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A fuzzy non-parametric time series model based on fuzzy data

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Abstract

Parametric time series models typically consists of model identification, parameter estimation, model diagnostic checking, and forecasting. However compared with parametric methods, nonparametric time series models often provide a very flexible approach to bring out the features of the observed time series. This paper suggested a novel fuzzy nonparametric method in time series models with fuzzy observations. For this purpose, a fuzzy forward fit kernel-based smoothing method was introduced to estimate fuzzy smooth functions corresponding to each observation. A simple optimization algorithm was also suggested to evaluate optimal bandwidths and autoregressive order. Several common goodness-of-fit criteria were also extended to compare the performance of the proposed fuzzy time series method compared to other fuzzy time series model based on fuzzy data. Furthermore, the effectiveness of the proposed method was illustrated through two numerical examples including a simulation study. The results indicate that the proposed model performs better than the previous ones in terms of both scatter plot criteria and goodness-of-fit evaluations.

Keywords: Goodness-of-fit criteria, additive, kernel function, fuzzy response, fuzzy time series, fuzzy smooth function.

1 Introduction

The time series prediction method has a wide range of applications in business, finance, computer science, engineering, medicine, physics, chemistry, and many interdisciplinary fields (e.g.[6, 7, 33, 40, 46, 52]). It worths noting that the traditional time series models lead to exact predictions. However, due to uncertain conditions of the future, it is better to predict a quantity via imprecise values such as fuzzy sets. In addition, the traditional time series models fail to deal with forecasting problems with vague or ambiguous observations represented by the linguistic concept. Such a shortcoming could be overcome by the fuzzy-based time series models. In this regard, fuzzy time series approaches, introduced by Song and Chissom [43], have successfully substituted the traditional ones. The fuzzy time series forecasting models have three main steps. In step 1, the exact data are reported. By identifying the fuzzy logical relationships, the forecasting values can be transformed via fuzzy quantities (based on their universe of discourse) in step 2. Step 3 offers a defuzzified approach [5, 13, 16, 28, 31, 34, 57] to convert the fuzzy quantities into exact values. The identifying fuzzy logical relationship techniques in step 2 mainly include fuzzy logical relationship groups and matrice [8, 11, 12, 15, 17, 37, 39, 42, 43, 45, 49, 50, 53], soft computing techniques [2, 3, 10, 19, 21, 41, 51, 54, 55, 56], and statistical techniques adopted with fuzzy logic [8, 16, 29, 38, 47]. Step 2 plays an important role in the prediction performance of the proposed model. Fuzzy time series models based on imprecise information have gained considerable attention in last decades due to their vast applications in statistics and engineering. Many researchers have studied a time series model based on imprecise information. Soft computing techniques used in this context are mainly a combination of fuzzy sets, artificial neural networks, rough sets and evolutionary computation. Such approaches have been widely employed for exact or fuzzy predictions based on exact historical data such as enrollment, stocks index prices, temperature, financial prediction and electricity load (for a comprehensive review on these methods, see recent works of [1, 14, 20, 22, 23, 30, 32, 35, 36]. Other fuzzy time series methods rely of fuzzy data. In this regards, [48] proposed a fuzzy seasonal ARIMA for non-fuzzy data in the cases where the future situations are predicted as fuzzy values.They suggested a fuzzy confidence region for

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future forecasting. In addition, [24] proposed a statistical time series models based on fuzzy data. They suggested a semi-parametric time series model with fuzzy data, non-fuzzy coefficients and fuzzy smooth functions. Also, Zarei et al. [58] applied a special version of Hesamian and Akbari model for triangular fuzzy data. The main contribution of this paper is the investigation of a fuzzy nonparametric time series model based on fuzzy data inspired by method of Hesamian and Akbari [26].

The purpose of this article is to extend a multiple nonlinear model for fuzzy time series data. For this purpose, an additive kernel smoothing approach is proposed to estimate unknown fuzzy smoothing nonlinear functions. The performance of the proposed method was also compared with the existing fuzzy time series techniques in terms of several goodness-of-fit criteria. For practical reasons, the proposed method was further evaluated through a real data and a simulation study. The numerical results indicate that the proposed method provides sufficiently accurate results in a fuzzy time series model compared to the other ones.

The rest of this paper is organized as follows: Section 2 reviews some concepts including α-values of fuzzy numbers and a generalized difference. In Section 3, an additive nonparametric fuzzy time series model with fuzzy smooth function is suggested. Section 4 illustrates two numerical examples to evaluate the performance of the proposed method in terms of some common performance measures. Finally, the main contributions of this paper will be summarized in Section 5.

2 Fuzzy numbers

This section reviews some basic definitions of fuzzy numbers which will be used in the proposed method. A fuzzy set $\widetilde{A}$ of $\mathbf{R}$ (the real line) is defined by its membership function $\mu_{\widetilde{A}} : \mathbf{R} \to [\mathbf{0}, \mathbf{1}]$. In addition, a fuzzy set $\widetilde{A}$ of $\mathbf{R}$ is called a fuzzy number (**FN**) if it is normal, i.e. there is a unique $x^*_{\widetilde{A}} \in \mathbf{R}$ so that $\mu_{\widetilde{A}}(x^*_{\widetilde{A}}) = 1$, and for every $\alpha \in [0, 1]$, the set $\widetilde{A}[\alpha] = \{x \in \mathbf{R} : \mu_{\widetilde{\mathbf{A}}}(\mathbf{x}) \geq \alpha\}$ is a nonempty compact interval in $\mathbf{R}$. This interval is denoted by $\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$, where $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}[\alpha]\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}[\alpha]\}$. To represent and handel **FN**s, several authors have captured the information contained in a (unimodal) **FN** using a functional parametric form called an LR-fuzzy number $\widetilde{A} = (a; l_a, r_a)_{LR}$ where $l_a, r_a > 0$ with the following membership function:

$$\widetilde{A}(x) = \begin{cases} L(\frac{a-x}{l_a}), & x \leq a, \\ R(\frac{x-a}{r_a}), & x > a, \end{cases} \tag{1}$$

where L and R are continuous and strictly decreasing functions from $[0, 1]$ to $[0, 1]$ satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. A commonly used type of LR-fuzzy number are triangular fuzzy numbers (**TFN**s) $\widetilde{A} = (a; l_a, r_a)_T$ with $L(x) = R(x) = 1 - x$ whose membership function is:

$$\widetilde{A}(x) = \begin{cases} \frac{x-(a-l_a)}{l_a} & a - l_a \leq x \leq a, \\ \frac{a+r_a-x}{r_a} & a \leq x \leq a + r_a, \\ 0 & x \in \mathbf{R} - [a - l_a, a + r_a]. \end{cases} \tag{2}$$

Definition 2.1. [27] *The α-values of a **FN** $\widetilde{A}$ are defined for each $\alpha \in [0, 1]$ as follows:*

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^L_{2\alpha} & \alpha \in [0, 0.5], \\ \widetilde{A}^U_{2(1-\alpha)} & \alpha \in (0.5, 1], \end{cases} \tag{3}$$

Remark 2.2. *It should be noted that the relationship between α-values and α-cuts of a **FN** $\widetilde{A}$ can be evaluated as follows:*

$$\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha] = [\widetilde{A}_{\alpha/2}, \widetilde{A}_{1-\alpha/2}]. \tag{4}$$

Therefore, according to Resolution identity [4], the membership degree of $\widetilde{A}$ at $x \in \mathbf{R}$ can be computed as:

$$\widetilde{A}(x) = \sup\{\alpha \in [0, 1] : x \in [\widetilde{A}_{\alpha/2}, \widetilde{A}_{1-\alpha/2}]\}.$$

Remark 2.3. *Let $\lambda \in \mathbf{R}$ and $\widetilde{A}$ and $\widetilde{B}$ be two **FN**s. Then, for any $\alpha \in [0, 1]$, the following relationships relevant to some arithmetic operations of **FN**s hold for α-values of $\widetilde{A}$ and $\widetilde{B}$:*

1. $(\widetilde{A} \oplus \widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha$.

2. $(\lambda \otimes \widetilde{A})_\alpha = \begin{cases} \lambda \widetilde{A}_\alpha, & \lambda \geq 0, \\ -\lambda \widetilde{A}_{1-\alpha}, & \lambda < 0. \end{cases}$

where $\oplus$ and $\otimes$ denote the addition and multiplication operators in fuzzy domain [4].

Definition 2.4. [25] *The generalized difference between two **FN**s $\widetilde{A}$ and $\widetilde{B}$ is defined as a **FN** $\widetilde{A} \ominus_G \widetilde{B}$ with the following α-values:*

$$(\widetilde{A} \ominus_G \widetilde{B})_\alpha = \begin{cases} \inf_{\beta \in [\alpha, 1-\alpha]} (\widetilde{A}_\beta - \widetilde{B}_\beta) & for \quad 0.0 \leq \alpha \leq 0.50, \\ \sup_{\beta \in [1-\alpha, \alpha]} (\widetilde{A}_\beta - \widetilde{B}_\beta) & for \quad 0.50 \leq \alpha \leq 1.0. \end{cases} \tag{5}$$

The arithmetic properties of $\ominus_G$ are employed in next section to provide a fuzzy nonparametric time series mode (for more, see [25]).

Furthermore, a square error distance between two **FN**s $\widetilde{A}$ and $\widetilde{B}$ [25] is employed in this to investigate the performance of the proposed fuzzy time series model. For $1 \leq p < \infty$, the L_p-distance between two **FN**s $\widetilde{A}$ and $\widetilde{B}$ is defined by $d_p(\widetilde{A}, \widetilde{B}) = (\int_0^1 g(\alpha) |\widetilde{A}_\alpha - \widetilde{B}_\alpha|^p d\alpha)^{1/p}$ where

$$g(\alpha) = \begin{cases} 4\alpha & 0 \leq \alpha \leq 0.5, \\ 4(1-\alpha) & 0.5 \leq \alpha \leq 1. \end{cases} \tag{6}$$

We will use this distance measure in our method for $p = 1, 2$.

Remark 2.5. *Let $\widetilde{A} \in \mathcal{F}(\mathbf{R})$ and f be a continuous function. The fuzzy function $f(\widetilde{A})$ according to (Zadeh's) Extension principle* [4] *is can be rewritten as follows:*

$$(f(\widetilde{A}))_\alpha = \begin{cases} \inf_{\beta \in [\alpha, 1-\alpha]} f(\widetilde{A}_\beta) & 0.0 \leq \alpha \leq 0.50, \\ \sup_{\beta \in [1-\alpha, \alpha]} f(\widetilde{A}_\beta) & 0.50 < \alpha \leq 1.0. \end{cases} \tag{7}$$

Furthermore, a common defuzzification method known as "center of gravity" [44] is employed to examine performance of the proposed fuzzy time series model using some common scatter plots. For a **TFN** $\widetilde{A} = (a; l_a, r_a)_T$, the center of gravity of $\widetilde{A}$ is evaluated as:

$$M_{\widetilde{A}} = a + (r_a - l_a)/3. \tag{8}$$

3 Fuzzy nonparametric time series model

In this section, a fuzzy nonparametric time series model based on fuzzy data is introduced.

Definition 3.1. [24] *The fuzzy-valued mapping $\widetilde{X} : \Omega \to \mathcal{F}(\mathbf{R})$ is called a fuzzy random variable if for any $\alpha \in [0, 1]$, the real-valued mapping $\widetilde{X}_\alpha : \Omega \to \mathbf{R}$ is a real-valued random variable on $(\Omega, \mathcal{A}, \mathcal{P})$. In addition, it is said that $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_T$ is a fuzzy time series random variables if for every $\alpha \in [0, 1]$, $(\widetilde{X}_1)_\alpha, (\widetilde{X}_2)_\alpha, \ldots, (\widetilde{X}_T)_\alpha$ is a sequence of ordinary random variables indexed by time t. The observed value of $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_T$ is denoted by $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_T$.*

Definition 3.2. *A fuzzy nonparametric time series model (**FNTSM**) is defined as follows:*

$$\widetilde{x}_i = \bigoplus_{l=1}^{p} f_l(\widetilde{x}_{i-l}) \oplus \widetilde{\epsilon}_i, \tag{9}$$

where

1. *$\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_T$ is a fuzzy time series,*
2. *$f_l(\widetilde{x}_{i-l})$ shows fuzzy smooth functions to be estimated, and*
3. *$\widetilde{\epsilon}_i$ indicates a fuzzy error term.*

In order to estimate the fuzzy smooth functions $f_1(\widetilde{x}_{i-1}), f_2(\widetilde{x}_{i-2}), ..., f_p(\widetilde{x}_{i-p})$, inspired by Hesamian and Akbari [26], a fuzzy forward fit time series method is suggested. At first step, consider the univariate nonparametric fuzzy time series model $\widetilde{x}_i = f_1(\widetilde{x}_{i-1}) \oplus \widetilde{\epsilon}_i^1$. Extending the Nadaraya-Watson estimator [18] in the fuzzy environment, $f_1(\widetilde{x}_{i-1})$ can be evaluated via the α-value method as follows:

$$(\widetilde{\widehat{f}}_1^K(\widetilde{x}_{i-1}))_\alpha = [\inf_{\beta\in[\alpha/2,1-\alpha/2]} g(\beta), \sup_{\beta\in[\alpha/2,1-\alpha/2]} g(\beta)], \tag{10}$$

where

$$g(\beta) = \sum_{j=p+1}^{T} \big(w_j^1(\widetilde{x}_{i-1}; h_1)(\widetilde{x}_j)_\beta\big),$$

and

$$w_j^1(\widetilde{x}_{i-1}; h_1) = \frac{K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-1})}{h_1}\right)}{\sum_{j=p+1}^{T} K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-1})}{h_1}\right)}.$$

in which $K(.)$ shows a kernel function, $h_l > 0$ is a bandwidth parameter. Since, $w_j^1(\widetilde{x}_{i-1}; h_1)$ are positive constants, it is easy to show that $\widetilde{\widehat{f}}_1^K(\widetilde{x}_{i-1})$ is a **FN** as:

$$\widetilde{\widehat{f}}_1^K(\widetilde{x}_{i-1}) = \bigoplus_{j=p+1}^{T} \big(w_j^1(\widetilde{x}_{i-1}; h_1) \otimes \widetilde{x}_j\big). \tag{11}$$

At the second step, by adding $f_2(\widetilde{x}_{i-2})$ to the univariate fuzzy time series model $\widetilde{x}_i = f_1(\widetilde{x}_{i-1}) \oplus \widetilde{\epsilon}_i^1$, we get a fuzzy time series model as $\widetilde{x}_i = f_1(\widetilde{x}_{i-1}) \oplus f_2(\widetilde{x}_{i-2}) \oplus \widetilde{\epsilon}_i^2$. Substituting $\widehat{f}_1^K(\widetilde{x}_{i-1})$ instead of $f_1(\widetilde{x}_{i-1})$ and applying the generalized difference $\ominus_G$ on both sides, we get

$$\widetilde{x}_i \ominus_G \widehat{f}_1(\widetilde{x}_{i-1}) = (\widehat{f}_1(\widetilde{x}_{i-1}) \oplus f_2(\widetilde{x}_{i-2}) \oplus \widetilde{\epsilon}_i^2) \ominus_G \widehat{f}_1(\widetilde{x}_{i-1}) = f_2(\widetilde{x}_{i-2}) \oplus \widetilde{\epsilon}_i^2,$$

Therefore, another univariate fuzzy time series model is obtained. Similar to the first step, the estimated value of $f_2(\widetilde{x})$ can be attained as:

$$\widehat{f}_2^K(\widetilde{x}_{i-2}) = \bigoplus_{j=p+1}^{T} \left(w_j^2(\widetilde{x}_{i-2}; h_2) \otimes \big(\widetilde{x}_j \ominus_G \widetilde{\widehat{f}}_1^K(\widetilde{x}_{i-1})\big)\right), \tag{12}$$

where

$$w_j^2(\widetilde{x}_{i-2}; h_2) = \frac{K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-2})}{h_2}\right)}{\sum_{j=p+1}^{T} K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-2})}{h_2}\right)}.$$

By continuing this procedure until $(p-1)^{th}$ step, the fuzzy smooth functions $\widehat{f}_1(\widetilde{x}_{i-1}), \widehat{f}_2(\widetilde{x}_{i-2}), ..., \widehat{f}_{p-1}(\widetilde{x}_{i-p})$ can be estimated as:

$$\widehat{f}_s^K(\widetilde{x}_{i-s}) = \bigoplus_{j=p+1}^{T} \left(w_j^s(\widetilde{x}_{i-s}; h_s) \otimes (\widetilde{x}_j \ominus_G (\bigoplus_{v=1}^{s-1} \widehat{f}_v^K(\widetilde{x}_{i-v})))\right),\ s = 1, 2, ..., p-1, \tag{13}$$

where $h_s > 0$ is a bandwidth parameter and

$$w_j^s(\widetilde{x}_{i-s}; h_s) = \frac{K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-s})}{h_s}\right)}{\sum_{j=p+1}^{T} K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-s})}{h_s}\right)}.$$

At the final step, consider the fuzzy multivariate nonlinear time series model $\widetilde{x}_i = \bigoplus_{l=1}^{p-1} \widetilde{f}_l(\widetilde{x}_{i-l}) \oplus \widetilde{f}_p(\widetilde{x}_{i-p}) \oplus \widetilde{\epsilon}_i$. Similar to the previous steps, the estimated value of $\widetilde{f}_p(\widetilde{x}_{i-p})$ can be then obtained as:

$$\widehat{f}_p^K(\widetilde{x}_{i-p}) = \bigoplus_{j=p+1}^{T} \left(w_j^p(\widetilde{x}_{i-p}; h_p) \otimes \big(\widetilde{x}_j \ominus_G (\bigoplus_{v=1}^{p-1} \widehat{f}_v^K(\widetilde{x}_{i-v}))\big)\right), \tag{14}$$

where

$$w_j^p(\widetilde{x}_{i-p}; h_p) = \frac{K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-p})}{h_p}\right)}{\sum_{j=p+1}^{T} K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-p})}{h_p}\right)}.$$

Remark 3.3. *It should be pointed out that various performance measures have been proposed in the literature to estimate and compare the forecasting accuracy in different models [9]. In this paper, some commonly used performance measures were extended to estimate and compare the forecasting accuracy of different models:*

1. *Mean Forecast Error:*

$$MFE = \frac{\sum_{i=p+1}^{T} d_2^2(\widetilde{\widehat{x}}_i, \widetilde{x}_i)}{T-p}. \tag{15}$$

2. *Mean Absolute Scaled Error:*

$$MASE = \frac{\sum_{i=p+1}^{T} q_i}{T-p}, \tag{16}$$

where

$$q_i = \frac{d_2(\widetilde{\widehat{x}}_i, \widetilde{x}_i)}{\frac{1}{T-p}\sum_{i=p+1}^{T} d_2^2(\widetilde{x}_i, \widetilde{x}_{i-1})}. \tag{17}$$

3. *Mean Absolute Percentage Error:*

$$MAPE = \frac{\sum_{i=p+1}^{T} \int |\frac{\widetilde{\widehat{x}}_i(x) - \widetilde{x}_i(x)}{\widetilde{x}_i(x)}| dx \times 100\%}{T-p}. \tag{18}$$

4. *Mean Similarity Measure:*

$$MSM = \frac{1}{T-p}\sum_{i=p+1}^{T} S_{UI}(\widetilde{\widehat{x}}_i, \widetilde{x}_i), \tag{19}$$

$$S_{UI}(\widetilde{\widehat{x}}_i, \widetilde{x}_i) = \frac{Card(\widetilde{\widehat{x}}_i \cap \widetilde{x}_i)}{Card(\widetilde{\widehat{x}}_i \cup \widetilde{x}_i)}, \tag{20}$$

*where $\cap$, $\cup$ denote the intersection and union operators on the space of **FNs**, respectively; and $Card(\widetilde{A})$ denotes the cardinal number of $\widetilde{A}$. Note that $MSM \in (0,1]$. Therefore, it is reasonable that those values of MSM greater than 0.5 indicate the degree of closeness between outputs and their corresponding estimations.*

For a good forecast, MFE, MASE and MAPE should be close to zero as much as possible. In addition, to examine the relation between $\widetilde{X}s$ and $\widetilde{\widehat{X}}s$, beside the mentioned goodness-of-fit measures, we also applied the center of gravity to convert $\widetilde{x}s$ and $\widetilde{\widehat{x}}s$ into the exact values $M_{\widetilde{x}}s$ and $M_{\widetilde{\widehat{x}}}s$. Then, according to the conventional regression models, relationship between center gravity of $M_{\widetilde{x}}s$ ($M_{\widetilde{\widehat{x}}}s$) were investigated according to their plots.

Remark 3.4. *To estimate the unknown components of the proposed **FNTSM** model, the bandwidth $\boldsymbol{h} = (h_1, h_2, \ldots, h_p)^\top$ and the autoregressive order p should be simultaneously estimated based on fuzzy time series $(\widetilde{x}_1, \ldots, \widetilde{x}_T)^\top$ and a specified kernel function K. In this regards, three popular kernels including Epanechnikov, triweight and gaussian kernels (Table 1) were employed to examine their effect on performance measures. Since all the above mentioned target functions are connected, a hybrid optimization algorithm is required to estimate such parameters. For this purpose, the optimal vector of bandwidth $\boldsymbol{h} = (h_1, h_2, \ldots, h_p)^\top$ is then evaluated by minimizing the following extended cross validation criterion:*

$$\widehat{\boldsymbol{h}} = arg \min_{h_1, h_2, \ldots, h_p} \sum_{i=p+1}^{T} d_2^2(\widetilde{x}_i, \widetilde{\widehat{x}}_i^{(i)}), \tag{21}$$

where $\widetilde{\widehat{x}}_i^{(i)}$ denotes the estimation of $\widetilde{x}_i$ based on the fuzzy data $\widetilde{x}_j$, $j = p+1, 2, \ldots, T$ when $j \neq i$. In addition, the optimal autoregressive order p_{opt} is evaluated as $p_{opt} = \max_{p \in \{1,2,\ldots\}} MSM_p^{\boldsymbol{h}}$. In this regard, the following iterative algorithm to the autoregressive order p and the optimal value of bandwidth $\boldsymbol{h} = (h_1, h_2, \ldots, h_p)^\top$ is suggested as follows:

1) Let $p = 1$.

2) Choose $\widehat{\boldsymbol{h}}_p$ as the optimal bandwidth throughout an extended cross-validation criterion:

$$\widehat{\boldsymbol{h}}_p = arg \min_{h_1, h_2, \ldots, h_p} \sum_{i=p+1}^{T} d_2^2(\widetilde{x}_i, \widetilde{\widehat{x}}_i^{(i)}),$$

where $\widetilde{\widehat{x}}_i^{(i)}$ denotes the estimation of $\widetilde{x}_i$ based on the fuzzy data $\widetilde{x}_j$, $j = p+1, 2, \ldots, T$ where $j \neq i$.

Table 1: Some common kernel functions.

Epanechnikov	$K(y)=\begin{cases}\frac{3}{4}(1-y^2), & \lvert y\rvert\le 1,\\ 0, & \lvert y\rvert>1.\end{cases}$
Triweight	$K(y)=\begin{cases}\frac{35}{32}(1-y^2)^3, & \lvert y\rvert\le 1,\\ 0, & \lvert y\rvert>1.\end{cases}$
Gaussian	$K(y)=\frac{1}{\sqrt{2\pi}}e^{-0.5y^2}$.

3) Let $p=p+1$ and return to step 2 until $|MSM_{p+1}^{\widehat{\boldsymbol{h}}_{p+1}}-MSM_{p}^{\widehat{\boldsymbol{h}}_{p}}|<\epsilon$ for a small value of $\epsilon>0$.

Therefore, $p^*=p+1$ and $\widehat{\boldsymbol{h}}^*=\widehat{\boldsymbol{h}}_{p+1}$ are the optimal autoregressive order and optimal bandwidth.

Remark 3.5. *A fuzzy semi parametric linear time series model (**FSPTSM**) introduced by Hesamian and Akbari [24] as:*

$$\widetilde{x}_i=\oplus_{l=1}^{p}(\theta_l\otimes\widetilde{x}_{i-l})\oplus\widetilde{f}(t_i)\oplus\widetilde{\epsilon}_i,\quad i=p+1,p+2,\ldots,T,$$

where

1. $\widetilde{x}_i=(x_i;l_{x_i},r_{x_i})_{LR}$,
2. $\widetilde{f}(t_i)=(f(t_i);l_{f(t_i)},r_{f(t_i)})_{LR}$,
3. θ_l*'s, for* $l=1,2,\ldots,p$, *are unknown real-valued coefficients which should be estimated,*
4. $\widetilde{\epsilon}_i=(\epsilon_i;l_{\epsilon_i},r_{\epsilon_i})_{LR}$*'s are fuzzy errors.*

For selecting the optimal bandwidth h and autoregressive order p an algorithm was presented which combines both cross-validation procedure and least square error estimation:

1) Choose a grid for $p\in\{1,2,\ldots\}$ and a grid for $h\in\{0.01,0.02,\ldots,3\}$,

2) compute:

$$CV_p(h)=\frac{1}{T-p}\sum_{i=p+1}^{T-p}d_2^2\big(\widetilde{\widehat{x}}_i^{\,p}(h),\widetilde{x}_i\big),$$

where $\widetilde{\widehat{x}}_i^{\,p}(h)$ denotes the estimation of $\widetilde{x}_i$, related to the autoregressive order p and bandwidth h, based on the fuzzy data $\widetilde{x}_j$, $j=p+1,2,\ldots,T$ when $j\neq i$, and

$$MD_h(p)=\frac{\sum_{i=p+1}^{T}d_2^2(\widetilde{\widehat{x}}_i^{\,p}(h),\widetilde{x}_i)}{T-p},$$

$$S_h(p)=\frac{1}{T-p}\sum_{i=p+1}^{T}S_{UI}(\widetilde{\widehat{x}}_i^{\,p}(h),\widetilde{x}_i),$$

3) Choose h_{opt} as the optimal bandwidth such that:

$$CV_p(h_{opt})=\min_{h\in\{0.01,0.02,\ldots,3\}}CV_p(h),$$

and the optimal autoregressive order as:

$$p_{opt}=\min_{p\in\{1,2,\ldots,P\}}g(p),$$

where $g(p)=\frac{MD_{h_{opt}}(p)}{S_{h_{opt}}(p)}$.

Compared to the proposed algorithm with the method of Hesamian and Akbari, it can be seen that: 1- The method of Hesamian and Akbari relies on a fuzzy linear time series model while the proposed method is a fuzzy nonlinear regression model which is often happens in many real applications, 2- In our method the only unknown parameters to be estimated are bandwidths and autoregressive parameter, and 3- The proposed method provides a simple calculation procedure to estimate the unknown components of the model compared to Hesamian and Akbari method.

Table 2: Performance measures of the proposed **FNTSM** in comparison with Hesamian and Akbari corresponding to some specific kernels in Example 4.1.

Method	Kernel		Results
FNTSM	gaussian	$\widehat{h}_1 = 0.08 \widehat{h}_2 = 0.11$	
		MFE	42.04
		$MAPE$	0.75
		$MASE$	21.01
		MSM	0.68
	triweight	$\widehat{h}_1 = 0.08, \widehat{h}_2 = 0.10$	
		MFE	**42.04**
		$MAPE$	**0.88**
		$MASE$	**13.66**
		MSM	**0.78**
	Epanechnikov	$\widehat{h}_1 = 0.11, \widehat{h}_2 = 0.09$	
		MFE	34.49
		$MAPE$	0.81
		$MASE$	17.461
		MSM	0.71
FSPTSM	triweight	$h_{opt} = 0.10$, $p = 4$	
		Coefficient estimation	$\widehat{\theta}_1 = 0.064, \quad \widehat{\theta}_2 = 0.053, \widehat{\theta}_3 = 0.011, \quad \widehat{\theta}_4 = 0.028$
		MFE	65.24
		$MAPE$	0.42
		$MASE$	34.146
		MSM	0.41
	gaussian	$h_{opt} = 0.08$, $p = 4$	
		Coefficient estimation	$\widehat{\theta}_1 = 0.041, \quad \widehat{\theta}_2 = -0.023, \widehat{\theta}_3 =, \quad \widehat{\theta}_4 = 0.026$
		MFE	69.53
		$MAPE$	0.40
		$MASE$	38.874
		MSM	0.39
	epanechnikov	$h_{opt} = 0.15$, $p = 4$	
		Coefficient estimation	$\widehat{\theta}_1 = 0.064, \quad \widehat{\theta}_2 = 0.053, \widehat{\theta}_3 = 0.011, \quad \widehat{\theta}_4 = 0.028$
		MFE	65.02
		$MAPE$	0.42
		$MASE$	36.978
		MSM	0.40

4 Numerical examples

In this section, feasibility and effectiveness of the proposed **FNTSM** was examined via two numerical examples. Moreover, the proposed **FNTSM** was also examined with a fuzzy semi-parametric time series model (**FSPTSM**) introduced by Hesamian and Akbari [24]. We remove the method Zarei et al.[58] in our computational procedure since it is a special case of Hesamian and Akbari model. In order to conduct a competitive study, the measures of MFE, $MAPE$, $MASE$ and MSM are applied to calculate the goodness-of-fit criteria. In addition, popular kernels including triweight, Epanechnikov and gaussian were applied to examine the model's performance.

Example 4.1. *Consider a set of simulated data set of size $n = 700$ generated according to the following **FNTSM**:*

$$\widetilde{x}_i = \bigoplus_{l=1}^{2} f_l(\widetilde{x}_{i-l}) \oplus \widetilde{\epsilon}_i, \tag{22}$$

where

1. $f_1(\widetilde{x}_i) = 40 \otimes \cos(8/35 \otimes \widetilde{x}_i)$ *and* $f_2(\widetilde{x}_i) = (-1)^i \otimes \sin(8/350 \otimes \widetilde{x}_i)$,
2. $\widetilde{x}_1 = (0; 0.1, 0.2)_T$ *and* $\widetilde{x}_2 = (0; 0.2, 0.1)_T$,
3. $\widetilde{\epsilon}_i = (\epsilon_i; l_{\epsilon_i}, r_{\epsilon_i})_T$ *where* $\epsilon_i \sim N(0, 16)$, l_{ϵ_i} *and* r_{ϵ_i} *are random variables taken from* $U(1, 2)$.

*The $M_{\widetilde{x}}$ values are plotted in Fig. 1 which shows that there is a nonlinear relationship between fuzzy time series data. For this case, the goodness-of-fit values corresponding to our **FNTSM** are summarized in Table 2 based on some specific kernels. It can be seen that the best results are relevant to triweight kernel in terms of the performance measures of $MSM = 0.78$, $MFE = 42.04$, $MAPE = 0.88$, $MASE = 13.66$. In order to compare our method with that of [24], the results of **FSPTSM** are also summarized in Table 2 for applied kernels. Regarding each applied kernel and goodness-of-fit criterion, the proposed fuzzy time series method provided more accurate results compared with Hesamian and Akbari's approach. On the other hand the triweight kernel led to the best performance measures in both methods. Performance of the proposed **FNTSM** was also compared with **FSPTSM** for triweight kernel (the model with the best performance) as shown in Fig. 2. It is clear that the values of $M_{\widehat{\widetilde{x}}}$ in our method are closer to $M_{\widetilde{x}}$ values compared to Hesamian and Akbari method. Therefore, it can be concluded that the proposed method performs better than **FSPTSM** in this example.*

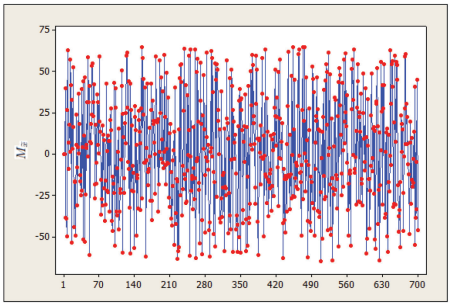


Figure 1: Time series plot of $M_{\widetilde{x}}$ values in Example 4.1.

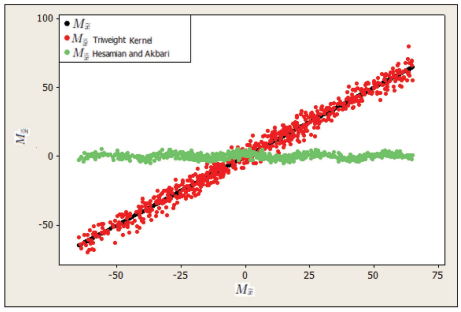


Figure 2: Comparing $M_{\widehat{\widetilde{x}}}$'s and $M_{\widetilde{x}}$'s based on triweight kernel in Example 4.1.

Example 4.2. *The main aim of this example is to examine daily time series analysis of an air pollutant in Isfahan City located in the center of Iran. The daily air pollution concentrations of ozone (O_3) were selected from March 2015 to March 2016 based on 350 days. Such data were reported by symmetric* ***TFNs*** *as $\widetilde{x}_i = (x_i; 0.05x_i)_T$. Time series plot of x_is is plotted in Fig. 3. For this case, the goodness-of-fit values corresponding to our* ***FNTSM*** *and* ***FSPTSM*** *introduced by Hesamian and Akbari are summarized in Table 3 for some popular kernels. Comparing the goodness-of-fit measurers of two methods, the proposed* ***FNTSM*** *gives better results corresponding to all kernels. However, the best results were evaluated for triweight kernel in terms of goodness-of-fit measures of $MSM = 0.73$, $MFE = 15.72$, $MAPE = 0.76$, $MASE = 15.04$. In this regards, the best fuzzy nonparametric time series model for prediction is relevant to triweight kernel:*

$$\widehat{\widetilde{x}}_i = \bigoplus_{l=1}^{2} \widehat{f}^K(\widetilde{x}_{i-l}),$$

$$\widetilde{\widehat{f}}_1^K(\widetilde{x}_{i-1}) = \bigoplus_{j=3}^{350} \left(w_j^1(\widetilde{x}_{i-1}; 0.1) \otimes \widetilde{x}_j\right).$$

and

$$\widehat{f}_2^K(\widetilde{x}_{i-2}) = \bigoplus_{j=3}^{350} \left(w_j^2(\widetilde{x}_{i-2}; 0.08) \otimes \left(\widetilde{x}_j \ominus_G \widetilde{\widehat{f}}_1^K(\widetilde{x}_{i-1})\right)\right),$$

in which

$$w_j^1(\widetilde{x}_{i-1}; 0.1) = \frac{K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-1})}{0.1}\right)}{\sum_{j=3}^{350} K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-1})}{0.1}\right)},$$

and

$$w_j^2(\widetilde{x}_{i-2}; 0.08) = \frac{K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-2})}{0.08}\right)}{\sum_{j=3}^{350} K\left(\frac{d_1(\widetilde{x}_j, \widetilde{x}_{i-2})}{0.08}\right)}.$$

*Performance of the proposed **FNTSM** relative to **FSPTSM** can be also investigated in Fig. 4. The results indicate that, $M_{\widetilde{\widehat{x}}}$ values in our method are closer to $M_{\widetilde{X}}$ than **FNTSM**. Therefore we conclude that the proposed **FNTSM** is also more efficient than **FSPTSM** in this example.*

Table 3: Performance measures of the proposed **FNTSM** in comparison with Hesamian and Akbari corresponding to some specific kernels in Example 4.2.

Method	Kernel		Results
FNTSM	gaussian	$\widehat{h}_1 = 0.1, \widehat{h}_2 = 0.15, \widehat{p} = 2$	
		MFE	20.01
		$MAPE$	0.655
		$MASE$	20.604
		MSM	0.63
	triweight	$\widehat{h}_1 = 0.1, \widehat{h}_2 = 0.08, \widehat{p} = 2$	
		MFE	**15.72**
		$MAPE$	**0.76**
		$MASE$	**15.04**
		MSM	**0.73**
	epanechnikov	$\widehat{h}_1 = 0.17, \widehat{h}_2 = 0.15, \widehat{p} = 2$	
		MFE	19.56
		$MAPE$	0.66
		$MASE$	17.07
		MSM	0.65
FSPTSM	triweight	$\widehat{h} = 1.85, \widehat{p} = 4$	
		Coefficient estimation	$\widehat{\theta}_1 = 0.216, \quad \widehat{\theta}_2 = 0.273, \widehat{\theta}_3 = 0.231, \quad \widehat{\theta}_4 = 0.265$
		MFE	30.23
		$MAPE$	0.476
		$MASE$	30.53
		MSM	0.47
	gaussian	$\widehat{h} = 1.65, \widehat{p} = 4$	
		Coefficient estimation	$\widehat{\theta}_1 = 0.247, \quad \widehat{\theta}_2 = 0.290, \widehat{\theta}_3 = 0.218, \quad \widehat{\theta}_4 = 0.230$
		MFE	30.512
		$MAPE$	0.472
		$MASE$	33.167
		MSM	0.46
	epanechnikov	$\widehat{h} = 2, \widehat{p} = 4$	
		Coefficient estimation	$\widehat{\theta}_1 = 0.188, \quad \widehat{\theta}_2 = 0.264, \widehat{\theta}_3 = 0.218, \quad \widehat{\theta}_4 = 0.230$
		MFE	23.512
		$MAPE$	0.472
		$MASE$	36.215
		MSM	0.46

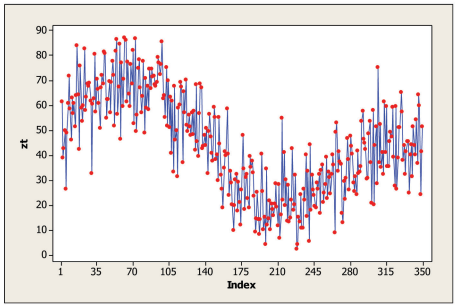


Figure 3: Time series plot of x values in Example 4.2.

5 Conclusion

The classical fuzzy time series models are often conducted based on parametric models. However, it may be too restrictive to suppose that the effect of such fuzzy data is a parametric relationship. The main aim of this paper was to propose a fuzzy nonparametric methodology for the time series model with fuzzy data and fuzzy smooth functions. For this purpose, a fuzzy forward fit kernel smoothing technique was introduced and discussed. In this content, the cross-validation and similarity measures based on some popular kernel functions were employed to estimate non-fuzzy bandwidths and autoregressive parameter. The effectiveness of the proposed fuzzy additive time series model was also examined and compared in terms of some common goodness-of-fit criteria. A common defuzzified method was also employed to compare the performance of the proposed method by some scatter plots. The results of the numerical examples clearly showed the better performances of the proposed method compared to others. The proposed approach showed potential effectiveness for fuzzy nonparametric time series model in real applications such as hydrology, agriculture, industry, economics and social studies. Although the triangular fuzzy numbers were used in the numerical

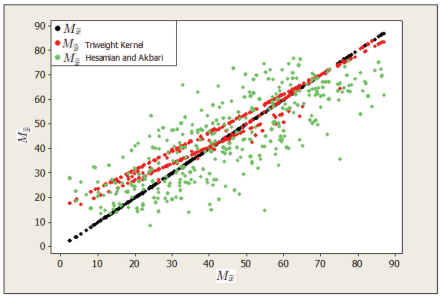


Figure 4: Comparing $M_{\widetilde{y}}$'s and $M_{\widetilde{x}}$'s based on triweight kernel in Example 4.2.

evaluations, the proposed fuzzy additive time series model is still applicable for all kinds of fuzzy numbers. Extending the proposed model for other types of statistical time series models such as seasonal time series are some potential topics for future research.

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