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Original paper

Semilinear logics with knotted axioms

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Abstract

Standard completeness, completeness on the real unit interval $[0,1]$, is one of important research areas in mathematical fuzzy logic. Recently, standard completeness for semilinear logics with knotted axioms has been investigated *proof-theoretically* by introducing and eliminating density rule. This paper introduces *model-theoretic* completeness for such logics. To this end, it is first shown that knotted axioms can be divided into left and right ones and then proved that mianorm-based logic systems with left and right knotted axioms are standard complete. This completeness is provided by embedding linearly ordered algebras into densely ordered ones and these algebras again into $[0,1]$. More exactly, mianorm-based systems with left and right knotted axioms and their algebraic structures are first discussed. After some examples of mianorms satisfying left and right knotted properties are introduced, standard completeness for those logics is established model-theoretically using the above construction. Finally, this investigation is extended to their corresponding involutive fixpointed systems.

Keywords: Knotted axioms, mianorm, semilinear logic, fuzzy logic, substructural logic.

1 Introduction

Fuzzy logic is well known as a logic dealing with *vagueness*. In general, the term 'fuzzy logic' has wide and narrow senses made by Zadeh [55]. In a wide sense, fuzzy logic "is fuzzily synonymous with the fuzzy set theory, FST, which is the theory of classes with unsharp boundaries." For instance, FST has a membership function on $[0,1]$ in place of a characteristic function on $\{0,1\}$ as its valuation. This valuation gives truth/membership degrees such as 0.3 and 0.7. In its narrow sense, "fuzzy logic, *FLn*, is a logical system which aims at a formalization of approximating reasoning. In this sense, FLn is an extension of multivalued logic."

One important trend in *FLn* is to introduce such logics with more general structures. In mathematical fuzzy logic, with the title "substructural fuzzy logic," those logics have been introduced as a subclass of substructural logics eliminating some structural axioms such as weakening. For instance, many substructural fuzzy logic systems deleting with structural axioms have been introduced [8, 9, 11, 49, 50, 51]. Especially, Cintula, Noguera, and Horčík [8, 9, 11] have studied fuzzy logics as a subclass of substructural logics with **SL**, the bounded version of **GL**[1], and introduced them as *semilinear* logics, which are complete with respect to (w.r.t.) linearly ordered models.[2] By $\mathbf{SL}^\ell$, they denote the *weakest* semilinear logic extending **SL**.

As the author mentioned in [52], semilinear logic has semantics based on algebraic structures on $[0,1]$ as its intended semantics. The completeness w.r.t. this is said to be *standard* completeness and semilinear logic systems complete w.r.t. these semantics are called *core* semilinear logic. Therefore, to establish *standard* completeness for a semilinear logic is very important in this field.

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[1]Galatos and Ono [20] introduced **GL**, the **FL** (Full Lambek logic) eliminating the associativity axiom, as the most basic substructural logic. This logic is characterized by the variety of residuated lattice-ordered unital groupoids.

[2]In Universal Algebra, a class of algebras is called 'semiX' if their subdirectly irreducible members are X. They introduced "semilinear" as a term inspired by this tradition. To get the more detailed reasons to take the term "semilinear" in place of "fuzzy", see [9, 10, 11].

The forerunner of this investigation is Hájek. He [24] introduced **BL** as the basic fuzzy logic characterized by *continuous* t-norms.[3] After two years, Cignoli et al. [7] proved its standard completeness. Since then, core semilinear logic systems with more general structures have been introduced as uninorm-, micanorm-, and mianorm-based logics.[4] The following are some basic examples: Esteva and Godo [16] introduced **MTL** (the **BL** deleting the divisibility axiom) as the most basic t-norm logic and Jenei and Montagna [28] proved its standard completeness over *left-continuous* t-norms. Metcalfe and Montagna [35] introduced the uninorm logic **UL**, the **MTL** eliminating the integrality axiom, and proved its standard completeness over *conjunctive left-continuous* uninorms. The author [49, 50] introduced the micanorm and mianorm logics **MICAL** and **MIAL** and proved their standard completeness on *conjunctive left-continuous* micanorms and on *conjunctive left-continuous* mianorms, respectively. Proof-theoretically, **MIAL** is the same as $\mathbf{SL}^{\ell}$.

Note that a key point in establishing standard completeness is to use *density* property, i.e., for standard completeness a device is needed to deal with *dense* linearly ordered elements. Associated with this, there are two approaches, i.e., *proof-theoretic* and *model-theoretic* methods, to treat such elements. The proof-theoretic method is to first introduce *density* rule and then eliminate it to a semilinear logic. This ensures that the logic without density rule is capable of covering dense linearly ordered elements. Using this method, Metcalfe and Montagna proved the standard completeness of **UL**. The model-theoretic one is to embed linearly ordered algebras into *dense* linearly ordered algebras. This assures that the corresponding logic can cover dense linearly ordered elements. Using this method, Jenei and Montagna proved the standard completeness of **MTL** and the author proved the standard completeness of **MICAL** and **MIAL**.

This paper aims to establish standard completeness for mianorm-based logics with knotted axioms as substructural core semilinear logic systems. We have at least three reasons to do it. The first is the reason to consider *mianorms*: As the author mentioned in [54], in theories and their applications several reasons to drop associativity from fusion operators have been introduced (see e.g. [3, 6, 12, 17, 25, 29, 34, 37, 38, 42, 43, 44, 45, 46]) and instead non-associative fusions have been extensively investigated in many areas such as subjective probabilities, statistics, quantum mechanics, psychology, linguistics, and medical science (see e.g. [5, 14, 15, 22, 23, 30, 31, 32, 33, 36]) as well as logic, mathematics, and computer science in general. In particular, the term 'mianorms' is, on the one hand, a generalization of theoretic concepts such as t-seminorms [21] (= semicopulas [13]) and micanorms [49] as well as the well-known concepts t-norms, t-conorms, and uninorms. On the other hand, it is a specification of the concept 'semi-uninorms' in the sense that mianorms are semi-uninorms on $[0, 1]$.[5] This means that the term *mianorms* is a basic concept for non-associative and/or non-commutative groupoid operations on $[0, 1]$. Therefore, mianorms can be interesting to researchers of non-associative fusions on $[0, 1]$ in theories and their applications.

The following second and third ones are more direct reasons related to Q1 and Q2 below. The second is the reason to introduce semilinear logics with *knotted axioms*: Many logicians have investigated extensively core semilinear logics with the n-potency axiom (p_n) $\varphi^{n-1} \leftrightarrow \varphi^n$.[6] In particular, after Wang [40] introduced $\mathbf{C}_n\mathbf{UL}$ (the **UL** with the n-potency axiom), Baldi [1], on the one hand, recalled Wang's $\mathbf{C}_n\mathbf{UL}$ as the **UL** with both the n-contraction axiom (c_n) $\varphi^{n-1} \to \varphi^n$ and the n-mingle axiom (m_n) $\varphi^n \to \varphi^{n-1}$, for given $n > 2$, and, on the other hand, introduced the **UL** with the knotted axiom $(kn_{n,k})$ $\varphi^k \to \varphi^n$, for $1 < n, k$. As he already mentioned in it, $(kn_{n,k})$ is not an axiom but a set of axioms and is a generalization of both n-contraction and n-mingle. Very recently, the author [52] noted that in the context of non-commutative logic the n-contraction and n-mingle axioms can be divided into left and right ones and introduced mianorm-based logics with left and right n-contraction and n-mingle axioms. However, mianorm-based logic with left and right knotted axioms were not considered in it and such logics have not yet been investigated. This gives rise to the following question.

Q1. Can we introduce mianorm-based logics with left and right knotted axioms?

The third is the reason to investigate *standard completeness*: As already mentioned above, standard completeness is important in the sense that this property is the criterion to judge whether a logic is a *core* one or not. Standard completeness for semilinear logics with knotted axioms has been provided proof-theoretically using the introduction and elimination of density rule [1, 2]. However, standard completeness for such logics has not yet been investigated *model-theoretically*. Note that the author [52] model-theoretically provided standard completeness for mianorm-based logics with left and right n-contraction and n-mingle axioms but for those logics with knotted axioms. Then, a natural question arises as follows.

Q2. Can we provide any model-theoretic standard completeness for mianorm-based logics with left and right knotted axioms?

[3] Triangular norms, briefly t-norms, are binary functions on the real unit interval $[0, 1]$ having identity 1 and satisfying increasingness, associativity, and commutativity.

[4] By taking identity $e \in [0, 1]$ in place of 1 from t-norms, we obtain uninorms. By deleting associativity from uninorms we obtain micanorms, and by dropping commutativity from micanorms we get mianorms (see [49, 50]).

[5] Mianorms and semi-uninorms both require identity and monotonicity. But a semi-uninorm is an operation on a complete lattice [34], whereas a mianorm is an operation on $[0, 1]$ [50].

[6] See note 5 in [53] for more details.

As an answer to both these questions, this paper establishes model-theoretic standard completeness for the logics mentioned in Q1 and Q2. It will imply that this paper provides a logical foundation of mianorms satisfying knotted properties. This itself is the basic motivation of the author to study such completeness. To this end, first note that, proof-theoretic standard completeness for semilinear logics with knotted axioms has to show that knotted axioms work for introducing and eliminating density rule, whereas model-theoretic standard completeness for those logics must show that knotted axioms work for dense linearly ordered sets. For this model-theoretic completeness, in Section 2, we first introduce axiomatic extensions of **MIAL** with left and right knotted axioms and their algebraic semantics. In Section 3 we then define mianorms satisfying left and right knotted properties and give some examples. In Section 4, using the model-theoretic construction introduced in [49, 50, 52], we provide standard completeness for those logics. This construction has two steps: the first step is the embedding of linearly ordered algebras into *dense* linearly ordered ones and the second step is the embedding again these algebras into standard ones (see Section 4 for more details). In Section 5, we expand this investigation to their fixpointed involutive extensions.

For convenience, notations and terminology similar to those in [8, 9, 11, 40, 49, 50, 51, 52] are adopted and the reader's familiarity with them, along with the results therein, are assumed.

2 Logics: Algebraic completeness

We base the system **MIAL** and its knotted extensions on a countable propositional language with a set of formulas Fm, built inductively from a set of propositional variables VAR, binary connectives $\&, \to, \rightsquigarrow, \vee, \wedge$ and constants $\bot, \top, \overline{1}, \overline{0}$. We further define $\varphi \leftrightarrow \psi$, $\varphi_{\overline{1}}$, ${}^{n}\varphi$ and φ^{n} as $(\varphi \to \psi) \wedge (\psi \to \varphi)$, $\varphi \wedge \overline{1}$, $\overbrace{\varphi\&(\varphi\& \cdots \&(\varphi\&\varphi)...))}^{n\varphi's}$ and $\overbrace{((...(\varphi\&\varphi)\& \cdots \&\varphi)\&\varphi}^{n\varphi's}$, respectively.

We first recall the axiomatization of **MIAL** and define mianorm-based logics with left and right knotted axioms.

Definition 2.1. *1.* [8, 9] *The following are axioms and rules for **MIAL**:*

$(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \psi) \to \psi$;
$((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$;
$\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$;
$((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$;
$\bot \to \varphi$;
$(\overline{1} \to \varphi) \leftrightarrow \varphi$;
$\varphi \to (\psi \to (\psi\&\varphi))$;
$\varphi \to (\psi \rightsquigarrow (\varphi\&\psi))$;
$(\psi\&(\varphi\&(\varphi \to (\psi \to \chi)))) \to \chi$;
$((\varphi\&(\varphi \rightsquigarrow (\psi \to \chi)))\&\psi) \to \chi$;
$((\varphi \to (\varphi\&(\varphi \to \psi)))\&(\psi \to \chi)) \to (\varphi \to \chi)$;
$((\varphi \rightsquigarrow ((\varphi \rightsquigarrow \psi)\&\varphi)))\&(\psi \to \chi)) \to (\varphi \rightsquigarrow \chi)$;
$(PL_{\alpha_{\delta,\varepsilon}})$ $(\varphi \to \psi)_{\overline{1}} \vee ((\delta\&\varepsilon) \to (\delta\&(\varepsilon\&(\psi \to \varphi)_{\overline{1}})))$;
$(PL_{\alpha'_{\delta,\varepsilon}})$ $(\varphi \to \psi)_{\overline{1}} \vee ((\delta\&\varepsilon) \to ((\delta\&(\psi \to \varphi)_{\overline{1}})\&\varepsilon))$;
$(PL_{\beta_{\delta,\varepsilon}})$ $(\varphi \to \psi)_{\overline{1}} \vee ((\delta \to (\varepsilon \to ((\varepsilon\&\delta)\&(\psi \to \varphi)_{\overline{1}})))$;
$(PL_{\beta'_{\delta,\varepsilon}})$ $(\varphi \to \psi)_{\overline{1}} \vee ((\delta \to (\varepsilon \rightsquigarrow ((\varepsilon\&\delta)\&(\psi \to \varphi)_{\overline{1}})))$;
$(\varphi_{\overline{1}}\&\psi_{\overline{1}}) \to (\varphi \wedge \psi)$;
$\varphi \to \psi, \varphi \vdash \psi$;
$\varphi \vdash \varphi_{\overline{1}}$;
$\varphi \vdash (\delta\&\varepsilon) \to (\delta\&(\varepsilon\&\varphi))$;
$\varphi \vdash (\delta\&\varepsilon) \to ((\delta\&\varphi)\&\varepsilon)$;
$\varphi \vdash \delta \to (\varepsilon \to ((\varepsilon\&\delta)\&\varphi))$;
$\varphi \vdash \delta \to (\varepsilon \rightsquigarrow ((\delta\&\varepsilon)\&\varphi))$.

2. Let $1 \leq k, n$.[7]

(a) $\boldsymbol{Kn^{r}_{n,k}MIAL}$ *is **MIAL** plus* $(kn^{r}_{n,k})$ $\varphi^{k} \to \varphi^{n}$ *right knotted axiom.*

(b) $\boldsymbol{Kn^{l}_{n,k}MIAL}$ *is **MIAL** plus* $(kn^{l}_{n,k})$ ${}^{k}\varphi \to {}^{n}\varphi$ *left knotted axiom.*

[7] Precisely, $(kn^{r}_{n,k})$ and $(kn^{l}_{n,k})$ are sets of axioms. For convenience, we henceforth call them axioms rather than sets of axioms. Note that knotted axioms are considered for given $1 < k, n$ in [1]. However, here we introduce left and right knotted axioms for given $1 \leq k, n$.

Example 2.2. *The following are some particular knotted logic systems. For $n \geq 2$,*

1. $\boldsymbol{C^r_n MIAL}$ *is* $\boldsymbol{MIAL}$ *plus* (c^r_n) $\varphi^{n-1} \to \varphi^n$ *right n-contraction.*
2. $\boldsymbol{C^l_n MIAL}$ *is* $\boldsymbol{MIAL}$ *plus* (c^l_n) ${}^{n-1}\varphi \to {}^{n}\varphi$ *left n-contraction.*
3. $\boldsymbol{M^r_n MIAL}$ *is* $\boldsymbol{MIAL}$ *plus* (m^r_n) $\varphi^{n} \to \varphi^{n-1}$ *right n-mingle.*
4. $\boldsymbol{M^l_n MIAL}$ *is* $\boldsymbol{MIAL}$ *plus* (m^l_n) ${}^{n}\varphi \to {}^{n-1}\varphi$ *left n-mingle.*
5. $\boldsymbol{P^r_n MIAL}$ *is* $\boldsymbol{MIAL}$ *plus* (p^r_n) $\varphi^{n} \leftrightarrow \varphi^{n-1}$ *right n-potency.*
6. $\boldsymbol{P^l_n MIAL}$ *is* $\boldsymbol{MIAL}$ *plus* (p^l_n) ${}^{n}\varphi \leftrightarrow {}^{n-1}\varphi$ *left n-potency.*

The first four systems were introduced as left and right n-contractive and n-mingle mianorm-based logics in [52] and the last two as left and right n-potent mianorm-based logics in [53].

For convenience, we use ambiguously '$\bot$,' '$\top$,' '$\vee$,' and '$\wedge$' as propositional constants and connectives and as special elements and algebraic operators.

Now we introduce algebraic structures characterizing the left and right knotted mianorm-based logics.

Definition 2.3. *1.* [19] *A* bounded pointed residuated lattice-ordered unital groupoid *is a structure* $(A, *, \backslash, /, \bot, \top, f, t, \vee, \wedge)$ *such that:*[8]

(a) $(A, \bot, \top, \vee, \wedge)$ *is a bounded lattice with bottom and top elements* $\bot$, $\top$.

(b) $(A, *, t)$ *is a unital groupoid.*

(c) f *is a point, i.e., an any element, of* A.

(d) $y \leq x\backslash z$ *if and only if (iff)* $x * y \leq z$ *iff* $x \leq z/y$*, for all* $x, y, z \in A$ *(residuation).*

2. [50] *Define* x_t *as* $x \wedge t$. *A* MIAL-algebra *is a bounded pointed residuated lattice-ordered unital groupoid satisfying: for all* $w, x, y, z \in A$,

(a) $(PL^{\mathcal{A}}_{\alpha_{\delta,\varepsilon}})$ $t \leq (x\backslash y)_t \vee ((w * z)\backslash(w * (z * (y\backslash x)_t)))$

(b) $(PL^{\mathcal{A}}_{\alpha'_{\delta,\varepsilon}})$ $t \leq (x\backslash y)_t \vee ((w * z)\backslash((w * (y\backslash x)_t) * z))$

(c) $(PL^{\mathcal{A}}_{\beta_{\delta,\varepsilon}})$ $t \leq (x\backslash y)_t \vee ((w\backslash(z\backslash((z * w) * (y\backslash x)_t)))$

(d) $(PL^{\mathcal{A}}_{\beta'_{\delta,\varepsilon}})$ $t \leq (x\backslash y)_t \vee ((w\backslash(((z * w) * (y\backslash x)_t)/z))$.

3. For all $x \in A$ *and* $1 \leq k, n$,

(a) A $Kn^r_{n,k}MIAL$*-algebra is a MIAL-algebra satisfying* $(kn^r_{n,k}{}^{\mathcal{A}})$ $x^k \leq x^n$;

(b) A $Kn^l_{n,k}MIAL$*-algebra is a MIAL-algebra satisfying* $(kn^l_{n,k}{}^{\mathcal{A}})$ ${}^{k}x \leq {}^{n}x$.

For convenience, all these algebras are called L-algebras.

For a set of formulas Fm and an L-algebra $\mathcal{A}$, we define an $\mathcal{A}$-*valuation* as a map $v : Fm \to \mathcal{A}$, which satisfies that $v(\#(\varphi_1, \ldots, \varphi_n)) = \#^{\mathcal{A}}(v(\varphi_1, \ldots, \varphi_n))$, where $\# \in \{\&, \to, \rightsquigarrow, \vee, \wedge, \bot, \top, \overline{0}, \overline{1}\}$ and $\#^{\mathcal{A}} \in \{*, \backslash, /, \vee, \wedge, \bot, \top, f, t\}$. We say that a formula φ is *valid* in $\mathcal{A}$ in case $t \leq v(\varphi)$ for all $\mathcal{A}$-valuation v and that an $\mathcal{A}$-valuation v is an $\mathcal{A}$-*model* of a theory T in case $t \leq v(\varphi)$ for all $\varphi \in T$.

For completeness, recall the below fact.

Fact 2.4. [11] Any axiomatic extension of a semilinear logic is semilinear too.

Since **MIAL** is a semilinear logic and L $\in \{\mathbf{Kn}^r_{n,k}\mathbf{MIAL}, \mathbf{Kn}^l_{n,k}\mathbf{MIAL}\}$ is an axiomatic extension of **MIAL**, we obtain completeness as its corollary.

Theorem 2.5. (*Strong completeness,* [11]) *Let* T *be a theory over* $L \in \{\boldsymbol{Kn^r_{n,k}MIAL}, \boldsymbol{Kn^l_{n,k}MIAL}\}$ *and* φ *be a formula.* $T \vdash_L \varphi$ *iff for all linearly ordered L-algebras* $\mathcal{A}$ *and an* $\mathcal{A}$*-valuation* v*, if* v *is an* $\mathcal{A}$*-model of* T*, then* $t \leq v(\varphi)$.

[8] Note that the system **SL** is characterized by the variety of pointed, bounded residuated lattice-ordered unital groupoids.

3 Knotted mianorms and examples

We henceforth denote $\bot, \top$, f, and t on $[0,1]$ by '0,' '1,' 'ξ,' and 'e,' respectively. First, note that an algebra $\mathcal{A}$ is said to be a *standard* algebra if its carrier set is $[0,1]$. Here we define mianorms with left and right knotted properties as L-mianorms.

Definition 3.1. *1.* (Mianorm, [50]) *A* mianorm *is a function* $\circ : [0,1]^2 \to [0,1]$ *such that for all* $x, y, z \in [0,1]$ *and for some* $e \in [0,1]$:

(a) $x \leq y$ *implies* $z \circ x \leq z \circ y$ *and* $x \circ z \leq y \circ z$ *(monotonicity), and*

(b) $x \circ e = e \circ x = x$ *(identity).*

2. (*L-mianorm*) *For* $L^{\mathcal{A}} \in \{kn^r_{n,k}{}^{\mathcal{A}}, kn^l_{n,k}{}^{\mathcal{A}}\}$, *a mianorm satisfying* $L^{\mathcal{A}}$ *is said to be an* L-mianorm.

A *micanorm* is a mianorm satisfying commutativity; a *uninorm* is a micanorm satisfying associativity; and a *t-norm* is a uninorm satisfying $e = 1$. We say that a mianorm $\circ$ is *conjunctive* in case $1 \circ 0 = 0 \circ 1 = 0$.

Next, we introduce some examples of left and right knotted mianorms. Note first the following two characteristic features of uninorms originated from Fodor–Yager–Rybalov [18]. For this, let $\circ$ be a uninorm operator. First, uninorms have contractive and expansive parts, i.e., for all $x \in [0,1]$, $x \leq x^2$ if $e \leq x$ and otherwise $x^2 \leq x$. Second, uninorms have two subalgebras, which are isomorphic to t-conorm and t-norm, i.e., for $e \in (0,1)$, $\circ : [0,1]\lceil_{[e,1]^2} \to [0,1]\lceil_{[e,1]}$ is isomorphic to a t-conorm and $\circ : [0,1]\lceil_{[0,e]^2} \to [0,1]\lceil_{[0,e]}$ to a t-norm (see [18, 52]).[9]

Now let $\circ$ be a mianorm operator. The first note still works for mianorms since $x = e \circ x \leq x \circ x$ if $e \leq x$ and otherwise $x = e \circ x \geq x \circ x$ by monotonicity, but the second one does not necessarily (see [52] for some such examples). Here we instead introduce some simple examples of knotted mianorms having subalgebras isomorphic to t-norm and t-conorm. By the term "right (left resp) (n,k)-knotted mianorm," we henceforth denote a mianorm satisfying $kn^r_{n,k}{}^{\mathcal{A}}$ ($kn^l_{n,k}{}^{\mathcal{A}}$ resp).

Example 3.2. *Let* $1 \leq k, n$.

1. The operator $\circ_1 : [0,1]^2 \to [0,1]$ *given by*

$$x \circ_1 y = \begin{cases} 1 & \textit{if } e < x, y; \\ 0 & \textit{if } x, y < e; \\ y & \textit{if } x = e < y; \\ x & \textit{otherwise,} \end{cases}$$

is a right (left) $(2,3)$-, $(3,2)$-*knotted, i.e.,* 3-*potent, mianorm operator.*

2. The operator $\circ_2 : [0,1]^2 \to [0,1]$ *given by*

$$x \circ_2 y = \begin{cases} 1 & \textit{if } e < x, y; \\ min(x,y) & \textit{if } x, y \leq e; \\ y & \textit{if } x = e < y; \\ x & \textit{otherwise,} \end{cases}$$

is a right (left) $(2,1)$-, $(2,3)$-, $(3,2)$-*knotted, i.e.,* 2-*contractive and* 3-*potent, mianorm operator.*

3. The operator $\circ_3 : [0,1]^2 \to [0,1]$ *given by*

$$x \circ_3 y = \begin{cases} max(x,y) & \textit{if } e \leq x, y; \\ 0 & \textit{if } x, y < e; \\ y & \textit{if } y < x = e; \\ x & \textit{otherwise,} \end{cases}$$

is a right (left) $(1,2)$-, $(2,3)$-, $(3,2)$-*knotted, i.e.,* 2-*mingle and* 3-*potent, mianorm operator.*

Remark 3.3. *The* $\circ$*'s in Example 3.2 have subalgebras isomorphic to the following t-norms and t-conorms.*

[9] $[0,1]\lceil_{[e,1]}$ and $[0,1]\lceil_{[e,1]^2}$ mean the $[0,1]$ restricted to $[e,1]$ and its square, i.e., $[0,1]\lceil_{[e,1]} \times [0,1]\lceil_{[e,1]}$, respectively, and similarly for $[0,1]\lceil_{[0,e]}$ and $[0,1]\lceil_{[0,e]^2}$.

1. *The mianorm $\circ_1$ in* 1 *has the subalgebras isomorphic to the drastic product t-norm and the drastic sum t-conorm.*[10]

2. *The mianorm $\circ_2$ in* 2 *has the subalgebras isomorphic to the Gödel t-norm and the drastic sum t-conorm and the mianorm $\circ_3$ in* 3 *the subalgebras isomorphic to the drastic product t-norm and the Gödel t-conorm.*[11]

4 Standard completeness

Here we establish standard completeness for L $\in \{\mathbf{Kn}^r_{n,k}\mathbf{MIAL}, \mathbf{Kn}^l_{n,k}\mathbf{MIAL}\}$ using the model-theoretic construction of the author in [49, 50, 52]. For this, we first explain the idea of accomplishing this completeness. The starting point is Theorem 2.5. This theorem says that if $T \nvdash_L \varphi$, then there is a linearly ordered L-algebra $\mathcal{A}$ and an $\mathcal{A}$-valuation v such that $t > v(\varphi)$ for any $\psi \in T$ such that $t \leq v(\psi)$. Then, for standard completeness, we need to show that this algebra can be embedded into a standard algebra. This embeddability is provided based on the following two steps:

1. We verify that every finite or countable linearly ordered L-algebra can be embedded into a countable *dense* linearly ordered L-algebra.

2. Based on the fact that every dense, linearly-ordered set is isomorphic to $(Q \cap [0,1], \leq)$ and this again to $([0,1], \leq)$, we verify that every countable linearly ordered L-algebra is embeddable into a standard L-algebra.

Recall the following fact.

Fact 4.1. (Standard completeness, [50]) *For* ***MIAL***, $T \vdash_{\boldsymbol{MIAL}} \varphi$ *iff for all standard* ***MIAL****-algebras and valuation* v, *if* $e \leq v(\psi)$ *for all* $\psi \in T$, *then* $e \leq v(\varphi)$.

We need to show that this theorem can be extended to the **MIAL** with knotted axioms.

For standard completeness, we first consider the *first step* 1, i.e., we prove that every finite or countable, linearly ordered L-algebra can be embedded into a countable *dense* linearly ordered L-algebra. Since these algebras are ordered algebras, we for convenience add the 'less than or equal to' relation symbol "$\leq$" to those algebras.

Proposition 4.2. *For every linearly ordered finite (or countable) L-algebra* $\boldsymbol{A} = (A, *, \backslash, /, \vee, \wedge, \bot, \top, f, t, \leq_A)$, *we can construct a countable ordered set* X, *a groupoid operation* $\circ$ *on* X, *and a map* h *from* A *into* X *satisfying the following conditions:*

1. *X has a* dense *order, a minimum* Min, *a maximum* Max, *and two special elements* ξ *and* e.

2. $(X, \circ, \preceq, e)$ *is a monotonic unital linearly ordered groupoid.*

3. $\circ$ *is left-continuous and conjunctive on* $(X, \preceq)$.

4. *h is an embedding of the structure* $(A, *, \vee, \wedge, \bot, \top, f, t, \leq_A)$ *into* $(X, \circ, max, min, Min, Max, \xi, e, \preceq)$, *and, for all* $l, m \in A$, $h(l\backslash m)$ *and* $h(m/l)$ *are the residuated pair of* $h(l)$ *and* $h(m)$ *in* $(X, \circ, max, min, Min, Max, \xi, e, \preceq)$.

5. $\circ$ *satisfies left and right knotted properties corresponding to* $*$.

Proof. For convenience, suppose A as a subset of $\mathbf{Q} \cap [0,1]$ with finite (or countable) elements, where 1 and 0 are the greatest and least elements. Define X as the set

$$\{(0,0)\} \cup \{(l,x) : l \in A \setminus \{0\ (=\bot)\} \text{ and } x \in \mathbf{Q} \cap (0,l]\}.$$

For $(l,x), (m,y) \in X$, we define $\preceq$ as follows.

[10] The drastic product t-norm and the drastic sum t-conorm are defined as follows:

$$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1)^2; \\ min(x,y) & \text{otherwise.} \end{cases}$$

$$S_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (0,1]^2; \\ max(x,y) & \text{otherwise.} \end{cases}$$

[11] Gödel t-norm and t-conorm are defined as follows.

$$T^G(x,y) = min\{x,y\} \quad \text{(Gödel t-norm)},$$
$$S^G(x,y) = max\{x,y\} \quad \text{(Gödel t-conorm)}.$$

$$(l,x) \preceq (m,y) \text{ iff } l <_A m \text{ or } l =_A m \text{ and } x \leq y.$$

For the sake of convenience, we henceforth eliminate the index A in $\leq_A$ and $=_A$, if we do not have to distinguish them.

For the operation $\circ$, first note that, for $\mathbf{Kn}^r_{n,k}\mathbf{MIAL}$ and $\mathbf{Kn}^l_{n,k}\mathbf{MIAL}$, $k \leq n$, we define $\circ$ as follows: for $(l,x),(m,y) \in X$,[12]

$$(l,x) \circ (m,y) = \begin{cases} min\{(l,x),(m,y)\} & \text{if } l * m = l \wedge m, \text{ and} \\ & (l,x) \preceq e(=(t,t)) \text{ or } (m,y) \preceq e; \\ max\{(l,x),(m,y)\} & \text{if } l * m = l \vee m, l \neq m, \text{ and} \\ & (l,x) \preceq e \text{ or } (m,y) \preceq e; \\ (l * m, l * m) & \text{otherwise.} \end{cases}$$

For $\mathbf{Kn}^r_{n,k}\mathbf{MIAL}$ and $\mathbf{Kn}^l_{n,k}\mathbf{MIAL}$, $n < k$, we define $\circ$ as follows: for (l,x), $(m,y) \in X$,[13]

$$(l,x) \circ (m,y) = \begin{cases} min\{(l,x),(m,y)\} & \text{if } l * m = l \wedge m, \text{ and} \\ & (l,x) \preceq e \text{ or } (m,y) \preceq e; \\ max\{(l,x),(m,y)\} & \text{if } l * m = l \vee m, \text{ and} \\ & (l,x) \succ e \text{ or } (m,y) \succ e; \\ (l * m, l * m) & \text{otherwise.} \end{cases}$$

Note that the conditions 1 to 4 are for dense linearly ordered MIAL-algebras and are proved in Proposition 2 in [50]. Hence, here we prove the condition 5, which is related to knotted properties introduced in Definition 2.3 3.

For $\mathbf{Kn}^r_{n,k}\mathbf{MIAL}$, we need to prove right knotted property, i.e., $(l,x)^k \preceq (l,x)^n$ for $1 \leq n,k$ and $(l,x) \in X$. We first prove it for $k \leq n$. Let $e \preceq (l,x)$. From the definition, it follows that $(l,x) \circ (l,x) = (l^2,l^2)$ and $(l^2,l^2) \circ (l,x) = (l^3,l^3)$. This implies that $(l,x)^k = (l^k,l^k)$ and $(l,x)^n = (l^n,l^n)$. Then, since $l^k \leq l^n$, we obtain that $(l,x)^k \preceq (l,x)^n$.

Otherwise, first consider the case that $l^2 < l$. Since $l^2 < l < t$, as above we also have that $(l,x) \circ (l,x) = (l^2,l^2)$ and $(l^2,l^2) \circ (l,x) = (l^3,l^3)$. Then, similarly we obtain that $(l,x)^k = (l^k,l^k)$ and $(l,x)^n = (l^n,l^n)$; therefore, $(l,x)^k \preceq (l,x)^n$ since $l^k \leq l^n$. Let $l \leq l^2$. This implies that $l = l^2$. Hence, we obtain that $(l,x) \circ (l,x) = (l,x)$. Then, since $l = l^2 = l^k = l^n$ and thus $(l,x)^k = (l,x)^n$, we obtain that $(l,x)^k \preceq (l,x)^n$.

We next prove it for $n < k$. Let $e \prec (l,x)$. First consider the case that $l < l^2$. Since $t < l < l^2$, as above we have that $(l,x) \circ (l,x) = (l^2,l^2)$ and $(l^2,l^2) \circ (l,x) = (l^3,l^3)$. Then, analogously we further obtain that $(l,x)^k = (l^k,l^k)$ and $(l,x)^n = (l^n,l^n)$. Then, since $l^k \leq l^n$, we have that $(l,x)^k \preceq (l,x)^n$. Let $l^2 \leq l$. This implies that $l^2 = l$ and thus $(l,x) \circ (l,x) = (l,x)$. Then, since $l = l^2 = l^k = l^n$ and thus $(l,x)^k = (l,x)^n$, we obtain that $(l,x)^k \preceq (l,x)^n$.

Next, consider the case that $l^2 < l$. Since $(l,x) \circ (l,x) = (l^2,l^2)$ and $(l^2,l^2) \circ (l,x) = (l^3,l^3)$, as above we can obtain that $(l,x)^k \preceq (l,x)^n$ since $l^k \leq l^n$.

Finally, consider the case that $l^2 = l$. Since $t \geq l = l^2$, we obtain that $(l,x) = (l,x) \circ (l,x)$ and thus $(l,x)^k = (l,x)^n$; therefore, $(l,x)^k \preceq (l,x)^n$.

For $\mathbf{Kn}^l_{n,k}\mathbf{MIAL}$, we need to prove left knotted property, i.e., ${}^k(l,x) \preceq {}^n(l,x)$ for $1 \leq n,k$ and $(l,x) \in X$. This proof is similar to that of the right knotted one. □

We next consider the *second step* 2, i.e., we prove that every countable linearly ordered L-algebra is embeddable into a *standard* L-algebra. It has two substeps: first, to embed a countable dense linearly ordered L-algebra into an algebra on $\mathbf{Q} \cap [0,1]$; second, to embed this algebra again into an algebra on $[0,1]$, i.e., a standard L-algebra.

Proposition 4.3. *Every linearly ordered countable L-algebra is embeddable into a standard algebra.*

Proof. Let $\mathbf{A}, X$, etc. be as in Proposition 4.2. First note that $(X, \preceq)$ is a dense, linearly-ordered countable set with minimum and maximum. This ensures that $(X, \preceq)$ is order isomorphic to $(\mathbf{Q} \cap [0,1], \leq)$. Take i as such an isomorphism and let for $a,b \in [0,1]$, $a \circ^+ b = i(i^{-1}(a) \circ i^{-1}(b))$ and for all $l \in A$, $h^+(l) = i(h(l))$. If 1, 2, 3, 4, and 5 in Proposition 4.2 hold true, then we have that $\mathbf{Q} \cap [0,1]$, 0, 1, ξ, e, $\circ^+$, $\leq$, and h^+ satisfy the conditions 1, 2, 3, 4, and 5 of Proposition 4.2 whenever $X, Min, Max, \xi(=(f,f)), e(=(t,t)), \circ, \preceq$, and h do. Then, we can assume that $X = \mathbf{Q} \cap [0,1]$, $\circ = \circ^+$, and $\preceq = \leq$ without loss of generality.

For $c,d \in [0,1]$, define:

$$c \mathbin{\bar{\circ}} d := sup_{x \in X : x \leq c} sup_{y \in X : y \leq d} x \circ y.$$

[12]This definition was first introduced in [47].
[13]This definition was first introduced in [48].

System	Algebra	Standard algebra
MIAL	MIAL-algebra	clc mianorm
$\mathbf{Kn}^r_{n,k}\mathbf{MIAL} = \mathbf{MIAL} + (kn^r_{n,k})$	$\mathrm{Kn}^r_{n,k}$MIAL-algebra	clc $\mathrm{Kn}^r_{n,k}$-mianorm
$\mathbf{Kn}^l_{n,k}\mathbf{MIAL} = \mathbf{MIAL} + (kn^l_{n,k})$	$\mathrm{Kn}^l_{n,k}$MIAL-algebra	clc $\mathrm{Kn}^l_{n,k}$-mianorm

Table 1: **MIAL** and its left and right knotted extensions

As easy consequences of the definition, we obtain the monotonicity, identity, and conjunctiveness of $\bar{\circ}$. The left-continuity of $\bar{\circ}$ is proved in Proposition 3 in [50]. This implies that $\bar{\circ}$ is a conjunctive left-continuous mianorm.

Here we prove the left and right knotted properties of $\bar{\circ}$. For the right knotted property of $\bar{\circ}$, assume $\langle c_i : i \in \mathbf{N}\rangle$ as an increasing sequence of reals on $[0,1]$, where $sup\{c_i : i \in \mathbf{N}\} = c$. It is obvious that $c^k = sup\{s^k : s \in \mathbf{Q} \cap [0,1], s \leq c\}$ and $c^n = sup\{s^n : s \in \mathbf{Q} \cap [0,1], s \leq c\}$. Then, we need to show that $c^k \leq c^n$, $1 \leq n, k$. Since $s^k \leq s^n$, we can obtain that $sup\{s^k : s \in \mathbf{Q} \cap [0,1], s \leq c\} \leq sup\{s^n : s \in \mathbf{Q} \cap [0,1], s \leq c\}$. Hence, we further have that $c^k \leq c^n$. The left knotted property of $\bar{\circ}$ can be proved analogously.

It directly follows from the definition that $\bar{\circ}$ extends $\circ$. Thus, by the conditions $1, 2, 3, 4$, and 5, we can ensure that h is an embedding of $(A, \bot, \top, f, t, \vee, \wedge, *, \leq_A)$ into $([0,1], 0, 1, \xi, e, max,\ min, \bar{\circ}, \leq)$. Moreover, it is proved by Proposition 3 in [50] that $\bar{\circ}$ has a residuated pair of implications. Therefore, $([0,1], 0, 1, \xi, e, max,\ min, \bar{\circ}, \leq)$ forms the corresponding standard algebra. □

Theorem 4.4. (*Standard completeness*) *For* $L \in \{\boldsymbol{Kn}^r_{n,k}\boldsymbol{MIAL},\ \boldsymbol{Kn}^l_{n,k}\boldsymbol{MIAL}\}$, $T \vdash_L \varphi$ *iff for every standard L-algebra and valuation* v, *if* $e \leq v(\psi)$ *for all* $\psi \in T$, *then* $e \leq v(\varphi)$.

Proof. The left-to-right direction is obvious. For the right-to-left direction, suppose that φ is a formula such that $T \nvdash_L \varphi$, $\mathbf{A}$ is a linearly ordered L-algebra, and v is a valuation in $\mathbf{A}$ such that for all $\psi \in T$, $t \leq v(\psi)$ but $t > v(\varphi)$. We further assume that h^+ is the embedding of $\mathbf{A}$ into the standard L-algebra as in Proposition 4.3. Then, we can obtain that that $h^+ \circ v$ is a valuation into the standard L-algebra such that $e \leq h^+ \circ v(\psi)$ but $e > h^+ \circ v(\varphi)$. □

This theorem assures that L is complete w.r.t. conjunctive left-continuous right or left knotted mianorms and their residua; therefore, L is a *core* semilinear logic.

Let us use *clc mianorm* as an abbreviation of conjunctive left-continuous mianorm. We summarize the logics introduced in Section 2 and their corresponding (standard) algebras in Table 1.

5 Fixpointed involutive extensions

For L $\in \{\mathbf{Kn}^r_{n,k}\mathbf{MIAL}, \mathbf{Kn}^l_{n,k}\mathbf{MIAL}\}$, we consider fixpointed involutive extensions of L and their standard completeness. For this, define two negation connectives $\neg, -$ as follows: $\neg\varphi := \varphi \to \overline{0}$ and $-\varphi := \varphi \rightsquigarrow \overline{0}$. We first define such logics as extensions of **IMIAL** (Involutive **MIAL**) and consider their algebraic semantics.

Definition 5.1.

1. [51] ***IMIAL*** *is* ***MIAL*** *plus (double negation elimination(1), DNE(1))* $-\neg\varphi \to \varphi$ *and (DNE(2))* $\neg - \varphi \to \varphi$.
2. [51] ***FIMIAL*** *is* ***IMIAL*** *plus (fixpoint, F)* $\overline{1} \leftrightarrow \overline{0}$.

Definition 5.2. *Let* $1 \leq n, k$ *and* $L \in \{\boldsymbol{Kn}^r_{n,k}\boldsymbol{MIAL},\ \boldsymbol{Kn}^l_{n,k}\boldsymbol{MIAL}\}$.

1. *IL is L plus (DNE(1)) and (DNE(2)). We denote the set of these systems by* $\{\boldsymbol{Kn}^r_{n,k}\boldsymbol{IMIAL},\ \boldsymbol{Kn}^l_{n,k}\boldsymbol{IMIAL}\}$ *and say that IL is an* involutive *extension of L.*
2. *For* $IL \in \{\boldsymbol{Kn}^r_{n,k}\boldsymbol{IMIAL}, \boldsymbol{Kn}^l_{n,k}\boldsymbol{IMIAL}\}$, *FIL is IL plus (F). We denote the set of these systems by*

$$\{\boldsymbol{Kn}^r_{n,k}\boldsymbol{FIMIAL}, \boldsymbol{Kn}^l_{n,k}\boldsymbol{FIMIAL}\},$$

and say that FIL is a fixpointed *extension of IL.*

Definition 5.3. *For* $x \in A$, *define* $\neg x$ *and* $-x$ *as* $x\backslash f$ *and* f/x, *respectively.*

1. [51] *An IMIAL-algebra is a MIAL-algebra satisfying:* $(DNE(1)^{\mathcal{A}})$ $-\neg x \leq x$ *and* $(DNE(2)^{\mathcal{A}})$ $\neg - x \leq x$.

2. [51] *A FIMIAL-algebra is an IMIAL-algebra satisfying: $(F^{\mathcal{A}})$ $t = f$.*

3. *IL-algebras are L-algebras satisfying $(DNE(1)^{\mathcal{A}})$ and $(DNE(2)^{\mathcal{A}})$ and FIL-algebras are IL-algebras satisfying $(F^{\mathcal{A}})$.*

The systems **IMIAL** and **FIMIAL** in Definition 5.1 are characterized by the varieties of IMIAL- and FIMIAL-algebras in Definition 5.3(1) and (2), respectively (see [51]). As in Section 2, let us define valuations and models. Then, we can show that IL and FIL introduced in Definition 5.2(1) and (2) are complete w.r.t. IL- and FIL-algebras, respectively, in Definition 5.3(3).

Since IL $\in \{\mathbf{Kn}^r_{n,k}\mathbf{IMIAL}, \mathbf{Kn}^l_{n,k}\mathbf{IMIAL}\}$ is an axiomatic extension of **MIAL**, as above we obtain completeness as a corollary of Fact 2.4.

Theorem 5.4. (*Strong completeness,* [11])

1. *Let T be a theory over $IL \in \{\boldsymbol{Kn}^r_{n,k}\boldsymbol{IMIAL}, \boldsymbol{Kn}^l_{n,k}\boldsymbol{IMIAL}\}$ and φ a formula. $T \vdash_{IL} \varphi$ iff for all linearly ordered IL-algebras $\mathcal{A}$ and an $\mathcal{A}$-valuation v, if v is an $\mathcal{A}$-model of T, then $t \leq v(\varphi)$.*

2. *Let T be a theory over $FIL \in \{\boldsymbol{Kn}^r_{n,k}\boldsymbol{FIMIAL}, \boldsymbol{Kn}^l_{n,k}\boldsymbol{FIMIAL}\}$ and φ a formula. $T \vdash_{FIL} \varphi$ iff for all linearly ordered FIL-algebras $\mathcal{A}$ and an $\mathcal{A}$-valuation v, if v is an $\mathcal{A}$-model of T, then $t \leq v(\varphi)$.*

Now we introduce some examples of fixpointed involutive knotted mianorms defined by negations. A function $n : [0, 1] \to [0, 1]$ is called a *negation* function (briefly a negation) iff n is non-increasing and satisfies $n(0) = 1$ and $n(1) = 0$. Consider a pair of negations $(\neg, -)$. The pair $(\neg, -)$ is called *involutive* if it satisfies $(\text{DNE}(1)^{\mathcal{A}'})$ $-\neg x = x$ and $(\text{DNE}(2)^{\mathcal{A}'})$ $\neg - x = x$; *fixpointed* if there is $x \in (0, 1)$ such that $\neg x = -x = x$, denoting this x by e; and *cyclic* if for all $x \in [0, 1]$, $\neg x = -x$. This means that if an involutive pair of negations is cyclic, then we can express it by only one negation, i.e., $\neg$ (or $-$). For easy understanding of this negation, we first introduce some known example of fixpointed involutive knotted mianorms defined by such negation.

Example 5.5. *Let $\neg$ be a fixpointed cyclic involutive negation, where identity $e \in (0, 1]$.*

1. [51] *A non-associative and non-commutative mianorm $\circ_4$ and its involutively residuated pair $(\backslash_4, /_4)$ are given by:*

$$x \circ_4 y = \begin{cases} min\{x, y\} & \text{if } x \leq \neg(y); \\ max\{x, y\} & \text{if } x > \neg(y) \text{ and } e \leq y; \\ x + y - e & \text{otherwise.} \end{cases}$$

$$x \backslash_4 y = \begin{cases} max\{\neg(x), y\} & \text{if } x \leq y; \\ min\{\neg(x), y\} & \text{if } x > y \text{ and } y \leq e; \\ e - x + y & \text{otherwise.} \end{cases}$$

$$y /_4 x = \begin{cases} min\{1, e - x + y\} & \text{if } x \leq e \leq y; \\ max\{\neg(x), y\} & \text{if } x \leq y \text{ and otherwise}; \\ min\{\neg(x), y\} & \text{otherwise.} \end{cases}$$

2. [51] *A non-associative and non-commutative mianorm $\circ_5$ and its involutively residuated pair $(\backslash_5, /_5)$ are given by:*

$$x \circ_5 y = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0; \\ max\{0, x + y - e\} & \text{if } x, y \neq 0 \text{ and } x \leq \neg(y); \\ min\{1, x + y - e\} & \text{if } x > \neg(y) \text{ and } e \leq y; \\ max\{x, y\} & \text{otherwise.} \end{cases}$$

$$x \backslash_5 y = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1; \\ min\{1, e - x + y\} & \text{if } 0 < x \leq y < 1; \\ max\{0, e - x + y\} & \text{if } x > y \text{ and } y \leq e; \\ min\{\neg(x), y\} & \text{otherwise.} \end{cases}$$

$$y /_5 x = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1; \\ max\{\neg(x), y\} & \text{if } x \leq e \leq y; \\ min\{1, e - x + y\} & \text{if } 0 < x \leq y < 1 \text{ and otherwise}; \\ max\{0, e - x + y\} & \text{otherwise.} \end{cases}$$

3. [52] *A left (right) n-contractive mianorm* $\circ_6$ *and its involutively residuated pair* $(\backslash_6, /_6)$ *are given by:*

$$x \circ_6 y = \begin{cases} min\{x, y\} & \text{if } x \leq \neg y; \\ min\{1, x + y - e\} & \text{if } x > \neg y \text{ and } x \leq y; \\ min\{1, max\{x, y, 2y - e\}\} & \text{otherwise.} \end{cases}$$

$$x \backslash_6 y = \neg(\neg y \circ_6 x) \quad \text{and} \quad y /_6 x = \neg(x \circ_6 \neg y).$$

4. [52] *A left (right) n-mingle mianorm* $\circ_7$ *and its involutively residuated pair* $(\backslash_7, /_7)$ *are given by:*

$$x \circ_7 y = \begin{cases} max\{x, y\} & \text{if } x > \neg y; \\ max\{0, x + y - e\} & \text{if } x \leq \neg y \text{ and } y \leq x; \\ max\{0, min\{x, y, 2x - e\}\} & \text{otherwise.} \end{cases}$$

$$x \backslash_7 y = \neg(\neg y \circ_7 x) \quad \text{and} \quad y /_7 x = \neg(x \circ_7 \neg y).$$

Recall that some examples of left and right knotted mianorms are introduced in Example 3.2 in Section 3. Here we introduce their corresponding fixpointed involutive examples.

Example 5.6. *Let the operation* $\neg$ *be a fixpointed, cyclic involutive negation, where fixpoint* $e \in (0, 1)$.

1. *A left (right)* $(2, 3)$-, $(3, 2)$-*knotted fixpointed mianorm* $\circ_{1'}$ *and its involutively residuated pair* $(\backslash_{1'}, /_{1'})$ *are given by:*

$$x \circ_{1'} y = \begin{cases} 1 & \text{if } y > \neg x \text{ and } e < x, y; \\ max\{x, y\} & \text{if } y > \neg x \text{ and otherwise}; \\ 0 & \text{if } y \leq \neg x \text{ and } x, y < e; \\ e & \text{if } y \leq \neg x \text{ and } 0 < y < e; \\ min\{x, y\} & \text{otherwise.} \end{cases}$$

$$x \backslash_{1'} y = \neg(\neg y \circ_{1'} x) \quad \text{and} \quad y /_{1'} x = \neg(x \circ_{1'} \neg y).$$

2. *A left (right)* $(2, 1)$-, $(2, 3)$-, $(3, 2)$-*knotted fixpointed mianorm* $\circ_{2'}$ *and its involutively residuated pair* $(\backslash_{2'}, /_{2'})$ *are given by:*

$$x \circ_{2'} y = \begin{cases} 1 & \text{if } y > \neg x \text{ and } e < x, y; \\ max\{x, y\} & \text{if } y > \neg x \text{ and otherwise}; \\ e & \text{if } y \leq \neg x \text{ and } 0 < y < e; \\ min\{x, y\} & \text{otherwise.} \end{cases}$$

$$x \backslash_{2'} y = \neg(\neg y \circ_{2'} x) \quad \text{and} \quad y /_{2'} x = \neg(x \circ_{2'} \neg y).$$

3. *A left (right)* $(1, 2)$-, $(2, 3)$-, $(3, 2)$-*knotted fixpointed mianorm* $\circ_{3'}$ *and its involutively residuated pair* $(\backslash_{3'}, /_{3'})$ *are given by:*

$$x \circ_{3'} y = \begin{cases} max\{x, y\} & \text{if } y > \neg x; \\ 0 & \text{if } y \leq \neg x \text{ and } x, y < e; \\ e & \text{if } y \leq \neg x \text{ and } 0 < y < e; \\ min\{x, y\} & \text{otherwise.} \end{cases}$$

$$x \backslash_{3'} y = \neg(\neg y \circ_{3'} x) \quad \text{and} \quad y /_{3'} x = \neg(x \circ_{3'} \neg y).$$

Finally, we establish standard completeness for FIL $\in \{\mathbf{Kn}^r_{n,k}\mathbf{FIMIAL}, \mathbf{Kn}^l_{n,k}\mathbf{FIMIAL}\}$.

Fact 5.7. [51] For every linearly ordered finite (or countable) **IMIAL**-algebra $\mathbf{A} = (A, *, \backslash, /, \bot, \top, 0, 1, \vee, \wedge, \leq_A)$, we can construct a countable ordered set X, a groupoid operation $\circ$ on X, and a map h from A into X satisfying the conditions 1 to 4 in Proposition 4.2 and the following condition hold:

6. For all $x \in X$, x satisfies (DNE(1)$^{\mathcal{A}}$) and (DNE(2)$^{\mathcal{A}}$).

Proposition 5.8. *For every linearly ordered finite (but at least three) or countable FIL-algebra $\boldsymbol{A} = (A, *, \backslash, /, \bot, \top, 0, 1, \vee, \wedge, \leq_A)$, we can construct a countable ordered set X, a groupoid operation $\circ$ on X, and a function h from A into X such that the conditions 1 to 5 in Proposition 4.2, 6 in Fact 5.7, and $(F^{\mathcal{A}})$ hold.*

Proof. By l^+, we denote the successor of l if it exists, otherwise $l^+ = l$, for all $l \in A$. First note that $\neg l$ and $-l$ in A are defined as $l\backslash\xi$ and ξ/l, respectively. Then, since this pair of negations is involutive, we obtain that: $l = (\neg m)^+$ iff $m = (\neg l)^+$; $(\neg(l^+))^+ = \neg l$ if $l < l^+$; and similarly for $-$. We define $(Y, \preceq)$ as a linearly ordered set defined by

$$Y = \{(l, x) : \exists l' \in A \text{ such that } l' < l = l'^+, \text{ and } x \in \mathbf{Q} \cap (0, l)\} \cup \{(l, l) : l \in A\},$$

with $\preceq$, which is the corresponding lexicographic ordering as above. Instead of the X above, we use this Y as a carrier set. We denote the first and second circles introduced in Proposition 4.2 as $\circ_{Y_1}$ and $\circ_{Y_2}$, respectively, so as to distinguish them. Here we define new groupoid operations $\oplus_{Y_1}$ and $\oplus_{Y_2}$ on Y, which are based on $\circ_{Y_1}$ and $\circ_{Y_2}$, respectively, as follows (but, for the sake of convenience, henceforth eliminating each index if we need not distinguish them):[14]

$$(l, x) \oplus (m, y) = \begin{cases} min\{\xi, (l, x) \circ (m, y)\} & \text{if } l = (\neg m)^+ \text{ and } \frac{p}{q} + \frac{p'}{q'} \leq 1, \\ & \text{where } x = l\frac{p}{q} \text{ and } y = m\frac{p'}{q'}, \text{ or} \\ & l < (\neg m)^+; \text{ or} \\ & \text{if } l = (-m)^+ \text{ and } \frac{p}{q} + \frac{p'}{q'} \leq 1, \\ & \text{where } x = l\frac{p}{q} \text{ and } y = m\frac{p'}{q'}, \text{ or} \\ & l < (-m)^+; \\ (l, x) \circ (m, y) & \text{otherwise.} \end{cases}$$

The operations $\oplus_{Y_1}$ and $\oplus_{Y_2}$ are for $\mathbf{Kn}^r_{n,k}$**FIMIAL** and $\mathbf{Kn}^l_{n,k}$**FIMIAL**, $k \leq n$, and for $\mathbf{Kn}^r_{n,k}$**FIMIAL** and $\mathbf{Kn}^l_{n,k}$**FIMIAL**, $k > n$, respectively. Here we define ${}^n(l, x)$ and $(l, x)^n$ as $\overbrace{(l, x) \oplus ((l, x) \oplus \cdots \oplus ((l, x) \oplus (l, x) ...))}^{n\varphi' s}$ and $\overbrace{((...((l, x) \oplus (l, x)) \oplus \cdots \oplus (l, x)) \oplus (l, x)}^{n\varphi' s}$, respectively. Since the conditions 1, 2, 3, 4 and 6 are proved in Proposition 2 in [51] and the condition $(F^{\mathcal{A}})$ is proved in Proposition 3 in [52], here we need to prove 5.

For $\mathbf{Kn}^r_{n,k}$**FIMIAL**, we prove right knotted property, i.e., $(l, x)^k \preceq (l, x)^n$ for $1 \leq n, k$ and $(l, x) \in X$. We first prove it for $k \leq n$. Consider the case that $l = (\neg l)^+$ and $2\frac{p}{q} \leq 1$, where $x = l\frac{p}{q}$, or $l < (\neg l)^+$. For $l^2 \geq l$, note that $l > t$ is not the case. Thus, we have that $l^2 = l \leq t = f < (\neg l)^+$. Then, from the definition it follows that $(l, x) \oplus_{Y_1} (l, x) = min\{\xi, (l, x) \circ_{Y_1} (l, x)\} = (l, x) \circ_{Y_1} (l, x) = (l, x)$. Therefore, since $l = l^2 = l^k = l^n$ and thus $(l, x)^k = (l, x)^n$, we obtain that $(l, x)^k \preceq (l, x)^n$. For $l^2 < l$, we need to show that $(l, x)^k \preceq (l, x)^n$ for $1 < k \leq n$. Since the condition implies that $l^2 < l < t$, we have that $(l, x) \oplus_{Y_1} (l, x) = min\{\xi, (l, x) \circ_{Y_1} (l, x)\} = (l, x) \circ_{Y_1} (l, x)$. Then, since $\oplus_{Y_1}$ can be reduced into $\circ_{Y_1}$, this ensures that $(l, x)^k \preceq (l, x)^n$ since $l^k \leq l^n$ for $1 < k \leq n$.

The proof for the case that $l = (-l)^+$ and $2\frac{p}{q} \leq 1$, where $x = l\frac{p}{q}$, or $l < (-l)^+$ is analogous. Otherwise, since $(l, x) \oplus_{Y_1} (l, x) = (l, x) \circ_{Y_1} (l, x)$, the proof can be reduced into that of the right knotted property for $\mathbf{Kn}^r_{n,k}$**MIAL**, where $k \leq n$, in Proposition 4.2.

We next prove it for $k > n$. As above, first consider the case that $l = (\neg l)^+$ and $2\frac{p}{q} \leq 1$, where $x = l\frac{p}{q}$, or $l < (\neg l)^+$. For $l \geq l^2$, as above, we have that $l^2 \leq l \leq t = f \leq (\neg l)^+$. Then, from the definition it follows that $(l, x) \oplus_{Y_2} (l, x) = min\{\xi, (l, x) \circ_{Y_2} (l, x)\} = (l, x) \circ_{Y_2} (l, x)$. Therefore, since $\oplus_{Y_2}$ can be reduced into $\circ_{Y_2}$ and $l^k \leq l^n$, we further obtain that $(l, x)^k \preceq (l, x)^n$. It is not the case that $l < l^2$.

The proof for the case that $l = (-l)^+$ and $2\frac{p}{q} \leq 1$, where $x = l\frac{p}{q}$, or $l < (-l)^+$ is analogous. Otherwise, as above, the proof can be reduced into that of the right knotted property for $\mathbf{Kn}^r_{n,k}$**MIAL**, where $k > n$, in Proposition 4.2.

For $\mathbf{Kn}^l_{n,k}$**FIMIAL**, we need to prove left knotted property, i.e., ${}^k(l, x) \preceq {}^n(l, x)$ for $1 \leq n, k$ and $(l, x) \in X$. This property is proved analogously. $\square$

Proposition 5.9. *Every linearly ordered countable FIL-algebra is embeddable into a standard algebra.*

Proof. The proof is similar to that of Proposition 4.3. $\square$

Theorem 5.10. (*Standard completeness*) *For FIL* $\in \{\boldsymbol{Kn}^r_{n,k}$***FIMIAL***, $\boldsymbol{Kn}^l_{n,k}$***FI- MIAL***$\}$, $T \vdash_{FIL} \varphi$ *iff for every standard FIL-algebra and valuation v, if $e \leq v(\psi)$ for all $\psi \in T$, then $e \leq v(\varphi)$.*

[14]This definition was first introduced in [51].

System	Algebra	Standard algebra
FIMIAL	FIMIAL-algebra	clc fi-mianorm
$\mathbf{Kn}^r_{n,k}$**FIMIAL**	$\mathrm{Kn}^r_{n,k}$FIMIAL-algebra	clc $\mathrm{Kn}^r_{n,k}$-fi-mianorm
$\mathbf{Kn}^l_{n,k}$**FIMIAL**	$\mathrm{Kn}^l_{n,k}$FIMIAL-algebra	clc $\mathrm{Kn}^l_{n,k}$-fi-mianorm

Table 2: **FIMIAL** and its left and right knotted extensions

Proof. The proof is similar to that of Theorem 4.4. □

This theorem assures that FIL is complete w.r.t. conjunctive left-continuous fixpointed involutive right or left knotted mianorms; therefore, FIL is a *core* semilinear logic. Finally, we note that the proof in Theorem 5.10 does not work for IL $\in \{\mathbf{Kn}^r_{n,k}\mathbf{IMIAL}, \mathbf{Kn}^l_{n,k}\mathbf{IMIAL}\}$.[15]

Let us use *fi-mianorm* as an abbreviation of fixpointed involutive mianorm. We summarize the logics introduced in this section and their corresponding (standard) algebras in Table 2.

6 Concluding remarks

We investigated standard completeness for semilinear logics based on mianorms satisfying left and right knotted properties and their fixpointed involutive extensions via the model-theoretic construction of the author. As mentioned above, we cannot use this construction for non-fixpointed involutive extensions. It is an open problem whether such logics are standard complete or not. Recently, Wang [41] proof-theoretically showed that **IUL** (Involutive uninorm logic) is standard complete. However, synthetic manipulations introduced there are complicate. Whether this method is applicable to our problem is a new research direction.

Note that logical and algebraic properties for substructural (semilinear) logics such as standard completeness, cut-elimination, local finiteness, finite embeddability property, finite model property, computational complexity, amalgamation, and decidability have been investigated (see [4, 26, 27, 39]). Among those properties we just treated standard completeness. To investigate other properties for mianorm-based logics with left and right knotted axioms also remains an another problem.

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[15] We can verify this using the examples introduced in Theorem 5 in [52].

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