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Solvability of fuzzy fractional stochastic Pantograph differential system

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Abstract

In this paper, a new type of equation namely fuzzy fractional stochastic Pantograph delay differential system (FSPDDS) is proposed. In our previous work, a first extension of fuzzy stochastic differential system into fuzzy fractional stochastic differential system by using Granular differentiability has been established. Here we study the existence and uniqueness results for the fuzzy FSPDDS which are obtained by using generalized Granular differentiability and contraction principle with weaker conditions. This kind of equation is used in many real world problems. Finally, we provide two numerical examples for the effectiveness of the theoretical results.

Keywords: Banach contraction, fuzzy fractional, fuzzy stochastic, granular differentiability.

1 Introduction

In the year 1965, Zadeh established the theory of fuzzy set [38]. The theory of fuzzy differential equations (DEs) has been developed to handle uncertain initial values in the systems and also to rectify the uncertain relationships between parameters, see [3, 9, 23]. In fuzzy set, the equations are replaced with real numbers in order to model the uncertain values. Fuzzy DEs can be classified into two (i) Random fuzzy DEs [13, 15, 26] and (ii) Stochastic fuzzy DEs [16, 25]. Existence results for fuzzy DEs with and without stochastic term have been studied by many authors in [8, 10, 14, 34]. Stochastic fuzzy DEs has two different sources of uncertainties one is randomness and the other is fuzziness. The most important applications on stochastic theory arise in population models. In particular, the birth rate is changed instantly with a change in population. Here the members of a population are thresholding by age before giving birth also time delay has to be introduced into the model. Let us consider the Malthusian time delay model $\mathscr{X}'(\theta) = (p-m)\mathscr{X}(\theta-\tau)$, $\mathscr{X}(0) = \mathscr{X}_0$ here p, m are the constants denote a reproduction and mortality coefficient, $\mathscr{X}_0$ represents the initial number of the individuals, the growth rate at time θ depend on the population at time $\theta-\tau$. The solution of the above is $\mathscr{X}(\theta) = \mathscr{X}_0 e^{a\theta}$, where $a = p - m$ and $a \neq 0$. Further one can transform the above model with some uncertainties exist in $\mathscr{X}(\theta)$. The state of population is depend on random factors but the observer can predict the state of population with linguistics terms like small, not tall, tall, big. Hence there are two types of uncertainties appear for the population growth one is randomness and the other is fuzziness. Hence both are incorporated in the dynamical systems some white noise captured in stochastic DEs and fuzzy stochastic DEs is another best modeling for phenomena in uncertainties.

Fractional differential equations (FDEs) and uncertainties play a vital role in real life. The concept of fuzzy fractional have been introduced by Agarwal et.al [2]. The impulsive effects are developed for studying the evolution process of gradual change in their states. The change of state or impulse estimate involves the perturbations like earthquake, hurricanes, harvesting, drug administration etc. Instead of having linear impulsive conditions one can transfer it into fuzzy impulsive conditions. Since they have many applications, while dealing with the uncertainty problems in control theory, biology, medical science and engineering. When solving the fuzzy impulsive FDEs analytically, it is difficult to

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obtain the exact solution. The concept of fuzzy impulsive FDEs have been studied by Najafi and Allahviranloo see [4, 20, 21].

Fuzzy delay DEs is one of the new branches of fuzzy mathematics. Fuzzy delay DEs have been studied by many researchers with different interpretations, one of them is the approach of $\mathscr{H}$ukuhara derivative. For fuzzy delay DEs, existence and uniqueness results have been proved by Lupulescu [7] using $\mathscr{H}$ukuhara derivative approach. In another way, the existence of fuzzy delay DEs have been derived in [11] by using generalized differentiability with two fuzzy solutions.

Moreover telegraph equations have tremendous applications in the fields such as mathematics, signal processing, wave propagation, random walk theory etc. It is used mainly in the production of designing the frequency. Recently fractional order with fuzzy uncertainty in telegraphic equation has been studied by Tapaswini and Behera [36].

Another application involves in electric trains, the device Pantograph is used for collecting electric current in the overload lines and it is the main source to move the train. Ockendon [24] has modeled this Pantograph equation. This equation plays an important role in applied mathematics, physics, control systems, quantum mechanics and electrodynamics etc. This type of equation normally occur with delay term. There are so many methods to solve this type of stochastic delay DEs namely block method, spectral method etc. Many researchers developed this equation into varied forms and introduced the aspect of such kind of problems modeled into fractional DEs. Also, it is more realistic to study the physical meaning with their mathematical models. The general Pantograph equation is

$$\mathscr{X}'(\theta) \quad = \quad a\mathscr{X}(\theta) + b\mathscr{X}(\mu\theta),\ 0 \leq \theta \leq T,\ \mathscr{X}(0) = \mathscr{X}_0 \text{ where } 0 < \mu < 1.$$

From the above discussions, we focus on the following fuzzy fractional Pantograph stochastic DEs with delay as

$$\begin{aligned} {}^{\mathscr{G}r}\mathcal{D}^q\mathscr{X}(\theta) &= A\mathscr{X}(\theta) + \lambda(\theta, \mathscr{X}(\theta - h)) + \Big\langle \int_0^\theta \Delta(s, \mathscr{X}(s-h))dw(s) \Big\rangle,\ t \in J := [0, T], \\ \mathscr{X}(\theta) &= \psi(\theta),\ \theta \in [-h, 0], \end{aligned}$$

where $0 < q < 1$, A, λ and Δ are defined later.

The study of fuzzy FDEs and integral equations have been seen in the literature [2, 37]. In [1, 4, 5, 6], the notions of fuzzy fractional integrals, fuzzy Riemann Liouville (RL) and fuzzy Caputo derivatives have been utilized. A new derivatives of neutrosophic RL and neutrosophic Caputo fractional derivatives have been studied by Son et.al [32]. Using the concept of $\mathscr{H}$ukuhara derivative in the fuzzy FDEs one can get two solutions with different geometric representation. It is the main drawback in this derivative but in most of the literature see [20, 21, 29, 37] $\mathscr{H}$ukuhara derivative has been used for finding the solution.

Initiated by the above, some new results on the fuzzy FSPDDS have been proved. In this paper, we use the concept of granular $(\mathscr{G}r)$-differentiability [18, 27, 35], Caputo fractional derivative and Mittag Leffler (ML) function to derive existence result. The concept is new and yet there is no literature available for dealing fuzzy FSPDDS, hence authors unable to undertake the comparative study .

The novelties and difficulties of this paper are described as follows:

(i) Fuzzy FPSDDS is new in finite dimensional stochastic settings.

(ii) Under a Holder's inequality, the existence and uniqueness results are derived through Banach contraction principle by imposing weaker conditions on nonlinear functions.

(iii) We use weaker condition of Holder's inequality instead of Lipschitz condition, a stronger one.

(iv) We have sorted out the difficulty raised in terms of ML function and expressed the solution embedded into $\mathcal{F}(\Re^d)$.

(iv) In order to apply Holder's inequality, one can take any value for A matrix but the norm value of A could be less than 1.

The paper is formulated as follows. Some basic definitions are summarized for fuzzy stochastic FDEs in Section 2. In Section 3, some sufficient conditions are derived for the existence and uniqueness of the fuzzy FSPDDS. A numerical simulation is provided in Section 4 to validate the theoretical results. Conclusion is drawn in Section 5.

Notations: Let $(\Omega, \mathcal{F}(\Re^d)) = (\Omega, \mathcal{A}, \{\mathcal{A}_\theta\}_{\theta \geq 0}, \mathcal{F}(\Re^d))$ be the complete probability space with filtration $\{\mathcal{A}_\theta : \theta \in J\}$ satisfying the usual conditions. Let $w(\theta)$ be a m-dimensional Brownian motion defined on the probability space

$(\Omega, \mathcal{F}(\Re^d))$. Let the collection of all strongly measurable square integrable $\mathcal{F}(\Re^d)$ valued random variables is denoted by $L_2(\Omega, \mathcal{F}(\Re^d))$ which is a complete metric space furnished with the following metric

$$D(\mathscr{X}, \mathscr{Y}) = (\mathbb{E}\ d_\infty^2(\mathscr{X}, \mathscr{Y}))^{\frac{1}{2}} = \sup_{\theta \in J_1} \{D^2(\mathscr{X}(\theta), \mathscr{Y}(\theta))\}^{\frac{1}{2}},$$

where $\mathbb{E}$ denotes the expectation.

For $J_1 = [-h, T]$, let $C(J_1, L_2(\Omega, \mathcal{F}(\Re^d)))$ be the Banach space of all continuous maps from J_1 to $L_2(\Omega, \mathcal{F}(\Re^d))$ satisfying the condition $\mathbb{E}\ d_\infty^2(\mathscr{X}, \mathscr{Y}) < \infty$. Let $\mathcal{B}_h$ be the closed bounded subspace of all continuous process $\mathscr{X}$ in $C(J_1, L_2(\Omega, \mathcal{F}(\Re^d)))$ consists of $\mathcal{A}_\theta$- adapted measurable process $\{\mathscr{X}(\theta),\ \theta \in J\}$ endowed with the norm

$$\mathbb{E}\ d_\infty^2(\mathscr{X}, \mathscr{Y}) = \sup_{\theta \in J} \mathbb{E}\ d_\infty^2(\mathscr{X}, \mathscr{Y}).$$

It is easy to verify that $\mathcal{B}_h$ endowed with the above metric norm is a complete metric space $(\mathcal{B}_h, d_\infty)$.

2 Preliminaries

Definition 2.1. [29] *Let $\mathcal{K}(\Re^d)$ be the family of all convex, compact and nonempty subsets of $\Re^d$, then the Hausdorff metric d_H which is defined by*

$$D := d_H(A, B) := \max\Big\{ \sup_{\mathscr{X} \in A} \inf_{\mathscr{Y} \in B} \|\mathscr{X} - \mathscr{Y}\|, \sup_{\mathscr{Y} \in B} \inf_{\mathscr{X} \in A} \|\mathscr{X} - \mathscr{Y}\| \Big\},\ A, B \in \mathcal{K}(\Re^d),$$

where $\|\cdot\|$ represents the usual Euclidean norm in $\Re^d$.

Definition 2.2. [28] *The metric space $(\mathcal{K}(\Re^d), D)$ is complete and the following properties for the metric D hold:*

(i) $D(\mathscr{X} + w, \mathscr{Y} + w) = D(\mathscr{X}, \mathscr{Y})$ *for all* $\mathscr{X}, \mathscr{Y}, w \in \mathcal{K}(\Re^d)$;

(ii) $D(\mathscr{X} + w, \mathscr{Y} + z) = D(\mathscr{X}, \mathscr{Y}) + D(w, z)$ *for all* $\mathscr{X}, \mathscr{Y}, z, w \in \mathcal{K}(\Re^d)$;

(iii) $D(k\mathscr{X}, k\mathscr{Y}) = |k| D(\mathscr{X}, \mathscr{Y})$ *for all* $\mathscr{X}, \mathscr{Y} \in \mathcal{K}(\Re^d)$.

Definition 2.3. [29] *Let X be a non-empty set. A fuzzy set ν in X is characterized by its membership function $\nu : X \to [0, 1]$. Then for each $\mathscr{X} \in X$, we interpret $\nu(\mathscr{X})$ as the degree of membership of the element $\mathscr{X}$ in the fuzzy set $\nu : \nu(\mathscr{X}) = 0$ corresponding to the nonmembership, $0 < \nu(\mathscr{X}) < 1$ to the partial membership and $\nu(\mathscr{X}) = 1$ to the full membership.*

Definition 2.4. [29] *Let $\mathcal{F}(\Re^d)$ denote the class of fuzzy subsets in the real axis $\nu : \Re^d \to [0, 1]$ which satisfy the following conditions:*

(1) *ν is normal (i.e) there exists $\mathscr{X}_0 \in \Re^d$ with $\nu(\mathscr{X}_0) = 1$.*

(2) *ν is a convex fuzzy set (i.e) $\nu(\lambda\mathscr{X} + (1 - \lambda)\mathscr{Y}) \geq \min\{\nu(\mathscr{X}) + \nu(\mathscr{Y})\}$ for all $\lambda \in [0, 1]$ and $\mathscr{X}, \mathscr{Y} \in \Re^d$.*

(3) *ν is upper semi continuous on $\Re^d$.*

(4) *$cl\{\mathscr{X} \in \Re^d : \nu(\mathscr{X}) > 0\}$ is compact, where cl denotes the closure of a set.*

We call $\mathcal{F}(\Re^d)$, the space of fuzzy numbers, obviously $\Re^d \subset \mathcal{F}(\Re^d)$.
The set $\Re^d$ can be embedded into $\mathcal{F}(\Re^d)$ by an embedding $\langle\cdot\rangle : \Re^d \to \mathcal{F}(\Re^d)$ defined as follows: for $r \in \Re^d$

$$\langle r\rangle(a) = \begin{cases} 1 \text{ if } a = r \\ 0 \text{ if } a \in \Re^d \backslash \{r\}. \end{cases}$$

Definition 2.5. [8, 14, 17] *Let $\mathcal{F}(\Re^d)$ represent the fuzzy set $\nu : \Re^d \to [0, 1]$ such that $[\nu]^\alpha \in \mathcal{K}(\Re^d)$ for every $\alpha \in [0, 1]$ where $[\nu]^\alpha = \{\mathscr{X} \in \Re^d;\ \nu(\mathscr{X}) \geq \alpha\}$ for $0 < \alpha \leq 1$, and $[\nu]^0 = cl\{\mathscr{X} \in \Re^d;\ \nu(\mathscr{X}) > 0\}$.*

The notation $[\nu]^\alpha = [\nu_l^\alpha, \nu_r^\alpha]$ denotes the α-level set of ν. We refer ν_l, ν_r as the left and right branches on ν, respectively.

Definition 2.6. [30] *The Hausdorff distance between fuzzy numbers is given as*

$$D(\mathscr{X},\mathscr{Y}) := \sup_{\alpha\in[0,1]} \max\{D(|\mathscr{X}_l^{\alpha}-\mathscr{Y}_l^{\alpha}|,|\mathscr{X}_r^{\alpha}-\mathscr{Y}_r^{\alpha})|\},$$

where $\mathscr{X}=[\mathscr{X}_l^{\alpha},\mathscr{X}_r^{\alpha}]$ *and* $\mathscr{Y}=[\mathscr{Y}_l^{\alpha},\mathscr{Y}_r^{\alpha}]$. *The metric space* $(\mathcal{F}(\Re^d),D)$ *is complete.*

We define $\langle 0\rangle\in\mathcal{F}(\Re^d)$ as $\langle 0\rangle := 1_{\{0\}}$, where for $\mathscr{X}\in\Re^d$, we have $1_{\{\mathscr{Y}\}}(\mathscr{X})=1$ if $\mathscr{X}=\mathscr{Y}$ and $1_{\{\mathscr{Y}\}}(\mathscr{X})=0$ if $\mathscr{X}\neq\mathscr{Y}$.

Definition 2.7. [5] *The generalised Hukuhara derivative of a two fuzzy-valued function* $f: J\to\mathcal{F}(\Re^d)$ *at* $\mathscr{X}_0$ *is defined as*

$$(f)'_{\mathscr{GH}}(\mathscr{X}_0)=\lim_{h\longrightarrow 0}\frac{f(\mathscr{X}_0+h)\ominus_{\mathscr{GH}} f(\mathscr{X}_0)}{h}.$$

If $(f)'_{\mathscr{GH}}(\mathscr{X}_0)\in\mathcal{F}(\Re^d)$, then f is said to be $\mathscr{GH}$-differentiable at $\mathscr{X}_0$. Also, f is $[(i)-\mathscr{GH}]$-differentiable at $\mathscr{X}_0$ if (i) $(f)'_{\mathscr{GH}}(\mathscr{X}_0;\alpha)=[(f_l^{\alpha})'(\mathscr{X}_0;\alpha),(f_r^{\alpha})'(\mathscr{X}_0;\alpha)]$ and f is $[(ii)-\mathscr{GH}]$-differentiable at $\mathscr{X}_0$ if (ii) $(f)'_{\mathscr{GH}}(\mathscr{X}_0;\alpha)=[(f_r^{\alpha})'(\mathscr{X}_0;\alpha),(f_l^{\alpha})'(\mathscr{X}_0;\alpha)]$, where $0\le\alpha\le 1$.

Definition 2.8. [18] *A fuzzy number* $\nu:[a,b]\to[0,1]$ *with its parametric form* $[\nu]^{\alpha}=[\nu_l^{\alpha},\nu_r^{\alpha}]$, *the horizontal membership function (*$\mathscr{H}MF$*)* $\nu^{\mathscr{G}r}=[0,1]\times[0,1]\to[a,b]$ *with* $\mathscr{X}=\nu^{\mathscr{G}r}(\alpha,\alpha_\nu)=\nu_l^{\alpha}+(\nu_l^{\alpha}-\nu_r^{\alpha})\alpha_\nu$ *indicates for the granule of information included in* $\mathscr{X}\in[a,b]$.

Definition 2.9. [18] *The* α*-level sets of* $\nu\in\mathcal{F}(\Re^d)$ *which are the span of the information granule is given by*

$$\mathscr{H}^{-1}(\nu^{\mathscr{G}r}(\alpha,\alpha_\nu))=[\nu]^{\alpha}=\Big[\inf_{\beta\ge\alpha}\min_{\alpha_\nu}\nu^{\mathscr{G}r}(\beta,\alpha_\nu),\sup_{\beta\ge\alpha}\max_{\alpha}\ \nu^{\mathscr{G}r}(\beta,\alpha_\nu)\Big].$$

Definition 2.10. [18] *The fuzzy valued function* $f:[a,b]\subset\Re\to\mathcal{F}(\Re^d)$ *is said to be granular differentiable or* $\mathscr{G}r$*-differentiable at a point* $\mathscr{X}_0\in[a,b]$ *if there exists an element* ${}^{\mathscr{G}r}\mathcal{D}\in\mathcal{F}(\Re^d)$ *such that the following limit,*

$${}^{\mathscr{G}r}\mathcal{D}f(\mathscr{X}_0)=\lim_{h\longrightarrow 0}\frac{f(\mathscr{X}_0+h)-f(\mathscr{X}_0)}{h},$$

exists for h *sufficiently near* 0. *In this case, we call* ${}^{\mathscr{G}r}\mathcal{D}$ *the granular derivative (*$\mathscr{G}r$*-derivative) of fuzzy valued function* f *at the point* $\mathscr{X}_0$. *We say that* f *is* $\mathscr{G}r$*-differentiable on* $[a,b]$ *if the* $\mathscr{G}r$*-derivative* ${}^{\mathscr{G}r}\mathcal{D}$ *exists for all points* $\mathscr{X}_0\in[a,b]$. *The fuzzy valued function* ${}^{\mathscr{G}r}\mathcal{D}:[a,b]\to\mathcal{F}(\Re^d)$ *is then called* $\mathscr{G}r$*-derivative of* f *on* $[a,b]$.
Then, we denote $C^1([a,b],\mathcal{F}(\Re^d))$ *the space of all continuously* $\mathscr{G}r$*-differentiable fuzzy valued functions on* $\mathbb{U}$.

Proposition 2.11. [18] *The fuzzy number* $F:[a,b]\to\mathcal{F}(\Re)$ *is granular differentiable and the function* $f(\theta)={}^{\mathscr{G}r}\mathcal{D}F(\theta)$ *is continuous on* $[a,b]$. *Then* $\int_a^b f(\theta)d\theta=F(b)\ominus^{\mathscr{G}r}F(a)$.

Definition 2.12. [8] *The fuzzy number* $F:[a,b]\to\mathcal{F}(\Re^d)$ *is granular differentiable and the function* $f(\theta)={}^{\mathscr{G}r}\mathcal{D}F(\theta)$ *is continuous on* $[a,b]$. *The fuzzy integral* $\int_a^b f(\theta)d\theta$, $0<\alpha\le 1$ *is defined by* $[\int_a^b[f(\theta)d\theta]]^{\alpha}=[\int_a^b f_l^{\alpha}(\theta)d\theta,\int_a^b f_r^{\alpha}(\theta)d\theta]$, *provided that the Lebesgue integrals on the right.*

Definition 2.13. [18] *The granular distance between* $\mathscr{X}$ *and* $\mathscr{Y}$ *in* $\mathcal{F}(\Re^d)$ *is the function* ${}^{\mathscr{G}r}D:\mathcal{F}(\Re^d)\times\mathcal{F}(\Re^d)\to[0,\infty)$ *given by*

$${}^{\mathscr{G}r}D(\mathscr{X},\mathscr{Y})=\sup_{\alpha}\max_{\alpha_{\mathscr{X}},\alpha_{\mathscr{Y}}}|\mathscr{X}^{\mathscr{G}r}(\alpha,\alpha_{\mathscr{X}})-\mathscr{Y}^{\mathscr{G}r}(\alpha,\alpha_{\mathscr{Y}})|.$$

We note that the space $\mathcal{F}(\Re^d)$ *endowed with supremum metric* ${}^{\mathscr{G}r}D$ *is a complete metric space.*

$$C(J,\mathcal{F}(\Re^d)):=\{g:J\to\mathcal{F}(\Re^d)\ \text{ is continuous}\}.$$

Definition 2.14. [22] *Let* $g:[a,b]\subset\Re^d\to\mathcal{F}(\Re^d)$, *for* $0<q\le 1$, *then the left and right sided fuzzy RL integral of fuzzy valued* f *is defined as*

$$\begin{aligned}{}^{\mathscr{G}r}I_{a+}^{q}g(\theta) &= \frac{1}{\Gamma(q)}\int_a^{\theta}(\theta-t)^{q-1}g(t)dt,\\ {}^{\mathscr{G}r}I_{b-}^{q}g(\theta) &= \frac{1}{\Gamma(q)}\int_{\theta}^{b}(t-\theta)^{q-1}g(t)dt.\end{aligned}$$

Definition 2.15. [22] *Let $g : [a,b] \subset \Re^d \to \mathcal{F}(\Re^d)$ for $0 < q \leq 1$, then the left and right sided fuzzy granular Caputo derivatives of fuzzy valued function is defined as*

$$\begin{aligned} {}^{\mathscr{g}_r}\mathcal{D}^q_{a+}g(\theta) &= \frac{1}{\Gamma(1-q)}\int_a^\theta \frac{(g'_{\mathscr{g}_r})(t)}{(\theta-t)^q}dt = {}^{\mathscr{g}_r}I^{1-q}_{a+}(g'_{\mathscr{g}_r}\theta), \\ {}^{\mathscr{g}_r}\mathcal{D}^q_{b-}g(\theta) &= -\frac{1}{\Gamma(1-q)}\int_a^\theta \frac{(g'_{\mathscr{g}_r})(t)}{(t-\theta)^q}dt = -{}^{\mathscr{g}_r}I^{1-q}_{b-}(g'_{\mathscr{g}_r}\theta). \end{aligned}$$

Definition 2.16. [19] *A two parameter function of the ML is defined by the series expansion*

$$M_{q,p}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(kq+p)},\ z \in C.$$

The Laplace integral of ML function is

$$\int_0^\infty e^{-s\theta}\theta^{q-1}M_{q,p}(\pm A\theta^q)d\theta = \frac{s^{q-p}}{(s^qI \mp A)}.$$

That is

$$L\{\theta^{q-1}M_{q,p}(\pm A\theta^q)\}(s) = \frac{s^{q-p}}{(s^qI \mp A)}.$$

Lemma 2.17. [17] *Let $p \geq 1$. If $\mathscr{X}, \mathscr{Y} \in L^p(J, \mathcal{F}(\Re^n))$, then for every $\theta \in J$ we have*

$$\mathbb{E}\ \sup_{s\in J} d^p_\infty\Big(\int_0^t \mathscr{X}(s)ds, \int_0^\theta \mathscr{Y}(s)ds\Big) \leq \theta^{p-1}\int_0^\theta \mathbb{E}\ d^p_\infty\big(\mathscr{X}(s), \mathscr{Y}(s)\big)ds.$$

3 Solution representation

Consider the following nonlinear fuzzy FSPDDS as follows

$$\begin{aligned} {}^{\mathscr{g}_r}\mathcal{D}^q\mathscr{X}(\theta) &= A\mathscr{X}(\theta) + \lambda(\theta, \mathscr{X}(\theta-h)) + \Big\langle \int_0^\theta \Delta(s, \mathscr{X}(s-h))dw(s)\Big\rangle,\ \theta \in J := [0,T], \\ \mathscr{X}(\theta) &= \psi(\theta),\ \theta \in [-h, 0], \end{aligned} \tag{1}$$

where $0 < q < 1$, A is $n \times n$ matrix, the nonlinear functions $\lambda : J \times C_h \to \mathcal{F}(\Re^d)$, $\Delta : J \times C_h \to \Re^d \times \Re^m$ are continuous on J and define for each $\mathscr{X} \in C_h$, $\mathscr{X}_\theta(s) = \mathscr{X}(\theta+s)$ for $s \in [-h, 0]$.

Taking the Laplace transform on (1) with respect to θ on both sides, we get

$$\begin{aligned} L\Big[{}^{\mathscr{g}_r}\mathcal{D}^q\mathscr{X}(\theta)\Big]^\alpha &= AL\Big[\mathscr{X}(\theta)\Big]^\alpha + L\Big[\lambda(\theta, \mathscr{X}(\theta-h))\Big]^\alpha + L\Big[\Big\langle \int_0^\theta \Delta(s, \mathscr{X}(s-h))dw(s)\Big\rangle\Big]^\alpha, \\ [s^q\hat{\mathscr{X}}(s) - s^{q-1}\psi(0)]^\alpha &= A[\hat{\mathscr{X}}(s)]^\alpha + [\hat{\lambda}(s)]^\alpha + [\hat{\Delta}(s)]^\alpha \\ [(s^qI - A)\hat{\mathscr{X}}(s)]^\alpha &= [s^{q-1}\psi(0)]^\alpha + [\hat{\lambda}(s)]^\alpha + [\hat{\Delta}(s)]^\alpha, \end{aligned}$$

where I is an identity matrix.

$$[\hat{\mathscr{X}}(s)]^\alpha = \frac{s^{q-1}}{(s^qI - A)}[\psi(0)]^\alpha + \frac{1}{(s^qI - A)}[\hat{\lambda}(s)]^\alpha + \frac{1}{(s^qI - A)}[\hat{\Delta}(s)]^\alpha.$$

Applying inverse Laplace transform on both sides, we have

$$\begin{aligned} L^{-1}\Big[\hat{\mathscr{X}}(s)\Big]^\alpha &= L^{-1}\Big\{(s^{q-1})(s^qI - A)^{-1}\Big\}[\psi(0)]^\alpha + L^{-1}\Big\{(s^qI - A)^{-1}\Big\} * L^{-1}\Big\{[\hat{\lambda}(s)]^\alpha\Big\} \\ &\quad + L^{-1}\Big\{(s^qI - A)^{-1}\Big\} * L^{-1}\Big\{[\hat{\Delta}(s)]^\alpha\Big\}. \end{aligned}$$

Finally, substituting fuzzy Laplace transformation of ML function, we get the solution of system (1) as

$$\begin{aligned}[\mathscr{X}(\theta)]^{\alpha} &= M_q(A\theta^q)[\psi(0)]^{\alpha} + \int_0^{\theta}(\theta-s)^{q-1}M_{q,q}(A(\theta-s)^q)[\lambda(s,\mathscr{X}(s-h))]^{\alpha}ds \\ &\quad + \int_0^{\theta}(\theta-s)^{q-1}M_{q,q}(A(\theta-s)^q)\Big\langle\int_0^{s}[\Delta(\tau,\mathscr{X}(\tau-h))]^{\alpha}dw(\tau)\Big\rangle ds,\end{aligned}$$

or

$$\begin{aligned}\mathscr{X}(\theta) &= M_q(A\theta^q)\psi(0) + \int_0^{\theta}(\theta-s)^{q-1}M_{q,q}(A(\theta-s)^q)\lambda(s,\mathscr{X}(s-h))ds \\ &\quad + \int_0^{\theta}(\theta-s)^{q-1}M_{q,q}(A(\theta-s)^q)\Big\langle\int_0^{s}\Delta(\tau,\mathscr{X}(\tau-h))dw(\tau)\Big\rangle ds,\end{aligned}$$

where $\mathscr{X}(\theta) = [\mathscr{X}(\theta)]^{\alpha} = [\mathscr{X}_l(\theta), \mathscr{X}_r(\theta)]$, $\psi(0) = [\psi(0)]^{\alpha} = [\psi(0)_l, \psi(0)_r]$,
$\lambda(s,\mathscr{X}(s-h)) = [\lambda(s,\mathscr{X}(s-h))]^{\alpha} = [\lambda(s,\mathscr{X}(s-h))_l, \lambda(s,\mathscr{X}(s-h))_r]$,
$\Delta(\tau,\mathscr{X}(\tau-h)) = [\Delta(\tau,\mathscr{X}(\tau-h))]^{\alpha} = [\Delta(\tau,\mathscr{X}(\tau-h))_l, \Delta(\tau,\mathscr{X}(\tau-h))_r]$. In order to prove the main result, we assume the following assumptions hold.

(A1): The functions λ and Δ are continuous and there exists a constant $\gamma > 1$ and the functions $M_{\lambda}(\cdot)$ and $M_{\Delta}(\cdot) \in L^{\gamma}(J,\Re^+)$ such that

(i) $d^2_{\infty}[\lambda(\theta,\mathscr{X}(\theta-h)),\lambda(\theta,\mathscr{Y}(\theta-h))] \le M_{\lambda}(\theta)\ d^2_{\infty}[\mathscr{X},\mathscr{Y}]$,

(ii) $\|\Delta(\theta,\mathscr{X}(\theta-h)) - \Delta(\theta,\mathscr{Y}(\theta-h))\|^2 \le M_{\Delta}(\theta)\ d^2_{\infty}[\mathscr{X},\mathscr{Y}]$.

(A2): There exist $N_1 \ge 1$, $N_2 \ge 1$ such that for $\theta \ge 0$, the one and two parameters ML functions satisfy

(i) $d^2_{\infty}[M_q(A\theta^q),\langle 0\rangle] \le N_1 e^{\|A\|\theta}$,

(ii) $d^2_{\infty}[M_{q,q}(A(\theta-s)^q),\langle 0\rangle] \le N_2 e^{\|A\|(\theta-s)}$.

In addition, set

$$\begin{aligned}S_1 &= 6\frac{T^{2q+1}}{q^2}N_2\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)^{\frac{1}{\beta}}\Big[\|M_{\lambda}\|_{L^{\gamma}(J,\Re^+)} + \|M_{\Delta}\|_{L^{\gamma}(J,\Re^+)}\Big], \\ S_2 &= 6\frac{T^{2q+1}}{q^2}N_2\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)^{\frac{1}{\beta}}[R_{\lambda} + R_{\Delta}].\end{aligned}$$

Main result:

Theorem 3.1. *Assume that assumptions (A1)-(A2) hold. Then the system (1) has at least one solution provided that*

$$P = 2\frac{T^{2q+1}}{q^2}N_2\left(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\right)^{\frac{1}{\beta}}\Big[\|M_{\lambda}\|_{L^{\gamma}(J,\Re^+)} + \|M_{\Delta}\|_{L^{\gamma}(J,\Re^+)}\Big] < 1,$$

where $\frac{1}{\beta} + \frac{1}{\gamma} = 1,\ \beta,\gamma > 1$.

Proof. For each positive number a, define $\mathcal{B}_a = \{\mathscr{X}(\theta) : \mathscr{X}(\theta) \in \mathcal{B}_h : \mathbb{E}\ d^2_{\infty}(\mathscr{X},\langle 0\rangle) \le a\}$. Then for each a, $\mathcal{B}_a$ is obviously a bounded, closed and convex set of $\mathcal{B}$. We set $R_{\lambda} = d^2_{\infty}[\lambda(\theta,\langle 0\rangle),\langle 0\rangle]$ and $R_{\Delta} = \sup_{\theta\in J}\mathbb{E}\|\Delta(\theta,\langle 0\rangle)\|^2$. Define $\Phi : \mathcal{B}_a \to \mathcal{B}_a$ as

$$\begin{aligned}(\Phi\mathscr{X})(\theta) &= M_q(A\theta^q)\psi(0) + \int_0^{\theta}(\theta-s)^{q-1}M_{q,q}(A(\theta-s)^q)\lambda(s,\mathscr{X}(s-h))ds \\ &\quad + \int_0^{\theta}(\theta-s)^{q-1}M_{q,q}(A(\theta-s)^q)\Big\langle\int_0^{s}\Delta(\tau,\mathscr{X}(\tau-h))dw(\tau)\Big\rangle ds.\end{aligned}$$

Step 1. We prove that there exist a positive number a such that $\Phi(\mathcal{B}_a) \subseteq \mathcal{B}_a$.

$$\begin{aligned}
\mathbb{E}\, d_\infty^2[(\Phi\mathscr{X})(\theta), \langle 0\rangle] &= \mathbb{E}\, d_\infty^2\Big[M_q(A\theta^q)\psi(0) + \int_0^\theta (\theta-s)^{q-1} M_{q,q}(A(\theta-s)^q)\lambda(s, \mathscr{X}(s-h))ds \\
&\quad + \int_0^\theta (\theta-s)^{q-1} M_{q,q}(A(\theta-s)^q)\Big\langle \int_0^s \Delta(\tau, \mathscr{X}(\tau-h))dw(\tau)\Big\rangle ds, \langle 0\rangle\Big] \\
&\leq 3\Big\{\mathbb{E}\, d_\infty^2[M_q(A\theta^q)\psi(0), \langle 0\rangle] + \mathbb{E}\, d_\infty^2\Big[\int_0^\theta (\theta-s)^{q-1} M_{q,q}(A(\theta-s)^q) \\
&\quad \times\lambda(s, \mathscr{X}(s-h))ds, \langle 0\rangle\Big] + \mathbb{E}\, d_\infty^2\Big[\int_0^\theta (\theta-s)^{q-1} M_{q,q}(A(\theta-s)^q) \\
&\quad \times\Big\langle \int_0^s \Delta(\tau, \mathscr{X}(\tau-h))dw(\tau)\Big\rangle ds, \langle 0\rangle\Big]\Big\} \\
&= 3[I_1 + I_2 + I_3].
\end{aligned}$$

Now using (A2) (i), one can estimate

$$\begin{aligned}
I_1 &= \mathbb{E}\, d_\infty^2[M_q(A\theta^q)\psi(0), \langle 0\rangle] \\
&\leq N_1 e^{2\|A\|\theta}\mathbb{E}\|\psi(0)\|^2.
\end{aligned}$$

Using (A1) (i), (A2) (i), Lemma 2.17 together with Holder's inequality, we have

$$\begin{aligned}
I_2 &= \mathbb{E}\, d_\infty^2\Big[\int_0^\theta (\theta-s)^{q-1} M_{q,q}(A(\theta-s)^q)\lambda(s, \mathscr{X}(s-h))ds, \langle 0\rangle\Big] \\
&\leq T\int_0^\theta \big((\theta-s)^{q-1}\big)^2 ds \int_0^\theta \Big(d_\infty^2[M_{q,q}(A(\theta-s)^q), \langle 0\rangle] \\
&\quad \times\mathbb{E}\, d_\infty^2[\lambda(s, \mathscr{X}(s-h)), \langle 0\rangle]\Big)ds \\
&\leq \frac{T^{2q+1}}{q^2} N_2 \int_0^\theta e^{2\|A\|(\theta-s)}\mathbb{E}\, \Big(d_\infty^2[\lambda(s, \mathscr{X}(s-h)), \lambda(s, \langle 0\rangle) + d_\infty^2(\lambda(s, \langle 0\rangle), \langle 0\rangle)]\Big)ds \\
&\leq 2\frac{T^{2q+1}}{q^2} N_2\Big\{\int_0^\theta e^{2\|A\|(\theta-s)}\Big(\mathbb{E}\, d_\infty^2[\lambda(s, \mathscr{X}(s-h)), \lambda(s, \langle 0\rangle)]ds \\
&\quad + \int_0^\theta e^{2\|A\|(\theta-s)}\mathbb{E}\, d_\infty^2[\lambda(s, \langle 0\rangle), \langle 0\rangle]\Big)ds\Big\} \\
&\leq 2\frac{T^{2q+1}}{q^2} N_2\Big\{\Big(\int_0^\theta e^{2\beta\|A\|(\theta-s)}ds\Big)^{\frac{1}{\beta}}\Big(\int_0^\theta M_\lambda^\gamma(s)ds\Big)^{\frac{1}{\gamma}}\, \mathbb{E}\, d_\infty^2[\mathscr{X}, \langle 0\rangle] + \Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)R_\lambda\Big\} \\
&\leq 2\frac{T^{2q+1}}{q^2} N_2\Big\{\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)^{\frac{1}{\beta}}\|M_\lambda\|_{L^\gamma(J,\Re^+)}a + \Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)R_\lambda\Big\}.
\end{aligned}$$

Using (A1) (ii), (A2) (ii), Lemma 2.17 and Holder's inequality, we obtain

$$\begin{aligned}
I_3 &= \mathbb{E}\, d_\infty^2\Big[\int_0^\theta (\theta-s)^{q-1} M_{q,q}(A(\theta-s)^q)\Big\langle \int_0^s \Delta(\tau, \mathscr{X}(\tau-h))dw(\tau)\Big\rangle ds, \langle 0\rangle\Big] \\
&\leq \int_0^\theta \big((\theta-s)^{q-1}\big)^2 ds \int_0^t \Big(d_\infty^2[M_{q,q}(A(\theta-s)^q), \langle 0\rangle]\ \mathbb{E}\, \Big\|\int_0^s \Delta(\tau, \mathscr{X}(\tau-h))dw(\tau)\Big\|^2 ds \\
&\leq \frac{T^{2q+1}}{q^2} N_2 \int_0^\theta e^{2\|A\|(\theta-s)}\Big(\mathbb{E}\|\Delta(s, \mathscr{X}(s-h)) - \Delta(s, \langle 0\rangle) + \Delta(s, \langle 0\rangle)\|^2\Big)ds \\
&\leq 2\frac{T^{2q+1}}{q^2} N_2\Big\{\int_0^\theta e^{2\|A\|(\theta-s)}\Big(\mathbb{E}\|\Delta(s, \mathscr{X}(s-h)) - \Delta(s, \langle 0\rangle)\|^2 \\
&\quad + \int_0^\theta e^{2\|A\|(\theta-s)}\mathbb{E}\|\Delta(s, \langle 0\rangle)\|^2\Big)ds\Big\}
\end{aligned}$$

$$\begin{aligned}
&\leq 2\frac{T^{2q+1}}{q^2}N_2\Big\{\Big(\int_0^\theta e^{2\beta\|A\|(\theta-s)}ds\Big)^{\frac{1}{\beta}}\Big(\int_0^\theta M_\Delta^\gamma(s)ds\Big)^{\frac{1}{\gamma}}\ \mathbb{E}\ d_\infty^2[\mathscr{X},\langle 0\rangle]+\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)R_\Delta\Big\}\\
&\leq 2\frac{T^{2q+1}}{q^2}N_2\Big\{\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)^{\frac{1}{\beta}}\|M_\Delta\|_{L^\gamma(J,\Re^+)}a+\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)R_\Delta\Big\}.
\end{aligned}$$

Altogether, we have

$$\begin{aligned}
\mathbb{E}\ d_\infty^2[(\Phi\mathscr{X})(\theta),\langle 0\rangle] &= 3\Big\{N_1e^{2\|A\|\theta}\mathbb{E}\|\psi(0)\|^2+2\frac{T^{2q+1}}{q^2}N_2\Big\{\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)^{\frac{1}{\beta}}\|M_\lambda\|_{L^\gamma(J,\Re^+)}a\\
&\quad+\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)R_\lambda\Big\}+2\frac{T^{2q+1}}{q^2}N_2\Big\{\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)^{\frac{1}{\beta}}\|M_\Delta\|_{L^\gamma(J,\Re^+)}a\\
&\quad+\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)R_\Delta\Big\}\Big\}\\
&\leq 3N_1e^{2\|A\|T}\mathbb{E}\|\psi(0)\|^2+S_1a+S_2:=a,
\end{aligned}$$

where

$$a = \frac{3N_1e^{2\|A\|T}\mathbb{E}\|\psi(0)\|^2+S_2}{1-S_1},\ S_1<1.$$

Hence, we obtain $\Phi(\mathcal{B}_a)\subseteq\mathcal{B}_a$.

Step 2. To prove Φ is a contraction map.

Let $\mathscr{X},\mathscr{Y}\in\mathcal{B}_a$, for each $\theta\in J$, we have

$$\begin{aligned}
\mathbb{E}\ d_\infty^2[(\Phi\mathscr{X})(\theta),(\Phi\mathscr{Y})(\theta)] &\leq 2\Big\{\mathbb{E}\ d_\infty^2\Big[\int_0^\theta(\theta-s)^{q-1}M_{q,q}(A(\theta-s)^q)[\lambda(s,\mathscr{X}(s-h),\lambda(s,\mathscr{Y}(s-h))]ds\Big]\\
&\quad+\mathbb{E}\ d_\infty^2\Big[\int_0^\theta(\theta-s)^{q-1}M_{q,q}(A(\theta-s)^q)\\
&\quad\times\Big\langle\int_0^s[\Delta(\tau,\mathscr{X}(\tau-h)),\Delta(\tau,\mathscr{Y}(\tau-h))]dw(\tau)\Big\rangle ds\Big]\\
&\leq 2\Big\{\frac{T^{2q+1}}{q^2}N_2\Big(\int_0^\theta e^{2\beta\|A\|(t-s)}ds\Big)^{\frac{1}{\beta}}\Big(\int_0^\theta M_\lambda^\gamma(s)ds\Big)^{\frac{1}{\gamma}}\ \mathbb{E}\ d_\infty^2(\mathscr{X},\mathscr{Y})\\
&\quad+\frac{T^{2q+1}}{q^2}N_2\Big(\int_0^\theta e^{2\beta\|A\|(t-s)}ds\Big)^{\frac{1}{\beta}}\Big(\int_0^\theta M_\Delta^\gamma(s)ds\Big)^{\frac{1}{\gamma}}\ \mathbb{E}\ d_\infty^2(\mathscr{X},\mathscr{Y})\\
&\leq 2\frac{T^{2q+1}}{q^2}N_2\Big\{\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)^{\frac{1}{\beta}}\|M_\lambda\|_{L^\gamma(J,\Re^+)}\mathbb{E}\ d_\infty^2(\mathscr{X},\mathscr{Y})\Big\}\\
&\quad+2\frac{T^{2q+1}}{q^2}N_2\Big\{\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)^{\frac{1}{\beta}}\|M_\Delta\|_{L^\gamma(J,\Re^+)}\mathbb{E}\ d_\infty^2(\mathscr{X},\mathscr{Y})\Big\}\\
&\leq 2\frac{T^{2q+1}}{q^2}N_2\Big(\frac{e^{2\beta\|A\|T}-1}{2\beta\|A\|}\Big)^{\frac{1}{\beta}}\Big[\|M_\lambda\|_{L^\gamma(J,\Re^+)}+\|M_\Delta\|_{L^\gamma(J,\Re^+)}\Big]\mathbb{E}\ d_\infty^2(\mathscr{X},\mathscr{Y})\\
&\leq P\ \mathbb{E}\ d_\infty^2(\mathscr{X},\mathscr{Y}).
\end{aligned}$$

Thus Φ is a contraction mapping, it has a unique fixed point $\mathscr{X}(\theta)\in\mathcal{B}_a$ which is a solution of (1). □

4 Computational examples

Example 4.1. *Consider the following fuzzy fractional pantograph stochastic delay DE described by*

$$\begin{aligned}
{}^{\mathscr{g}r}\mathcal{D}^q\mathscr{X}(t) &= 0.1\mathscr{X}(t)+0.6\mathscr{Y}(t)+\frac{\mathscr{X}(t)\cos(t-h)}{5}+\Delta_1\mathscr{X}(t)dw_1(t),\\
{}^{\mathscr{g}r}\mathcal{D}^q\mathscr{Y}(t) &= -0.2\mathscr{X}(t)-0.5\mathscr{Y}(t)+\frac{\mathscr{Y}(t)\sin(t-h)}{10}+\Delta_2\mathscr{Y}(t)dw_1(t),\ t\in[0,1],\\
\mathscr{X}(t) &= \psi(t),\ t\in[-0.2,0].
\end{aligned}\tag{2}$$

Here

$$A=\begin{pmatrix} 0.1 & 0.6 \\ -0.2 & -0.5 \end{pmatrix},\quad \lambda(t,\mathscr{X}(t),\mathscr{Y}(t))=\begin{pmatrix} \frac{\cos(t-h)}{5} \\ \frac{\sin(t-h)}{10} \end{pmatrix},\quad \Delta(t,\mathscr{X}(t),\mathscr{Y}(t))=\begin{pmatrix} \Delta_1 dw_1(t) \\ \Delta_2 dw_1(t) \end{pmatrix},$$

where $t=0.5,\ h=0.2,\ q=0.9,\ \ \Delta_1=\Delta_2=\frac{1}{100}$.
Fuzzy initial conditions are given by

$$\begin{aligned}\mathscr{X}(0,r) &= [\underline{\mathscr{X}}(0),\bar{\mathscr{X}}(0)],\\ \mathscr{Y}(0,r) &= [\underline{\mathscr{Y}}(0),\bar{\mathscr{Y}}(0)],\ r\in[0,1].\end{aligned}$$

Using r-cut, the initial condition becomes

$$\begin{aligned}\mathscr{X}(0,r) &= [r(\mathfrak{b}_1-\mathfrak{a}_1)+\mathfrak{a}_1,\mathfrak{c}_1-r(\mathfrak{c}_1-\mathfrak{b}_1)],\\ \mathscr{Y}(0,r) &= [r(\mathfrak{b}_2-\mathfrak{a}_2)+\mathfrak{a}_2,\mathfrak{c}_2-r(\mathfrak{c}_2-\mathfrak{b}_2)].\end{aligned}$$

$\mathfrak{a}_1=0.1,\ \mathfrak{b}_1=0.2,\ \mathfrak{c}_1=0.3$ *and* $\mathfrak{a}_2=0.3,\ \mathfrak{b}_2=0.6,\ \mathfrak{c}_2=0.9$.

Table 1: Fuzzy solution for $q=0.9$

$t\to$	$r=0.1$	$r=0.2$	$r=0.3$	$r=0.4$	$r=0.5$
$[\underline{x}(t,r),\bar{x}(t,r)]$	$[0.11,0.29]$	$[0.12,0.28]$	$[0.13,0.27]$	$[0.14,0.26]$	$[0.15,0.25]$
$[\underline{y}(t,r),\bar{y}(t,r)]$	$[0.33,0.87]$	$[0.36,0.84]$	$[0.39,0.81]$	$[0.42,0.78]$	$[0.45,0.75]$

Table 2: Fuzzy solution for $q=0.9$

$t\to$	$r=0.6$	$r=0.7$	$r=0.8$	$r=0.9$	$r=1$
$[\underline{x}(t,r),\bar{x}(t,r)]$	$[0.16,0.24]$	$[0.17,0.23]$	$[0.18,0.22]$	$[0.19,0.21]$	$[0.2,0.2]$
$[\underline{y}(t,r),\bar{y}(t,r)]$	$[0.48,0.72]$	$[0.51,0.69]$	$[0.54,0.66]$	$[0.57,0.63]$	$[0.6,0.6]$

Also, it is easy to verify that for any $\mathscr{X},\mathscr{Y}\in\Re^2$

$$\begin{aligned}\mathbb{E}\ d^2_\infty[\lambda(t,\mathscr{X}(t-h)),\lambda(t,\mathscr{Y}(t-h))] &\le \frac{1}{25}\cos(0.7)\ \mathbb{E}\ d^2_\infty[\mathscr{X},\mathscr{Y}],\\ \mathbb{E}\|\Delta(t,\mathscr{X}(t-h))-\Delta(t,\mathscr{Y}(t-h))\|^2 &\le \frac{1}{100}\ \mathbb{E}\ d^2_\infty[\mathscr{X},\mathscr{Y}].\end{aligned}$$

By simple calculation, one can obtain that, $\|M_\lambda(t)\|_{L^\gamma(J,\Re^+)}=0.03,\ \ \|M_\Delta(t)\|_{L^\gamma(J,\Re^+)}=0.3$.
Hence choose $N_2=1,\ \beta=1.2$, *one can get*

$$P=2.469\Big(\frac{6.027}{1.9536}\Big)^{\frac{1}{1.2}}[0.03+0.01]=0.2525<1.$$

Thus all the conditions of Theorem 3.1 holds, (2) admits a unique solution.

Remark 4.2. *In order to get the solution to be a fuzzy number, the r-cut values should be taken in the interval* $[0,1]$. *And also A matrix norm should be less than 1. Further, it should be mentioned that the considered system is itself a fuzzy system its r-cut values always be a fuzzy number. Hence the solution will always be a fuzzy number.*

Remark 4.3. *Impulsive fuzzy fractional DEs have been studied in* [37]. *Several authors derived the fuzzy fractional and fuzzy stochastic DEs see* [5, 14, 29]. *We utilize the approach of* $\mathscr{G}r$*-differentiability to get a unique solution see* [22, 27, 31]. *Then we obtained the existence and uniqueness of fuzzy FSPDDS by using Holder's inequality in Banach Contraction principle. We made a first attempt to study the fuzzy FSPDDS.*

Remark 4.4. *Figures 1 and 2 indicated that the stochastic behavior of the fuzzy system (2) with r-cut representation where r takes the values* $r=0.1,0.2,\ldots,0.9$. *Figures 1(a) and 2(a) represent the lower and upper level of the system with time responses of* $x(t)$ *and* $y(t)$, *Figures 1(b) and 2(b) shown the clear view of fuzziness and crispness for the proposed system (2).*

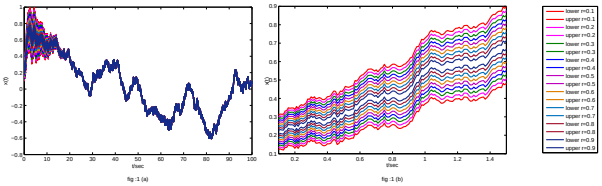


Figure 1: Time response of x(t) with fuzziness trajectory of the system (2)

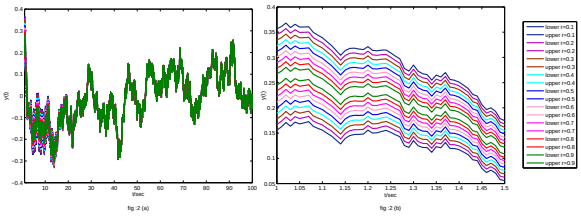


Figure 2: Time response of y(t) with fuzziness trajectory of the system (2)

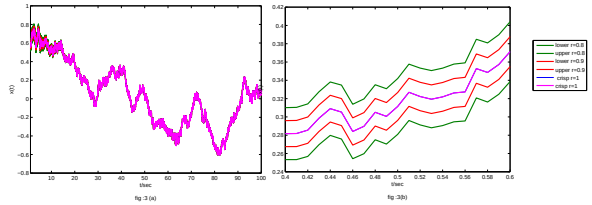


Figure 3: Time response of x(t) with fuzziness and crisp trajectory of the system (2)

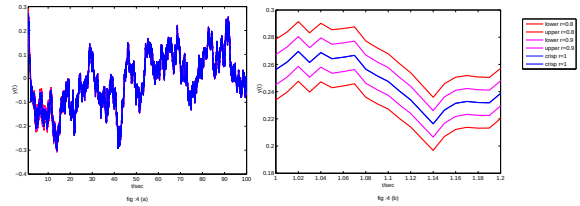


Figure 4: Time response of y(t) with fuzziness and crisp trajectory of the system (2)

Remark 4.5. *Figures 3 and 4 indicate the stochastic behavior of the fuzzy and crisp related system (2) with r-cut representation where r takes values r=0.8, 0.9 and 1. From the Figures 3(a) and 4(a), one can conclude that both lower and upper bounds of the fuzzy solutions are the same for* $r = 1$*, which is a crisp solution.The crisp solution lies between the lower and upper bounds of the fuzzy solutions. One can verify that for* $r = 1$*, the fuzzy initial condition is converted to crisp initial conditions. Figures 3 (b) and 4 (b) shown the clear view of fuzziness and crispness for the proposed system. (2)*

Example 4.6. *Consider the following fuzzy fractional stochastic delay DE described by*

$$\begin{aligned} {}^{g_r}\mathcal{D}^q \mathscr{X}(t) &= \mathcal{A}\mathscr{X}(t) + \mathcal{B}\mathscr{X}(t-\tau) + \lambda(t, \mathscr{X}(t), \mathscr{Y}(t)) + \Delta(t, \mathscr{X}(t), \mathscr{Y}(t)), \\ {}^{g_r}\mathcal{D}^q \mathscr{Y}(t) &= \mathcal{A}\mathscr{Y}(t) + \mathcal{B}\mathscr{Y}(t-\tau) + \lambda(t, \mathscr{X}(t), \mathscr{Y}(t)) + \Delta(t, \mathscr{X}(t), \mathscr{Y}(t)), \\ \mathscr{X}(t) &= \mathscr{X}_0(t), \ t \in [0, 1]. \end{aligned} \tag{3}$$

Here

$$\mathcal{A} = \begin{pmatrix} 0.05 & 0.3 \\ -0.1 & -0.25 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0.05 & 0.3 \\ -0.1 & -0.25 \end{pmatrix}, \quad \lambda(t, \mathscr{X}(t), \mathscr{Y}(t)) = \begin{pmatrix} \frac{\cos(t-h)}{5} \\ \frac{\sin(t-h)}{10} \end{pmatrix},$$

$$\Delta(t, \mathscr{X}(t), \mathscr{Y}(t)) = \begin{pmatrix} \frac{1}{100} dw_1(t) \\ \frac{1}{100} dw_1(t) \end{pmatrix},$$

where $t = 0.5, \tau = 200,\ h = 0.2,\ q = 0.9.$

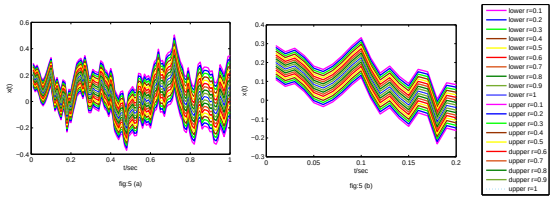


Figure 5: Time response of x(t) with fuzziness and crisp trajectory of the system (3) with delay

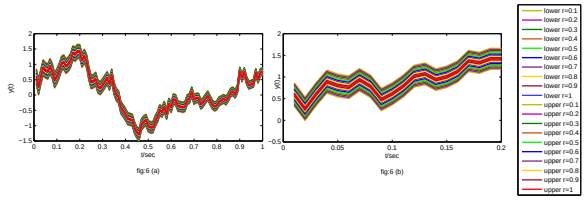


Figure 6: Time response of y(t) with fuzziness and crisp trajectory of the system (3) with delay

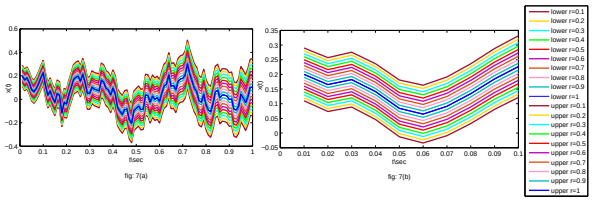


Figure 7: Time response of x(t) with fuzziness and crisp trajectory of the system (3) without delay

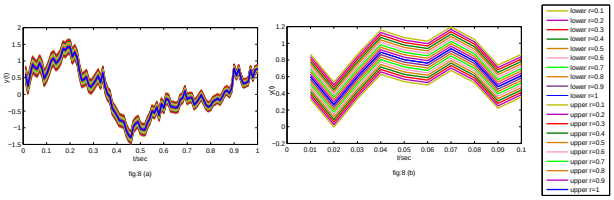


Figure 8: Time response of y(t) with fuzziness and crisp trajectory of the system (2) without delay

Remark 4.7. *Figures 5, 6, 7, 8 indicate that the stochastic behavior of the fuzzy system (3) with r-cut representation where r takes the values* $r = 0.1, 0.2, \ldots, 1$. *Figures 5(a) and 6(a) represent the lower and upper level of the system with time responses of* $x(t)$ *and* $y(t)$ *with delay. Figures 7(a) and 8(a) represent the lower and upper level of the system with time responses of* $x(t)$ *and* $y(t)$ *without delay. Figures 5(b), 6(b), 7(b) and 8(b) shown the clear view of fuzziness and crispness for the proposed system.*

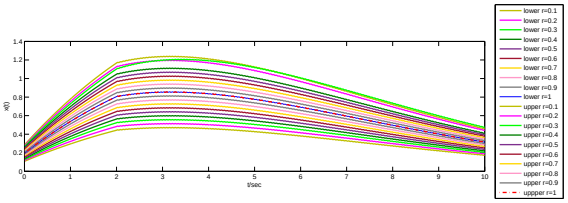


Figure 9: Time response of x(t) with fuzziness and crisp trajectory of the system (3) without stochastic

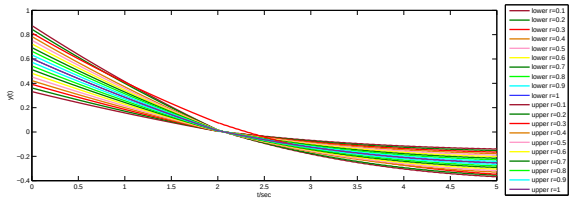


Figure 10: Time response of y(t) with fuzziness and crisp trajectory of the system (3) without stochastic

Remark 4.8. *Figures 9 and 10 indicate the r-cut representation of the fuzzy and crisp of the system (3) where r takes values* $r = 0.1, 0.2, \ldots 1$. *Due to nature of stochasticity, from Figures 5(a), 6(a), 7(a) and 8 (a), one can observe that the nature of the system response is not smooth when compared to Figures 9 and 10.*

Remark 4.9. *From the Figures 5-10, we conclude that both lower and upper bounds of the fuzzy solutions are the same for* $r = 1$, *which is a crisp solution.The crisp solution lies between the lower and upper bounds of the fuzzy solutions. One can see that for* $r = 1$, *the fuzzy initial condition is converted to crisp initial conditions.*

Remark 4.10. *For the neutrosophic control systems, the necessary and sufficient conditions have been established by Son et.al in* [33]. *Here, it guarantees the controllability and stabilizability of the system by the connection of granular control systems via horizontal membership functions. The concept of single valued neutrosophic functions related with granular calculus has been introduced in* [34]. *Using granular approach one can see some applications like Spring Mass Damper and the inverted pendulum exist for the system as described in* [34]. *Clustering is one of the important area for application in along with a fuzzy set theory. Based on neutrosophic association matrix fuzzy clustering have been derived in* [12]. *Inspired by the above works, we will propose to study neutrosophic systems with granular differentiability in future.*

5 Conclusion

This paper dealt with a fuzzy FSPDDS in the sense of $\mathscr{G}r$ differentiability and we derived the existence and uniqueness of solutions by Banach contraction principle with some weaker conditions. Finally the effectiveness of the theoretical result has been demonstrated through the numerical simulation. We have seen that the lower and upper solutions for $r = 1$ are equal as we have expected. It is noted that when we increase the fractional order lower and upper bounds of the uncertain solutions of $\mathscr{X}$ and $\mathscr{Y}$ lead respectively gradual increase and decrease in time.

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