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Ordinal sum constructions for aggregation functions on the real unit interval

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Abstract

We discuss ordinal sums as one of powerful tools in the aggregation theory serving, depending on the context, both as a construction method and as a representation, respectively. Up to recalling of several classical results dealing with ordinal sums, in particular dealing, e.g., with continuous t-norms, copulas, or recent results, e.g., concerning uninorms with continuous underlying functions, we present also several new results, such as the uniqueness of the link between t-norms or t-conorms, and related Archimedean components, problems dealing with the cardinality of the considered index sets in ordinal sums, or infinite ordinal sums of aggregation functions covering by one type of ordinal sums both t-norms and t-conorms ordinal sums.

Keywords: Ordinal sum, t-norm, t-conorm, uninorm.

1 Introduction

The idea of ordinal sums appeared in the study of algebraic structures around 1940. In particular, for two disjoint sets X_1, X_2 such that $(X_1, *_1)$ and $(X_2, *_2)$ are semigroups, Climescu in 1946 [11] has introduced their ordinal sum $(X, *)$, where $X = X_1 \cup X_2$ and

$$x * y = \begin{cases} x *_1 y & \text{if } x, y \in X_1, \\ x *_2 y & \text{if } x, y \in X_2, \\ x & \text{if } x \in X_1 \text{ and } y \in X_2, \\ y & \text{if } x \in X_2 \text{ and } y \in X_1. \end{cases}$$

Then $(X, *)$ is a semigroup. Obviously, $(X, *)$ is a commutative semigroup if and only if both $(X_1, *_1)$ and $(X_2, *_2)$ are commutative.

Next, Birkhoff [8] in 1940 has introduced an ordinal sum of posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$ as a poset $(X, \leq)$, where again $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ and

$$x \leq y \text{ if } \begin{cases} \text{either } x, y \in X_1 \text{ and } x \leq_1 y, \\ \text{or } x, y \in X_2 \text{ and } x \leq_2 y, \\ \text{or } x \in X_1 \text{ and } y \in X_2. \end{cases}$$

Clearly, $(X, \leq)$ is a chain if and only if both $(X_1, \leq_1)$ and $(X_2, \leq_2)$ are chains. By induction we can immediately extend ordinal sums (of semigroups or of posets) to any finite number of summands and it is not difficult to see that

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there is a direct generalization to any system of disjoint semigroups $((X_t, *_t))_{t\in I}$ (disjoint posets $((X_t, \leq_t))_{t\in I}$), where I is a chain. Formally, the constraint of disjointness of ordinal sum summands for semigroups (posets) can be relaxed, compare Theorem 2.1, though this relaxed construction does not yield any new semigroups (posets) up to some very special cases, see, e.g., Example 5.1.

For naturally ordered semigroups, both types of ordinal sums were merged together by Clifford [10] in 1954, see also [14]. Though again one can consider any chain as an index set, we present Clifford's approach for the ordinal sum of 2 naturally ordered semigroups. Let $(X_i, \leq_i, *_i)$, $i = 1, 2$, be two naturally ordered semigroups, i.e., $\leq_i$ is a total order on X_i and $*_i$ is an associative binary operation on X_i which is increasing with respect to $\leq_i$. Suppose either $X_1 \cap X_2 = \emptyset$, or, if $X_1 \cap X_2 \neq \emptyset$ then $X_1 \cap X_2 = \{v\}$ is a singleton such that v is the top element of $(X_1, \leq_1)$ and the neutral element of $(X_1, *_1)$, and it is also the bottom element of $(X_2, \leq_2)$ and the annihilator of $(X_2, *_2)$. Then their ordinal sum $(X, \leq, *)$ is also a naturally ordered semigroup, where

$$x \leq y \text{ if } \begin{cases} \text{either } x, y \in X_1 \text{ and } x \leq_1 y, \\ \text{or } x, y \in X_2 \text{ and } x \leq_2 y, \\ \text{or } x \in X_1 \text{ and } y \in X_2, \end{cases}$$

and

$$x * y = \begin{cases} x *_1 y & \text{if } x, y \in X_1, \\ x *_2 y & \text{if } x, y \in X_2, \\ x & \text{if } x \in X_1 \text{ and } y \in X_2, \\ y & \text{if } x \in X_2 \text{ and } y \in X_1. \end{cases}$$

Our aim is to discuss ordinal sums related to particular aggregation functions on $[0, 1]$. For more details concerning aggregation functions we recommend [7] and [15]. The considered real unit interval is equipped by the standard total order $\leq$ of reals, and we will always deal with subsets X_t of $[0, 1]$. Clearly, for any such X_t, $(X_t, \leq)$ is a chain and the related ordinal sum of such chains is again a chain $(X, \leq)$, where $X = \bigcup_{t\in I} X_t$. We will not construct a new ordering if $X = \bigcup_{t\in I} X_t \subseteq [0, 1]$, as then we will always consider the standard order $\leq$ of reals (for several details concerning ordinal sums and t-norms on $[0, 1]$ we recommend [19]).

Therefore we will not focus on ordinal sums of posets (chains), but on ordinal sums related to some other properties. In particular, we will consider associative aggregation functions, and thus the ordinal sums of semigroups. Note that in the case of associative aggregation functions, their binary form univocally determines also their n-ary form for any $n > 2$.

The paper is organized as follows. In the next section, we recall several basic notions which we will use in the paper. In Section 3, we discuss ordinal sums in the framework of continuous t-norms. Note that commonly accepted idea of representation of continuous t-norms deals with at most countably many summands in the related ordinal sum representation. We clarify the situation and give examples showing the falsity of necessity of such a claim. As a corollary, similar results for continuous t-conorms and for continuous nullnorms are obtained. In Section 4, we discuss the uniqueness of t-norms with fixed Archimedean components. Section 5 deals with uninorms and n-uninorms with continuous underlying functions. In Section 6, we generalize ordinal sums of finitely many aggregation functions introduced by De Baets and Mesiar [6, 22] and stress their link to ordinal sums of t-norms (t-conorms). Finally, some concluding remarks are added.

2 Basic notions

For a fixed $n \in \mathbb{N}$, $n \geq 2$, the mapping $A\colon [0, 1]^n \longrightarrow [0, 1]$ is called an aggregation function (see [15]) whenever it is increasing and $A(0, \ldots, 0) = 0$, $A(1, \ldots, 1) = 1$.

A triangular norm (see [18]) is a binary function $T\colon [0, 1]^2 \longrightarrow [0, 1]$ which is commutative, associative, increasing in both variables and 1 is its neutral element. Hence, T is a binary aggregation function. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}} [0, 1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm (see [18]) is a binary function $S\colon [0, 1]^2 \longrightarrow [0, 1]$ which is commutative, associative, increasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm S by the equation

$$S(x, y) = 1 - T(1 - x, 1 - y),$$

and vice-versa.

Next, we recall a fundamental result of Clifford [10], as formulated in [18].

Theorem 2.1. *Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Put $X = \bigcup_{\alpha\in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

The ordinal sum construction for t-norms and t-conorms reduces to the following proposition [18].

Proposition 2.2. *Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k\in K}$ $((]c_k, d_k[)_{k\in K})$ be a system of open, disjoint subintervals of $[0,1]$. Let $(T_k)_{k\in K}$ $((S_k)_{k\in K})$ be a system of t-norms (t-conorms). Then the ordinal sum $T = (\langle a_k, b_k, T_k\rangle \mid k \in K)$ $(S = (\langle c_k, d_k, S_k\rangle \mid k \in K))$ given by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k)T_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & \text{if } (x,y) \in [a_k, b_k[^2 \text{ for some } k \in K, \\ \min(x,y) & \text{else} \end{cases}$$

and

$$S(x,y) = \begin{cases} c_k + (d_k - c_k)S_k\left(\frac{x-c_k}{d_k-c_k}, \frac{y-c_k}{d_k-c_k}\right) & \text{if } (x,y) \in]c_k, d_k]^2 \text{ for some } k \in K, \\ \max(x,y) & \text{else} \end{cases}$$

is a t-norm (t-conorm). The t-norm T (t-conorm S) is continuous if and only if all summands T_k (S_k) for $k \in K$ are continuous.

Note that due to the total order $\leq$ on $[0,1]$, one can always introduce a total order on the set K, namely $k_1 \leq k_2$ if and only if $\frac{a_{k_1}+b_{k_1}}{2} \leq \frac{a_{k_2}+b_{k_2}}{2}$ $(\frac{c_{k_1}+d_{k_1}}{2} \geq \frac{c_{k_2}+d_{k_2}}{2})$. Moreover, the ordinal sum of t-norms (t-conorms) is not an ordinal sum in the sense of Clifford, with respect to the index set K, whenever $\bigcup_{k\in K} [a_k, b_k] \neq [0,1]$.

Each continuous t-norm (t-conorm) can be represented as an ordinal sum of continuous Archimedean t-norms (t-conorms), see [21]. Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm S) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e., there exists $(x,y) \in]0,1[^2$ such that $T(x,y) = 0$ $(S(x,y) = 1)$. Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator, which is uniquely determined up to a positive multiplicative constant. More details on t-norms and t-conorms can be found in [2, 18].

Similarly, one can introduce the ordinal sums of copulas and quasi-copulas [24], which are then not only construction but also a representation, in the same way as it is in the case of continuous t-norms.

A uninorm (introduced in [35]) is a binary function $U\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, increasing in both variables and has a neutral element $e \in [0,1]$ (see also [13]). Evidently, if $e = 1$ $(e = 0)$ then we retrieve a t-norm (t-conorm).

For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called conjunctive (disjunctive) if $U(1,0) = 0$ $(U(1,0) = 1)$. For each uninorm U with the neutral element $e \in]0,1[$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm T_U on $[0,1]^2$, $U(x,y) = e \cdot T_U(\frac{x}{e}, \frac{y}{e})$. Similarly, the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e., a linear transformation of some t-conorm S_U, $U(x,y) = e + (1-e)\cdot S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e})$. Moreover, $\min(x,y) \leq U(x,y) \leq \max(x,y)$ for all $(x,y) \in [0,e]\times[e,1] \cup [e,1]\times[0,e]$.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [13]). A uninorm which possesses a continuous additive generator is called representable. Note that in [34] (see also [25]) it was shown that a uninorm is representable if and only if it is continuous on $[0,1]^2 \setminus \{(0,1),(1,0)\}$.

A uninorm U is called internal if $U(x,y) \in \{x,y\}$ for all $(x,y) \in [0,1]^2$; and it is called idempotent if $U(x,x) = x$ for all $x \in [0,1]$. Observe that if a uninorm U is internal then it is also idempotent and vice-versa. Moreover, U is called d-internal if it is internal and there exists a continuous and strictly decreasing function $g_U\colon [0,1] \longrightarrow [0,1]$ such that $U(x,y) = \min(x,y)$ if $y < g_U(x)$ and $U(x,y) = \max(x,y)$ if $y > g_U(x)$.

Uninorms with continuous underlying functions were completely characterized in [29, 30]. In [30] it was shown that the set of all points of discontinuity of a uninorm U with continuous underlying functions T_U and S_U is a subset of the graph of the characterizing set-valued function of such a uninorm. In [29] it was claimed that each uninorm U with continuous underlying functions T_U and S_U can be decomposed into an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator). More precisely, each uninorm with continuous underlying functions can be decomposed into an ordinal sum of a countable number of trivial semigroups and semigroups characterized by the Definition 2.3.

For any $0 \le a < b \le c < d \le 1$, $v \in [b, c]$ and a uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ with the neutral element $e \in]0,1[$ assume the transformation $f\colon [0,1] \longrightarrow [a,b[\cup \{v\} \cup]c,d]$, given by

$$f(x) = \begin{cases} (b-a)\cdot \frac{x}{e} + a & \text{if } x \in [0,e[\,, \\ v & \text{if } x = e, \\ d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise.} \end{cases} \tag{1}$$

Then f is linear on $[0,e[$ and on $]e,1]$ and thus it is an increasing, piece-wise linear bijection of $[0,1]$ to $([a,b[\cup \{v\} \cup]c,d])$ which preserves the commutativity, the associativity, the monotonicity and the neutral element. Therefore the binary function $U_v^{a,b,c,d}\colon ([a,b[\cup \{v\} \cup]c,d])^2 \longrightarrow ([a,b[\cup \{v\} \cup]c,d])$ given by

$$U_v^{a,b,c,d}(x,y) = f(U(f^{-1}(x), f^{-1}(y))), \tag{2}$$

is a commutative, associative, increasing function with neutral element v, which will be called a uninorm on $([a,b[\cup \{v\} \cup]c,d])^2$. The backward transformation f^{-1} then transforms a uninorm defined on $([a,b[\cup \{v\} \cup]c,d])^2$ to a uninorm defined on $[0,1]^2$. Note that in the case when $b = c = v$ then f is a continuous, piece-wise linear transformation from $[0,1]$ to $[a,d]$ such that $f(e) = v$.

We define the following semigroups related to continuous t-norms, t-conorms and representable uninorms.

Definition 2.3. *Let $a, b, c, d \in [0,1]$ with $a < b < c < d$, $v \in [b,c]$. Then*

(i) a semigroup $(]a,b[\cup \{v\} \cup]c,d[\,, *)$ will be called a representable *semigroup if $*$ is isomorphic via (2) to a restriction of a representable uninorm on $[0,1]^2$ to $]0,1[^2$,*

(ii) a semigroup $(]a,b[\,, *)$ will be called a t-strict *semigroup if $*$ is linearly isomorphic to a restriction of a strict t-norm on $[0,1]^2$ to $]0,1[^2$,*

(iii) a semigroup $(]c,d[\,, *)$ will be called an s-strict *semigroup if $*$ is linearly isomorphic to a restriction of a strict t-conorm on $[0,1]^2$ to $]0,1[^2$,*

(iv) a semigroup $([a,b[\,, *)$ will be called a t-nilpotent *semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-norm on $[0,1]^2$ to $[0,1[^2$,*

(v) a semigroup $(]c,d]\,, *)$ will be called an s-nilpotent *semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-conorm on $[0,1]^2$ to $]0,1]^2$,*

(vi) a semigroup $(]a,b[\cup]c,d[\,, *)$ will be called a d-internal *semigroup if $*$ is isomorphic via (2) to a restriction of an d-internal uninorm on $[0,1]^2$ to $(]0,1[\setminus \{e\})^2$,*

(vii) a semigroup $(]a,b[\,, *)$ will be called a t-internal *semigroup if $* = \min$,*

(viii) a semigroup $(]c,d[\,, *)$ will be called an s-internal *semigroup if $* = \max$.*

From [29, Proposition 11] and [29, Definition 7] we obtain the following result.

Theorem 2.4. *Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm with continuous underlying functions. Then $([0,1], U)$ is an ordinal sum of a countable number of eight types of semigroups from Definition 2.3 and trivial semigroups.*

Observe that this result was based on an incorrect assumption that the set of accumulation points of a countable set is countable. This, however, is not true, it is enough to assume the set of rational numbers in the unit interval. We will show that the correct result is that each uninorm with continuous underlying functions can be decomposed into an ordinal sum of a countable number of semigroups from Definition 2.3 and a possibly uncountable number of trivial semigroups (see Example 5.2 and Theorem 5.3).

Another important class of associative aggregation functions is formed by nullnorms [9]. Recall that an associative aggregation function $A\colon [0,1]^2 \longrightarrow [0,1]$ possessing a zero-element $a \in]0,1[$ is a nullnorm if and only if there is a t-norm T and a t-conorm S, both acting on $[0,1]$ such that

$$A(x,y) = \begin{cases} a \cdot S(\frac{x}{a}, \frac{y}{a}) & \text{if } \max(x,y) < a, \\ a + (1-a) \cdot T(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } \min(x,y) > a, \\ a & \text{otherwise.} \end{cases}$$

Clearly, the nullnorm A is continuous if and only if the related t-norm T and t-conorm S are continuous, and thus they can be expressed as ordinal sums of continuous Archimedean t-norms (t-conorms).

Now let us recall the definition of an n-uninorm (see [1]).

Definition 2.5. *Assume an $n \in \mathbb{N} \setminus \{1\}$. Let $V\colon [0,1]^2 \longrightarrow [0,1]$ be a commutative binary function. If for $0 = z_0 < z_1 < \cdots < z_n = 1$ and $e_i \in [z_{i-1}, z_i]$, $i = 1, \ldots, n$ we have $V(e_i, x) = x$ for all $x \in [z_{i-1}, z_i]$ then $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$ is called an n-neutral element of V.*

Definition 2.6. *A binary function $U^n\colon [0,1]^2 \longrightarrow [0,1]$ is an n-uninorm if it is commutative, associative, increasing in each variable and has an n-neutral element $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$.*

The basic structure of n-uninorms was described by Akella in [1] and the characterizations of the main five classes of 2-uninorms was given in [36].

Each n-uninorm has the following building blocks around the main diagonal.

Proposition 2.7. [36] *Let U^n be an n-uninorm with the n-neutral element $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$. Then*

(i) U^n restricted to $[z_{i-1}, e_i]^2$, for $i = 1, \ldots, n$, is a linear transformation of a t-norm. We will denote this t-norm by T_i.

(ii) U^n restricted to $[e_i, z_i]^2$ for $i = 1, \ldots, n$, is a linear transformation of a t-conorm. We will denote this t-conorm by S_i.

(iii) U^n restricted to $[z_{i-1}, z_i]^2$ for $i = 1, \ldots, n$, is a linear transformation of a uninorm. We will denote this uninorm by U_i.

(iv) U^n restricted to $[z_i, z_j]^2$ for $i, j \in \{0, 1, \ldots, n\}$, $i < j$, is a linear transformation of a $(j-i)$-uninorm.

Moreover, U^n restricted to $[e_i, e_{i+1}]^2$ for $i = 1, \ldots, n-1$, is a linear transformation of a nullnorm.

Finally, since we will use ordinal sums of trivial semigroups, let us recall that there exists only one operation on a trivial semigroup, namely the function $\mathrm{Id}\colon \{x\}^2 \longrightarrow \{x\}$, which is simply defined by $\mathrm{Id}(x,x) = x$.

3 Ordinal sums yielding t-norms

First we recall Archimedean components of a t-norm.

Definition 3.1. [20] *Let $T\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm. Two elements $x, y \in [0,1]$ are called Archimedean equivalent if there is an $n \in \mathbb{N}$ such that $x_T^{(n)} \le y \le x$ or $y_T^{(n)} \le x \le y$, where $x_T^{(n)} = T(x, x_T^{(n-1)})$ and $x_T^{(0)} = 1$. For each $x \in [0,1]$, the equivalence class I_x containing x is called an Archimedean class of T and it is a convex subset of $[0,1]$. For each $x \in [0,1]$ the pair $(I_x, T|_{I_x^2})$ is a subsemigroup of $([0,1], T)$, and it is called an Archimedean component of T.*

For each t-norm the set of all Archimedean classes forms a partition of $[0,1]$. Moreover, if for a non-empty subset $A \subseteq [0,1]$ we put $I_A = \bigcup\limits_{x \in A} I_x$ then $(I_A, T|_{I_A^2})$ is a totally ordered abelian semigroup whose semigroup operation is bounded from above by the minimum. For each $x \in [0,1]$, $(I_x, T|_{I_x^2})$ is the maximal Archimedean subsemigroup containing x. Note that a semigroup is called Archimedean if all its points are Archimedean equivalent, i.e., it has only one Archimedean class.

Observe that each continuous t-norm T is uniquely determined by its non-trivial Archimedean components. From [20, Lemma 4.1] we know that if $(I, *)$ is a non-trivial Archimedean component of some continuous t-norm T then either $I = [a, b[$ or $I =]a, b[$ for some $a, b \in [0,1]$. Note that in the case of non-continuous t-norms $I = [a, b]$ can be included, as well [17].

In the previous section we have seen that an ordinal sum of t-norms (see Proposition 2.2) is defined by a countable number of intervals, while the remaining part of the unit interval is completed by the minimum. This definition is based on the fact that the only idempotent t-norm is the minimum. When Ling [21], following the representation of

I-semigroups due to Mostert and Shields [33], showed that each continuous t-norm can be expressed as an ordinal sum of a countable number of Archimedean t-norms, she used the ordinal sum defined in Proposition 2.2 (which is due to Ling [21]). We would like to see whether the same result can be obtained also using the original definition for ordinal sum construction given by Clifford (see Theorem 2.1).

It is easy to show that if a non-trivial Archimedean component $(I,*)$ of a t-norm T (i.e., such that I is not a singleton) can be expressed as an ordinal sum (in the sense of Clifford) of two (or more) semigroups then one of the summands is equal to $(I,*)$ and all other summands are singletons. Note that then necessarily $I = [a,b[$ for some $a,b \in [0,1]$ and singleton summands are defined on $\{a\}$. We will call such an ordinal sum a trivial ordinal sum. Thus we can say that Archimedean components of a continuous t-norm are irreducible with respect to ordinal sum construction (in the sense of Clifford).

The question which we want to answer is whether each continuous t-norm can be expressed as an ordinal sum, in the sense of Clifford, of a countable number of irreducible semigroups, i.e., semigroups that cannot be expressed as a non-trivial ordinal sum. This is evidently not true since, for example, for the minimum t-norm the irreducible semigroups are the trivial semigroups $(\{x\},\mathrm{Id})_{x\in[0,1]}$. Indeed, the minimum t-norm can be expressed as an ordinal sum of these trivial semigroups and their number is uncountable.

On the other hand, it is obvious that the number of non-trivial Archimedean components of T is countable since the related Archimedean classes are mutually disjoint and each non-trivial Archimedean component contains an interval, i.e., at least one rational number. As we mentioned above, each continuous t-norm is uniquely determined by its non-trivial Archimedean components and it can be expressed as an ordinal sum of t-norms on the respective non-trivial Archimedean classes in the sense of Proposition 2.2. Observe that in this ordinal sum the remaining parts of the unit square are filled by the minimum. Inspired by this approach, if we want to express a continuous t-norm as an ordinal sum of a countable number of basic semigroups, in the sense of Clifford, we can take the minimum as a basic semigroup. Thus the main question here is whether each continuous t-norm can be expressed as an ordinal sum, in the sense of Clifford, of a countable number of Archimedean (representable) and idempotent semigroups.

Example 3.2. *Assume a t-norm T which is an ordinal sum of the product t-norm on the carrier $\left[0,\frac{1}{2}\right]$. This t-norm has Archimedean classes $\{0\}, \left]0,\frac{1}{2}\right[$ and $\{x\}$ for $x \in \left[\frac{1}{2},1\right]$. Therefore the ordinal sum decomposition into Archimedean components of this t-norm is uncountable. However, it can be expressed as an ordinal sum of $G_1 = (\{0\},\mathrm{Id})$, $G_2 = (\left]0,\frac{1}{2}\right[,*)$ and $G_3 = (\left[\frac{1}{2},1\right],\min)$, where $*$ denotes the linear transformation of the standard product acting on $]0,1[$ to the interval $\left]0,\frac{1}{2}\right[$. Thus this t-norm can be expressed as an ordinal sum of three semigroups which are either Archimedean or idempotent.*

As we have mentioned above, the number of non-trivial semigroups in the ordinal sum has to be countable since each interval contains at least one rational number. However, the closure of a countable set need not to be countable, which is the case of rational numbers from the unit interval. Another prominent example is the Cantor set. The construction of the Cantor set is as follows: from the unit interval we successively in steps delete open intervals (so-called middles)

1. $p_1^1 = \left]\frac{1}{3},\frac{2}{3}\right[$,
2. $p_1^2 = \left]\frac{1}{9},\frac{2}{9}\right[$, $p_2^2 = \left]\frac{7}{9},\frac{8}{9}\right[$,
3. $p_1^3 = \left]\frac{1}{27},\frac{2}{27}\right[$, $p_2^3 = \left]\frac{7}{27},\frac{8}{27}\right[$, $p_3^3 = \left]\frac{19}{27},\frac{20}{27}\right[$, $p_4^3 = \left]\frac{25}{27},\frac{26}{27}\right[$,

and so on. Assuming $C_0 = [0,1]$ and $C_n = \frac{C_{n-1}}{3} \cup (\frac{2}{3} + \frac{C_{n-1}}{3})$ for $n \in \mathbb{N}$, the Cantor set C can be obtained by $C = \bigcap_{n=1}^{\infty} C_n$. The Cantor set does not contain any interval of a non-zero length, however, this set is not countable. Moreover, since each non-trivial interval contains a rational number, the number of removed open intervals is countable. Thus there are uncountably many points in the Cantor set which are not end-points of the deleted intervals.

Example 3.3. *Assume intervals p_n^i from the definition of the Cantor set with $n \in \mathbb{N}$ and $i \in \{1,\dots,2^{n-1}\}$ and let $p_n^i \le p_m^j$ if $\sup p_n^i \le \inf p_m^j$ for all $n,m \in \mathbb{N}$ and $i \in \{1,\dots,2^{n-1}\}$, $j \in \{1,\dots,2^{m-1}\}$. Then the set $A = \{(n,i) \mid n \in \mathbb{N}, i \in \{1,\dots,2^{n-1}\}\}$ is linearly ordered by $\le$. Assume an ordinal sum $T = (\langle \inf p_n^i, \sup p_n^i, T_{\mathbf{P}}\rangle \mid n \in \mathbb{N}, i \in \{1,\dots,2^{n-1}\})$, where $T_{\mathbf{P}}$ is the product t-norm. Then T is an ordinal sum of continuous t-norms, i.e., it is a continuous t-norm. Each non-trivial Archimedean component of this t-norm corresponds to one of the intervals p_n^i. Thus the set of all trivial Archimedean components corresponds exactly to the Cantor set. We would like to express T as an ordinal sum of Archimedean and idempotent semigroups in the sense of Clifford. Since the Cantor set does not contain any interval of non-zero length, between each two points x,y from the Cantor set, $x<y$, there is a point z which belongs to one of the non-trivial Archimedean components of T. Therefore $T(x,z) = x$ and $T(y,z) = z$, which shows us that x and y cannot*

be merged to one semigroup in our ordinal sum construction. Thus each trivial Archimedean component should be taken separately. Then T is an ordinal sum in the sense of Clifford of a countable number of Archimedean semigroups and an uncountable number of trivial semigroups.

From the duality, all these results on t-norms can be shown also for t-conorms. Observe that in the case of nullnorms, the ordinal sum can be used only on subareas of the unit square, i.e., a nullnorm cannot be expressed as a non-trivial ordinal sum of semigroups. Here we can use an ordinal sum construction only to construct the underlying t-norm and t-conorm, for which we can then apply the results of this section. Note that a nullnorm can be expressed as a non-trivial z-ordinal sum of semigroups, see [31].

4 Uniqueness of t-norms with fixed Archimedean components

In [20, Proposition 3.4] it was shown that an ordinal sum of Archimedean components of a t-norm T is the strongest t-norm with the same Archimedean components as T. Therefore if T is uniquely given by its Archimedean components then it is equal to the ordinal sum of its Archimedean components. We would like to investigate the cases when this situation occurs. As a first example let us mention that each continuous t-norm is determined by its Archimedean components (see [20]). However, this is no longer true for non-continuous t-norms.

Example 4.1. *(i) Let $T_1\colon [0,1]^2 \longrightarrow [0,1]$ be given by*

$$T_1(x,y) = \begin{cases} \frac{(2\cdot x-1)\cdot(2\cdot y-1)+1}{2} & \text{if } x,y \in \left[\frac{1}{2},1\right], \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to check that T_1 is a t-norm with non-trivial Archimedean components $(\left[0,\frac{1}{2}\right[,*)$ and $(\left]\frac{1}{2},1\right[,\cdot)$, where $\cdot$ denotes the linear transformation of the standard product acting on $]0,1[$ to the interval $\left]\frac{1}{2},1\right[$, and $x*y=0$ for all $x,y\in\left[0,\frac{1}{2}\right[$. However, an ordinal sum of Archimedean components of T_1 is a t-norm $T_1^\diamond\colon[0,1]^2\longrightarrow[0,1]$ given by

$$T_1^\diamond(x,y) = \begin{cases} \frac{(2\cdot x-1)\cdot(2\cdot y-1)+1}{2} & \text{if } x,y \in \left[\frac{1}{2},1\right], \\ 0 & \text{if } x,y \in \left[0,\frac{1}{2}\right[, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

which is different from T_1.
(ii) Let $T_2\colon [0,1]^2 \longrightarrow [0,1]$ be given by

$$T_2(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y)=1, \\ 2\cdot\min(x,\frac{1}{4})\cdot\min(y,\frac{1}{4}) & \text{if } x,y \in \left[0,\frac{1}{2}\right], \\ \frac{(2\cdot x-1)\cdot(2\cdot y-1)+1}{2} & \text{if } x,y \in \left]\frac{1}{2},1\right[, \\ \min(x,y,\frac{1}{4}) & \text{otherwise.} \end{cases}$$

Then T_2 is a t-norm with non-trivial Archimedean components $(\left]0,\frac{1}{2}\right],*)$ and $(\left]\frac{1}{2},1\right[,\cdot)$, where $\cdot$ denotes the linear transformation of the standard product acting on $]0,1[$ to the interval $\left]\frac{1}{2},1\right[$, and $x*y = 2\cdot\min(x,\frac{1}{4})\cdot\min(y,\frac{1}{4})$ for all $x,y\in\left[0,\frac{1}{2}\right]$. However, an ordinal sum of Archimedean components of T_2 is a t-norm $T_2^\diamond\colon[0,1]^2\longrightarrow[0,1]$ given by

$$T_2^\diamond(x,y) = \begin{cases} 2\cdot\min(x,\frac{1}{4})\cdot\min(y,\frac{1}{4}) & \text{if } x,y \in \left[0,\frac{1}{2}\right], \\ \frac{(2\cdot x-1)\cdot(2\cdot y-1)+1}{2} & \text{if } x,y \in \left]\frac{1}{2},1\right[, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

which is different from T_2.

On the other hand, if for an Archimedean component $(I,*)$ of a t-norm T there is $I=\left]a,b\right[$, or $I=\left[a,b\right[$, then [20, Lemma 4.3] and [20, Theorem 4.4] gives us the following.

Lemma 4.2. *Let $(I,*)$ be an Archimedean component of a t-norm T and let $I=\left]a,b\right[$, or $I=\left[a,b\right[$. Assume that for each $x\in\left]a,b\right[$ there is a $y\in\left]a,b\right[$ such that $x*y>a$ and that $s*t=u*t>a$ implies $s=u$ for all $s,t,u\in I$. Then $T(b,x)=x$ for all $x\in I$.*

Theorem 4.3. *Let T be a t-norm and suppose that each of its non-trivial Archimedean components satisfies the hypothesis of Lemma 4.2. Then there is no other t-norm T_2 having the same Archimedean components as T, i.e., T is the ordinal sum of its Archimedean components.*

For example, if each non-trivial Archimedean component $(I, *)$ of a t-norm T is cancellative and $\sup I < 1$ implies $\sup I \notin I$ then T is uniquely determined as an ordinal sum of its Archimedean components.

However, this is no longer true if some of non-trivial Archimedean components of T is defined on a right-closed interval I, where $\sup I < 1$.

Example 4.4. *For this example we will use the t-norm defined in* [16]. *Each $x \in \,]0, 1]$ can be represented by $x \sim (x^*, i)$, where $x \in \left]\frac{1}{2^{i+1}}, \frac{1}{2^i}\right]$, $i \in \mathbb{N}_0$ and x^* is the relative position of x in the interval $\left]\frac{1}{2^{i+1}}, \frac{1}{2^i}\right]$, i.e., $x^* = 2^{i+1}x - 1$. Define $T\colon [0,1]^2 \longrightarrow [0,1]$ by $T(x,y) = z$, where $z \sim (x^* \cdot y^*, i+j)$ if $x \sim (x^*, i), y \sim (y^*, j)$ and $T(x,y) = 0$ if $\min(x,y) = 0$. Then T is a cancellative, left-continuous t-norm, which has two non-trivial Archimedean components defined on $\left]0, \frac{1}{2}\right]$ and on $\left]\frac{1}{2}, 1\right[$. We can easily observe that this t-norm has cancellative Archimedean components, however, it is not equal to the ordinal sum of its Archimedean components.*

In the case when some non-trivial Archimedean component is defined on an interval which contains its right end-point we have to add the continuity. Then we get the following result.

Proposition 4.5. *Let $(I, *)$ be an Archimedean component of a t-norm T such that $*$ is continuous and $I = \,]a, b]$, or $I = [a, b]$. Assume that for each $x \in \,]a, b]$ there is a $v \in \,]a, b]$ such that $x * v > a$ and that $s * t = u * t > a$ implies $s = u$ for all $s, t, u \in I$. Then $T(x,y) = x$ for all $x \in I$ and $y > b$.*

Proof. Let $f\colon I \longrightarrow I$ be given by $f(x) = \lim_{y \longrightarrow b^+} T(x,y)$. Then $a \le f(x) \le x$ for all $x \in I$ and f is increasing. Further, since for each $x \in \,]a, b]$ there exists a $v \in \,]a, b]$ such that $T(x, v) > a$ we have $f(x) > a$ for $x > a$.

The definition of the function f implies that for any $t \in I$ and any $\varepsilon > 0$ (small enough) there exists a $\delta > 0$ such that $f(t) \le T(t, b+\delta) \le f(t) + \varepsilon$ and thus for any $t, z \in I$ we get

$$T(z, f(t)) \le T(z, T(t, b+\delta)) \le T(z, f(t) + \varepsilon).$$

Since T is continuous on I^2 and $\varepsilon \longrightarrow 0$ implies $\delta \longrightarrow 0$ we get

$$\lim_{\delta \longrightarrow 0} T(z, T(t, b+\delta)) = T(z, f(t)).$$

Similarly we can show that $\lim_{\delta \longrightarrow 0} T(z, T(t, b+\delta)) = \lim_{\delta \longrightarrow 0} T(t, T(z, b+\delta)) = T(t, f(z))$, using the commutativity and the associativity of T. Thus for all $t, z \in I$ there is

$$T(z, f(t)) = T(t, f(z)).$$

Next we will show that f is continuous on $I \setminus \{b\}$. If $a \in I$ then evidently f is right-continuous in a. Assume $x \in \,]a, b[$ and let $\lim_{t \longrightarrow x^-} f(t) = g < h = \lim_{t \longrightarrow x^+} f(t)$. Then $g > a$ and there exists $q \in \,]a, b]$ such that $T(q, g) > a$.

Since T is continuous on I^2 we get: $a < T(q, g) = \lim_{t \longrightarrow x^-} T(q, f(t)) = \lim_{t \longrightarrow x^-} T(t, f(q))$. Similarly, $a < T(q, h) = \lim_{t \longrightarrow x^+} T(q, f(t)) = \lim_{t \longrightarrow x^+} T(t, f(q))$.

The continuity of T on I^2 further implies $\lim_{t \longrightarrow x^-} T(t, f(q)) = \lim_{t \longrightarrow x^+} T(t, f(q))$ and thus $a < T(q, g) = T(q, h)$. The conditional cancellativity then implies $g = h$, which is a contradiction. Thus f is continuous on $I \setminus \{b\}$.

If $f(x) = x$ then the monotonicity of T and $T(x, 1) = x$ implies $T(x, y) = x$ for all $y > b$. Assume that $f(x) = v < x$ for some $x \in I \setminus \{b\}$. Then for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $v \le T(x, b+\delta) \le v + \varepsilon$.

Since $b \in I$ there is also $b < T(y_1, y_2)$ for all $y_1, y_2 \in \,]b, 1]$. Thus for all $0 < \delta < 1 - b$, $0 < \delta_1 < 1 - b$, there is $b < T(b+\delta, b+\delta_1) \le \min(b+\delta, b+\delta_1)$, i.e., $T(b+\delta, b+\delta_1) = b + \delta_2$ for some $\delta_2 \in \,]0, \min(\delta, \delta_1)]$.

Then for any $\delta_1 > 0$ (small enough) we have $T(b+\delta_1, v+\varepsilon) \ge T(b+\delta_1, T(x, b+\delta)) = T(x, T(b+\delta, b+\delta_1)) = T(x, b+\delta_2) \ge v$. Thus $f(v+\varepsilon) \ge v$ for all $\epsilon > 0$ such that $v + \varepsilon \le b$. Since $v < x < b$ and f is continuous on $I \setminus \{b\}$ then $f(v) = v$.

Further, there exists a $w \in \,]a, b]$ such that $T(w, v) > a$. Then $a < T(w, v) = T(w, f(v)) = T(v, f(w))$ and $a < T(w, v) = T(w, f(x)) = T(x, f(w))$.

Thus $a < T(v, f(w)) = T(x, f(w))$ and since T is conditionally cancellative on I^2 we get $v = x$. Therefore $f(x) = x$ for all $x \in I \setminus \{b\}$. Since $f(b) \le b$ and f is increasing we get also $f(b) = b$. Then $T(x, y) = x$ for all $x \in I$ and $y > b$. □

Thus we get the following result.

Theorem 4.6. *Let T be a t-norm and suppose that each of its non-trivial Archimedean components satisfies either the hypothesis of Lemma 4.2, or the hypothesis of Proposition 4.5. Then there is no other t-norm T_1 different from T having the same Archimedean components as T.*

5 Uninorms and n-uninorms with continuous underlying functions

Each t-norm (t-conorm), which can be expressed as an ordinal sum of semigroups can be also expressed as an ordinal sum of disjoint semigroups. This is no longer true in the case of uninorms.

Example 5.1. *Assume the Lukasiewicz t-norm $T_L\colon [0,1]^2 \longrightarrow [0,1]$ given by $T_L(x,y) = \max(0, x+y-1)$ and any representable uninorm U. Let G_1, G_2 be two semigroups such that $G_1 = (\left[\frac{1}{2}, \frac{3}{4}\right], *)$ and $*$ is the linear transformation of the Lukasiewicz t-norm to the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$, $G_2 = (\left[0, \frac{1}{2}\right] \cup \left]\frac{3}{4}, 1\right], U^*)$, where U^* is a transformation of U via (2), i.e., $\frac{1}{2}$ is the annihilator of G_1 and the neutral element of G_2 such that for all $x \in \left]0, \frac{1}{2}\right[$ there exists a $y \in \left]\frac{3}{4}, 1\right[$ such that $U^*(x,y) = \frac{1}{2}$. Then the ordinal sum of G_1 and G_2 with the order $1 < 2$ on the index set $A = \{1,2\}$ is a uninorm V. However, since U is representable and since T_L is nilpotent we cannot remove the point $\frac{1}{2}$ from neither of these two semigroups. Therefore V cannot be expressed as an ordinal sum of disjoint semigroups.*

Based on the ordinal sum of t-norms the ordinal sum of uninorms was defined also just for a countable index set (see [26, 28]). However, in [27] we have seen that ordinal sums with respect to an uncountable index set can also be used in the case of uninorms.

The result of [29] shows that each uninorm with continuous underlying functions can be expressed as an ordinal sum of a countable number of eight types of semigroups from Definition 2.3 and trivial semigroups defined on singletons. However, this result is based on an incorrect observation made in [29, Remark 3, Definition 7] and [29, Proposition 9] which claims that each trivial (singleton) semigroup is always between two non-trivial semigroups, or it is an accumulation point of the set of their end points, which ensures that the number of summands is countable. It is true that the number of non-trivial semigroups is countable since the support of each such a semigroup contains at least one rational number. However, the closure of a countable set is not always countable, which is for example the case of the set of rational numbers from the unit interval. The corrected result can be found in Theorem 5.3.

Example 5.2. *Assume the Cantor set C and let $D = \frac{C}{2}$. Denote $r_n^i = \frac{p_n^i}{2}$, $s_n^i = 1 - \frac{p_n^i}{2}$, $R_n^i = (r_n^i \cup s_n^i)^2$, for all $n \in \mathbb{N}$, $i \in \{1, \ldots, 2^{n-1}\}$, where p_n^i is the corresponding open interval from the construction of the Cantor set. Assume a representable uninorm U and let $e = \frac{1}{2}$, $A = [0,1]^2 \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{i=1}^{2^{n-1}} R_n^i$. We define a binary function $V\colon [0,1]^2 \longrightarrow [0,1]$ for $x \leq y$ using (1) by*

$$V(x,y) = \begin{cases} U_v^{a,b,c,d} & \text{if } (x,y) \in R_n^i \text{ for some } n \in \mathbb{N},\ i \in \{1,\ldots,2^{n-1}\}, \\ \min(x,y) & \text{if } (x,y) \in A,\ x+y \leq 1, \\ \max(x,y) & \text{if } (x,y) \in A,\ x+y > 1, \end{cases}$$

where $a = \inf r_n^i$, $b = \sup r_n^i$, $c = 1-b$, $d = 1-a$ and $v = b$. For $y < x$ we define $V(x,y) = V(y,x)$. Then V is evidently commutative and since $e = \frac{1}{2} \in D \cap (1-D)$ for all $x \in [0,1]$ there is $V(x,e) = \min(x,e)$ if $x+e \leq 1$, i.e., $x \leq \frac{1}{2}$ which implies $V(x,e) = \min(x,e) = x$ and $V(x,e) = \max(x,e)$ if $x+e > 1$, i.e., $x > \frac{1}{2}$ which implies $V(x,e) = \max(x,e) = x$. Thus e is the neutral element of V. Further observe that V on A coincide with the idempotent uninorm U_1 given by $U_1(x,y) = \min(x,y)$ if $x+y \leq 1$ and $U_1(x,y) = \max(x,y)$ if $x_y > 1$. To show the associativity of V we will distinguish for $x,y,z \in [0,1]$ three cases. In the first case $x,y,z \in r_n^i \cup s_n^i$ for some $n \in \mathbb{N}$, $i \in \{1,\ldots,2^{n-1}\}$. Then the associativity follows from the associativity of the representable uninorm U. In the second case there does not exist such $n \in \mathbb{N}$ and $i \in \{1,\ldots,2^{n-1}\}$ that at least two elements from $\{x,y,z\}$ belong to $r_n^i \cup s_n^i$. Then $(x,y),(y,z),(x,z) \in A$ and the associativity follows from the associativity of U_1.

In the third case there exist $n \in \mathbb{N}$ and $i \in \{1,\ldots,2^{n-1}\}$ such that exactly two elements from $\{x,y,z\}$ belong to $r_n^i \cup s_n^i$. Without loss of generality we will assume that $x,y \in r_n^i \cup s_n^i$, $z \notin r_n^i \cup s_n^i$ and we will compare all three values $V(V(x,y),z)$, $V(x,V(y,z))$ and $V(V(x,z),y)$. Observe that $V(x,y) \in r_n^i \cup s_n^i$ and since $z \notin r_n^i \cup s_n^i$ there is $V(x,y) + z \neq 1$. Similarly, $x+z \neq 1$ and $y+z \neq 1$.

1. *If $x,y \in r_n^i$. Then either $z < \min(x,y)$ and then $z < V(x,y)$, or $z > \max(x,y)$ and then $z > V(x,y)$. There is $x+z < 1$ if and only if $y+z < 1$ and then also $V(x,y)+z < 1$. Similarly, there is $x+z > 1$ if and only if $y+z > 1$ and then also $V(x,y)+z > 1$. Therefore we have four possibilities.*

- *If* $z < \min(x, y)$ *and* $x + z < 1$*, then* $V(V(x, y), z) = z$*,* $V(x, V(y, z)) = V(x, z) = z$ *and* $V(V(x, z), y) = V(z, y) = z$*.*
- *If* $z < \min(x, y)$ *and* $x + z > 1$*, then* $V(V(x, y), z) = V(x, y)$*,* $V(x, V(y, z)) = V(x, y)$ *and* $V(V(x, z), y) = V(x, y)$*.*
- *If* $z > \max(x, y)$ *and* $x + z < 1$*, then* $V(V(x, y), z) = V(x, y)$*,* $V(x, V(y, z)) = V(x, y)$ *and* $V(V(x, z), y) = V(x, y)$*.*
- *If* $z > \max(x, y)$ *and* $x + z > 1$*, then* $V(V(x, y), z) = z$*,* $V(x, V(y, z)) = V(x, z) = z$ *and* $V(V(x, z), y) = V(z, y) = z$*.*

2. *If* $x, y \in s_n^i$*, then the associativity can be shown similarly as in the previous case.*

3. *If* $x \in r_n^i$ *and* $y \in s_n^i$*, then* $x \le V(x, y) \le y$ *and we can obtain the following: If* $x < z$*, then* $1 - c = b < z$*, i.e.,* $1 < c + z < y + z$*. Similarly,* $x > z$ *implies* $1 > y + z$*,* $y < z$ *implies* $1 < x + z$ *and* $y > z$ *implies* $x + z < 1$*. Thus we have the following three possibilities:*

- *If* $z < x < y$*, then* $y + z < 1$ *and* $x + z < 1$ *and then also* $V(x, y) + z < 1$ *and* $z < V(x, y)$*. Then* $V(V(x, y), z) = z$*,* $V(x, V(y, z)) = V(x, z) = z$ *and* $V(V(x, z), y) = V(z, y) = z$*.*
- *If* $x < z < y$*, then* $x + z < 1 < y + z$ *and* $b < z < 1 - b$*. There is* $V(V(x, z), y) = V(x, y)$*,* $V(x, V(y, z)) = V(x, y)$*. If* $V(x, y) \le b$*, then* $V(x, y) < z \le 1 - b \le 1 - V(x, y)$*, i.e.,* $z + V(x, y) \le 1$ *and* $V(V(x, y), z) = V(x, y)$*. If* $V(x, y) > c$*, then* $V(x, y) > c \ge z > 1 - c > 1 - V(x, y)$*, i.e.,* $z + V(x, y) > 1$ *and* $V(V(x, y), z) = V(x, y)$*.*
- *If* $x < y < z$*, then* $y + z > 1$ *and* $x + z > 1$ *and then also* $V(x, y) + z > 1$ *and* $z > V(x, y)$*. Then* $V(V(x, y), z) = z$*,* $V(x, V(y, z)) = V(x, z) = z$ *and* $V(V(x, z), y) = V(z, y) = z$*.*

4. *If* $x \in s_n^i$ *and* $y \in r_n^i$ *then the associativity can be shown similarly as in the previous case.*

To show the monotonicity, due to the commutativity, it is enough to show that the partial function $V(x, \cdot)$ *is increasing for all* $x \in [0, 1]$*. If* $x \in D$ *or* $1 - x \in D$ *then* $V(x, y) = \min(x, y)$ *for all* $y \le 1 - x$ *and* $V(x, y) = \max(x, y)$ *for all* $y > 1 - x$*, i.e.,* $V(x, \cdot)$ *is increasing. Further suppose that* $x \in r_n^i$ *for some* $n \in \mathbb{N}$*,* $i \in \{1, \ldots, 2^{n-1}\}$ *(the case when* $x \in s_n^i$ *is analogous). Then* $V(x, y) = y$ *for* $y \le a$*. If* $y \in r_n^i$ *then* $V(x, y) \in [a, x]$ *and* $V(x, \cdot)$ *is increasing on* r_n^i*. Further,* $V(x, y) = x$ *for* $y \in [b, c]$*. If* $y \in s_n^i$ *then* $V(x, y) \in [x, y]$ *and* $V(x, \cdot)$ *is increasing on* s_n^i*. Finally, if* $y \ge d > 1 - x$ *there is* $x + y > 1$ *and* $V(x, y) = y$*. Summarizing, if* $x \in r_n^i$ *then* $V(x, \cdot)$ *is increasing.*

Thus V *is commutative, associative, increasing and has a neutral element* e *and therefore it is a uninorm. Further, since its underlying functions are isomorphic to an ordinal sum of continuous t-norms (continuous t-conorms)* V *has continuous underlying functions. Observe that* V *is an ordinal sum of trivial semigroups from* D *and* $1 - D$ *and non-trivial semigroups* $(r_n^i \cup \{b\} \cup s_n^i, U_b^{a,b,c,d})$ *for* $n \in \mathbb{N}$*,* $i \in \{1, \ldots, 2^{n-1}\}$*, which are isomorphic to the representable uninorm* U *via* (1)*.*

However, V *cannot be expressed as an ordinal sum of a countable number of semigroups from Definition 2.3 and trivial semigroups (compare Example 3.3).*

Therefore the main result from [29] *should be corrected to the following theorem.*

Theorem 5.3. *Let* $U \colon [0, 1]^2 \longrightarrow [0, 1]$ *be a uninorm with continuous underlying functions. Then* U *can be expressed as an ordinal sum of a countable number of semigroups from Definition 2.3 and a possibly uncountable number of trivial semigroups.*

Proof. The proof is similar to the proof of [29, Proposition 11]. The only correction has to be done in the partition of the unit interval which was given in [29, Definition 7], where the sets A and D are possibly uncountable and then also the index sets M_3 and O_3 are possibly uncountable. □

The result with a countable number of summands was from [29] incorrectly transferred to [32], where the main result of the paper states that each n-uninorm with continuous underlying functions can be expressed as a z-ordinal sum of a countable number of semigroups from Definition 2.3 and trivial semigroups (for the definition of the z-ordinal sum construction see [32, 31]). Similarly as in the case of uninorms this result is not correct. As an example we can take a z-ordinal sum of three semigroups $G_1 = ([0, \frac{1}{2}], V^{[0, \frac{1}{2}]})$, $G_2 = ([\frac{1}{2}, 1], V^{[\frac{1}{2}, 1]})$ and $G_3 = (\{\frac{1}{2}\}, \mathrm{Id})$ with respect to the set $A = \{3\}$ and a partial order $\preceq$ given by $1 \wedge 2 = 3$, where $V^{[a,b]}$ is a linear transformation of the uninorm V from Example 5.2 to the interval $[a, b]$. Then this z-ordinal sum is a 2-uninorm W from Class 1, i.e., $W(x, y) = V^{[0, \frac{1}{2}]}(x, y)$ if $x, y \in [0, \frac{1}{2}]$, $W(x, y) = V^{[\frac{1}{2}, 1]}(x, y)$ if $x, y \in [\frac{1}{2}, 1]$ and $W(x, y) = \frac{1}{2}$ otherwise. Here $([0, 1], W)$ can be expressed as a

z-ordinal sum of a countable number of non-trivial semigroups and an uncountable number of trivial semigroups. Note that supports of the corresponding trivial semigroups cover the transformation of the Cantor set to intervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$. This 2-uninorm cannot be expressed as a z-ordinal sum of semigroups from Definition 2.3 and trivial semigroups with a countable number of summands.

Similarly as in the case of uninorms, the correct statement of [32, Theorem 10] is the following.

Theorem 5.4. *Let $U^n\colon [0,1] \longrightarrow [0,1]$ be an n-uninorm, $U^n \in \mathcal{U}_n$. Then U^n can be expressed as a z-ordinal sum of a countable number of semigroups from Definition 2.3 and a possibly uncountable number of trivial semigroups, where $A \sim \{z_1, \ldots, z_{n-1}\}$ and $(C, \preceq)$ has a tree structure.*

The proof is exactly the same as the proof of [32, Theorem 10], since the mistake occurs in results for uninorms. Thus the only difference here is that we should use the corrected result from Theorem 5.3.

6 Ordinal sums of aggregation functions

Ordinal sums of t-norms and t-conorms, see Proposition 2.2, differ in the role of min and max in their definition. To unify both approaches to ordinal sums, observe that both t-norms and t-conorms are aggregation functions. Ordinal sum of aggregation functions related to finitely many non-overlapping intervals covering the unit interval $[0,1]$ was proposed by De Baets and Mesiar [6, 22]. We generalize their approach in the spirit of Proposition 2.2.

Theorem 6.1. *Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k\in K}$ be a system of open, disjoint subintervals of $[0,1]$, $\bigcup_{k\in K} [a_k, b_k] = [0,1]$. Let $(A_k)_{k\in K}$ be a system of binary aggregation functions. Then the ordinal sum $A = (\langle a_k, b_k, A_k\rangle \mid k \in K)$ given by*

$$A(x,y) = \sum_{k\in K} (b_k - a_k) \cdot A_k(\min(1, \max(0, \frac{x-a_k}{b_k-a_k})), \min(1, \max(0, \frac{y-a_k}{b_k-a_k}))). \tag{3}$$

is an aggregation function.

Proof. The monotonicity of A is obvious. Further, $A(0,0) = \sum_{k\in K}(b_k - a_k)A_k(0,0) = 0$ and $A(1,1) = \sum_{k\in K}(b_k - a_k)A_k(1,1) = 1$, and thus A is an aggregation function. □

Obviously, A is commutative (continuous) if and only if all summands A_k are commutative (continuous).

It is not difficult to check that if $(x,y) \in [a_k, b_k]^2$ then $A(x,y) = a_k + (b_k - a_k) \cdot A_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k})$.

Lemma 6.2. *Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k\in K}$ be a system of open, disjoint subintervals of $[0,1]$, $\bigcup_{k\in K} [a_k, b_k] = [0,1]$. Let $(A_k)_{k\in K}$ be a system of binary aggregation functions such that each A_k has a neutral element $e = 1$ (and thus $a = 0$ is its annihilator). Then the ordinal sum $A = (\langle a_k, b_k\rangle \mid k \in K)$ can be written as*

$$A(x,y) = \begin{cases} a_k + (b_k - a_k) \cdot A_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) & \text{if } (x,y) \in [a_k, b_k[^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Proof. Let $x \in [a_s, b_s]$ and $y \in [a_r, b_r]$, where $a_s < a_r$. Then

$$\begin{aligned} A(x,y) &= \sum_{\substack{k\in K\\ a_k<a_s}} (b_k - a_k) \cdot A_k(1,1) + (b_s - a_s) \cdot A_s(\frac{x-a_s}{b_s-a_s}, 1) + \sum_{\substack{k\in K\\ a_k>a_s}} (b_k - a_k) \cdot A_k(0, \min(1, \max(0, \frac{y-a_k}{b_k-a_k}))) \\ &= a_s + (b_s - a_s) \cdot \frac{x-a_s}{b_s-a_s} = x. \end{aligned}$$

Similarly, $A(y,x) = y$ if $a_s > a_r$ and then the ordinal sum $A = (\langle a_k, b_k\rangle \mid k \in K)$ can be written as

$$A(x,y) = \begin{cases} a_k + (b_k - a_k) \cdot A_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) & \text{if } (x,y) \in [a_k, b_k[^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

□

As an easy consequence we see that if all summands A_k are t-norms (copulas, quasi-copulas) then also A is a t-norm (copula, quasi-copula). Similarly, one can show that the same formula (3) yields a t-conorm A whenever all summands A_k are t-conorms.

Note that Theorem 6.1 does not cover Proposition 2.2 whenever $\bigcup_{k\in K} [a_k, b_k] \neq [0,1]$.

7 Concluding remarks

We have recalled and discussed the topic of ordinal sums in the framework of aggregation functions on $[0,1]$. Note that some of ordinal sums were already discussed in different contexts. For example, Moster and Shields [33] in 1957 presented an ordinal sum-based representation of I-semigroups, i.e., topological semigroups acting on some closed real interval I. Their results were restricted to the case $I = [0,1]$ by Ling [21] in 1965, yielding the representation of continuous t-norms and t-conorms, compare Proposition 2.2. All mentioned ordinal sums have considered associativity as a basic property of considered summands and resulting ordinal sums, and thus their binary forms were studied only, with no loss of generality. With the extension to n-dimensional case, one should stress the important role of the top (or the bottom) input elements. In particular, the ordinal sums of t-norms and t-conorms discussed in Proposition 2.2 lead to the following n-ary form of T (S) for $n > 2$:

$$T(x_1,\dots,x_n) = \begin{cases} a_k + (b_k - a_k)\cdot T_k(\varphi(\frac{x_1-a_k}{b_k-a_k}),\dots,\varphi(\frac{x_n-a_k}{b_k-a_k})) & \text{if } \min(x_1,\dots,x_n) \in [a_k,b_k[\,, \\ \min(x_1,\dots,x_n) & \text{else} \end{cases}$$

and

$$S(x_1,\dots,x_n) = \begin{cases} c_k + (d_k - c_k)\cdot S_k(\varphi(\frac{x_1-c_k}{d_k-c_k}),\dots,\varphi(\frac{x_n-c_k}{d_k-c_k})) & \text{if } \max(x_1,\dots,x_n) \in \,]a_k,b_k]\,, \\ \max(x_1,\dots,x_n) & \text{else,} \end{cases}$$

where $\varphi\colon \mathbb{R} \longrightarrow [0,1]$ is given by $\varphi(x) = \min(1,\max(0,x))$.

Observe that a similar construction and representation for n-copulas was shown by Mesiar and Sempi in [24].

Not going into details, note that the ordinal sum of overlap functions was introduced by Dimuro and Bedregal in 2014 in [12]. Based on the representation of continuous t-norms, an ordinal sum of residual implications was proposed by De Baets and Mesiar in [5]. Several other types of ordinal sums of fuzzy implications were discussed in Drygaś et al. in [3, 4] and in [23].

Our paper can be seen as an overview and summarization of several types of ordinal sums related to the aggregation and related functions on $[0,1]$. Some of our original results either expand few already known results, or stress the better understanding of ordinal sums, in particular when considering the n-dimensional case with $n > 2$, or when discussing the possible cardinality of the corresponding index set for particular ordinal sums.

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References

[1] P. Akella, *Structure of n-uninorms*, Fuzzy Sets and Systems, **158**(15) (2007), 1631-1651.

[2] C. Alsina, M. J. Frank, B. Schweizer, *Associative functions: Triangular norms and copulas*, World Scientific, Singapore, 2006.

[3] M. Baczyński, P. Drygaś, A. Król, R. Mesiar, *New types of ordinal sum of fuzzy implications*, 2017 IEEE International Conference on Fuzzy Systems, (2017), 1-6.

[4] M. Baczyński, P. Drygaś, R. Mesiar, *Monotonicity in the construction of ordinal sums of fuzzy implications*, In Proc. of AGOP 2017, (2017), 189-199.

[5] B. De Baets, R. Mesiar, *Residual implicators of continuous t-norms*, In Proc. of EUFIT '96, Zimmermann H. J. (editor), ELITE, (Aachen 1996), (1996), 27-31.

[6] B. De Baets, R. Mesiar, *Ordinal sums of aggregation operators*, In: Bouchon-Meunier B et al. (eds), Technologies for Constructing Intelligent Systems 2, Physica-Verlag, Heidelberg, (2002), 137-148.

[7] G. Beliakov, A. Pradera, T. Calvo, *Aggregation functions: A guide for practitioners*, New York, Springer-Verlag, 2007.

[8] G. Birkhoff, *Lattice theory*, American Mathematical Society Colloquium Publications, **25**, 1940.

[9] T. Calvo, B. De Baets, J. Fodor, *The functional equations of Frank and Alsina for uninorms and nullnorms*, Fuzzy Sets and Systems, **120**(3) (2001), 385-394.

[10] A. H. Clifford, *Naturally totally ordered commutative semigroups*, American Journal of Mathematics, **76** (1954), 631-646.

[11] A. C. Climescu, *Sur l'équation fonctionelle de l'associativité*, Bulletin de l'École Polytechnique de Jassy, 1, (1946), 1-16.

[12] G. P. Dimuro, B. Bedregal, *Archimedean overlap functions: The ordinal sum and the cancellation, idempotency and limiting properties*, Fuzzy Sets and Systems, **252** (2014), 39-54.

[13] J. C. Fodor, R. R. Yager, A. Rybalov, *Structure of uninorms*, International Journal of Uncertainty, Fuzziness and Knowledge-based Systems, **5** (1997), 411-127.

[14] L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, Oxford, 1963.

[15] M. Grabisch, J. L. Marichal, R. Mesiar, E. Pap, *Aggregation functions*, Cambridge University Press, 2009.

[16] P. Hájek, *Observations on the monoidal t-norm logic*, Fuzzy Sets and Systems, **132** (2002), 107-112.

[17] S. Jenei, *A note on the ordinal sum theorem and its consequence for the construction of triangular norms*, Fuzzy Sets and Systems, **126**(2) (2002), 199-205.

[18] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[19] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms as ordinal sums in the sense of A.H. Clifford*, Semigroup Forum, **65** (2002), 71-82.

[20] E. P. Klement, R. Mesiar, E. Pap, *Archimedean components of triangular norms*, Journal of the Australian Mathematical Society, **78** (2005), 239-255.

[21] C. M. Ling, *Representation of associative functions*, Publicationes Mathematicae Debrecen, **12** (1965), 189-212.

[22] R. Mesiar, B. De Baets, *Ordinal sums of aggregation operators*, In proc. of AGGREGATION '99, Calvo T., Mesiar R. (eds.), UIB Palma de Mallorca, (1999), 133-143.

[23] R. Mesiar, A. Mesiarová, *Residual implications and left-continuous t-norms which are ordinal sums of semigroups*, Fuzzy Sets and Systems, **143** (2004), 47-57.

[24] R. Mesiar, C. Sempi, *Ordinal sums and idempotents of copulas*, Aequationes Mathematicae, **79**(1-2) (2010), 39-52.

[25] A. Mesiarová-Zemánková, *Multi-polar t-conorms and uninorms*, Information Sciences, **301** (2015), 227-240.

[26] A. Mesiarová-Zemánková, *Ordinal sum construction for uninorms and generalized uninorms*, International Journal of Approximate Reasoning, **76** (2016), 1-17.

[27] A. Mesiarová-Zemánková, *A note on decomposition of idempotent uninorms into an ordinal sum of singleton semigroups*, Fuzzy Sets and Systems, **299** (2016), 140-145.

[28] A. Mesiarová-Zemánková, *Ordinal sums of representable uninorms*, Fuzzy Sets and Systems, **308** (2017), 42-53.

[29] A. Mesiarová-Zemánková, *Characterization of uninorms with continuous underlying t-norm and t-conorm by means of the ordinal sum construction*, International Journal of Approximate Reasoning, **83** (2017), 176-192.

[30] A. Mesiarová-Zemánková, *Characterization of uninorms with continuous underlying t-norm and t-conorm by their set of discontinuity points*, IEEE Transactions on Fuzzy Systems, **26**(2) (2018), 705-714.

[31] A. Mesiarová-Zemánková, *Characterization of idempotent n-uninorms*, Fuzzy Sets and Systems, (2020), Doi: 10.1016/j.fss.2020.12.019.

[32] A. Mesiarová-Zemánková, *Characterization of n-uninorms with continuous underlying functions via z-ordinal sum construction*, International Journal of Approximate Reasoning, **133** (2021), 60-79.

[33] P. S. Mostert, A. L. Shields, *On the structure of semi-groups on a compact manifold with boundary*, Annals of Mathematics, Series II, **65** (1957), 117-143.

[34] D. Ruiz, J. Torrens, *Distributivity and conditional distributivity of a uninorm and a continuous t-conorm*, IEEE Transactions on Fuzzy Systems, **14**(2) (2006), 180-190.

[35] R. R. Yager, A. Rybalov, *Uninorm aggregation operators*, Fuzzy Sets and Systems, **80** (1996), 111-120.

[36] W. Zong, Y. Su, H. W. Liu, B. De Baets, *On the structure of* 2*-uninorms*, Information Sciences, **467** (2018), 506-527.