) = \mathcal{N}_x(A_1) \wedge \mathcal{N}_x(A_2)$;
$(FN4)$ $\mathcal{N}_x(A) = \bigvee_{x \in B \subseteq A} \bigwedge_{y \in B} \mathcal{N}_y(B)$.

A continuous mapping from a fuzzifying neighborhood space $(X, \mathcal{N}^X)$ to a fuzzifying neighborhood space $(Y, \mathcal{N}^Y)$ is a mapping $f : X \to Y$ such that $\forall x \in X, \forall B \in 2^Y$, $\mathcal{N}^Y_{f(x)}(B) \leq \mathcal{N}^X_x(f^{\leftarrow}(B))$. The category of fuzzifying neighborhood spaces and their continuous mappings is denoted by FY-**NBS**. The category FY-**Top** is isomorphic to the category of FY-**NBS** [24].

Remark 2.7. [20] *Let $\mathcal{N} = \{\mathcal{N}_x \mid x \in X\}$ be a set satisfying conditions (FN1)-(FN3). Then (FN4) is equivalent to the following condition:*
*(FN4)** $\mathcal{N}_x(A) = \bigvee_{x \in B \subseteq A} (\mathcal{N}_x(B) \wedge \bigwedge_{y \in B} \mathcal{N}_y(A))$.

Definition 2.8. [22] *A fuzzifying uniformity on X is a mapping $\mathcal{U} : 2^{X \times X} \to [0,1]$ satisfying the following conditions:*
$(FU1)$ $\mathcal{U}(\emptyset) = 0$, $\mathcal{U}(X \times X) = 1$;
$(FU2)$ $\mathcal{U}(V) > 0 \Rightarrow \Delta \subseteq V$, *where Δ denotes the diagonal of $X \times X$.*
$(FU3)$ $\mathcal{U}(V) = \mathcal{U}(V^{-1})$, *where* $V^{-1} = \{(x,y) \mid (y,x) \in V\}$.
$(FU4)$ $\mathcal{U}(V \cap W) = \mathcal{U}(V) \wedge \mathcal{U}(W)$;
$(FU5)$ $\mathcal{U}(V) \leq \bigvee\{\mathcal{U}(W) \mid W \in 2^{X \times X}, W \circ W \subseteq V\}$, *where* $W \circ W = \{(x,z) \mid \exists y \in X, s.t., (x,y) \in W, (z,y) \in W\}$.

A continuous map from a fuzzifying uniform space $(X, \mathcal{U}^X)$ to a fuzzifying uniform space $(Y, \mathcal{U}^Y)$ is a map $f : X \to Y$ such that $\forall V \in 2^{Y \times Y}$, $\mathcal{U}^Y(V) \leq \mathcal{U}^X(f^{\Leftarrow}(V))$, where $f^{\Leftarrow}(V) = \{(x_1, x_2) \in X^2 \mid (f(x_1), f(x_2)) \in V\}$.

In the following, the concepts of fuzzifying closure operator and fuzzifying interior operator are presented and the category of fuzzifying closure spaces and the category of fuzzifying interior spaces are all isomorphic to the category FY-**Top**.

Definition 2.9. [17, 18] *A fuzzifying closure operator on X is a mapping $cl : 2^X \to [0,1]^X$ satisfying the following conditions:*
$(FC1)$ $\forall x \in X, cl(\emptyset)(x) = 0$;
$(FC2)$ $\forall x \in A, cl(A)(x) = 1$;
$(FC3)$ $cl(A_1 \cup A_2) = cl(A_1) \vee cl(A_2)$;
$(FC4)$ $cl(A)(x) = \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} cl(B)(y)$.

A continuous mapping from a fuzzifying closure space (X, cl^X) to a fuzzifying closure space (Y, cl^Y) is a mapping $f : X \to Y$ such that $\forall x \in X, \forall A \in 2^X$, $cl^X(A)(x) \leq cl^Y(f^{\to}(A))(f(x))$, where $f^{\to}(A) = \{f(x) \mid x \in A\}$. The category of fuzzifying closure spaces and their continuous mappings is denoted by FY-**CS**.

3 Fuzzy k-pseudo metric space

In this section, a new definition of a fuzzy k-pseudo metric and its relevant examples are introduced. This new definition is inspired by the idea that we can equivalently regard a crisp k-pseudo metric as a mapping $\chi_d : X \times X \times [0,\infty) \to \{0,1\}$ defined by

$$\chi_d(x,y,t) = \begin{cases} 1, & t > d(x,y); \\ 0, & t \leq d(x,y). \end{cases}$$

It satisfies the following conditions:
(1) $\chi_d(x,y,0)=0$;
(2) $\forall t>0, \chi_d(x,x,t)=1$;
(3) $\chi_d(x,y,t)=\chi_d(y,x,t)$;
(4) $\chi_d(x,y,t)\wedge\chi_d(y,z,s)\le\chi_d(x,z,k(t+s))$;
(5) $\bigvee_{s<t}\chi_d(x,y,s)=\chi_d(x,y,t)$;
(6) $\bigvee_{t>0}\chi_d(x,y,t)=1$.

The condition (5) means χ_d is left-continuous, which is a generalization of the property of nonnegative real numbers: "$d(x,y)<s, s<t\Rightarrow d(x,y)<t$". It also implies the monotonicity: $t_1\le t_2\Rightarrow\chi_d(x,y,t_1)\le\chi_d(x,y,t_2)$. And, the condition (6) means $lim_{t\to 0}\ \chi_d(x,y,t)=1$ introduced by Grabiec, which is a generalization of the property: "$\exists t>0\ \ s.t.\ d(x,y)<t$ ".

Both (5) and (6) are natural hidden properties. However, if we want to generalize the definition of a k-pseudo metric to the fuzzy case, these conditions should not be omitted, even very important in some conclusions.

In [19], Šostak generalized the definition of a k-pseudo metric to fuzzy setting. Now, we shall introduce a new definition of a fuzzy k-pseudo metric as follows.

Definition 3.1. *Let $k\ge 1$ be a fixed constant. A* **fuzzy k-pseudo metric space** *is a triple (X,M,T), where T is a t-norm and $M:X\times X\times[0,\infty)\to[0,1]$ is a mapping satisfying: $\forall x,y,z\in X$, $\forall t,s\in[0,\infty)$,*
$(FM1)$ $M(x,y,0)=0$;
$(FM2)$ $\forall t>0,\ M(x,x,t)=1$;
$(FM3)$ $M(x,y,t)=M(y,x,t)$;
$(FM4)$ $T(M(x,y,t),M(y,z,s))\le M(x,z,k(t+s))$;
$(FM5)$ $\bigvee_{s<t}M(x,y,s)=M(x,y,t)$;
$(FM6)$ $\bigvee_{t>0}M(x,y,t)=1$.
The mapping M is called a **fuzzy k-pseudo metric**. *If M also satisfies:*
$(FM2)^*$ $\forall t>0,\ M(x,y,t)=1\Rightarrow x=y$.
then (X,M,T) is called a fuzzy k-metric space.

A mapping $f:(X,M_X,T_X)\to(Y,M_Y,T_Y)$ is called contractive if $\forall\varepsilon>0$, there exists $\delta_\varepsilon>0$ such that $\forall x,y\in X$, $M_X(x,y,\delta_\varepsilon)\le M_Y(f(x),f(y),\varepsilon)$. It is easy to check that fuzzy k-pseudo metric spaces and their contractive mappings forms a category, denoted by FY-**KPMS**.

Remark 3.2. (1) *If $[0,1]$ is reduced to $\{0,1\}$, then M returns to the definition of a crisp k-pseudo metric.*
(2) *$(FM5)$ implies the function $M(x,y,-):[0,\infty)\to[0,1]$ is non-decreasing.*

Throughout this paper, unless otherwise stated, the definition of fuzzy k-pseudo metrics is adopted to take the t-norm $\wedge$.

Example 3.3. *A k-pseudo metric space $(X,d_{\rho\varphi})$ constructed in Example 2.3, then the mapping $M_{d_{\rho\varphi}}:X\times X\times[0,\infty)\to[0,1]$ defined by $\forall x,y\in X$, $\forall t\in[0,\infty)$,*

$$M_{d_{\rho\varphi}}(x,y,t)=\begin{cases}\dfrac{t}{t+(\varphi\circ\rho)(x,y)}, & \text{if } t>0;\\ 0, & \text{if } t=0.\end{cases}$$

is a fuzzy k-pseudo metric for any t-norm.

Example 3.4. *Define a mapping $M_{\|\ \|}:X\times X\times[0,\infty)\to[0,1]$ by $\forall x,y\in X$, $\forall t\in[0,\infty)$,*

$$M_{\|\ \|}(x,y,t)=\begin{cases}e^{-\frac{\|x-y\|^2}{t}}, & \text{if } t>0;\\ 0, & \text{if } t=0.\end{cases}$$

Then it is a fuzzy 2-metric for the product t-norm and then for any weaker t-norm (see [19]). We only need to prove (FM4), other conditions are trivial. Since $\|x-z\|^2\le 2(\|x-y\|^2+\|y-z\|^2)$, we know

$$\|x-z\|^2\le 2\left(\left(\frac{t+s}{t}\right)\|x-y\|^2+\left(\frac{t+s}{s}\right)\|y-z\|^2\right),$$

for all $x, y, z \in X$ *and* $s, t > 0$. *It follows that*

$$\frac{\|x-z\|^2}{2(t+s)} \leq \frac{\|x-y\|^2}{t} + \frac{\|y-z\|^2}{s}.$$

Hence

$$e^{-\frac{\|x-z\|^2}{2(t+s)}} \geq e^{-\frac{\|x-y\|^2}{t}} \cdot e^{-\frac{\|y-z\|^2}{s}}.$$

Therefore $M_{\|\ \|}(x,z,2(t+s)) \geq M_{\|\ \|}(x,y,t) \cdot M_{\|\ \|}(y,z,s)$.

At the end of this section, we shall illustrate the difference between fuzzy metrics and fuzzy k-metrics. Also, illustrating that a fuzzy k-metric is actually a generalization of a fuzzy metric.

Example 3.5. *Let* $X = [1,4]$, $T = \wedge$. *Define* $M : X \times X \times [0,\infty) \to [0,1]$

$$M(x,y,t) = \begin{cases} 0, & \text{if } t = 0; \\ 1, & \text{if } x = y, t \neq 0; \\ 1, & \text{if } x \neq y, t > x \cdot y; \\ 0, & \text{if } x \neq y, t \leq x \cdot y. \end{cases}$$

Then M *is a fuzzy* 2*-metric, not a fuzzy metric.*

Proof. **Step 1.** We need to check M is a fuzzy 2-pseudo metric.

It suffices to prove $M(x,y,t) \wedge M(y,z,s) \leq M(x,z,2(t+s))$, whenever $x \neq y,\ t > xy$ and $y \neq z,\ s > yz$. If $x = z$, then $M(x,z,2(t+s)) = 1$. If $x \neq z$ and

$$xz \leq (xy)(yz) \leq \left(\frac{xy+yz}{2}\right)^2 < \left(\frac{t+s}{2}\right)^2 = \left(\frac{t+s}{4}\right)(t+s).$$

Suppose $\left(\frac{t+s}{4}\right) > 2$, i.e., $t+s > 8$. Since $xz \leq 16$ and $2(t+s) > 16$, it is obvious that $xz < 2(t+s)$. Suppose $\left(\frac{t+s}{4}\right) \leq 2$, then $xz < 2(t+s)$. Hence $M(x,z,2(t+s)) = 1$. No matter what situation is, $M(x,y,t) \wedge M(y,z,s) \leq M(x,z,2(t+s))$ holds.

Step 2. M is not a fuzzy pseudo metric.

Let $x = 4,\ y = 1,\ z = 3$ and $t = 4.1 > xy,\ t = 3.1 > yz$. Then $t+s = 7.2 < xz$ and $2(t+s) = 14.4 > xz$. This implies $M(x,y,t) = 1,\ M(y,z,s) = 1$ and $M(x,z,2(t+s)) = 1$. But $M(x,z,t+s) = 1 - \frac{1}{8.2} \neq 1$. Hence $M(x,y,t) \wedge M(y,z,s) \leq M(x,z,2(t+s))$ and $M(x,y,t) \wedge M(y,z,s) \nleq M(x,z,(t+s))$. □

4 Fuzzifying structures induced by fuzzy k-pseudo metrics

In this section, various kinds of fuzzifying structures induced by a fuzzy k-pseudo metrics are constructed, including a fuzzifying neighborhood system, a fuzzifying topology, a fuzzifying closure operator, a fuzzifying interior operator and a fuzzifying uniformity.

Firstly, we recall some conclusions in crisp k-pseudo metric spaces. Let (X,d) be a crisp k-pseudo metric space. Define the open ball $B(x,r) = \{y \in X \mid d(x,y) < r\}$. Then the set $\mathcal{N}_d = \{(\mathcal{N}_d)_x \mid x \in X\}$ is a neighborhood system, where $(\mathcal{N}_d)_x = \{A \subseteq X \mid \exists r > 0,\ B(x,r) \subseteq A\}$. Also $\tau_d = \{A \subseteq X \mid \forall x \in A,\ \exists r > 0,\ B(x,r) \subseteq A\}$ is a topology and $\tau_d = \tau_{\mathrm{N}_d}$.

However $\sigma_d = \{A \subseteq X \mid A = \bigcup_{i \in I} B(x_i, \varepsilon_i)\}$ is not a topology, it is a pre-topology and $\tau_d \subsetneq \sigma_d$. The reason is that every open ball should not to be an open set in τ_d (Because of the triangle inequality in k-metric spaces). Readers can refer to the following counterexample.

Example 4.1. [19] *Let* $X = \{a\} \cup [b,c]$ *and the length of* $[b,c]$ *is* s. *Let* $d_t \in [b,c]$ *with* $d_t - b = t$ *for any* $t \in (0,s)$. *The distance on* $[b,c]$ *is the usual Euclidean metric and define* $d(a,b) = s$, $d(a,c) = 2s$, $d(a,d_t) = 2s - t$. *Then* d *is a* 2*-metric. However,* $B(b,\delta) \nsubseteq B(a, s+\varepsilon)$ *for any* $\varepsilon > 0$ *and* $\delta > 0$.

Next, we shall introduce a fuzzifying neighborhood system induced by a fuzzy k-pseudo metric in the following theorem.

Theorem 4.2. *Let M be a fuzzy k-pseudo metric on X. Define a mapping $\mathcal{N}_x^M : 2^X \to [0,1]$ by $\forall A \in 2^X$,*

$$\mathcal{N}_x^M(A) = \bigvee_{r>0} \bigwedge_{y \notin A} (1 - M(x,y,r)) .$$

Then $\mathcal{N}^M = \{\mathcal{N}_x^M \mid x \in X\}$ is a fuzzifying neighborhood system on X.

Proof. We need check the conditions (FN1)-(FN3) and (FN4)*.
(FN1) $\mathcal{N}_x^M(X) = \bigwedge \emptyset = 1$. By (FM2), we have $\mathcal{N}_x^M(\emptyset) \leq \bigvee_{r>0}(1 - M(x,x,r)) = 0$. Then $\mathcal{N}_x^M(\emptyset) = 0$.
(FN2) If $x \notin A$, then $\mathcal{N}_x^M(A) \leq \bigvee_{r>0}(1 - M(x,x,r)) = 0$. Hence $\mathcal{N}_x^M(A) = 0$.
(FN3) It is obvious that $\mathcal{N}_x^M(A_1) \leq \mathcal{N}_x^M(A_2)$ whenever $A_1 \subseteq A_2$. Then $\mathcal{N}_x^M(A_1 \cap A_2) \leq \mathcal{N}_x^M(A_1) \wedge \mathcal{N}_x^M(A_2)$. It suffices to prove $\mathcal{N}_x^M(A_1) \wedge \mathcal{N}_x^M(A_2) \leq \mathcal{N}_x^M(A_1 \cap A_2)$. Since the function $M(x,y,-) : [0,\infty) \to [0,1]$ is non-decreasing, we have

$$\begin{aligned} & \mathcal{N}_x^M(A_1) \wedge \mathcal{N}_x^M(A_2) \\ = & \left(\bigvee_{s>0} \bigwedge_{y_1 \notin A_1} (1 - M(x,y_1,s))\right) \wedge \left(\bigvee_{t>0} \bigwedge_{y_2 \notin A_2} (1 - M(x,y_2,t))\right) \\ = & \bigvee_{s>0,t>0} \bigwedge_{y_1 \notin A_1, y_2 \notin A_2} (1 - M(x,y_1,s)) \wedge (1 - M(x,y_2,t)) \\ \leq & \bigvee_{r>0} \bigwedge_{y \notin A_1 \cap A_2} (1 - M(x,y,r)) = \mathcal{N}_x^M(A_1 \cap A_2). \end{aligned}$$

(FN4)* $\mathcal{N}_x^M(A) \geq \bigvee_{x \in B \subseteq A}(\mathcal{N}_x^M(B) \wedge \bigwedge_{y \in B} \mathcal{N}_y^M(A))$ is trivial, since the map $\mathcal{N}_x^M(\cdot)$ is order-preserving. The key of the proof is that

$$\mathcal{N}_x^M(A) \leq \bigvee_{x \in B \subseteq A} (\mathcal{N}_x^M(B) \wedge \bigwedge_{y \in B} \mathcal{N}_y^M(A)).$$

Take any $a \in (0,1)$ such that $a < \mathcal{N}_x^M(A) = \bigvee_{r>0} \bigwedge_{y \notin A} (1 - M(x,y,r))$. Then there exists $r_0 > 0$ such that $a \leq 1 - M(x,y,r_0)$ for all $y \notin A$. Let

$$B = \{y \in X \mid M(x,y,\frac{r_0}{2k}) > 1 - a\}.$$

Then $x \in B \subseteq A$. (For any $y \notin A$, we have $M(x,y,\frac{r_0}{2k}) \leq M(x,y,r_0) \leq 1 - a$, which means $y \notin B$. So $B \subseteq A$.) Note that

$$\mathcal{N}_x^M(B) = \bigvee_{r>0} \bigwedge_{y \notin B} (1 - M(x,y,r)) \geq \bigwedge_{y \notin B} (1 - M(x,y,\frac{r_0}{2k})) \geq a,$$

and

$$\bigwedge_{y \in B} \mathcal{N}_y^M(A) = \bigwedge_{y \in B} \bigvee_{r>0} \bigwedge_{z \notin A} (1 - M(y,z,r)) \geq \bigwedge_{y \in B} \bigwedge_{z \notin A} (1 - M(y,z,\frac{r_0}{2k})).$$

Next, we show that $a \leq \bigwedge_{y \in B} \mathcal{N}_y^M(A)$. It suffices to show $a \leq \bigwedge_{y \in B} \bigwedge_{z \notin A}(1 - M(y,z,\frac{r_0}{2k}))$. For all $y \in B$, we know $M(x,y,\frac{r_0}{2k}) > 1 - a$. For all $z \notin A$, we get $M(x,z,r_0) \leq 1 - a$. By (FM4), we have

$$M(x,z,r_0) \geq T(M(x,y,\frac{r_0}{2k}), M(y,z,\frac{r_0}{2k})) = M(x,y,\frac{r_0}{2k}) \wedge M(y,z,\frac{r_0}{2k}).$$

This implies $M(y,z,\frac{r_0}{2k}) \leq 1 - a$. Hence $a \leq \bigwedge_{y \in B} \bigwedge_{z \notin A}(1 - M(y,z,\frac{r_0}{2k})) \leq \bigwedge_{y \in B} \mathcal{N}_y^M(A)$. Therefore $a \leq \mathcal{N}_x^M(B) \wedge \bigwedge_{y \in B} \mathcal{N}_y^M(A)$. By the arbitrariness of a, we obtain $\mathcal{N}_x^M(A) \leq \bigvee_{x \in B \subseteq A}(\mathcal{N}_x^M(B) \wedge \bigwedge_{y \in B} \mathcal{N}_y^M(A))$. □

Theorem 4.3. *If $f : (X, M_X, \wedge) \to (X, M_Y, \wedge)$ is a contractive between fuzzy k-pseudo metric spaces, then $f : (X, \mathcal{N}^{M_X}) \to (Y, \mathcal{N}^{M_Y})$ is also continuous between fuzzifying neighborhood spaces induced by fuzzy k-pseudo metrics.*

Proof. We need show $\forall x \in X, \forall B \in 2^Y$, $\mathcal{N}_{f(x)}^{M_Y}(B) \leq \mathcal{N}_x^{M_X}(f^{\leftarrow}(B))$, i.e.,

$$\bigvee_{r>0} \bigwedge_{z \notin B} (1 - M_Y(f(x),z,r)) \leq \bigvee_{s>0} \bigwedge_{y \notin f^{\leftarrow}(B)} (1 - M_X(x,y,s)) ,$$

For any $r > 0$, since $f : (X, M_X, T_X) \to (X, M_Y, T_Y)$ is contractive, there exists $\delta_r > 0$ such that $M_X(x_1, x_2, \delta_r) \leq M_Y(f(x_1), f(x_2), r))$. Then

$$\bigwedge_{z \notin B} (1 - M_Y(f(x), z, r)) \leq \bigwedge_{y \notin f^{\leftarrow}(B)} (1 - M_Y(f(x), f(y), r)) \leq \bigwedge_{y \notin f^{\leftarrow}(B)} (1 - M_X(x, y, \delta_r)) \leq \mathcal{N}_x^{M_X}(f^{\leftarrow}(B)).$$

Hence $\mathcal{N}_{f(x)}^{M_Y}(B) \leq \mathcal{N}_x^{M_X}(f^{\leftarrow}(B))$. □

In [24], it is shown that the category FY-**NBS** is isomorphic to the category FY-**Top**. That is, if $\mathcal{T}$ is a fuzzifying topology, then $\mathcal{N}^{\mathcal{T}} = \{\mathcal{N}_x^{\mathcal{T}} \mid x \in X\}$ is a fuzzifying neighborhood system, where $\mathcal{N}_x^{\mathcal{T}} = \bigvee_{x \in B \subseteq A} \mathcal{T}(B)$. Conversely, if $\mathcal{N} = \{\mathcal{N}_x \mid x \in X\}$ is a fuzzifying neighborhood system, then $\mathcal{T}^{\mathcal{N}}(A) = \bigwedge_{x \in A} \mathcal{N}_x(A)$ is a fuzzifying topology. As a consequence, we can get a fuzzifying topology induced by a fuzzy k-pseudo metric.

Theorem 4.4. *Let M be a fuzzy k-pseudo metric on X. Define a mapping $\mathcal{T}^M : 2^X \to [0, 1]$ by $\forall A \in 2^X$,*

$$\mathcal{T}^M(A) = \bigwedge_{x \in A} \bigvee_{r > 0} \bigwedge_{y \notin A} (1 - M(x, y, r)).$$

Then $\mathcal{T}^M$ is a fuzzifying topology on X.

Theorem 4.5. *If $f : (X, M_X, \wedge) \to (X, M_Y, \wedge)$ is contractive between fuzzy k-pseudo metric spaces, then $f : (X, \mathcal{T}^{M_X}) \to (Y, \mathcal{T}^{M_Y})$ is also continuous between fuzzifying topological spaces induced by fuzzy k-pseudo metrics.*

Šostak proved that a k-pseudo metric can induce a topology. However, Šostak did not consider whether a k-pseudo metric can induce a uniformity or not. Actually, we have the following lemma.

Lemma 4.6. *Let (X, d) be a k-pseudo metric space. Define*

$$U_r = \{(x, y) \in X \times X \mid d(x, y) < r\},$$
$$\mu_d = \{V \in 2^{X \times X} \mid \exists r > 0, s.t.\ U_r \subseteq V\}.$$

Then μ_d is a uniformity on X.

Proof. Let $\mathcal{B} = \{U_r \mid r \in (0, \infty)\}$. In order to prove μ_d is a uniformity, we only need to prove that $\mathcal{B}$ is a uniform base of μ_d.
i) $\forall U_r \in \mathcal{B}$, $\Delta = \{(x, x) \in X \times X \mid x \in X\} \subseteq U_r$, since $d(x, x) = 0 < r$.
ii) $\forall U_r \in \mathcal{B}$, $(U_r)^{-1} = U_r$, since $d(x, y) = d(y, x)$.
iii) $\forall U_{r_1}, U_{r_2} \in \mathcal{B}$, there exists $W = U_{r_1 \wedge r_2}$ such that $W \subseteq U_{r_1} \cap U_{r_2}$.
iv) $\forall U_r \in \mathcal{B}$, let $V = U_{\frac{r}{2k}}$. Then $V \circ V \subseteq U_r$.
In fact, for any $(x, y) \in V \circ V$, there exists $z \in X$ such that $(x, z) \in V$ and $(z, y) \in V$. This shows $d(x, z) < \frac{r}{2k}$ and $d(z, y) < \frac{r}{2k}$. By (D4), we have $d(x, y) \leq k(d(x, z) + d(z, y)) < k(\frac{r}{2k} + \frac{r}{2k}) = r$. Hence $(x, y) \in U_r$. □

Base on Lemma 4.6, the following theorem shows the construction of a fuzzifying uniformity induced by a fuzzy k-pseudo metric.

Theorem 4.7. *Let M be a fuzzy k-pseudo metric on X. Define a mapping $\mathcal{U}^M : 2^{X \times X} \to [0, 1]$ by $\forall V \in 2^{X \times X}$,*

$$\mathcal{U}^M(V) = \bigvee_{r > 0} \bigwedge_{(x, y) \notin V} (1 - M(x, y, r)).$$

Then $\mathcal{U}^M$ is a fuzzifying uniformity on X.

Proof. We need to check (FU1)-(FU5). At first, (FU1)-(FU3) can be easily proved from (FM2) and (FM3). The proof of (FU4) is similar to that of (FN3) in Theorem 4.2. What remains is to prove (FU5).

Take any $a \in (0, 1)$ with $a < \mathcal{U}^M(V)$. Then there exists some $r_0 > 0$ such that $a \leq 1 - M(x, y, r_0)$ for all $(x, y) \notin V$. Let

$$W = \{(u, v) \in X \times X \mid M(u, v, \frac{r_0}{2k}) > 1 - a\}.$$

Then $W \circ W \subseteq V$. (In fact, for any $(x, y) \in W \circ W$, there exists $z \in X$ such that $(x, z) \in W$ and $(z, y) \in W$. By the definition of W, we have $M(x, z, \frac{r_0}{2k}) > 1 - a$ and $M(z, y, \frac{r_0}{2k}) > 1 - a$. From (FM4), we know that $M(x, y, r_0) \geq M(x, z, \frac{r_0}{2k}) \wedge M(z, y, \frac{r_0}{2k}) > 1 - a$. So $(x, y) \in V$.) Further, $M(u, v, \frac{r_0}{2k}) \leq 1 - a$ for all $(u, v) \notin W$. Hence $a \leq \bigvee_{r > 0} \bigwedge_{(u, v) \notin W} (1 - M(u, v, r)) = \mathcal{U}^M(W)$. By the arbitrariness of a, we obtain $\mathcal{U}^M(V) \leq \bigvee_{W \in 2^{X \times X}, W \circ W \subseteq V} \mathcal{U}^M(W)$. □

Theorem 4.8. *If $f:(X,M_X,\wedge)\to(X,M_Y,\wedge)$ is contractive between fuzzy k-pseudometric spaces, then $f:(X,\mathcal{U}^{M_X})\to(Y,\mathcal{U}^{M_Y})$ is also continuous between fuzzifying uniform spaces induced by fuzzy k-pseudo metrics.*

Proof. We need to show that $\forall V\in 2^{Y\times Y}$, $\mathcal{U}^{M_Y}(V)\le\mathcal{U}^{M_X}(f^{\Leftarrow}(V))$, i.e.,

$$\bigvee_{r>0}\bigwedge_{(y_1,y_2)\notin V}(1-M_Y(y_1,y_2,r))\le\bigvee_{s>0}\bigwedge_{(x_1,x_2)\notin f^{\Leftarrow}(V)}(1-M_X(x_1,x_2,s)),$$

For any $r>0$, since $f:(X,M_X,T_X)\to(X,M_Y,T_Y)$ is contractive, there exists $\delta_r>0$ such that $M_X(x_1,x_2,\delta_r)\le M_Y(f(x_1),f(x_2),r))$. Then

$$\bigwedge_{(y_1,y_2)\notin V}(1-M_Y(y_1,y_2,r))\le\bigwedge_{(x_1,x_2)\notin f^{\Leftarrow}(V)}(1-M_Y(f(x_1),f(x_2),r))\le\bigwedge_{(x_1,x_2)\notin f^{\Leftarrow}(V)}(1-M_X(x_1,x_2,\delta_r))\le\mathcal{U}^{M_X}(f^{\Leftarrow}(V)).$$

Hence $\mathcal{U}^{M_Y}(V)\le\mathcal{N}^{M_X}(f^{\Leftarrow}(V))$. □

Due to the category FY-**NBS** is isomorphic to the category FY-**CS** [18, 24]. In what follows, a fuzzifying closure operator and a fuzzifying interior operator induced by a fuzzy k-pseudo metric are given.

Theorem 4.9. *Let M be a fuzzy k-pseudo metric on X. Define a mapping $cl^M:2^X\to[0,1]^X$ by $\forall A\in 2^X$, $\forall x\in X$,*

$$cl^M(A)(x)=\bigwedge_{r>0}\bigvee_{y\in A}M(x,y,r).$$

Then cl^M is a fuzzifying closure operator on X.

Theorem 4.10. *If $f:(X,M_X,\wedge)\to(X,M_Y,\wedge)$ is contractive between fuzzy k-pseudo metric spaces, then $f:(X,cl^{M_X})\to(Y,cl^{M_Y})$ is also continuous between fuzzifying closure spaces induced by fuzzy k-pseudo metrics.*

5 Relationships between nests of k-pseudo metrics and fuzzy k-pseudo metrics

In this section, given a fuzzy k-pseudo metric, we can construct a nest of k-pseudo metrics. Conversely, a fuzzy k-pseudo metric can be obtained by a nest of k-pseudo metrics.

Definition 5.1. *A nest of k-pseudo metrics is a set of k-pseudo metrics $\mathcal{D}=\{d_a\mid a\in(0,1)\}$ satisfying: $\forall a\in(0,1)$, $d_a=\bigwedge_{b>a}d_b$.*

Remark 5.2. *From Definition 5.1, we know $a\le b\Rightarrow d_a\le d_b$.*

Lemma 5.3. *Let $(X,M,\wedge)$ be a fuzzy k-pseudo metric space. For any $a\in(0,1)$, define a mapping $d_a^M:X\times X\to[0,\infty)$ by $\forall x,y\in X$,*

$$d_a^M(x,y)=\bigvee\{t\in[0,\infty)\mid M(x,y,t)\le a\}.$$

Then

(1) $M(x,y,t)\le a\Leftrightarrow d_a^M(x,y)\ge t$, *i.e.*, $M(x,y,t)>a\Leftrightarrow d_a^M(x,y)<t$.

(2) $d_a^M(x,y)=\bigwedge\{t\in[0,\infty)\mid M(x,y,t)>a\}$.

Proof. Since the constructions of d_a^M is similar to that of in [12], the proofs are also similar and omitted here. □

Next, we shall show that $\forall a\in(0,1)$, d_a^M is a crisp k-pseudo metric.

Theorem 5.4. *Let $(X,M,\wedge)$ be a fuzzy k-pseudo metric space. Then d_a^M is a k-pseudo metric for any $a\in(0,1)$. And, the set $\mathcal{D}^M=\{d_a^M\}_{a\in(0,1)}$ is a nest of k-pseudo metrics.*

Proof. We need to check the conditions (D1)-(D3). (D1)-(D2) are trivial. We only need check d_a^M fulfills (D3). For all $x,y,z\in X$, let $r>0$ with

$$r>k(d_a^M(x,y)+d_a^M(y,z))=k\left(\bigwedge\{t\mid M(x,y,t)>a\}+\bigwedge\{s\mid M(y,z,s)>a\}\right).$$

Then there exist $t \geq 0$, $s \geq 0$ such that $M(x, y, t) > a$, $M(y, z, s) > a$ and $r > k(t+s)$. By (FM4), we know

$$a < M(x, y, t) \wedge M(y, z, s) \leq M(x, z, k(t+s)) \leq M(x, z, r).$$

It follows from Theorem 5.3 that $d_a^M(x, z) < r$. By the arbitrariness of r, we obtain $d_a^M(x, z) \leq k(d_a^M(x, y) + d_a^M(y, z))$. □

Up to now, we have obtained a nest of k-pseudo metrics. In the following, we will consider the converse problem: whether a fuzzy k-pseudo metric can be obtained from a nest of k-pseudo metrics.

Lemma 5.5. *Let $\mathcal{D} = \{d_a \mid a \in (0,1)\}$ be a nest of k-pseudo metrics. Define a mapping $M^{\mathcal{D}}: X \times X \times [0, \infty) \to [0, 1]$ by $\forall x, y \in X$, $\forall t \in [0, \infty)$,*

$$M^{\mathcal{D}}(x, y, t) = \bigvee \{a \in (0,1) \mid d_a(x, y) < t\}.$$

Then

(1) $M^{\mathcal{D}}(x, y, t) > a \Leftrightarrow d_a(x, y) < t$, *i.e.*, $M^{\mathcal{D}}(x, y, t) \leq a \Leftrightarrow d_a(x, y) \geq t$.

(2) $M^{\mathcal{D}}(x, y, t) = \bigwedge \{a \in (0,1) \mid d_a(x, y) \geq t\}$.

Proof. The proofs are similar to that of in [12] and omitted here. □

Theorem 5.6. *Let $\mathcal{D} = \{d_a \mid a \in (0,1)\}$ be a nest of k-pseudo metrics. Then $M^{\mathcal{D}}$ is a fuzzy k-pseudo metric.*

Proof. We need check (FM1)-(FM6). Firstly, (FM1)-(FM3) are trivial.

(FM4) For all $x, y, z \in X$ and $s, t \in [0, \infty)$, let $r \in (0, 1)$ with

$$r < M^{\mathcal{D}}(x, y, t) \wedge M^{\mathcal{D}}(y, z, s) = \bigvee \{a_1 \mid d_{a_1}(x, y) < t\} \wedge \bigvee \{a_2 \mid d_{a_2}(y, z) < s\}.$$

Then there exist $a_1, a_2 \in (0, 1)$ such that $d_{a_1}(x, y) < t$, $d_{a_2}(y, z) < s$ and $r < a_1 \wedge a_2$. So $d_r(x, y) < t$ and $d_r(y, z) < s$. Since $d_r(x, z) \leq k(d_r(x, y) + d_r(y, z)) < k(t+s)$, it follows that $M^{\mathcal{D}}(x, z, k(t+s)) > r$. From the arbitrariness of r, we obtain $M^{\mathcal{D}}(x, y, t) \wedge M^{\mathcal{D}}(y, z, s) \leq M^{\mathcal{D}}(x, z, k(t+s))$.

(FM5) can be proved by the following equations:

$$\bigvee_{s<t} M^{\mathcal{D}}(x, y, s) = \bigvee_{s<t} \bigvee \{a \mid d_a(x, y) < s\} = \bigvee \{a \mid d_a(x, y) < t\} = M^{\mathcal{D}}(x, y, t).$$

(FM6) For any $\varepsilon \in (0, 1)$, there exist some $t > 0$ such that $d_{1-\varepsilon}(x, y) < t$. By Lemma 5.5, we know $M^{\mathcal{D}}(x, y, t) > 1 - \varepsilon$. So $\bigvee_{t>0} M^{\mathcal{D}}(x, y, t) \geq 1 - \varepsilon$. Hence $\bigvee_{t>0} M^{\mathcal{D}}(x, y, t) = 1$. □

Let us consider the following families:

$$\mathscr{M} = \{M \mid M \text{ is a fuzzy } k\text{-pseudo metric under the } t\text{-norm } \wedge\}.$$

$$\mathscr{D} = \{\mathcal{D} \mid \mathcal{D} = \{d_a \mid a \in (0,1)\} \text{ is a nest of } k\text{-pseudo metrics}\}.$$

By Theorem 5.3 and Theorem 5.6, there exists a bijection $f : \mathscr{M} \to \mathscr{D}$ defined by $f(M) = \mathcal{D}^M = \{d_a^M\}_{a \in (0,1)}$ for all $M \in \mathscr{M}$, and there exists a bijection $g : \mathscr{D} \to \mathscr{M}$ defined by $g(\mathcal{D}) = M^{\mathcal{D}}$ for all $\mathcal{D} \in \mathscr{D}$.

In the following theorem, we shall show that there is a one-to-one correspondence between fuzzy k-pseudo metrics under the t-norm $\wedge$ and nests of k-pseudo metrics.

Theorem 5.7. *Let M be a fuzzy k-pseudometric under the t-norm $\wedge$ and $\mathcal{D} = \{d_a \mid a \in (0,1)\}$ be a nest of k-pseudo metrics. Then $M^{\mathcal{D}^M} = M$ and $\mathcal{D}^{M^{\mathcal{D}}} = \mathcal{D}$.*

Proof. (1) $M^{\mathcal{D}^M} = M$ can be proved from the following equations.

$$\begin{aligned} M^{\mathcal{D}^M}(x, y, t) &= \bigvee \{a \mid d_a^M(x, y) < t\} = \bigvee \{a \mid \bigwedge \{s \mid M(x, y, s) > a\} < t\} \\ &= \bigvee \{a \mid \exists s < t, \ s.t.\ M(x, y, s) > a\} = \bigvee \{M(x, y, s) \mid s < t\} = M(x, y, t). \end{aligned}$$

(2) In order to prove $\mathcal{D}^{M^{\mathcal{D}}} = \mathcal{D}$, we need to prove $\forall a \in (0, 1)$, $d_a^{M^{\mathcal{D}}} = d_a$

$$\begin{aligned} d_a^{M^{\mathcal{D}}}(x, y) &= \bigwedge \{t \mid M^{\mathcal{D}}(x, y, t) > a\} = \bigwedge \{t \mid \bigvee \{b \mid d_b(x, y) < t\} > a\} \\ &= \bigwedge \{t \mid \exists b > a, \ s.t.\ d_b(x, y) < t\} = \bigwedge \{d_b(x, y) < t \mid b > a\} = d_a(x, y). \end{aligned}$$

□

Finally, we shall discuss the relations between fuzzifying structures induced by a nest of k-pseudo metrics and fuzzifying structures induced by a fuzzy k-pseudo metric. The following lemma will be useful afterwards.

Lemma 5.8. *Let (X, d_a) and (X, d_b) be k-pseudo metric spaces and $d_a \le d_b$. Then $\tau_{d_a} \subseteq \tau_{d_b}$ and $\mu_{d_a} \subseteq \mu_{d_b}$.*

Proof. The proofs are easily to be proved and are omitted here. □

Given a nest of k-pseudo metrics $D = \{d_a \mid a \in (0,1)\}$, we can obtain a family of non-decreasing topologies $\{\tau_{d_a}\}_{a\in(0,1)}$ and a family of non-decreasing uniformities $\{\mu_{d_a}\}_{a\in(0,1)}$ from Lemma 5.8.

Then a fuzzifying topology $\mathcal{T}^{\mathcal{D}}$ can be generated by this family of crisp topologies according to [24], that is, $\mathcal{T}^{\mathcal{D}} : 2^X \to [0,1]$ is defined by $\forall A \in 2^X$,

$$\mathcal{T}^{\mathcal{D}}(A) = \bigvee\{1 - a \mid A \in \tau_{d_a}\}.$$

And a fuzzifying uniformity $\mathcal{U}^{\mathcal{D}}$ can also be generated according to [2, 3], that is, $\mathcal{U}^{\mathcal{D}} : 2^{X\times X} \to [0,1]$ is defined by $\forall V \in 2^{X\times X}$,

$$\mathcal{U}^{\mathcal{D}}(V) = \bigvee\{1 - a \mid V \in \mu_{d_a}\}.$$

As we have been discussed in Section 4, a fuzzifying topology $\mathcal{T}^{M^{\mathcal{D}}}$ can be generated by $M^{\mathcal{D}}$, that is,

$$\mathcal{T}^{M^{\mathcal{D}}}(A) = \bigwedge_{x\in A}\bigvee_{r>0}\bigwedge_{y\notin A}\left(1 - M^{\mathcal{D}}(x,y,r)\right) = \bigwedge_{x\in A}\bigvee_{r>0}\bigwedge_{y\notin A}\bigwedge_{b\in(0,1)}\{1 - b \mid d_b(x,y) < r\}.$$

And a fuzzifying uniformity $\mathcal{U}^{M^{\mathcal{D}}}$ can be generated by $M^{\mathcal{D}}$,

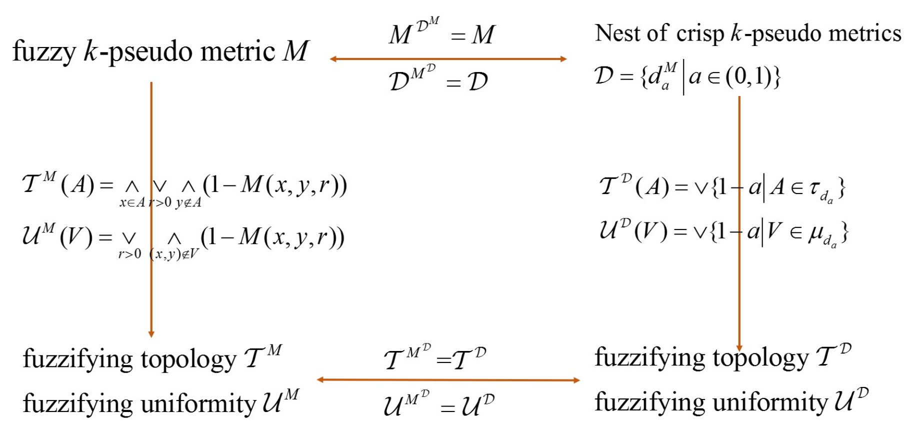
$$\mathcal{U}^{M^{\mathcal{D}}}(V) = \bigvee_{r>0}\bigwedge_{(x,y)\notin V}\left(1 - M^{\mathcal{D}}(x,y,r)\right) = \bigvee_{r>0}\bigwedge_{(x,y)\notin V}\bigwedge_{b\in(0,1)}\{1 - b \mid d_b(x,y) < r\}.$$

For the relation between $\mathcal{T}^{\mathcal{D}}$ and $\mathcal{T}^{M^{\mathcal{D}}}$, and the relation between $\mathcal{U}^{\mathcal{D}}$ and $\mathcal{U}^{M^{\mathcal{D}}}$, we have the following theorem.

Theorem 5.9. *Let $D = \{d_a \mid a \in (0,1)\}$ be a nest of k-pseudo metrics. Then $\mathcal{T}^{\mathcal{D}} = \mathcal{T}^{M^{\mathcal{D}}}$ and $\mathcal{U}^{\mathcal{D}} = \mathcal{U}^{M^{\mathcal{D}}}$.*

Proof. The proofs are similar to that of in [13] and are omitted here. □

At the end of this section, we have the following summary figure.



6 Conclusions

In this paper, a new definition of fuzzy k-pseudo metric and various fuzzifying structures induced by this new fuzzy k-pseudo metrics were introduced. Besides, we showed that there is a one-to-one correspondence between fuzzy k-pseudo metrics and nests of k-pseudo metrics.

The notion of partial metric introduced by S.G. Matthews is an important generalization of metric. It would be our interest in the future to generalize fuzzy k-pseudo metrics to fuzzy partial k-pseudo metrics and study the similar results of this paper on fuzzy partial k-pseudo metrics.

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