Iranian Journal of Fuzzy Systems

Volume 19, Number 2, (2022), pp. 31-48

Original paper

Commutative, associative and non-decreasing functions continuous around diagonal

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Abstract

We characterize all functions that can be obtained as a z-ordinal sum of semigroups related to continuous t-norms, t-conorms, representable uninorms and idempotent semigroups. We show that this class of functions is bigger than the class of n-uninorms with continuous underlying functions. Vice versa, we show that the characterization of n-uninorms with continuous underlying functions via z-ordinal sum can be extended to any commutative, associative and non-decreasing binary function on the unit interval, which has continuous Archimedean components and is continuous on the diagonal.

Keywords: n-uninorm, ordinal sum, z-ordinal sum, representable uninorm, t-norm, t-conorm, nullnorm.

1 Introduction

The associativity equation [1] has attracted a big interest in the last century in the field of functional equations, since associative functions can be uniquely extended for an arbitrary number of inputs. Especially associative, monotone non-decreasing and commutative (AMC) functions such as t-norms, t-conorms and their generalizations uninorms and nullnorms (see [3, 5, 10, 25]) were applied in many domains as for example in control systems, image processing and soft computing. The t-norms and t-conorms are used for modeling of logical connectives in fuzzy logic, however, they are widely used also in expert systems, neural networks, multi-criteria decision making and many other fields. Since any uninorm (nullnorm) can be taken as a bipolar t-conorm (t-norm) these extensions of t-norms and t-conorms can be considered in the bipolar logic and bipolar aggregation.

Among several construction methods for AMC functions the prominent is the construction via ordinal sum [6]. The z-ordinal sum construction (defined in [22]) extends the ordinal sum construction of Clifford, which is defined for linearly ordered families of semigroups, to partially ordered families of semigroups, where the partial order and the related index set correspond to a meet semi-lattice. Similarly as uninorms with continuous underlying functions can be expressed as an ordinal sum of semigroups related to continuous Archimedean t-norms and t-conorms, representable uninorms and idempotent semigroups, n-uninorms with continuous underlying functions can be expressed as a z-ordinal sum of these semigroups.

Since the class of n-uninorms contains the classes of t-norms, t-conorms, nullnorms and uninorms, the characterization of n-uninorms with continuous underlying functions via the z-ordinal sum construction covers also the characterization of continuous t-norms, continuous t-conorms, nullnorms and uninorms with continuous underlying functions as special cases. All these functions can be expressed as a z-ordinal sum of semigroups related to continuous Archimedean t-norms and t-conorms, representable uninorms and idempotent semigroups. The question which we want to answer in this paper is the following: what are the most general non-decreasing functions that can be expressed as a z-ordinal sum of such semigroups? We will show that these are just commutative, associative and non-decreasing functions on the unit interval, which have continuous Archimedean components and are continuous on the diagonal.

Corresponding Author: A. Mesiarová-Zemánková

Received: October 2021; Revised: **; Accepted: January 2022.

https://doi.org/10.22111/IJFS.2022.6786

The paper is organized as follows. In the next section we introduce all necessary notions and results. In Section 3 we will study functions on the unit interval which are constructed via z-ordinal sum of Archimedean, representable and idempotent semigroups. Section 4 is dedicated to the study of Archimedean components of commutative, associative and non-decreasing functions on the unit interval and in Section 5 we show their decomposition into Archimedean, representable and idempotent semigroups. Our conclusions can be found in Section 6.

2 Basic notions

A triangular norm [10] is a binary function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm [10] is a binary function $S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element.

Each continuous t-norm (t-conorm) can be expressed as an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm S) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e., there exists $(x,y) \in\,]0,1[^2$ such that $T(x,y) = 0$ ($S(x,y) = 1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator, which is uniquely determined up to a positive multiplicative constant.

Definition 2.1. [11] *Let $T\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm. Two elements $x, y \in [0,1]$ are called Archimedean equivalent if there is an $n \in \mathbb{N}$ such that $x_T^{(n)} \leq y \leq x$ or $y_T^{(n)} \leq x \leq y$, where $x_T^{(n)} = T(x, x_T^{(n-1)})$ for all $n \in \mathbb{N}$ and $x_T^{(0)} = 1$. For each $x \in [0,1]$, the equivalence class I_x containing x is called an Archimedean class of T and it is a convex subset of $[0,1]$. For each $x \in [0,1]$ the pair $(I_x, T|_{I_x^2})$ is a subsemigroup of $([0,1], T)$, and it is called an Archimedean component of T.*

For each t-norm the set of all Archimedean components forms a partition of $[0,1]$. Moreover, if for a non-empty subset $A \subseteq [0,1]$ we put $I_A = \bigcup\limits_{x\in A} I_x$ then $(I_A, T|_{I_A^2})$ is a totally ordered abelian semigroup. For each $x \in [0,1]$, $(I_x, T|_{I_x^2})$ is the maximal Archimedean subsemigroup containing x and there is no idempotent point in the interior of I_x.

For t-conorms we can define Archimedean components in a similar way, however, since t-conorms are disjunctive the respective inequalities from Definition 2.1 change to $y \leq x \leq y_S^{(n)}$ or $x \leq y \leq x_S^{(n)}$.

More details on t-norms and t-conorms can be found in [3, 10].

A uninorm (introduced in [25]) is a binary function $U\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and have a neutral element $e \in [0,1]$ (see also [8]). Evidently, if $e = 1$ ($e = 0$) then we retrieve a t-norm (t-conorm).

For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called conjunctive (disjunctive) if $U(1,0) = 0$ ($U(1,0) = 1$). For each uninorm U with the neutral element $e \in\,]0,1[$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e., a linear transformation of some t-conorm S_U. Moreover, $\min(x,y) \leq U(x,y) \leq \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [8]). A uninorm which possesses a continuous additive generator is called representable. Note that in [24] it was shown that a uninorm is representable if and only if it is continuous on $[0,1]^2 \setminus \{(0,1),(1,0)\}$.

A uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ is called internal if $U(x,y) \in \{x,y\}$ for all $(x,y) \in [0,1]^2$; and it is called idempotent if $U(x,x) = x$ for all $x \in [0,1]$.

Observe that if a uninorm U is internal then it is also idempotent and vice-versa (see [7]).

Definition 2.2. *A uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ is called d-internal if it is internal and there exists a continuous and strictly decreasing function $g_U\colon [0,1] \longrightarrow [0,1]$ such that $U(x,y) = \min(x,y)$ if $y < g_U(x)$ and $U(x,y) = \max(x,y)$ if $y > g_U(x)$.*

A binary function $V\colon [0,1]^2 \longrightarrow [0,1]$ is called a nullnorm (see [5, 15, 14]) if it is commutative, associative, non-decreasing in each variable and has an annihilator $z \in [0,1]$ such that $V(0,x) = x$ for all $x \leq z$ and $V(1,x) = x$ for all $x \geq z$. If $z = 0$ ($z = 1$) then V is a t-norm (t-conorm). For each nullnorm V with annihilator $z \in\,]0,1[$, the restriction of V to $[0,z]^2$ is a t-conorm on $[0,z]^2$, and the restriction of V to $[z,1]^2$ is a t-norm on $[z,1]^2$, while $V(x,y) = z$ on $[0,z] \times [z,1]$ and on $[z,1] \times [0,z]$.

Now let us recall the definition of an n-uninorm (see [2]).

Definition 2.3. *Assume $n \in \mathbb{N}\setminus\{1\}$. Let $V\colon [0,1]^2 \longrightarrow [0,1]$ be a commutative binary function. Then $\{e_1,\dots,e_n\}_{z_1,\dots,z_{n-1}}$ is called an n-neutral element of V if for $0 = z_0 < z_1 < \cdots < z_n = 1$ and $e_i \in [z_{i-1}, z_i]$, $i = 1,\dots,n$ we have $V(e_i, x) = x$ for all $x \in [z_{i-1}, z_i]$.*

A binary function $U^n\colon [0,1]^2 \longrightarrow [0,1]$ is an n-uninorm if it is associative, non-decreasing in each variable, commutative and has an n-neutral element $\{e_1,\dots,e_n\}_{z_1,\dots,z_{n-1}}$. The basic structure of n-uninorms was described by Akella in [2] and the characterizations of the main five classes of 2-uninorms was given in [26].

Each n-uninorm has the following building blocks around the main diagonal.

Proposition 2.4. *Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm with the n-neutral element $\{e_1,\dots,e_n\}_{z_1,\dots,z_{n-1}}$. Then*

(i) U^n restricted to $[z_{i-1}, e_i]^2$, for $i = 1,\dots,n$, is a linear transformation of a t-norm. We will denote this t-norm by T_i.

(ii) U^n restricted to $[e_i, z_i]^2$ for $i = 1,\dots,n$, is a linear transformation of a t-conorm. We will denote this t-conorm by S_i.

(iii) U^n restricted to $[z_{i-1}, z_i]^2$ for $i = 1,\dots,n$, is a linear transformation of a uninorm. We will denote this uninorm by U_i.

(iv) U^n restricted to $[z_i, z_j]^2$ for $i, j \in \{0,1,\dots,n\}$, $i < j$, is a linear transformation of a $(j-i)$-uninorm.

Moreover, U^n restricted to $[e_i, e_{i+1}]^2$ for $i = 1,\dots,n-1$, is a linear transformation of a nullnorm.

In the following we recall the z-ordinal sum construction. Note that a meet semi-lattice (or lower semi-lattice) is a partially ordered set which has a meet (or greatest lower bound) for any non-empty finite subset (see [4]). Since the existence of the meet is required only for non-empty finite subsets this is equivalent to the existence of the meet between all pairs of arguments.

Theorem 2.5. *Let A and B be two index sets such that $A \cap B = \emptyset$ and $C = A \cup B \neq \emptyset$. Let $(G_\alpha)_{\alpha\in C}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups and let the set C be partially ordered by the binary relation $\preceq$ such that $(C, \preceq)$ is a meet semi-lattice. Further suppose that each semigroup G_α for $\alpha \in A$ possesses an annihilator z_α, and for all $\alpha, \beta \in C$ such that α and β are incomparable there is $\alpha \wedge \beta \in A$. Assume that for all $\alpha, \beta \in C$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$. In the second case suppose that for all $\gamma \in C$ which is incomparable with $\alpha \wedge \beta$ there is $\alpha \wedge \gamma = \beta \wedge \gamma$ and for each $\gamma \in C$ with $\alpha \wedge \beta \prec \gamma \prec \alpha$ or $\alpha \wedge \beta \prec \gamma \prec \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Further,*

(i) in the case that $\alpha \wedge \beta \in A$ then $x_{\alpha,\beta} = z_{\alpha\wedge\beta}$ is the annihilator of both G_β and G_α;

(ii) in the case that $\alpha \wedge \beta = \alpha \in B$ then $x_{\alpha,\beta}$ is both the annihilator of G_β and the neutral element of G_α.

Put $X = \bigcup_{\alpha\in C} X_\alpha$ and define the binary operation $*$ on X by

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta, textand\ \alpha \wedge \beta = \alpha \in B, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta,\ \text{and } \alpha \wedge \beta = \beta \in B, \\ z_\gamma & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta,\ \text{and } \alpha \wedge \beta = \gamma \in A. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in C$ the semigroup G_α is commutative.

The set A from the previous theorem will be called the branching set. Observe that sets X_α in z-ordinal sum construction need not to be disjoint, however, in [22] it was shown that $*$ is well defined and thus in order to obtain the value $x * y$ for $x \in X_\alpha \cap X_\beta$ we can select any of the two semigroups.

Before we proceed with the main results of the paper we will recall several notions that we will use. Since we will use ordinal sums of trivial semigroups, let us recall that there exists only one operation on a trivial semigroup, namely the function $\mathrm{Id}\colon \{x\}^2 \longrightarrow \{x\}$, which is simply defined by $\mathrm{Id}(x,x) = x$.

Further, for any $0 \leq a < b \leq c < d \leq 1$, $v \in [b,c]$ and a uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ with the neutral element $e \in]0,1[$ we will use the transformation $f\colon [0,1] \longrightarrow [a,b[\cup \{v\} \cup]c,d]$, given by

$$f(x) = \begin{cases} (b-a)\cdot \frac{x}{e} + a & \text{if } x \in [0,e[, \\ v & \text{if } x = e, \\ d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise.} \end{cases} \tag{1}$$

Then f is linear on $[0,e[$ and on $]e,1]$ and thus it is an increasing, piece-wise linear isomorphism of $[0,1]$ to $([a,b[\cup\{v\}\cup]c,d])$ which preserves the commutativity, the associativity, the monotonicity and the neutral element; and the binary function $U_v^{a,b,c,d}\colon ([a,b[\cup\{v\}\cup]c,d])^2 \longrightarrow ([a,b[\cup\{v\}\cup]c,d])$ given by

$$U_v^{a,b,c,d}(x,y) = f(U(f^{-1}(x), f^{-1}(y))), \tag{2}$$

will be called a uninorm on $([a,b[\cup\{v\}\cup]c,d])^2$. The backward transformation f^{-1} then transforms a commutative, associative and non-decreasing function with the neutral element v, defined on $([a,b[\cup\{v\}\cup]c,d])^2$, to a uninorm defined on $[0,1]^2$.

Uninorms with continuous underlying functions were completely characterized in [19] using the following semigroups.

Definition 2.6. *Let $a,b,c,d \in [0,1]$ with $a < b < c < d$, $v \in [b,c]$. Then*

(*i*) *a semigroup $(]a,b[\cup\{v\}\cup]c,d[,*)$ will be called a* representable *semigroup if $*$ is isomorphic via* (2) *to a restriction of a representable uninorm on $[0,1]^2$ to $]0,1[^2$,*

(*ii*) *a semigroup $(]a,b[,*)$ will be called a* t-strict *semigroup if $*$ is linearly isomorphic to a restriction of a strict t-norm on $[0,1]^2$ to $]0,1[^2$,*

(*iii*) *a semigroup $(]c,d[,*)$ will be called an* s-strict *semigroup if $*$ is linearly isomorphic to a restriction of a strict t-conorm on $[0,1]^2$ to $]0,1[^2$,*

(*iv*) *a semigroup $([a,b[,*)$ will be called a* t-nilpotent *semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-norm on $[0,1]^2$ to $[0,1[^2$,*

(*v*) *a semigroup $(]c,d],*)$ will be called an* s-nilpotent *semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-conorm on $[0,1]^2$ to $]0,1]^2$,*

(*vi*) *a semigroup $(]a,b[\cup]c,d[,*)$ will be called a* d-internal *semigroup if $*$ is isomorphic via* (2) *to a restriction of an d-internal uninorm on $[0,1]^2$ to $(]0,1[\setminus\{e\})^2$,*

(*vii*) *a semigroup $(]a,b[,*)$ will be called a* t-internal *semigroup if $* = \min$,*

(*viii*) *a semigroup $(]c,d[,*)$ will be called an* s-internal *semigroup if $* = \max$.*

In [19] it was shown (and corrected in [23]) that each uninorm with continuous underlying functions can be decomposed into an ordinal sum of a countable number of semigroups from Definition 2.6 and a possibly uncountable number of trivial semigroups. Similarly, in [20] it was shown (and corrected in [23]) that each n-uninorm with continuous underlying functions can be decomposed into a z-ordinal sum of a countable number of semigroups from Definition 2.6 and a possibly uncountable number of trivial semigroups, where the branching set A corresponds to the set of division points $\{z_1,\ldots,z_{n-1}\}$ and $(C,\preceq)$ has a tree structure.

3 Z-ordinal sum of Archimedean, representable and idempotent semigroups

As we mentioned above, the characterization of n-uninorms with continuous underlying functions via the z-ordinal sum covers also characterization of continuous t-norm, t-conorms, uninorms and nullnorms with continuous underlying functions. However, a z-ordinal sum of semigroups from Definition 2.6 (and trivial semigroups) can yield also functions that are not n-uninorms. A trivial example is a function that is not monotone. If we require an example of a non-decreasing function we have the following.

Example 3.1. *Let $F\colon [0,1]^2 \longrightarrow [0,1]$ be given by*

$$F(x,y) = \begin{cases} \min(x,y) & \text{if } \min(x,y) \le \frac{1}{2}, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

This function can be expressed as an ordinal sum of trivial semigroups $\{(\{x\},\mathrm{Id})\}_{x\in[0,1]}$ with respect to linear order given by

$$x \preceq y \text{ if } \begin{cases} \text{either } x \in \left[0,\frac{1}{2}\right],\ y\in[0,1] \text{ and } x \le y, \\ \text{or } x,y \in \left]\frac{1}{2},1\right] \text{ and } x \ge y. \end{cases}$$

The function F has no neutral element and $[0,1]$ cannot be divided into a finite number of intervals with local neutral elements, i.e., it is not an n-uninorm.

At first we will focus on idempotent functions. It is easy to observe that if for a set X and a function $F\colon X^2 \longrightarrow X$, the function F can be expressed as a z-ordinal sum of trivial semigroups $(\{x\}, \mathrm{Id})_{x\in X}$ with respect to a partial order $\preceq^*$ on X and $A = X$, then F is commutative, associative, idempotent and $F(x, y) = x \wedge^* y$ for all $x, y \in X$.

From [21] we then have the following results.

Theorem 3.2. *Let $F\colon X^2 \longrightarrow X$ be a commutative, associative, idempotent function. Then the relation $\preceq_F$ given by $x \preceq_F y$ if $F(x, y) = x$ for all $x, y \in X$ is a partial order and $(X, \preceq_F)$ is a meet semi-lattice. Further, $x \wedge_F y = F(x, y)$.*

The previous results show us that each commutative, associative, idempotent function can be expressed as a z-ordinal sum of trivial semigroups. It is also easy to see that for a commutative, associative, idempotent function with the neutral element $e \in X$ and the partial order from the previous theorem, e is the top element of the meet semi-lattice $(X, \preceq_F)$.

Theorem 3.3. [21] *Let $(L, \leq, 0_L, 1_L)$ be a bounded chain and let $F\colon L^2 \longrightarrow L$ be a commutative, associative, idempotent function non-decreasing in both variables. Then for all $a, x, y \in L$ there is $F(x, y) = a$ if and only if $F(x, a) = a = F(y, a)$ and $a \in [\min(x, y), \max(x, y)]$.*

The monotonicity of F from the previous result then implies that $F(t, a) = a$ for all $t \in [\min(x, y), \max(x, y)]$ and if $x < a < y$ then

$$a = F(s, a) \leq F(s, t) \leq F(a, t) = a,$$

i.e., $F(s, t) = a$ for all $s \in [x, a]$ and all $t \in [a, y]$. In the corresponding z-ordinal sum of trivial semigroups then the point a belongs to the branching set A, while s is incomparable with t for all $s \in [x, a]$ and all $t \in [a, y]$.

Now we will move from idempotent functions to more general functions. The properties of functions on the unit interval that can be expressed as a z-ordinal sum of semigroups from Definition 2.6 and trivial semigroups are summarized in the following result.

Proposition 3.4. *Let $F\colon [0,1]^2 \longrightarrow [0,1]$ be a function such that $([0,1], F)$ can be expressed as a z-ordinal sum of semigroups from Definition 2.6 and trivial semigroups. Then F is associative, commutative, it has a continuous diagonal and if $a, b \in [0,1]$ are idempotent points of F such that there is no idempotent point of F in $]a, b[$ then F is continuous on the smallest set that contains $]a, b[^2$ and is closed under F.*

Proof. Theorem 2.5 implies that F is commutative and associative. Further, since there is no idempotent point of F in $]a, b[$ all points from $]a, b[$ belong to the same semigroup from Definition 2.6. If $]a, b[^2$ is closed under F then evidently F is continuous on $]a, b[^2$. Otherwise either $[a, b[^2$ or $]a, b]^2$ is closed under F and then all points from $[a, b[^2$ or $]a, b]^2$, respectively, belong to the same semigroup from Definition 2.6. In both cases it is easy to show that F is continuous on the corresponding set.

Assume the diagonal $d\colon [0,1] \longrightarrow [0,1]$ of F given for all $t \in [0,1]$ by $d(t) = F(t, t)$. If $x \in [0,1]$ is not an idempotent point of F then x belongs to a summand, which corresponds to a non-trivial semigroup and thus x is a point of continuity of the diagonal d. Assume that $x > 0$ is an idempotent point of F, i.e., $F(x, x) = x$. Then either there is an idempotent point of F in $]x - \delta, x[$ for all $0 < \delta < x$, or there exists and idempotent point a of F, $a < x$ such that there is no idempotent point in $]a, x[$. In the first case we get $\lim\limits_{t \longrightarrow x^-} d(t) = x$. In the second case F is on $]a, x[$ isomorphic to a continuous t-norm, or t-conorm and thus in both cases $\lim\limits_{t \longrightarrow x^-} d(t) = x$. Similarly we can show that $\lim\limits_{t \longrightarrow x^+} d(t) = x$ for all $x < 1$. Therefore F is continuous on the diagonal. □

As we mentioned above, a z-ordinal sum of semigroups from Definition 2.6 and trivial semigroups can yield a function F which is not monotone, in which case we can say that the partial order $\preceq$ is not compatible with the standard order on $[0,1]$. However, due to lack of space, in this work we will focus only on non-decreasing functions and we will leave the study of compatibility of $\preceq$ with the standard order on $[0,1]$ for the future work where we will study the conditions under which z-ordinal sum yields a non-decreasing function. Note that in the case of idempotent functions this compatibility is characterized by Theorems 4.6 and 5.1 from [21].

In the following sections we will try to show that if $F\colon [0,1]^2 \longrightarrow [0,1]$ is a commutative, associative, non-decreasing function, with a continuous diagonal and continuous Archimedean components then $([0,1], F)$ can be expressed as a z-ordinal sum of a countable number of semigroups from Definition 2.6 and a possibly uncountable number of trivial semigroups.

4 Archimedean components of commutative, associative and non-decreasing functions

In this section we will first define Archimedean components of a commutative, associative and non-decreasing function on the unit interval and then we will study properties of these Archimedean components under assumption of their continuity.

4.1 Definition of Archimedean components

In the case of a commutative, associative and non-decreasing function we cannot define its Archimedean components directly, as in the case of t-norms and t-conorms, since it can have a mixture of conjunctive and disjunctive behaviour. However, since there is no idempotent point in the interior of an Archimedean class we can define the Archimedean classes accordingly.

Definition 4.1. *Assume a commutative, associative and non-decreasing function* $F\colon [0,1]^2 \longrightarrow [0,1]$ *and let* D *be the set of its idempotent points, i.e.,*

$$D = \{x \in [0,1] \mid F(x,x) = x\}.$$

Let $[0,1] \setminus D = \bigcup_{k \in K} W_k$*, where the sets* W_k *are components of* $[0,1] \setminus D$ *with respect to connectedness. Then for all* $k \in K$ *and all* $x \in W_k$ *there is* $F(x,x) \neq x$ *and the sets* W_k *will be called Archimedean sets.*

Lemma 4.2. *Assume a commutative, associative and non-decreasing function* $F\colon [0,1]^2 \longrightarrow [0,1]$ *with a continuous diagonal and its Archimedean sets* W_k *for* $k \in K$*. If* F *is continuous on* W_k^2 *for some* $k \in K$ *then* $F(x,x) < x$ *for some* $x \in W_k$ *implies* $F(x,y) \leq \min(x,y)$ *for all* $x,y \in W_k$ *and* $F(x,x) > x$ *for some* $x \in W_k$ *implies* $F(x,y) \geq \max(x,y)$ *for all* $x,y \in W_k$*.*

Proof. Assume $k \in K$ and let F be continuous on W_k^2. Since W_k is a connected subset of $[0,1]$ it is an interval. Denote $\inf W_k = a$ and $\sup W_k = b$. Then either $F(a,a) = a$ or there is an idempotent point of F in every neighbourhood of a and the continuity of the diagonal of F then implies $F(a,a) = a$, which is a contradiction. Similarly we can show that $F(b,b) = b$ and therefore $W_k = \,]a,b[\,$. Since F is continuous on the diagonal and there is no idempotent element of F in W_k there is either $F(x,x) < x$ for all $x \in W_k$, or $F(x,x) > x$ for all $x \in W_k$. We will suppose the first case as the second is analogous. Then for all $x \in W_k$ there is $\lim\limits_{n \longrightarrow \infty} x_F^{(n)} = a$. Indeed, since $F(x,x) < x$ for all $x \in W_k$, the sequence $\{x_F^{(2^n)}\}$ is decreasing and thus it has a limit. Further, the monotonicity of F and $F(a,a) = a$ imply $F(x,x) \geq a$ for all $x \in W_k$. If $\lim\limits_{n \longrightarrow \infty} x_F^{(n)} = \lim\limits_{n \longrightarrow \infty} x_F^{(2^n)} = p > a$ then the continuity of F on W_k ensures that p is an idempotent point of F, which is a contradiction. Thus $\lim\limits_{n \longrightarrow \infty} x_F^{(n)} = a$.

If $F(x,y) \leq \min(x,y)$ for all $x,y \in W_k$ then the proof is finished. Suppose that there exist $x,y \in W_k$ such that $F(x,y) > x$. Then the monotonicity of F implies $y > x$ and the continuity of F on W_k implies that there exists a $z \in \,]x,y[$ such that $F(x,z) = x$. Then the associativity of F implies

$$x = F(x,z) = F(x, z_F^{(2)}) = \cdots = F(x, z_F^{(n)}),$$

for all $n \in \mathbb{N}$. Then, however, there exists an $m \in \mathbb{N}$ such that $z_F^{(m)} < x$ and we get $x = F(x, z_F^{(m)}) \leq F(x,x) < x$, which is a contradiction. Therefore $F(x,y) \leq \min(x,y)$ for all $x,y \in W_k$. In the case when $F(x,x) > x$ for some $x \in W_k$ we can analogously show that $F(x,y) \geq \max(x,y)$ for all $x,y \in W_k$. □

Remark 4.3. *If in the previous result* $F(x,y) \leq \min(x,y)$ *for all* $x,y \in W_k$ *then the monotonicity implies* $a = F(a,a) \leq F(a,x) \leq a$ *and thus* $F(a,x) = a$ *for all* $x \in W_k$*. Then either* W_k *is closed under* F*, i.e.,* $F(x,y) \in W_k$ *for all* $x,y \in W_k$ *or* $F(x,y) \leq a$ *for some* $x,y \in W_k$*. In the second case we have* $a = F(a,y) \leq F(x,y) \leq a$ *and therefore if* W_k *is not closed under* F *then* $W_k \cup \{a\}$ *is closed under* F*. In this case it is easy to show that if* F *is continuous on* W_k^2 *then it is also continuous on* $(W_k \cup \{a\})^2$*.*

Similarly, if $F(x,y) \geq \max(x,y)$ *for all* $x,y \in W_k$ *then* $F(b,x) = b$ *for all* $x \in W_k$ *and if* W_k *is not closed under* F *then* $W_k \cup \{b\}$ *is closed under* F*. In the second case, if* F *is continuous on* W_k^2 *then it is also continuous on* $(W_k \cup \{b\})^2$*.*

Definition 4.4. *Assume a commutative, associative and non-decreasing function $F\colon [0,1]^2 \longrightarrow [0,1]$ with a continuous diagonal and let F be continuous on Archimedean sets W_k for all $k \in K$. We define sets V_k for $k \in K$ such that*

$$V_k = \begin{cases} W_k & \text{if } W_k \text{ is closed under } F, \\ W_k \cup \{\inf W_k\} & \text{if } W_k \text{ is not closed under } F \text{ and} \\ & F(x,y) \leq \min(x,y) \text{ for all } x,y \in W_k, \\ W_k \cup \{\sup W_k\} & \text{if } W_k \text{ is not closed under } F \text{ and} \\ & F(x,y) \geq \max(x,y) \text{ for all } x,y \in W_k. \end{cases}$$

The sets V_k will be called non-trivial Archimedean classes of F and semigroups $(V_k, F|_{V_k^2})$ will be called non-trivial Archimedean components of F. The set of all non-trivial Archimedean classes of F will be denoted by $\mathcal{AC}$. Further, for each $d \in [0,1]$ such that $F(d,d) = d$ and $d \notin V_k$ for all $k \in K$ the set $\{d\}$ will be called a trivial Archimedean class and $(\{d\}, \mathrm{Id})$ will be called a trivial Archimedean component of F.

We denote by $\mathcal{F}$ the set of all commutative, associative and non-decreasing functions on the unit interval, with continuous Archimedean components and a continuous diagonal. Further, for simplicity we will write (X, F) instead of $(X, F|_{X^2})$.

From Lemma 4.2 we know that F from $\mathcal{F}$ is on each of its non-trivial Archimedean components a linear transformation of a restriction of some t-subnorm (t-superconorm) (see [9]). In the following we show that F on each of its Archimedean components is in fact a linear transformation of a restriction of a continuous t-norm (t-conorm) to open unit interval – in the case of a strict t-norm (t-conorm) or to a right-open (left-open) unit interval – in the case of a nilpotent t-norm (t-conorm). From [18, Corollary 1] and [16, Lemma 1] we get the following result.

Lemma 4.5. *Let $M\colon [0,1]^2 \longrightarrow [0,1]$ be a t-subnorm continuous on $[0,1[^2$ and let $\lim\limits_{x\longrightarrow 1^-} M(x,x) = 1$. Then M is a continuous t-norm.*

Therefore a function F from $\mathcal{F}$ is on each of its non-trivial Archimedean components equal to a linear transformation of one of the following: a strict t-norm restricted to open unit square, a nilpotent t-norm restricted to $[0,1[^2$, a strict t-conorm restricted to open unit square, a nilpotent t-conorm restricted to $]0,1]^2$.

In the following we will write $I_{a,b}$ for a non-trivial Archimedean class which is defined on the interval $]a,b[$, or $[a,b[$, or $]a,b]$ for some $a,b \in [0,1]$, $a < b$. From the construction of Archimedean classes it then follows that a and b are idempotent points. We will also write I_x for an Archimedean class which contains the point $x \in [0,1]$.

Lemma 4.6. *Let $F \in \mathcal{F}$ and let $I_{a,b} \in \mathcal{AC}$. Then $F(b,x) \geq x$ and $F(a,x) \leq x$.*

Proof. We already know that $(I_{a,b}, F)$ is linearly isomorphic to a restriction of a continuous Archimedean t-norm or a continuous Archimedean t-conorm. In the first case we get $F(b,x) \geq \lim\limits_{t\longrightarrow b^-} F(t,x) = x$ and $F(a,x) \leq \lim\limits_{t\longrightarrow a^+} F(t,x) = a$. In the second case we get $F(b,x) \geq \lim\limits_{t\longrightarrow b^-} F(t,x) = b$ and $F(a,x) \leq \lim\limits_{t\longrightarrow a^+} F(t,x) = x$. In summary we get $F(b,x) \geq \min(x,b) = x$ and $F(a,x) \leq \max(a,x) = x$. □

4.2 Interaction of idempotent points with Archimedean components

In the following few lemmas we will examine the value of $F(x,y)$ in the case when at least one of these points is an idempotent point of F.

Lemma 4.7. *Let $F \in \mathcal{F}$ and let $c,d \in [0,1]$ be two idempotent points of F. Then also $F(c,d)$ is an idempotent point of F.*

Proof. The commutativity and the associativity of F imply that

$$F(F(c,d), F(c,d)) = F(F(c,c), F(d,d)) = F(c,d),$$

i.e., $F(c,d)$ is an idempotent point of F. □

Lemma 4.8. *Let $F \in \mathcal{F}$, $I_{a,b} \in \mathcal{AC}$ and let $d \in [0,1]$ be an idempotent point of F. Then for all $x \in I_{a,b}$ either $F(x,d) = x$, or $F(x,d)$ is an idempotent point of F.*

Proof. If $x = d$ or $F(x, d) \in \{x, d\}$ the claim is obvious. Assume $x \neq d$ and $F(x, d) = q$, where $q \notin \{x, d\}$. If q is an idempotent point of F the proof is finished. Further we will suppose that q is not an idempotent point of F. If $x \in \{a, b\}$ then it is an idempotent point and by Lemma 4.7 also q is an idempotent point, which is a contradiction. Thus $x \in \,]a, b[$. We will assume $x < d$ as the case when $x > d$ is analogous. Then $b \leq d$ and $F(x, d) = q$ implies

$$q = F(x, F(d, d)) = F(F(x, d), d) = F(q, d).$$

Moreover,

$$a \leq F(x, x) \leq F(x, d) \leq F(d, d) = d.$$

If $a \notin I_{a,b}$, then $a < F(x, x) \leq q$. Thus $q \in I_{a,b} \cup [b, d]$. If $q \geq b$ then $F(x, d) = q = F(q, d)$ implies $F(b, d) = q$ and since b and d are idempotent points also q is an idempotent point, which is a contradiction. Therefore $q \in I_{a,b} \setminus \{a, b\}$.

Since $x, q \in I_{a,b}$ are not idempotent points and F is isomorphic to (a restriction of) a continuous Archimedean t-norm (t-conorm) on $I^2_{a,b}$ there exists a $p \in I_{a,b} \setminus \{a, b\}$ such that either $F(q, p) = x$, or $F(x, p) = q$. In the first case

$$q = F(d, x) = F(d, F(q, p)) = F(F(d, q), p) = F(q, p) = x,$$

which is a contradiction. Therefore $F(x, p) = q$. Then

$$q = F(d, q) = F(d, F(x, p)) = F(F(d, x), p) = F(q, p).$$

However, since F is isomorphic to a (restriction of a) continuous Archimedean t-norm (t-conorm) on $I^2_{a,b}$ we get $q \in \{a, b\}$, i.e., q is an idempotent point of F, which is a contradiction. Summarizing, either $F(x, d) = x$, or $F(x, d)$ is an idempotent point of F. □

Remark 4.9. *Assume $F \in \mathcal{F}$ and let $b, d \in [0, 1]$, $b < d$, be two idempotent points of F. Then $u = F(b, d)$ is the annihilator of F on $[b, d]$. Indeed,*

$$u = F(b, d) = F(F(b, b), d) = F(b, F(b, d)) = F(b, u),$$

and similarly $F(d, u) = u$ and then the monotonicity of F implies $F(x, u) = u$ for all $x \in [b, d]$. Further, $b = F(b, b) \leq u \leq F(d, d) = d$, i.e., $u \in [b, d]$ and by Lemma 4.7 u is an idempotent point of F. We will often use this fact in the following proofs.

Lemma 4.10. *Let $F \in \mathcal{F}$, $I_{a,b} \in \mathcal{AC}$ and let $d \in [0, 1]$ be an idempotent point of F. If $F(x, d) = z \neq x$ for some $x \in I_{a,b}$ then $b \leq d$ implies $z = F(b, d)$ and $d \leq a$ implies $z = F(a, d)$.*

Proof. We will suppose that $b \leq d$ as the case when $d \leq a$ is analogous. If $F(x, d) = z \neq x$ then by Lemma 4.8 z is an idempotent element. Further, $a = F(a, a) \leq F(x, d) \leq F(d, d) = d$, i.e., $z \in \{a\} \cup [b, d]$. If $z = a$ then $a = F(a, a) \leq F(x, b) \leq F(x, d) = a$, i.e., $F(x, b) = a$. Lemma 4.6 implies $F(x, b) \geq x$, which means that $x = a$. Then, however, $F(x, d) = x$, which is a contradiction. Thus $z \in [b, d]$.

Since b and d are idempotent points of F then $F(b, d)$ is the annihilator of F on $[b, d]$. We have $z = F(x, d) \leq F(b, d)$ and $F(d, F(b, d)) = F(b, d)$. Then

$$F(x, F(b, d)) = F(x, F(d, F(b, d))) = F(F(x, d), F(b, d)) = F(z, F(b, d)) = F(b, d),$$

and since $F(b, d) = F(x, F(b, d)) \leq F(x, F(d, d)) = F(x, d) = z$ we get $z = F(b, d)$. □

Lemma 4.11. *Let $F \in \mathcal{F}$, $I_{a,b} \in \mathcal{AC}$ and let $d \in [0, 1]$ be an idempotent point of F. Then either $F(s, d) = s$ for all $s \in I_{a,b}$, or $F(s, d) = F(b, d)$ ($F(s, d) = F(a, d)$) for all $s \in I_{a,b}$.*

Proof. We will again suppose that $b \leq d$ as the proof for the case when $d \leq a$ is analogous. Denote $z = F(b, d)$ and assume any $x \in I_{a,b} \setminus \{a, b\}$.

1. First we will suppose that $F(x, d) = x$. Then for all $y \in I_{a,b} \setminus \{a, b, x\}$ there exists a $p \in I_{a,b} \setminus \{a, b\}$ such that either $F(p, x) = y$, or $F(p, y) = x$. In the first case we get $y = F(p, x) = F(p, F(x, d)) = F(F(p, x), d) = F(y, d)$. Suppose the second case and let $F(y, d) \neq y$. Then by Lemma 4.8 $F(y, d) = z$. We get

$$F(p, y) = x = F(x, d) = F(F(p, y), d) = F(p, F(y, d)) = F(p, z).$$

Since $F(p, z) = x \in I_{a,b} \setminus \{a, b\}$ we know that $F(p, z)$ is not an idempotent point and thus by Lemma 4.8 $F(p, y) = x = F(p, z) = p$. However, since $p \notin \{a, b\}$ and $y \notin \{a, b\}$ there is $F(p, y) \neq p$, which is a contradiction. Therefore $F(y, d) = y$ for all $y \in I_{a,b} \setminus \{a, b\}$.

If $a \in I_{a,b}$ (and similarly for $b \in I_{a,b}$) then $F(x, p_1) = a$ for some $p_1 \in I_{a,b} \setminus \{a, b\}$ and $a = F(p_1, x) = F(p_1, F(x, d)) = F(a, d)$. Therefore $F(y, d) = y$ for all $y \in I_{a,b}$.

2. Now assume that $F(x,d) = z$. If $F(s,d) = s$ for some $s \in \,]a,b[$ then similarly as above we can show that $F(x,d) = x$, which is a contradiction. Thus for all $s \in \,]a,b[$ Lemma 4.10 implies $F(s,d) = z$. Then

$$F(d,z) = F(d,F(d,s)) = F(F(d,d),s) = F(d,s) = z,$$

and $F(s,z) = F(s,F(d,z)) = F(F(s,d),z) = F(z,z) = z$ for all $s \in \,]a,b[$. If $a \in I_{a,b}$ (and similarly if $b \in I_{a,b}$) then $F(x,p_1) = a$ for some $p_1 \in I_{a,b} \setminus \{a,b\}$ and we get

$$F(a,d) = F(F(p_1,x),d) = F(p_1,F(x,d)) = F(p_1,z) = z.$$

Thus $F(s,d) = z$ for all $s \in I_{a,b}$.

□

Remark 4.12. *From the previous result for $F \in \mathcal{F}$, $I_{a,b} \in \mathcal{AC}$ and an idempotent point d of F we see that if $a \in I_{a,b}$ then $F(a,d) = a$ implies that either $F(x,d) = x$ for all $x \in I_{a,b}$, or $F(x,d) = a$ for all $x \in I_{a,b}$. Similar observation can be done in the case when $b \in I_{a,b}$.*

4.3 Interaction of two non-trivial Archimedean components

Lemma 4.13. *Let $F \in \mathcal{F}$, $I_{a,b}, I_{c,d} \in \mathcal{AC}$, $b \leq c$ and $z = F(b,c)$. Then $F(v,z) \in \{v,z\}$ for all $v \in I_{a,b} \cup I_{c,d}$.*

Proof. Assume a non-idempotent point $s \in I_{a,b}$. Then by Lemma 4.10 either $F(s,z) = s$, or $F(s,z) = F(b,z)$. In the second case $F(b,z)$ is the annihilator of F on $[b,z]$ and since z is the annihilator of F on $[b,c]$ we get $F(b,z) = z$. Thus $F(s,z) \in \{s,z\}$ for all $s \in I_{a,b}$. Similarly we can show that $F(t,z) \in \{t,z\}$ for all $t \in I_{c,d}$. □

Lemma 4.14. *Let $F \in \mathcal{F}$, $I_{a,b}, I_{c,d} \in \mathcal{AC}$, $b \leq c$ and $z = F(b,c)$. Then $I_{a,b} \cup \{z\} \cup I_{c,d}$ is closed under F.*

Proof. Assume $x \in I_{a,b}$ and $y \in I_{c,d}$. Since $F(x,b) \geq x$ and $F(y,c) \leq y$ we know that $F(x,y) \in [x,y]$.

If $F(x,y) \notin I_{a,b} \cup I_{c,d}$ then $F(x,y) \in [b,c]$ and Lemma 4.13 implies $F(x,z) \in \{x,z\}$, $F(y,z) \in \{y,z\}$.

Then we have the following possibilities:

(i) If $F(x,z) = z = F(y,z)$ then $z = F(x,z) \leq F(x,y) \leq F(z,y) = z$, i.e., $F(x,y) = z$.

(ii) If $F(x,z) = x$ and $F(y,z) = z$ then $F(x,y) = F(F(x,z),y) = F(x,F(z,y)) = F(x,z) = x$, i.e., $F(x,y) = x$.

(iii) If $F(x,z) = z$ and $F(y,z) = y$ then $F(x,y) = F(x,F(z,y)) = F(F(x,z),y) = F(z,y) = y$, i.e., $F(x,y) = y$.

(iv) If $F(x,z) = x$ and $F(y,z) = y$. Then $F(x,y) \in [b,c]$ implies

$$F(x,y) = F(F(z,x),y) = F(z,F(x,y)) = z,$$

i.e., $F(x,y) = z$.

Summarising, in all cases $F(x,y) \in I_{a,b} \cup \{z\} \cup I_{c,d}$. Since $F(z,z) = z$ and $F(w,z) \in \{w,z\}$ for all $w \in I_{a,b} \cup I_{c,d}$ we see that $I_{a,b} \cup \{z\} \cup I_{c,d}$ is closed under F. □

Lemma 4.15. *Let $F \in \mathcal{F}$ and $I_{a,b}, I_{c,d} \in \mathcal{AC}$, $b \leq c$. If $F(x,y) \notin I_{a,b} \cup I_{c,d}$ for some $x, y \in I_{a,b} \cup I_{c,d}$ then $F(x,y) = z = F(b,c)$ and there is either $F(v,z) = v$ for all $v \in I_{a,b} \cup I_{c,d}$, or $F(v,z) = z$ for all $v \in I_{a,b} \cup I_{c,d}$.*

Proof. Similarly as in the proof of Lemma 4.14 we can show that $F(x,y) = z \notin I_{a,b} \cup I_{c,d}$ for some $x, y \in I_{a,b} \cup I_{c,d}$ implies $z = F(b,c)$ and thus z is the annihilator of F on $[b,c]$. Further, $F(v,z) \in \{v,z\}$ for all $v \in I_{a,b} \cup I_{c,d}$. Without loss of generality we can assume that $x \in I_{a,b}$ and $y \in I_{c,d}$. Since $z \notin I_{a,b} \cup I_{c,d}$ we know that $x \neq z$, $y \neq z$.

• If $F(x,z) = x$. In the case that $x = a$ and $F(s,z) = a$ for all $s \in I_{a,b}$ we get $s \leq F(s,b) \leq F(s,z) = a$ for all $s \in I_{a,b}$, which is a contradiction. If $x = b$ we get $F(b,z) = b \neq z$, which is a contradiction since z is the annihilator of F on $[b,c]$. Therefore $F(x,z) = x$ implies $F(s,z) = s$ for all $s \in I_{a,b}$ and $F(y,z) = z$ gives us

$$z = F(x,y) = F(F(x,z),y) = F(x,F(z,y)) = F(x,z) = x,$$

which is a contradiction. Thus $F(y,z) = y$. If $y = d$ and $F(t,z) = d$ for all $t \in I_{c,d}$ we get $d = F(t,z) \leq F(t,c) \leq t$, for all $t \in I_{c,d}$, which is a contradiction. If $y = c$ we have $F(c,z) = c$ which implies $c = z$ since z is the annihilator of F on $[b,c]$. Then, however, $z \in I_{c,d}$, which is a contradiction. Therefore $F(v,z) = v$ for all $v \in I_{a,b} \cup I_{c,d}$.

• If $F(x,z) = z$ then $F(s,z) = z$ for all $s \in I_{a,b}$ and $F(y,z) = y$ implies

$$y = F(z,y) = F(F(x,z),y) = F(x,F(z,y)) = F(x,y) = z,$$

which is a contradiction. Thus $F(v,z) = z$ for all $v \in I_{a,b} \cup I_{c,d}$.

□

Lemma 4.16. *Let $F \in \mathcal{F}$, $I_{a,b}, I_{c,d} \in \mathcal{AC}$, $b \le c$. If for $z = F(b,c)$ there is $F(v,z) = v$ for all $v \in I_{a,b} \cup I_{c,d}$ then F on $(I_{a,b} \cup \{z\} \cup I_{c,d})^2$ is isomorphic via* (2) *to a uninorm with continuous underlying functions, possibly restricted to the open, or half-open unit square.*

Proof. Due to Lemma 4.6 we know that $F(s,b) \ge s$ for all $s \in I_{a,b}$ and $F(t,c) \le t$ for all $t \in I_{c,d}$. Thus $F(s,z) = s$ for all $s \in I_{a,b} \cup I_{c,d}$ implies $F(s,b) = s$ for all $s \in I_{a,b}$ and $F(t,c) = t$ for all $t \in I_{c,d}$. Then $F(x,y) \le F(x,b) = x$ and thus $F(x,y) \le \min(x,y)$ for all $x,y \in I_{a,b}$ and similarly $F(x,y) \ge \max(x,y)$ for all $x,y \in I_{c,d}$. Therefore $b \notin I_{a,b}$ and $c \notin I_{c,d}$. Further, $a = F(a,a) \le F(a,z) \le F(s,z) = s$ for all $s \in I_{a,b}$ and thus $F(a,z) = a$. Similarly, $F(d,z) = d$.

Now we will show that $[a,b[\cup \{z\} \cup]c,d]$ is closed under F. If $I_{a,b} \cup I_{c,d}$ is closed under F then obviously also $I_{a,b} \cup \{z\} \cup I_{c,d}$ is closed under F. If $I_{a,b} \cup I_{c,d}$ is not closed under F then by Lemma 4.14 the set $I_{a,b} \cup \{z\} \cup I_{c,d}$ is closed under F. Further, $F(a,x) = a$ for $x \in [a,b[$, $F(a,z) = a$ and for $y \in]c,d]$ there is $F(a,y) \le F(z,y) = y$, i.e., $F(a,y) \in \{a,y\} \cup [b,c]$. However, if $F(a,y) \in [b,c]$ we get

$$F(a,y) = F(a, F(y,z)) = F(F(a,y), z) = z,$$

and thus $F(a,y) \in [a,b[\cup \{z\} \cup]c,d]$. Similarly we can show that $F(d,x) \in [a,b[\cup \{z\} \cup]c,d]$ for all $x \in [a,b[\cup \{z\} \cup]c,d]$. Therefore the set $[a,b[\cup \{z\} \cup]c,d]$ is closed under F.

It is easy to see that z is the neutral element of F on $([a,b[\cup \{z\} \cup]c,d])^2$. Therefore F is on $([a,b[\cup \{z\} \cup]c,d])^2$ isomorphic via the transformation (1) to a commutative, associative, non-decreasing operation $U\colon [0,1]^2 \longrightarrow [0,1]$ with the neutral element $e \in]0,1[$, i.e., to a uninorm. Moreover, since F is on $[a,b[^2$ $(]c,d]^2)$ isomorphic to a restriction of a continuous t-norm (t-conorm) and $\lim_{s \longrightarrow b^-} F(x,s) = x$ for all $x \in [a,b[$ $(\lim_{t \longrightarrow c^+} F(y,t) = y$ for all $y \in]c,d])$ we see that U is a uninorm with continuous underlying functions. □

If for $F \in \mathcal{F}$, $I_{a,b}, I_{c,d} \in \mathcal{AC}$, $b \le c$, $z = F(b,c)$ there is $F(s,z) = s$ for all $s \in I_{a,b} \cup I_{c,d}$ and $F(s,t) = z$ for some $s \in I_{a,b}$, $t \in I_{c,d}$ then we say that $I_{a,b}$ and $I_{c,d}$ are paired. The previous result and the structure of uninorms with continuous Archimedean underlying functions described in [12] show that if $I_{a,b}$ and $I_{c,d}$ are paired, for some $I_{a,b}, I_{c,d} \in \mathcal{AC}$, $b \le c$, then F on $(I_{a,b} \cup \{F(b,c)\} \cup I_{c,d})^2$ is isomorphic to a representable uninorm (restricted to open unit square), i.e., $I_{a,b} =]a,b[$ and $I_{c,d} =]c,d[$.

Lemma 4.17. *Let $F \in \mathcal{F}$, $I_{a,b}, I_{c,d} \in \mathcal{AC}$, $b \le c$. If $F(s,t) \in I_{a,b} \cup I_{c,d}$ for all $s \in I_{a,b}$ and $t \in I_{c,d}$ then one of the following cases hold:*

- *$b \in I_{a,b}$ and $F(s,t) = b$ for all $s \in I_{a,b}$ and $t \in I_{c,d}$, $b = F(b,c)$,*
- *$F(s,t) = s$ for all $s \in I_{a,b}$ and $t \in I_{c,d}$,*
- *$c \in I_{c,d}$ and $F(s,t) = c$ for all $s \in I_{a,b}$ and $t \in I_{c,d}$, $c = F(b,c)$,*
- *$F(s,t) = t$ for all $s \in I_{a,b}$ and $t \in I_{c,d}$.*

Proof. Let $x \in]a,b[$ and $y \in]c,d[$. Assume $F(x,y) \in I_{a,b}$ (the case when $F(x,y) \in I_{c,d}$ is analogous) and denote $z = F(b,c)$. Then

$$x \le F(x,b) \le F(x,z) \le F(x,y) \in I_{a,b},$$

i.e., $F(x,z) \in I_{a,b}$. Due to Lemma 4.8 there is either $F(x,z) = x$, or $b \in I_{a,b}$ and $F(x,z) = b$.

• If $b \in I_{a,b}$ and $F(x,z) = b$. Here $F(b,z) = b$, i.e., $z = b$, $F(x,b) = b$ and $F(x,y) \in I_{a,b}$ implies $F(x,y) = b$. Thus

$$F(y,b) = F(y, F(x,b)) = F(F(y,x), b) = F(b,b) = b.$$

Then Lemma 4.11 implies $F(t,b) = b$ for all $t \in I_{c,d}$ and similarly $F(s,b) = b$ for all $s \in I_{a,b}$.

Therefore $b = F(s,b) \le F(s,t) \le F(b,t) = b$ and then $F(s,t) = b$ for all $s \in I_{a,b}$ and $t \in I_{c,d}$.

• If $F(x,z) = x$ then Lemma 4.11 implies that $F(s,z) = s$ for all $s \in I_{a,b}$. On the other hand, since z is the annihilator of F on $[b,c]$ we have either $F(z,t) = z$ for all $t \in I_{c,d}$ or $F(z,t) = t$ for all $t \in I_{c,d}$. In the first case we get

$$F(s,t) = F(F(s,z), t) = F(s, F(z,t)) = F(s,z) = s,$$

for all $s \in I_{a,b}$ and $t \in I_{c,d}$. Assume the second case, i.e., $F(z,t) = t$ for all $t \in I_{c,d}$. Then by Lemma 4.16 there is $b \notin I_{a,b}$, $c \notin I_{c,d}$ and F on $(I_{a,b} \cup \{z\} \cup I_{c,d})^2$ is isomorphic to a uninorm with continuous Archimedean underlying functions, possibly restricted to open, or half-open unit square. Since $F(s,t) \in I_{a,b} \cup I_{c,d}$ for all $s \in I_{a,b}$, $t \in I_{c,d}$ this uninorm is not representable and $F(x,y) \in I_{a,b}$ implies that the corresponding uninorm coincides inside of the unit square with a uninorm from $\mathcal{U}_{\min}$ (see [12]). Therefore $F(s,t) = s$ for all $s \in]a,b[$ and all $t \in]c,d[$. If $a \in I_{a,b}$ then for $s_1, s_2 \in I_{a,b}$ such that $F(s_1,s_2) = a$ we get $F(a,t) = F(F(s_1,s_2),t) = F(s_1, F(s_2,t)) = F(s_1,s_2) = a$ for all $t \in]c,d[$. Similarly, if $d \in I_{c,d}$ then $F(d,s) = s$ for all $s \in [a,b[$. □

Lemma 4.18. *Let $F \in \mathcal{F}$, $I_{a,b}, I_{c,d} \in \mathcal{AC}$, $b \le c$. Assume that there exists $x \in I_{a,b}$ and $y \in I_{c,d}$ such that $F(x,y) \notin \{x,y\}$ is not an idempotent point of F. Then $I_{a,b}$ and $I_{c,d}$ are paired.*

Proof. Since $F(x,y) \notin \{x,y\}$ is not an idempotent point of F we know that nor x neither y is an idempotent point of F. Assume $z = F(b,c)$. If $F(x,z) = F(y,z) = z$ then $F(x,y) = z$, which is a contradiction since z is an idempotent point of F. If $F(x,z) = x$ and $F(y,z) = z$ then $F(x,y) = x$, which is a contradiction. If $F(x,z) = z$ and $F(y,z) = y$ then $F(x,y) = y$, which is a contradiction. Thus $F(x,z) = x$ and $F(y,z) = y$. Then $F(v,z) = v$ for all $v \in I_{a,b} \cup I_{c,d}$, i.e., F on $(I_{a,b} \cup \{z\} \cup I_{c,d})^2$ is isomorphic to a uninorm with continuous Archimedean functions. From [12] we know that the only uninorm with continuous Archimedean underlying functions for which $F(x,y)$ is not an idempotent point is a representable uninorm. Therefore $I_{a,b}$ and $I_{c,d}$ are paired. □

4.4 Interaction of paired Archimedean components with other points

Lemma 4.19. *Let $F \in \mathcal{F}$, $I_{a,b}, I_{c,d} \in \mathcal{AC}$, $b \le c$, $z = F(b,c)$, and let $I_{a,b}$ and $I_{c,d}$ be paired. Then for all idempotent points $w \in [0,1]$, $w \notin I_{a,b} \cup \{z\} \cup I_{c,d}$ one of the following holds:*

- *$F(v,w) = v$ for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$,*
- *$F(v,w) = F(z,w) \ne z$ for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$.*

Proof. Assume any idempotent point $w \in [0,1]$ of F, $w \notin I_{a,b} \cup \{z\} \cup I_{c,d}$. Since $I_{a,b}$ and $I_{c,d}$ are paired there is $I_{a,b} =]a,b[$ and $I_{c,d} =]c,d[$. Then $F(z,w) = u$ is an idempotent point of F which is an annihilator of F on $[\min(z,w), \max(z,w)]$.

• If $F(z,w) = z$ then $F(v,w) = F(F(v,z),w) = F(v,F(z,w)) = F(v,z) = v$ for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$.

• If $F(z,w) = u \ne z$. In the case when $F(w,z) \in [b,c]$ then $z = F(F(w,z),z) = F(w,F(z,z)) = F(w,z)$, which is a contradiction. Similarly, if $w \in [b,c]$ we get $F(w,z) = z$, which is a contradiction. Thus either $w \le a$ or $d \le w$. We will suppose $w \le a$ as the other case is analogous. Then $F(z,w) \notin [b,c]$ implies $F(z,w) < b$, i.e., $w \le u \le a$ and

$$u = F(z,w) = F(F(z,z),w) = F(z,F(z,w)) = F(z,u),$$

and $F(w,u) = F(w,F(z,u)) = F(F(w,z),u) = F(u,u) = u$. Due to Lemma 4.11 for $s \in I_{a,b}$ there is $F(s,u) \in \{s, F(a,u)\}$. Moreover, due to the monotonicity $F(a,z) = a$. Thus $F(a,w) = F(F(a,z),w) = F(a,F(z,w)) = F(a,u) = u$ since $a \in [w,z]$ and u is the annihilator of F on $[w,z]$.

If $F(s,u) = s$ for some $s \in]a,b[$ then for $t_s \in]c,d[$ such that $F(s,t_s) = z$ there is $u = F(z,u) = F(F(t_s,s),u) = F(t_s,F(s,u)) = F(t_s,s) = z$, which is a contradiction. Therefore $F(s,u) = u$ for all $s \in]a,b[$.

For $t \in]c,d[$ and $s_t \in]a,b[$ such that $F(t,s_t) = z$ we get $u = F(z,u) = F(F(t,s_t),u) = F(t,F(s_t,u)) = F(t,u)$, i.e., $F(t,u) = u$. Thus for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$ there is $F(v,u) = u$. Then $u = F(w,u) \le F(w,v) \le F(u,v) = u$, i.e., $F(w,v) = u$ for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$. □

Lemma 4.20. *Let $F \in \mathcal{F}$, $I_{a,b}, I_{c,d} \in \mathcal{AC}$, $b \le c$, $z = F(b,c)$, and let $I_{a,b}$ and $I_{c,d}$ be paired. Then for all non-idempotent points $w \in [0,1]$, $w \notin I_{a,b} \cup \{z\} \cup I_{c,d}$ one of the following holds:*

- *$F(v,w) = v$ for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$,*
- *$F(v,w) = w$ for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$,*
- *$F(v,w) = u = F(v,u) = F(w,u)$ for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$, where u is an idempotent point of F, $u \notin \{z,w\}$.*

Proof. If w is not an idempotent point then w belongs to a non-trivial Archimedean class, which we will denote I_{a_w,b_w}.

- If $b_w \le a$ then Lemma 4.11 implies that $F(z,w) \in \{w, F(b_w,z)\}$. Since $F(b_w,z)$ is the annihilator of F on $[b_w,z]$ we have $z = F(b_w,z)$ if $F(b_w,z) \in [b,c]$ and $F(b_w,z) = F(b_w,a) = u$ if $F(b_w,z) \in [b_w,a]$. Thus $F(w,z) \in \{w,z,u\}$.

    1. If $F(w,z) = w$ then since $F(w,b_w) \ge w$ the monotonicity of F implies $F(s,w) = w$ for all $s \in]a,b[$ and for all $t \in]c,d[$ and $s_t \in]a,b[$ such that $F(t,s_t) = z$ we get $w = F(w,z) = F(w,F(s_t,t)) = F(F(w,s_t),t) = F(w,t)$. Thus $F(w,v) = w$ for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$.
    2. If $F(w,z) = z$ then for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$ there is $F(v,w) = F(F(v,z),w) = F(v,F(z,w)) = F(v,z) = v$, i.e., $F(v,w) = v$.
    3. If $F(w,z) = u \notin \{z,w\}$ then $F(z,u) = u$ and $F(w,u) = F(w,F(z,u)) = F(F(w,z),u) = F(u,u) = u$. Further, $F(v,w) = F(F(v,z),w) = F(v,F(z,w)) = F(v,u)$ for all $v \in I_{a,b} \cup I_{c,d}$ and from Lemma 4.19 we know that $F(z,u) = u \ne z$ implies $F(v,u) = F(z,u) = u$. Thus $F(v,w) = u$ for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$.

- If $d \le a_w$ then the proof is analogous to that of the previous case.
- If $b \le a_w$, $b_w \le z$ then since z is the annihilator of F on $[b,c]$ there is $F(a_w,z) = z = F(z,b_w)$ and then the monotonicity implies $F(z,w) = z$. Therefore
$$F(w,v) = F(w,F(z,v)) = F(F(w,z),v) = F(z,v) = v,$$
for all $v \in I_{a,b} \cup \{z\} \cup I_{c,d}$.
- If $z \le a_w$, $b_w \le c$ then the proof is analogous to that of the previous case.

□

The previous result shows that for each non-trivial Archimedean class of F there exists at most one different non-trivial Archimedean class of F such that these two classes are paired.

5 Decomposition via z-ordinal sum

In this section we will define summands for our z-ordinal sum construction, show their basic properties and then show that F can be expressed as a z-ordinal sum of Archimedean, representable and idempotent semigroups.

Let us observe that Archimedean classes of the function F need not be disjoint. Such a situation can happen when there are two Archimedean components $(]a,b],F)$ and $([b,c[,F)$ which are isomorphic to a nilpotent t-conorm and a nilpotent t-norm, respectively. In such a case the monotonicity of F implies $F(x,y) = b$ for all $x \in]a,b]$ and all $y \in [b,c[$. From Theorem 2.5 we then see that these two Archimedean components have to be incomparable in the corresponding partial order of the respective z-ordinal sum and the point b have to be the annihilator of the semigroup from the branching set A which corresponds to the meet of our two Archimedean components.

We denote
$$H = \{d \in [0,1] \mid \text{ there exist } x,y \in [0,1] \text{ such that } x < d < y, F(x,d) = d, F(y,d) = d\}.$$
Observe that the monotonicity then implies $d = F(x,d) \le F(x,y) \le F(d,y) = d$, i.e., $F(x,y) = d$ and $d = F(x,d) \le F(d,d) \le F(y,d) = d$ implies $F(d,d) = d$, i.e., d is an idempotent point of F. Moreover, $d = F(s,d) = F(t,d) = F(s,t)$ for all $s \in [x,d]$ and all $t \in [d,y]$. Since $d \in]x,y[$ points x and y cannot belong to the same Archmedean class. Thus classes I_x and I_y cannot be comparable. Further we denote
$$E = \{d \in [0,1] \mid F(d,d) = d, d \notin H, d \notin V_k \text{ for all } k \in K\}.$$

Now we will define summands for our z-ordinal sum construction. We denote
$$K_1 = \{k \in K \mid \text{ there exists } m \in K \text{ such that } k \text{ and } m \text{ are paired}, \sup V_k \le \inf V_m\},$$
and for each $k \in K_1$, which is paired with $m \in K$, where $V_k = I_{a,b} =]a,b[$, $V_m = I_{c,d} =]c,d[$, we denote
$$X_k =]a,b[\cup \{F(b,c)\} \cup]c,d[.$$
Further we denote
$$K_2 = \{k \in K \mid \text{ there is no } m \in K \text{ such that } k \text{ and } m \text{ are paired}\},$$
and for $k \in K_2$ we define $X_k = V_k$. Moreover, we will assume the set K_3 isomorphic with the set E via the isomorphism $\varphi\colon K_3 \longrightarrow E$, such that $K \cap K_3 = \emptyset$, and for $k \in K_3$ we define $X_k = \{\varphi(k)\}$. Finally, we will assume the set K_4 isomorphic with the set H via the isomorphism $\psi\colon K_4 \longrightarrow H$, such that $K_4 \cap (K_3 \cup K) = \emptyset$ and for $k \in K_4$ we define $X_k = \{\psi(k)\}$.

We define $C = K_1 \cup K_2 \cup K_3 \cup K_4$, $A = K_4$ and $B = C \setminus A$.

Lemma 5.1. *Let $F \in \mathcal{F}$ and assume $m,k \in K_1$. Then $k \ne m$ implies $X_k \cap X_m = \emptyset$.*

Proof. Assume $X_k \cap X_m \ne \emptyset$. Since Archimedean sets are disjoint we obtain for $X_k =]a_k,b_k[\cup \{z_k\} \cup]c_k,d_k[$ and $X_m =]a_m,b_m[\cup \{z_m\} \cup]c_m,d_m[$, where $z_k = F(b_k,c_k)$ and $z_m = F(b_m,c_m)$, that $z_k = z_m$ and $X_m \cap X_k = \{z_m\}$. Thus $[b_k,c_k] \cap [b_m,c_m] \ne \emptyset$ and therefore z_m is the annihilator of F on $[\min(b_k,b_m), \max(c_k,c_m)]$. Since $m \ne k$ there is $b_m \ne b_k$ and $c_m \ne c_k$. Assume that $b_k < b_m$ (the case when $b_k > b_m$ is analogous). Then z_m is annihilator of F on $[b_k,b_m]$, which is a contradiction since F on X_m is isomorphic to a representable uninorm (restricted to open unit square), i.e., $F(z_m,t) = t$ for all $t \in]a_m,b_m[$. Thus for all $k,m \in K_1$ there $k \ne m$ implies $X_k \cap X_m = \emptyset$. □

Remark 5.2. *Assume that $X_m \cap X_k \neq \emptyset$ for some $m, k \in C$, $m \neq k$. Then $m \notin K_3$, $k \notin K_3$. From previous result we know that if $m, k \in K_1$ then $X_m \cap X_k = \emptyset$. Further we have the following.*

- *If $k, m \in K_4$ then both X_m and X_k are singletons, i.e., $X_m = X_k$ which implies $m = k$, which is a contradiction.*
- *If $k \in K_4$, $m \in K_1$ (the case when $k \in K_1$, $m \in K_4$ is analogous) then $X_k = \{d\}$ for some $d \in H$ and thus $X_m \cap X_k = \{d\}$, where d is the neutral element of X_m. Then $F(v, d) = v$ for all $v \in X_m$.*
- *If $k \in K_4$, $m \in K_2$ (the case when $k \in K_2$, $m \in K_4$ is analogous) then $X_k = \{d\}$ for some $d \in H$, $X_m \cap X_k = \{d\}$ and d is the annihilator of F on X_m. Then $F(v, d) = d$ for all $v \in X_m$.*
- *If $k, m \in K_2$, without loss of generality we can assume $\sup X_k \leq \inf X_m$. Then $X_m \cap X_k = \{d\}$, where d is an idempotent point and $F(d, s) = d = F(t, d)$ for all $s \in X_k$ and all $t \in X_m$. Therefore $d \in H$ is the annihilator of F on X_k and X_m.*
- *If $k \in K_1$, $m \in K_2$ (the case when $k \in K_2$, $m \in K_1$ is analogous) then $X_m \cap X_k = \{d\}$, where for $X_k =]a_k, b_k[\cup \{F(b_k, c_k)\} \cup]c_k, d_k[$ we have $d = F(b_k, c_k)$. Then $X_m \subset [b_k, c_k]$ and d is the annihilator of F on X_m and the neutral element of F on X_k, i.e., $F(s, d) = s$ and $F(t, d) = d$ for all $s \in X_k$ and all $t \in X_m$.*

From Lemma 4.14 and the definition of Archimedean classes we can easily see the following result.

Lemma 5.3. *Let $F \in \mathcal{F}$. Then X_k is closed under F for all $k \in C$.*

Lemma 5.4. *Let $F \in \mathcal{F}$, $k \in C$ and assume an idempotent point d of F. Then either $F(s, d) = s$ for all $s \in X_k$, or $F(s, d) = u$ for some idempotent point u of F for all $s \in X_k$.*

Proof. If $k \in K_3 \cup K_4$ the result is obvious. If $k \in K_2$ the result follows from Lemma 4.11. If $k \in K_1$ the result follows from Lemma 4.19. □

Lemma 5.5. *Let $F \in \mathcal{F}$ and assume $k, m \in C$. If $F(s, t) \in X_k \cup X_m$ for all $s \in X_k$ and $t \in X_m$ then one of the following holds:*

- *$F(s, t) = s$ for all $s \in X_k$ and $t \in X_m$,*
- *$F(s, t) = t$ for all $s \in X_k$ and $t \in X_m$,*
- *$F(s, t) = u \in X_k \cup X_m$ for all $s \in X_k$ and $t \in X_m$.*

Proof. If $k, m \in K_2 \cup K_3 \cup K_4$ then the claim follows from Lemmas 4.11 and 4.17. Assume $k \in K_1$ (the case when $m \in K_1$ is analogous), $X_k =]a_k, b_k[\cup \{z_k\} \cup]c_k, d_k[$, where $z_k = F(b_k, c_k)$. If $m \in K_3 \cup K_4$ then the claim follows from Lemmas 4.11 and 5.4.

• Suppose $m \in K_2$, $X_m = I_{a_m, b_m}$. For $w \in]a_m, b_m[$ there is $F(z_k, w) \in \{z_k, w, u\}$ for some idempotent point $u \notin \{w, z_k\}$ and since $F(s, t) \in X_k \cup X_m$ for all $s \in X_k$ and all $t \in X_m$ there is $u \in \{a_m, b_m\}$, $u \in X_m$. If $F(z_k, w) = w$ then $F(z_k, t) = t$ for all $t \in X_m$ and Lemma 4.20 implies $F(v, t) = t \in X_m$ for all $v \in X_k$ and all $t \in X_m$. If $F(z_k, w) = z_k$ then $F(z_k, t) = z_k$ for all $t \in X_m$ and Lemma 4.20 implies $F(v, t) = v \in X_k$ for all $v \in X_k$ and all $t \in X_m$. If $F(z_k, w) = u \in \{a_m, b_m\}$ then $F(z_k, t) = u$ for all $t \in X_m$ and Lemma 4.20 implies $F(v, t) = u \in X_m$ for all $v \in X_k$ and all $t \in X_m$.

• Suppose $m \in K_1$, $X_m =]a_m, b_m[\cup \{z_m\} \cup]c_m, d_m[$. Then $F(z_k, z_m)$ is an idempotent point and $F(z_k, z_m) \in X_k \cup X_m$ implies $F(z_k, z_m) \in \{z_k, z_m\}$. Without loss of generality we will suppose that $F(z_k, z_m) = z_k$. Since $m \neq k$ Lemma 5.1 implies $z_k \neq z_m$. Lemma 4.19 then implies $F(z_k, t) = z_k$ for all $t \in X_m$ and we have $F(s, t) = F(F(s, z_k), t) = F(s, F(z_k, t)) = F(s, z_k) = s \in X_k$ for all $s \in X_k$, and all $t \in X_m$. □

Lemma 5.6. *Let $F \in \mathcal{F}$ and assume $k, m \in C$. If $F(x, y) = z \notin X_k \cup X_m$ for some $x \in X_k$ and $y \in X_m$ then $F(s, t) = z$ for all $s \in X_k$ and $t \in X_m$ and z is the annihilator of F on $X_k \cup \{z\} \cup X_m$.*

Proof. If $m, k \in K_3 \cup K_4$ the claim is obvious. If $m \in K_3 \cup K_4$, or $k \in K_3 \cup K_4$ the claim follows from Lemmas 4.11 and 4.19.

- Assume $k, m \in K_2$. Since m and k are not paired Lemma 4.15 implies $F(s, t) = z$ and $F(x, z) = z$ for all $s \in X_k$, $t \in X_m$ and $x \in X_k \cup X_m$.

- If $m,k \in K_1$, $X_k =]a_k,b_k[\cup\{z_k\}\cup]c_k,d_k[$, $X_m =]a_m,b_m[\cup\{z_m\}\cup]c_m,d_m[$. Then $z = F(x,y) = F(F(z_k,x),y) = F(z_k,F(x,y)) = F(z_k,z)$ and similarly $F(z_m,z) = z$. Then Lemma 4.19 implies $F(s,z) = z = F(t,z)$ for all $s \in X_k$ and $t \in X_m$, i.e., z is the annihilator of F on $X_k \cup \{z\} \cup X_m$. Further, since $F(x,y) = z$ Lemma 4.20 implies $F(s,t) = z$ for all $s \in X_k$ and $t \in X_m$.

- If $k \in K_1$, $m \in K_2$, $X_k =]a_k,b_k[\cup\{z_k\}\cup]c_k,d_k[$, $X_m = I_{a_m,b_m}$. If $[a_m,b_m] \subseteq [b_k,c_k]$ then $F(z_k,t) = z_k$ for all $t \in X_m$ and then $F(v,t) = v$ for all $v \in X_k$ and all $t \in X_m$, which is a contradiction. Thus either $b_m \le a_k$, or $d_k \le a_m$. We will suppose $b_m \le a_k$ as the other case is analogous. Then $z = F(b_m,a_k)$. Similarly as above $F(x,y) = z$ implies $F(z_k,z) = z$ and by Lemma 4.19 there is $F(v,z) = z$ for all $v \in X_k$. Further, $F(y,z) = F(y,F(x,z)) = F(z,z) = z$ and thus $F(t,z) = z$ for all $t \in X_m$. Therefore z is the annihilator of F on $X_m \cup \{z\} \cup X_k$. Then $z = F(t,z) \le F(t,v) \le F(z,v) = z$, i.e., $F(t,v) = z$ for all $v \in X_k$ and all $t \in X_m$.

- If $k \in K_2$, $m \in K_1$ the proof is analogous to the previous case.

□

Definition 5.7. *On the set C we define a relation $\preceq$ such that for $k,m \in C$ there is*

- *$k \preceq m$ if $k = m$.*
- *$k \preceq m$ if $k \neq m$ and $F(s,t) = s$ for all $s \in X_k$ and all $t \in X_m$.*

Next we show that $\preceq$ is a partial order.

Lemma 5.8. *Let $F \in \mathcal{F}$ and assume the relation $\preceq$ on C from Definition 5.7. Then $\preceq$ is a partial order.*

Proof. In order to show that $\preceq$ is a partial order we have to show that it is reflexive, antisymmetric and transitive.

- Reflexivity follows directly from the definition.

- Antisymmetry. Assume that $k \preceq m$ and $m \preceq k$ for some $m,k \in C$, $m \neq k$. Then $k \preceq m$ implies $F(s,t) = s$ for all $s \in X_k$, $t \in X_m$ and $m \preceq k$ implies $F(t,s) = t$ for all $s \in X_k$, $t \in X_m$. The commutativity of F then implies $X_k = X_m$, which is possible only if $m = k$.

- Transitivity. Assume that $k \preceq m$ and $m \preceq v$ for some $m,k,v \in C$. If $k = m$, or $m = v$, or $k = v$ the transitivity clearly holds. Suppose that k,m,v are all different. Then $F(s,w) = F(F(s,t),w) = F(s,F(t,w)) = F(s,t) = s$ for all $s \in X_k$, $t \in X_m$, $w \in X_v$. Thus $F(s,w) = s$ for all $s \in X_k$, $w \in X_v$, i.e., $k \preceq v$.

Therefore $\preceq$ is a partial order.

□

Lemma 5.9. *Let $F \in \mathcal{F}$ and assume $k,m \in C$ and the partial order $\preceq$ from Definition 5.7. If k and m are incomparable then there exists a $z \in [0,1]$ such that either $\sup X_k \le z \le \inf X_m$, or $\sup X_m \le z \le \inf X_k$ and $F(s,t) = z = F(s,z) = F(t,z)$ for all $s \in X_k$, $t \in X_m$, i.e., $z \in H$.*

Proof. Assume that m and k are incomparable. Then $m \neq k$ and thus Lemma 5.5 and Lemma 5.6 imply $F(s,t) = z = F(s,z) = F(t,z)$ for all $s \in X_k$, $t \in X_m$, for some $z \in [0,1]$, where z is an idempotent point of F by Lemma 4.15 and Lemma 4.17. The monotonicity of F with Lemma 4.6 imply $z \in [\min(s,t), \max(s,t)]$ for all $s \in X_k$ and all $t \in X_m$. First we will show that either $\sup X_k \le \inf X_m$, or $\sup X_m \le \inf X_k$.

If $k,m \notin K_1$ evidently either $\sup X_k \le \inf X_m$, or $\sup X_m \le \inf X_k$.

- If $m,k \in K_1$, $X_k =]a_k,b_k[\cup\{z_k\}\cup]c_k,d_k[$, $X_m =]a_m,b_m[\cup\{z_m\}\cup]c_m,d_m[$. If $t \in [b_k,c_k]$ for some $t \in X_m$ then $z \in [b_k,t]$ and also $z \in [t,c_k]$, which implies $t = z$. Then, however, $z = F(z,z_k) = z_k$ and thus $t \in X_k$, which is a contradiction by Lemma 5.1. Thus $t \notin [b_k,c_k]$ for all $t \in X_m$ and similarly $s \notin [b_m,c_m]$ for all $s \in X_k$, i.e., either $\sup X_k \le \inf X_m$, or $\sup X_m \le \inf X_k$.

- If $k \in K_1$, $m \notin K_1$ then similarly as in the previous case we can show that $t \in [b_k,c_k]$ for some $t \in X_m$ implies $t = z_k$. If $m \in K_3 \cup K_4$ then $k \preceq m$, which is a contradiction. If $m \in K_2$ then $F(z_k,t) = z_k$ for all $t \in X_m$ and $F(z_k,s) = s$ for all $s \in X_k$. Then, however, $F(s,t) = F(F(s,z_k),t) = F(s,F(z_k,t)) = F(s,z_k) = s$ for all $s \in X_k$ and all $t \in X_m$, i.e., $k \preceq m$, which is a contradiction. Thus either $\sup X_k \le \inf X_m$, or $\sup X_m \le \inf X_k$.

- If $k \notin K_1$, $m \in K_1$ the claim can be shown as in the previous case.

Suppose that $\sup X_k \leq \inf X_m$ (the case when $\sup X_m \leq \inf X_k$ is analogous). Then either $F(s,t) = z = F(z,t)$ for some $s < z < t$, i.e., $z \in H$, or one of X_k and X_m is equal to $\{z\}$. However, if $X_k = \{z\}$ (and similarly if $X_m = \{z\}$) then $F(z,t) = z$ for all $t \in X_m$, i.e., $k \preceq m$, which is a contradiction.

□

Proposition 5.10. *Let $F \in \mathcal{F}$ and assume the partial order $\preceq$ from Definition 5.7. Then $(C, \preceq)$ is a meet semi-lattice.*

Proof. We will show that each pair of points from C have the meet. Assume $k, m \in C$. If $k \preceq m$ or $m \preceq k$ the proof is finished. Suppose that k and m are incomparable. Then Lemma 5.9 implies that there exists $z \in H$ such that $F(s,t) = z = F(s,z) = F(t,z)$ for all $s \in X_k$ and all $t \in X_m$. Then $\psi^{-1}(z) \preceq k$ and $\psi^{-1}(z) \preceq m$. Suppose that there is an $l \in C$ such that $l \preceq k$ and $l \preceq m$. Then for all $r \in X_l$, $s \in X_k$ and $t \in X_m$ there is $F(r,s) = r = F(r,t)$ and thus

$$F(r,z) = F(r, F(s,t)) = F(F(r,s),t) = F(r,t) = r.$$

Therefore $l \preceq \psi^{-1}(z)$ and $k \wedge m = \psi^{-1}(z)$.

Now we will show the main result. Let us remark that if $X_k \cap X_p \neq \emptyset$ for some $k, p \in C$, $k \neq p$ then $X_k \cap X_p = \{x\}$ and if k and p are incomparable then by Lemma 5.9 $\sup X_k \leq z \leq \inf X_m$, or $\sup X_m \leq z \leq \inf X_k$, which implies $z = x$. Therefore in all cases $X_k \cap X_p = \{x\}$ implies $x \in X_{k \wedge p}$.

Theorem 5.11. *Let $F \in \mathcal{F}$ and assume the partial order $\preceq$ from Definition 5.7. Then F can be expressed as a z-ordinal sum of semigroups $\{(X_k, F)\}_{k \in C}$ with respect to the partial order $\preceq$ and the branching set $A = K_4$.*

Proof. First we have to verify that all conditions of Theorem 2.5 are satisfied. We have index sets A and B such that $A \cap B = \emptyset$ and $C = A \cup B \neq \emptyset$. We also have a family of semigroups $(G_k)_{k \in C}$ with $G_k = (X_k, F)$. From Proposition 5.10 we know that $(C, \preceq)$ is a meet semi-lattice. Each semigroup G_k for $k \in A$ trivially has an annihilator and if k is incomparable with m for some $k, m \in C$ then Lemma 5.9 implies that $k \wedge m \in A$. Further assume that $X_k \cap X_m = \{x\}$ for some $k, m \in C$. Remark 5.2 implies that if $k \wedge m \in A$ then $x = \psi(k \wedge m)$ is the annihilator of both X_k and X_m and if $k \wedge m \notin A$ then x is both the annihilator of $X_{k \vee m}$ and the neutral element of $X_{k \wedge m}$.

Assume that $k \wedge m \prec v \prec k$ for some $v \in C$. Then $v \prec k$ implies $F(s,x) = s$ for all $s \in X_v$ and $k \wedge m \prec v$ implies $F(x,s) = x$ since $x \in X_{k \wedge m}$. Thus $s = x$ for all $s \in X_v$, i.e., $X_v = \{x\}$. Similarly, if $k \wedge m \prec v \prec m$ for some $v \in C$ we get $X_v = \{x\}$.

Further assume that v is incomparable with $k \wedge m$. Since $x \in X_{k \wedge m}$ there exists $r \in X_v$ such that $F(x,r) = q \neq r$, for some idempotent point $q \in H$. If $q = x$ then either v is incomparable with k, or $k \prec v$. In the later case, however, $k \wedge m \preceq k \prec v$, which is a contradiction. Thus v is incomparable with k and similarly v is incomparable with m and $v \wedge k = \psi^{-1}(q) = v \wedge m$. Therefore all conditions from Theorem 2.5 are verified.

Assume that $([0,1], G)$ is a z-ordinal sum of semigroups $\{(X_k, F)\}_{k \in C}$ with respect to the partial order $\preceq$ and the set $A = K_4$. Evidently, the function G is commutative. We want to show that $F(x,y) = G(x,y)$ for all $x, y \in [0,1]$.

- If $x, y \in X_k$ for some $k \in C$ then evidently $F(x,y) = G(x,y)$.
- If $x \in X_k$, $y \in X_m$, $m \neq k$, $k \preceq m$. Then $k \preceq m$ implies $F(x,y) = x$. Further we have to distinguish two cases, when $k \in A$ and when $k \in C \setminus A$. In the first case $k \in K_4$, i.e, $X_k = \{q\}$ for some $q \in H$ and $G(x,y) = G(q,y) = q = x$. In the second case $G(x,y) = x$. Thus in both cases $F(x,y) = G(x,y)$.
- If $x \in X_k$, $y \in X_m$, $m \neq k$, $m \preceq k$. This case can be shown similarly as the previous case.
- If $x \in X_k$, $y \in X_m$ and k and m are incomparable. Then Lemma 5.9 implies that $F(s,t) = z = F(s,z) = F(t,z)$ for all $s \in X_k$ and all $t \in X_m$ for some $z \in H$. Similarly as in Proposition 5.10 we can show that $\psi(k \wedge m) = z$, i.e. $G(x,y) = z = F(x,y)$.

Summarizing, in all cases $F(x,y) = G(x,y)$, i.e., F can be expressed as a z-ordinal sum of semigroups $\{(X_k, F)\}_{k \in C}$ with respect to the partial order $\preceq$ and the branching set $A = K_4$. □

Further we will show that $(C, \preceq)$ has for a function $F \in \mathcal{F}$ a tree structure, i.e., if k and m are incomparable for some $k, m \in C$ then there is no element $p \in C$ such that $k \preceq p$ and $m \preceq p$.

Lemma 5.12. *Let $F \in \mathcal{F}$ and assume the partial order $\preceq$ from Definition 5.7. If k and m are incomparable for some $k, m \in C$ then k and m have no upper bound, i.e., there is no $p \in C$ such that $k \preceq p$ and $m \preceq p$.*

Proof. If k and m are incomparable then Lemma 5.9 implies $k \wedge m \in A = K_4$, i.e., for $z = \psi(k \wedge m)$ there is $F(s,t) = z$ for all $s \in X_k$ and all $t \in X_m$, where either $\sup X_k \leq z \leq \inf X_m$, or $\sup X_m \leq z \leq \inf X_k$. We will suppose $\sup X_k \leq z \leq \inf X_m$, as the other case is analogous. Assume that there exists $p \in C$ such that $k \preceq p$ and $m \preceq p$. Then $X_p \neq \{z\}$ and for any $r \in X_p$, $r \neq z$ we have

- if $r < z$ then for any $t \in X_m \setminus \{z\}$ there is $t = F(r,t) \leq F(z,t) = z < t$, which is a contradiction;
- if $r > z$ then for any $s \in X_k \setminus \{z\}$ there is $s = F(r,s) \geq F(z,s) = z > s$, which is a contradiction.

Therefore, if k and m are incomparable then k and m have no upper bound. □

Lemma 5.13. *Let $F \in \mathcal{F}$ and let $\preceq$ be the partial order from Definition 5.7. Then for each $p \in A = K_4$ with $X_p = \{z\}$ the set $Q = \{k \in C \mid p \prec k\}$ can be divided into two sets Q_1 and Q_2 such that $k_1 \wedge m_1 \in Q_1$ for all $k_1, m_1 \in Q_1$, $k_2 \wedge m_2 \in Q_2$ for all $k_2, m_2 \in Q_2$ and either $\sup(\bigcup_{k \in Q_1} X_k) \leq z \leq \inf(\bigcup_{k \in Q_2} X_k)$, or $\sup(\bigcup_{k \in Q_2} X_k) \leq z \leq \inf(\bigcup_{k \in Q_1} X_k)$.*

Proof. For all $x \in \bigcup_{k \in Q} X_k$ there is $F(x,z) = z$. First we will show that for each $k \in Q$ there is either $x \leq z$ for all $x \in X_k$, or $z \leq x$ for all $x \in X_k$. Assume $k \in Q$ and $x_1, x_2 \in X_k$ such that $x_1 < z < x_2$. Then $k \notin K_2 \cup K_3 \cup K_4$, i.e., $k \in K_1$. Let $X_k = I_{a,b} \cup \{z_k\} \cup I_{c,d}$. If $z = z_k$ then $F(x,z) = x$ for all $x \in X_k$, which is a contradiction. Thus $z \neq z_k$, $z \in [b,c]$. Then, however, $F(z_k, z) = z_k$ since z_k is the annihilator of F on $[b,c]$. On the other hand, $F(z, z_k) = z$ since $p \prec k$, i.e., $z = z_k$, which is a contradiction.

We denote $Q_1 = \{k \in Q \mid x \leq z \text{ for all } x \in X_k\}$ and $Q_2 = \{k \in Q \mid z \leq x \text{ for all } x \in X_k\}$.

Assume $k, m \in Q_1$. If k and m are comparable then evidently $k \wedge m \in Q_1$. Suppose that k and m are incomparable. Then $p \preceq k \wedge m \in A$, $q = \psi(k \wedge m) \in H$. If $k \wedge m \in Q_1$ the proof is finished. Suppose the opposite, i.e., $k \wedge m \in Q_2 \cup \{p\}$. Then $z \leq q$ and for $x \in X_k \setminus \{q\}$ and $y \in X_m \setminus \{q\}$, $x \neq y$, there is $F(x,y) = q$. We will assume $x < y$ as the case for $y < x$ is analogous. Then $q = F(x,y) \leq F(y,y) \leq F(z,z) \leq F(q,q) = q$, i.e., $F(y,y) = q > y$. Thus $m \notin K_3 \cup K_4$, i.e., $m \in K_1 \cup K_2$. However, if $m \in K_1$ then $F(y,y)$ is an idempotent point of F only if y is the neutral element of G_m and then $F(y,y) = y$, which is a contradiction. Therefore $m \in K_2$. Then $x < t$ for all $t \in X_m$ and $q \in X_m$. Then, however, $q = F(x,t) \leq F(t,t) \leq F(q,q) = q$, implies $F(t,t) = q$ for all $t \in X_m$, which is impossible since F on X_m is isomorphic to a restriction of a nilpotent t-conorm to the left-open unit square. Thus $k \wedge m \in Q_1$ for all $k, m \in Q_1$. Similarly we can show that $k \wedge m \in Q_2$ for all $k, m \in Q_2$. □

Remark 5.14. *From previous results we can see that similarly as in the case of n-uninorms with continuous underlying functions, the meet semi-lattice $(C, \preceq)$ resembles a binary tree, where nodes correspond to points from A.*

The monotonicity of F implies that above each point $p \in A$ (i.e., $p \in K_4$) there are at most two branches, one branch contains only semigroups with points smaller than (or equal to) $\psi(p)$ and the second branch contains only semigroups with points greater than (or equal to) $\psi(p)$.

Our results cover the z-ordinal sum decomposition of continuous t-norms, continuous t-conorms (see [10, 13, 23]), nullnorms with continuous underlying functions [20], uninorms with continuous underlying functions [19] and n-uninorms with continuous underlying functions [20]. However, they can be applied also for the function F from Example 3.1. Similar examples of functions from $\mathcal{F}$ can be constructed from partial orders on C: in the case when the top element of some branch in the respective tree structure on $(C, \preceq)$ has no neutral element then F is not an n-uninorm. □

6 Conclusions

We have characterized all commutative, associative, non-decreasing functions with continuous Archimedean components, which are continuous around the diagonal. We have shown that the class of these functions coincides with the class of functions that can be constructed from semigroups from Definition 2.6 and trivial semigroups via z-ordinal sum construction.

In this work we have considered that the function F has a continuous diagonal and that all its Archimedean components are continuous. This result can be made even more general when we consider the conditions under which a t-norm (t-conorm) with a continuous diagonal is continuous (see [17]). Similarly as in [18], the continuity on Archimedean components can be partially replaced by the cancellativity.

Acknowledgment

This work was supported by grants VEGA 1/0006/19, APVV-16-0073 and Program Fellowship of SAS.

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