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Ranking of generalized fuzzy numbers based on accuracy of comparison

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Abstract

Ranking generalized fuzzy numbers plays an important role in many applied models and, in particular, decision-making procedures. In ranking process of two generalized fuzzy numbers, it is natural to compare the sets of values in support of two generalised fuzzy numbers. Accordingly, the comparison of a real number and a generalised fuzzy number as well as two generalised fuzzy numbers have to be considered. On the other hand, it is seen that a definitive process of comparison of a real number and a generalised fuzzy number, as well as two generalised fuzzy numbers, is not possible. So in this study, a method for comparing a real number and a generalised fuzzy number with a degree of accuracy (between a zero and one) is defined and then the method is generalized to compare two generalised fuzzy numbers. In general, an index to rank a real number and generalised fuzzy number is constructed. Eventually, this index is extended to rank two generalised fuzzy numbers based on the concept of accuracy of comparison. The advantage of our method is that it can compare two generalised fuzzy numbers with an accuracy of comparison. Also, a definition is introduced to make a definitive comparison. Finally, the proposed method is illustrated by some numerical examples.

Keywords: Fuzzy set, generalized, ranking fuzzy numbers.

1 Introduction

In many problems in the fields of engineering and applied sciences it is essential to construct a mathematical model. However, the parameters of the models are often qualitative. In such case, it is not possible to solve the mathematical models with classical logic or classical set theory. The mathematical models with classical logic often do not respond suitably or lead to an inappropriate results. With the introduction of Zadeh's theory of fuzzy logic in 1965, which is a generalization of classical logic, mathematical modeling of this kind of problem was made possible [60].

Fuzzy mathematics has been extensively applied in different field of mathematics. For instance, fuzzy data with real number ranges were introduced by Dubois and Prade as fuzzy numbers for the first time [23]. Following it, numerous studies using fuzzy data and fuzzy number have been conducted in different filed of sciences. For instance, metric spaces [31, 51], differential equations [4, 11, 20], decision making [18, 19, 45, 48, 52], risk assessment [10, 43, 59], clustering [36, 46, 63], system equations [3, 27, 49], artificial neural network [25, 26, 47], agriculture [21, 42], data envelopment analysis [7, 44, 50], genetic algorithms [30, 38, 54], control [24, 41, 53, 62] and etc.

In many instances, while solving these problems or the outputs of most of these problems, it is necessary to compare the fuzzy numbers in order to reach at the final decision, which is called fuzzy number ranking. Generally, ranking of fuzzy numbers is divided into three categories. Namely,

(i) Defuzzification methods: In which case, fuzzy numbers are converted to a crisp number by means of a function as utilized in ranking them.

(ii) Comparison with reference set methods: In which case, a reference set is constructed and a distance measure between the reference set and the fuzzy numbers is constructed and utilised in ranking the fuzzy numbers.

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(iii) Pairwise comparison methods: In which case, the preference of fuzzy numbers is determined by comparing two fuzzy numbers with the preference function.

There are numerous studies that can be classified under Category (i). Namely, Suneela and Chakraverty [49] modified the Yager ranking formula and presented a function for ranking fuzzy numbers. Jiang et al. [35] used the weight of centroid points, fuzziness and spread and constructed a score of the fuzzy number and constructed the ranking index. In another study, Chi and Vincent [15] defined a rank index for generalized fuzzy number (GFN) using a centroid point and height of fuzzy numbers. Adabitabar Firozja et al. [28] defined a functions of vague and value corresponding to each fuzzy number and used them as ranking index. Gu and Xuan [32] used possibilistic mean and possibilistic standard deviation for each fuzzy number and used them ranking the fuzzy numbers. Allahviranloo and Saneifard [8] introduced a ranking index to rank various fuzzy numbers.

Some of the studies under Category (ii) are as followed. Chai et al. in [12] used Dempster-Shafer theory and designed a generalized method to rank fuzzy numbers which uses the Murphy's combination rule. In another study, the maximizing and minimizing sets of fuzzy numbers that are a generalization of the Chen's [13] method are defined by Chu and Nguyen [16]. Allahviranloo and Firozja [6] defined a continuous metric on fuzzy numbers and performed the ranking by calculating the relative distance of each fuzzy number to two real numbers.

Further, some of the studies grouped under Category (iii) are as follows. Adabitabar Firozja et al. [2] proposed a ranking function for ranking real numbers and fuzzy number with acceptance rate larger or acceptance rate smaller and then extended it for ranking two fuzzy numbers. Molinari [39] defined a preference relation using the weighted mean of the left and right functions for ranking triangular fuzzy numbers. Hierro et al. [34], introduced a new fuzzy binary relation in the whole set of fuzzy numbers and used in practice to compare two distinct fuzzy numbers. A fuzzy probabilistic preference relation between two fuzzy numbers was defined by Zhang et al. [61] and used in comparing fuzzy numbers. Hesamian and Akbari [33] developed a preference index for comparing two trapezoidal fuzzy numbers. Allahviranloo and et al. in [5] proposed a modified new weighted distance method to rank fuzzy numbers.

It is obvious that in most cases a definite comparison is not possible, so ranking has to be done based on some degree of accuracy. For this purpose, a function of correct degree with values in the interval $[0, 1]$ is defined, where the number 1 the threshold of correct and number 0 the incorrect threshold, which determines the proximity to the threshold. Further, if the degree function is closer to the threshold, it is considered correct, and if the degree function is remote to the threshold, it is incorrect; and finally if both are equal, then the numbers identical.

This motivates to develop a new method of ranking generalised fuzzy number, as such the main contribution of this study are summarized as follows. A function of degree of correctness for comparing a fuzzy number with a real number is introduced and then the function is generalized to compare two GFNs. Finally, a rank degree function as a ratio of the set fuzzy integer less (more) than that number with membership rate to the set of fuzzy integers is defined to calculate the degree of correctness in larger (smaller) real number with a fuzzy number.

The paper is organized as follows. Section 2, refers to the background concept of fuzzy numbers and its related concepts. In Section 3.1 the ranking index to rank a real number and a GFN is proposed. In Section 3.2, the concept of ranking real number and GFN is extended to rank GFNs. In Section 4, the proposed method is demonstrated through some numerical examples. Finally, in Section 5, conclusion is drawn based on the findings.

2 Background

This section reviews the basic concept of fuzzy number, the arithmetic operations for ranking GFNs and used symbols.

2.1 The basic concept of fuzzy number

Definition 2.1. *A fuzzy set A in the universe of discourse X, which is a space of points, is defined as the following set $A = \{(x,\ \mu_A(x))\,|x \in X\}$, and characterized by a membership function $\mu_A(x) : X \to [0,\ 1]$ in which a generic element x in X is a real number in the closed interval $[0,\ 1]$. The values at x indicate the grade of membership of x in A.*

Definition 2.2. *A fuzzy set A, described as any fuzzy subset on the space of real number $\mathbb{R}$, is said to be a fuzzy number if its membership function $\mu_A(x)$ is convex, normalized, and piecewise continuous* [23].

Definition 2.3. *A GFN A is described as any fuzzy subset of the universe set $\mathbb{R}$ with membership function μ_A defined as follows:*

(a) μ_A is a continuous mapping from closed interval $[a, d]$ to the closed internal $[0, \omega]$, $0 < \omega \leq 1$;

(b) μ_A^L is strictly increasing on interval $[a, b]$;

(c) $\mu_A(x) = \omega$ *for all* $x \in [b, c]$*;*

(d) μ_A^R *is strictly decreasing on interval* $[c, d]$.

A GFN can be represented by $A = (a, b, c, d, \omega)$ where ω is the height of the GFN and the membership function of A can be expressed as

$$\mu_A(x) = \begin{cases} \mu_A^L(x), & \text{if } a \leq x \leq b, \\ \omega, & \text{if } b \leq x \leq c, \\ \mu_A^R(x), & \text{if } c \leq x \leq d, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $\mu_A^L : [a, b] \longrightarrow [0, \omega]$ and $\mu_A^R : [c, d] \longrightarrow [0, \omega]$.

Based on the basic theories of fuzzy numbers, A is a normal fuzzy number if $\omega = 1$.

Definition 2.4. *The GFN A is called generalized L-R fuzzy number if* $\mu_A^L(x) = L(\omega\frac{b-x}{b-a}), a \leq x \leq b$ *and* $\mu_A^R(x) = R(\omega\frac{x-c}{d-c}), c \leq x \leq d$*, where L and* $\mathbb{R}$ *are strictly decreasing functions defined on* $[0, \omega]$*.*

Generalized trapezoidal fuzzy numbers are special cases of generalized L-R fuzzy number with $L(t) = R(t) = 1 - t$. For a Generalized trapezoidal fuzzy number A, the support of A is defined as $\text{supp}(A) = \{x \in X : \mu_A(x) > 0\} = [a, d]$.

2.2 Arithmetic operations

Definition 2.5. *A* α*-level interval of GFN A is denoted as:*

$$[A]^\alpha = \left[A_\omega^L(\alpha), A_\omega^R(\alpha)\right] = \left[\mu_A^{L^{-1}}\left(\frac{\alpha}{\omega}\right), \mu_A^{R^{-1}}\left(\frac{\alpha}{\omega}\right)\right]. \tag{2}$$

Theorem 2.6. [29] *Suppose* $f : X \times X \to X$ *is a continuous function and A, B are fuzzy numbers. Then*

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha). \tag{3}$$

Notation 2.7. *Suppose* $f : X \times X \to X$*,* $f(x, y) = x * y$ *and* $* \in \{+, -, *, \div\}$*, A, B are two fuzzy numbers,* $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ *and* $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$*. Then,*

$$[A * B]^\alpha = [A]^\alpha * [B]^\alpha. \tag{4}$$

3 Ranking of two GFNs

In this section, the proposed method is being discussed thoroughly. Initially, a method to compare a real number and a GFN with a degree of accuracy (a value between zero and one) is constructed, and then the method is generalised to compare GFNs.

3.1 Accuracy of comparison a real number with a GFN

In this subsection, an index to rank a real number and GFN has been defined based on the concept of the area.

Definition 3.1. *Let A be a GFN and* $x \in \mathbb{R}$*. Let* $Deg(A \leq x) : \mathbb{R} \to [0, 1]$ *be a real function to measure greater x of A and* $Deg(x \leq A) : \mathbb{R} \to [0, 1]$ *be a real function to measure smaller x of A. The functions* $Deg(A \leq x)$ *and* $Deg(x \leq A)$ *are defined as follows:*

$$Deg(A \leq x) = \frac{\int_{-\infty}^{x} \mu_A(t)dt}{\int_{-\infty}^{+\infty} \mu_A(t)dt}, \tag{5}$$

and

$$Deg(x \leq A) = \frac{\int_{x}^{+\infty} \mu_A(t)dt}{\int_{-\infty}^{+\infty} \mu_A(t)dt}. \tag{6}$$

respectively.

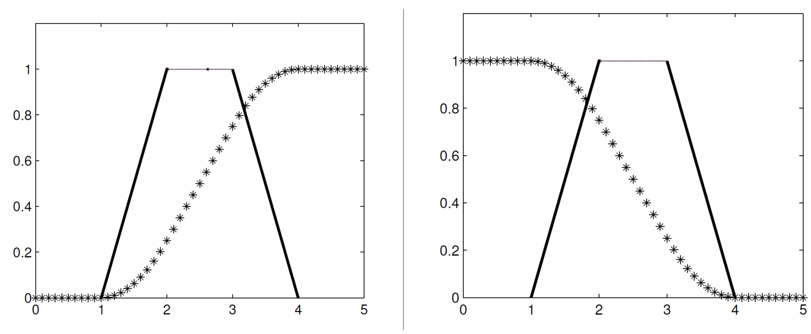
Theorem 3.2. *The function Deg($A \leq x$) is increasing and Deg($x \leq A$) is decreasing.*

Proof. Let $x_1 < x_2$, then

$$\int_{-\infty}^{x_1} \mu_A(t)dt < \int_{-\infty}^{x_2} \mu_A(t)dt \Rightarrow \frac{\int_{-\infty}^{x_1} \mu_A(t)dt}{\int_{-\infty}^{\infty} \mu_A(t)dt} < \frac{\int_{-\infty}^{x_2} \mu_A(t)dt}{\int_{-\infty}^{\infty} \mu_A(t)dt} \Rightarrow \text{Deg}(A \leq x_1) < \text{Deg}(A \leq x_2).$$

Thus, Deg($A \leq x$) is increasing. Similarly, it can be shown that Deg($x \leq A$) decreasing. □

The function Deg($A \leq x$) is a characteristic function that shows, the accuracy of comparison grater than x of A and the characteristic function Deg($x \leq A$) gives the accuracy of comparison smaller x of A. Let $A = (1, 2, 3, 4)$ be a GFN. The Figure 1 shows the characteristic functions Deg($A \leq x$) and Deg($x \leq A$). For instance, Deg($A \leq x$) is the



(a) Deg($A \leq x$) $******$ (b) Deg($x \leq A$) $******$

Figure 1: Characteristic functions Deg($A \leq x$) and Deg($x \leq A$) of the fuzzy number A.

accuracy of comparison larger x of A which is the ratio of the greater area than half of the total area of A by the total area of A. Deg($x \leq A$) is the accuracy of comparison smaller x of A which is the ratio of the smaller area than half of the total area of A by the total area of A. This concept is described by Figure 2, where A is a GFN and $x \in \text{supp}(A)$ such that area from left foot of A to x is greater than half of the total area of A be S_1 and the area from x to right foot of A be S_2. Let the total area be $S(= S_1 + S_2)$. Then, accuracy of comparison larger x of A is Deg($A \leq x$) $= \frac{S_1}{S}$ and accuracy of comparison smaller x of A is Deg($x \leq A$) $= \frac{S_2}{S}$.

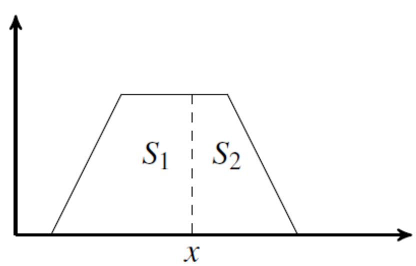


Figure 2: Index Deg($A \leq x$) and Deg($x \leq A$).

Here are some of the properties of the functions Deg($A \leq x$) and Deg($x \leq A$) of the GFN A.

Proposition 3.3. *Let $x \in \mathbb{R}$ and A be a GFN, then $0 \leq Deg(A \leq x) \leq 1$ and $0 \leq Deg(x \leq A) \leq 1$.*

Proposition 3.4. *Let A be GFN and for $x \in \mathbb{R}$, $Deg(A \leq x) + Deg(x \leq A) = 1$.*

Proposition 3.5. *Let A be GFN and for $x, \lambda \in \mathbb{R}$*

$$Deg(\lambda A \leq \lambda x) = \begin{cases} Deg(A \leq x) & 0 \leq \lambda \\ Deg(x \leq A) & \lambda < 0 \end{cases}, \qquad Deg(\lambda x \leq \lambda A) = \begin{cases} Deg(x \leq A) & 0 \leq \lambda \\ Deg(A \leq x) & \lambda < 0. \end{cases}$$

Proposition 3.6. *Let A be GFN and for $x, \lambda \in \mathbb{R}$, then $Deg(A+\lambda \leq x+\lambda) = Deg(A \leq x)$ and $Deg(x+\lambda \leq A+\lambda) = Deg(x \leq A)$.*

Proposition 3.7. *Let A be a GFN and $x, y \in \mathbb{R}$. If $x < y$, then $Deg(A \leq x) < Deg(A \leq y)$ and $Deg(x \leq A) > Deg(y \leq A)$.*

Definition 3.8. *Based on the index $Deg(A \leq x)$ and $Deg(x \leq A)$ where A is a GFN and $x \in \mathbb{R}$, the following decision can be made*

(1) *If $Deg(A \leq x) = Deg(x \leq A)$, then $A \approx x$.*

(2) *If $0 < Deg(x \leq A) < \frac{1}{2}$ or $\frac{1}{2} < Deg(A \leq x) < 1$, then $A \preceq x$.*

(3) *If $\frac{1}{2} < Deg(x \leq A) < 1$ or $0 < Deg(A \leq x) < \frac{1}{2}$, then $x \preceq A$.*

(4) *If $Deg(x \leq A) = 1$ or $Deg(A \leq x) = 0$, then $x < A$.*

(5) *If $Deg(x \leq A) = 0$ or $Deg(A \leq x) = 1$, then $A < x$.*

Corollary 3.9. *From the Proposition 3.5, the following result is valid. $A \preceq x \Leftrightarrow -x \preceq -A$ and $x \preceq A \Leftrightarrow -A \preceq -x$.*

3.2 Accuracy of comparison of two GFNs

In this subsection, an index to rank two GFN has been defined based on the concept of the area.

Definition 3.10. *Let A and B be two GFNs, then the ranking index $Deg(A \leq B)$ is defined as*

$$Deg(A \leq B) = \frac{\int_{-\infty}^{+\infty} \{Deg(A \leq x)\mu_B(x) + Deg(x \leq B)\mu_A(x)\}\,dx}{\int_{-\infty}^{+\infty} \{\mu_A(x) + \mu_B(x)\}\,dx}, \tag{7}$$

where $Deg(A \leq x)$ and $Deg(x \leq B)$ are defined in Eqations 5 and 6 respectively and $Deg(A \leq B)$ is accuracy of comparison smaller A of B.

In fact, in order to calculate $\text{Deg}(A \leq B)$, the larger of the x in $\text{supp}(B)$ are compared with B and the smaller of the x in $\text{supp}(A)$ are compared with B.

Proposition 3.11. *Let A and B be two GFNs, then $0 \leq Deg(A \leq B) \leq 1$.*

Proposition 3.12. *Let A and B be two GFNs, then $Deg(A \leq B) + Deg(B \leq A) = 1$.*

Proof. From the Equation 7 and Proposition 3.4, it follows as

$$\begin{aligned}
&\text{Deg}(A \leq B) + \text{Deg}(B \leq A)\\
=&\frac{\int_{-\infty}^{+\infty} \{\text{Deg}(A \leq x)\mu_B(x) + \text{Deg}(x \leq B)\mu_A(x)\}\,dx}{\int_{-\infty}^{+\infty} \{\mu_A(x) + \mu_B(x)\}\,dx} + \frac{\int_{-\infty}^{+\infty} \{\text{Deg}(B \leq x)\mu_A(x) + \text{Deg}(x \leq A)\mu_B(x)\}\,dx}{\int_{-\infty}^{+\infty} \{\mu_A(x) + \mu_B(x)\}\,dx}\\
=&\frac{1}{\int_{-\infty}^{+\infty} (\mu_A(x) + \mu_B(x))\,dx} \int_{-\infty}^{+\infty} \{(\text{Deg}(A \leq x) + \text{Deg}(x \leq A))\mu_B(x) + (\text{Deg}(x \leq B) + \text{Deg}(B \leq x))\mu_A(x)\}\,dx\\
=&\frac{\int_{-\infty}^{+\infty} \{\mu_B(x) + \mu_A(x)\}\,dx}{\int_{-\infty}^{+\infty} \{\mu_A(x) + \mu_B(x)\}\,dx} = 1.
\end{aligned}$$

□

Proposition 3.13. *Let A and B be two GFNs and $\lambda \in \mathbb{R}$, then*

$$Deg(\lambda A \leq \lambda B) = \begin{cases} Deg(A \leq B), & 0 \leq \lambda, \\ Deg(B \leq A), & \text{otherwise.} \end{cases}$$

Proof. From the Equations 5, 6 and 7 and the Proposition 3.5. For $\lambda \geq 0$,

$$\mathrm{Deg}(\lambda A \leq \lambda B) = \frac{\int_{-\infty}^{+\infty} \{\mathrm{Deg}(\lambda A \leq x)\mu_{\lambda B}(x) + \mathrm{Deg}(x \leq \lambda B)\mu_{\lambda A}(x)\}\, dx}{\int_{-\infty}^{+\infty} \{\mu_{\lambda A}(x) + \mu_{\lambda B}(x)\}\, dx}$$

If $x = \lambda t$, then $dx = \lambda dt$, therefore

$$\begin{aligned}\mathrm{Deg}(\lambda A \leq \lambda B) =& \frac{\int_{-\infty}^{+\infty} \{\mathrm{Deg}(\lambda A \leq \lambda t)\mu_{\lambda B}(\lambda t) + \mathrm{Deg}(\lambda t \leq \lambda B)\mu_{\lambda A}(\lambda t)\}\, \lambda dt}{\int_{-\infty}^{+\infty} \{\mu_{\lambda A}(\lambda t) + \mu_{\lambda B}(\lambda t)\}\, \lambda dt} \\ =& \frac{\int_{-\infty}^{+\infty} \{\mathrm{Deg}(A \leq t)\mu_B(t) + \mathrm{Deg}(t \leq B)\mu_A(t)\}\, dt}{\int_{-\infty}^{+\infty} \{\mu_A(t) + \mu_B(t)\}\, dt} \\ =& \mathrm{Deg}(A \leq B).\end{aligned}$$

The proof is similar to the case $\lambda < 0$. □

Proposition 3.14. *Let A and B be two GFNs and $\lambda \in \mathbb{R}$, then Deg$(\lambda + A \leq \lambda + B) =$ Deg$(A \leq B)$.*

Proof. By the Equation 7 and the Proposition 3.6

$$\mathrm{Deg}(\lambda + A \leq \lambda + B) = \frac{\int_{-\infty}^{+\infty} \left\{\mathrm{Deg}(\lambda + A \leq x)\mu_{(\lambda+B)}(x) + \mathrm{Deg}(x \leq \lambda + B)\mu_{(\lambda+A)}(x)\right\} dx}{\int_{-\infty}^{+\infty} \left\{\mu_{(\lambda+A)}(x) + \mu_{(\lambda+B)}(x)\right\} dx}.$$

If $x = \lambda + t$, then $dx = dt$, hence

$$\begin{aligned}\mathrm{Deg}(\lambda + A \leq \lambda + B) =& \frac{1}{\int_{-\infty}^{+\infty} \left\{\mu_{(\lambda+A)}(\lambda + t) + \mu_{(\lambda+B)}(\lambda + t)\right\} dt} \times \\ & \int_{-\infty}^{+\infty} \left\{\mathrm{Deg}(\lambda + A \leq \lambda + t)\mu_{(\lambda+B)}(\lambda + t) + \right. \\ & \left. \mathrm{Deg}(\lambda + t \leq \lambda + B)\mu_{(\lambda+A)}(\lambda + t)\right\} dt \\ =& \frac{\int_{-\infty}^{+\infty} \{\mathrm{Deg}(A \leq t)\mu_B(t) + \mathrm{Deg}(t \leq B)\mu_A(t)\}\, dt}{\int_{-\infty}^{+\infty} \{\mu_A(t) + \mu_B(t)\}\, dt} \\ =& \mathrm{Deg}(A \leq B).\end{aligned}$$

□

Definition 3.15. *Let A and B be two GFNs, following conclusions can be made based on the ranking index Deg$(A \leq B)$:*

(1) *If $0 <$ Deg$(A \leq B) < \frac{1}{2}$, then $A \succeq B$.*

(2) *If $\frac{1}{2} <$ Deg$(A \leq B) < 1$, then $A \preceq B$.*

(3) *If Deg$(A \leq B) = 1$, then $A < B$.*

(4) *If Deg$(A \leq B) = 0$, then $A > B$.*

(5) *If Deg$(A \leq B) = \frac{1}{2}$ then $A \approx B$.*

The proposed method of ranking two GFNs A and B is represented graphically by a flowchart as shown in Figure 3.

Definition 3.15 refer to the comparison of two GFNs with accuracy of comparison. Thus in order to make a definitive comparison, the Definition 3.15 can be used efficiently.

The proposed method can also efficiently rank the images of GFNs which is evident from the Corollary 3.16 which follows immediately from the Proposition 3.13.

Corollary 3.16. $A \preceq B \Leftrightarrow -B \preceq -A$ *and* $B \preceq A \Leftrightarrow -A \preceq -B$.

Notation 3.17. *If $A \preceq B$, then $A \prec B$ or $A \approx B$.*

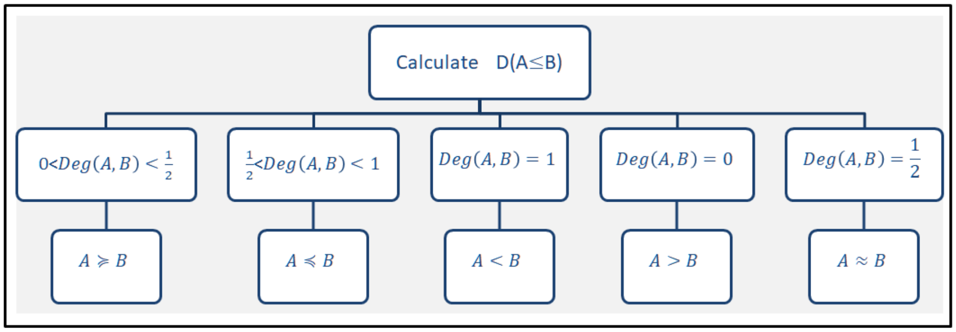


Figure 3: Graphical representation of the proposed method.

It is often reasonable to check, whether a method follows the reasonable properties proposed by Wang and Kerre [55]. In the next proposition, these reasonable properties are verified by the proposed ranking method.

Proposition 3.18. *Let T_1 and T_2 be the set of fuzzy numbers that the proposed ranking method $Deg(\cdot)$ can be applied and U is a finite subset of T_1. For A, B and C belong to $U \subseteq T_1$, we verify the following properties. Let T_1 and T_2 be the set of fuzzy numbers that the proposed ranking method $Deg(\cdot)$ can be applied and U is a finite subset of T_1. For A, B and C belong to $U \subseteq T_1$, we verify the following properties.*

(K_1) $A \preceq A$.

(K_2) *If* $A \succeq B$ *and* $B \succeq A$, *then* $A \approx B$.

(K_3) *If* $A \succeq B$ *and* $B \succeq C$, *then* $A \succeq C$.

(K_4) *If* $\inf supp(A) > \sup supp(B)$, *then* $A \succeq B$.

(K_4') *If* $\inf supp(A) > \sup supp(B)$, *then* $A \succ B$.

(K_5) *Let* A *and* B *be* $T_1 \cap T_2$. *If* $A \succ B$ *on* T_1, *then* $A \succ B$ *on* T_2.

(K_6) *Let* A, B, $A + C$ *and* $B + C$ *to be elements of* T_1. *If* $A \succeq B$, *then* $A + C \succeq B + C$.

(K_6') *Let* A, B, $A + C$, $B + C$ *and* $C \neq \emptyset$ *to be elements of* T_1. *If* $A \succeq B$, *then* $A + C \succ B + C$.

(K_7) *Let* A, B, AC, BC *and* $C \succeq 0$ *to be elements of* T_1. *If* $A \succeq B$, *then* $AC \succeq BC$.

Proof. (K_1) According to Definition 3.10, $\text{Deg}(A \preceq A) = \frac{1}{2}$. Then, from Definition 3.15 and Remark 3.17, it follows that $A \preceq A$.

(K_2) $A \succeq B$, then by Remark 3.17, it follows that $A > B$ or $A \approx B$. Therefore, $\text{Deg}(A \succeq B) = 1$ or $\text{Deg}(A \succeq B) = \frac{1}{2}$. Similarly, if $B \succeq A$, then by Remark 3.17, it follows that $B > A$ or $B \approx A$. Therefore, $\text{Deg}(B \succeq A) = 1$ or $\text{Deg}(B \succeq A) = \frac{1}{2}$. Thus, $\text{Deg}(A \succeq B) = \text{Deg}(B \succeq A) = \frac{1}{2}$. Therefore, by Definition 3.15, it follows that $A \approx B$.

(K_3) If $A \succeq B$, then $\text{Deg}(A \succeq B) > \frac{1}{2}$. Meanwhile

$$\frac{\int_{-\infty}^{+\infty} \{\text{Deg}(B \leq x)\mu_A(x) + \text{Deg}(x \leq A)\mu_B(x)\}\, dx}{\int_{-\infty}^{+\infty} \{\mu_A(x) + \mu_B(x)\}\, dx} > \frac{1}{2}. \tag{8}$$

If $B \succeq C$, then $\text{Deg}(B \succeq C) > \frac{1}{2}$. Meanwhile

$$\frac{\int_{-\infty}^{+\infty} \{\text{Deg}(C \leq x)\mu_B(x) + \text{Deg}(x \leq B)\mu_C(x)\}\, dx}{\int_{-\infty}^{+\infty} \{\mu_B(x) + \mu_C(x)\}\, dx} > \frac{1}{2}. \tag{9}$$

Then, from the sum of inequalities 8 and 9, it follows

$$\int_{-\infty}^{+\infty} \{\mu_A(x) + 2\mu_B(x) + \mu_C(x)\}\, dx < 2\int_{-\infty}^{+\infty} \text{Deg}(B \leq x)\mu_A(x) + \text{Deg}(x \leq A)\mu_B(x) + \text{Deg}(C \leq x)\mu_B(x) + \text{Deg}(x \leq B)\mu_C(x) dx.$$

It is enough to show that $\text{Deg}(x \leq A) + \text{Deg}(C \leq x) \leq 1$. Suppose such a condition is not true (reductio ad absurdum) $\text{Deg}(x \leq A) + \text{Deg}(C \leq x) > 1$.

Based on Proposition 3.12

$$1 = \text{Deg}(x \leq C) + \text{Deg}(C \leq x) > \text{Deg}(x \leq A) + \text{Deg}(C \leq x) > 1.$$

That is a contradiction.

(K_4) $\inf supp(A) > \sup supp(B)$, then with Definition 3.1 for all $x \in supp(A); \text{Deg}(x > B) = 1$ and for all $x \in supp(B); \text{Deg}(A > x) = 1$. Therefore,

$$\text{Deg}(B \leq A) = \frac{\int_{-\infty}^{+\infty} \{\text{Deg}(B \leq x)\mu_A(x) + \text{Deg}(x \leq A)\mu_B(x)\}\, dx}{\int_{-\infty}^{+\infty} \{\mu_B(x) + \mu_A(x)\}\, dx} = \frac{\int_{-\infty}^{+\infty} \{\mu_A(x) + \mu_B(x)\}\, dx}{\int_{-\infty}^{+\infty} \{\mu_B(x) + \mu_A(x)\}\, dx} = 1.$$

(K_4') This proof is similar to K_4.

(K_5) This method of ranking GFNs is a pairwise comparison method. So, by interchanging the comparison space, the ranking order will not change.

(K_6 & K_6') Proof is established by the Proposition 3.14.

(K_7) Proof is established by the Proposition 3.13.

□

Using the Proposition 3.13, the following generalization is obvious from K_7.

Notation 3.19. *Let A, B, AC, BC and $C \prec 0$ to be elements of T_1. If $A \succeq B$, then $AC \preceq BC$.*

4 Demonstration of the proposed method

In this section, some numerical examples are presented to demonstrated the proposed ranking method.

Example 4.1. *Consider the fuzzy number $A = (0.5, 0.7, 0.9)$ and the real number $x = 0.8$ as shown in Figure 4 . Then Deg$(A \leq x) = 0.916667$. Therefore, $A \preceq x$, with accuracy of comparison 0.916667 (or $A \succeq x$, with accuracy of comparison 0.0833333).*

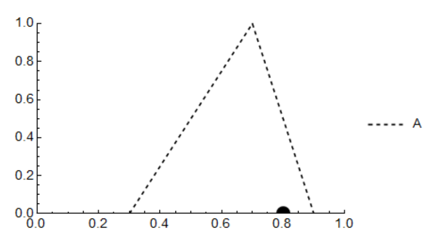


Figure 4: Figure of the fuzzy number $A = (0.5, 0.7, 0.9)$ and the real number $x = 0.8$.

Example 4.2. *Consider the GFN $A_{0.7} = (0.2, 0.4, 0.8, 1.0)$ and the real number $x = 0.3$ as shown in Figure 5 . Then, Deg$(A \leq x) = 0.0416667$. Therefore, $x \preceq A$, with accuracy of comparison 0.958333 (or $x \succeq A$, with accuracy of comparison 0.0416667).*

Example 4.3. *Consider the GFN $A_{0.3} = (0.3, 0.7, 0.9)$ and the GFN $B_{0.8} = (0.3, 0.4, 0.7, 0.9)$ as shown in Figure 6. Then, Deg$(A \leq B) = 0.381944$; therefore, $A \succeq B$, with accuracy of comparison 0.618056 (or $A \preceq B$, with accuracy of comparison 0.381944).*

Example 4.4. *Consider the following fuzzy numbers as shown in Figure 7* [58]:

Set 1: $A = (0.4, 0.5, 1)$, $B = (0.4, 0.7, 1)$, $C = (0.4, 0.9, 1)$.

Set 2: $A = (0.3, 0.4, 0.7, 0.9)$, $B = (0.3, 0.7, 0.9)$, $C = (0.5, 0.7, 0.9)$.

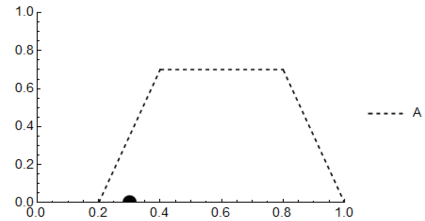


Figure 5: The GFN $A_{0.7} = (0.2, 0.4, 0.8, 1.0)$ and the real number $x = 0.3$.

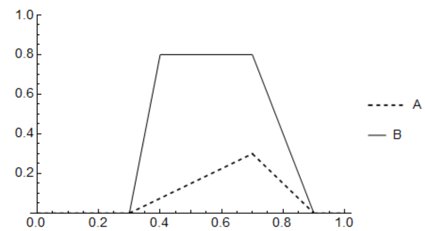


Figure 6: The GFN $A_{0.3} = (0.3, 0.7, 0.9)$ and the GFN $B_{0.8} = (0.3, 0.4, 0.7, 0.9)$.

Table 1: A comparison of the proposed method and other existing methods in the Example 4.4.

Methods	Set 1	Set 2	Set 3	Set 4
Abbasbandy & Asady [1]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$
Asady & Zendehnam [9]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$
Cheng distance [14]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Cheng CV uniform [14]	$A \prec C \prec B$	$A \prec B \prec C$	$B \prec C \prec A$	$A \prec C \prec B$
Chu & Tsao [17]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Deng et al. [22]	$A \prec B \prec C$	$B \prec C \prec A$	$A \prec C \prec B$	$C \prec B \prec A$
Nejad & Mashinchi [40]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Wang et al. [56]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Yager [57]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
de Hierro et al. [34]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Proposed method	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$

Set 3: $A = (0.3, 0.5, 0.7)$, $B = (0.3, 0.5, 0.8, 0.9)$, $C = (0.3, 0.5, 0.9)$.

Set 4: $A = (0, 0.4, 0.7, 0.8)$, $B = (0.2, 0.5, 0.9)$, $C = (0.1, 0.6, 0.8)$.

A comparative study of these sets with various existing method and the proposed method is presented in Table 1.

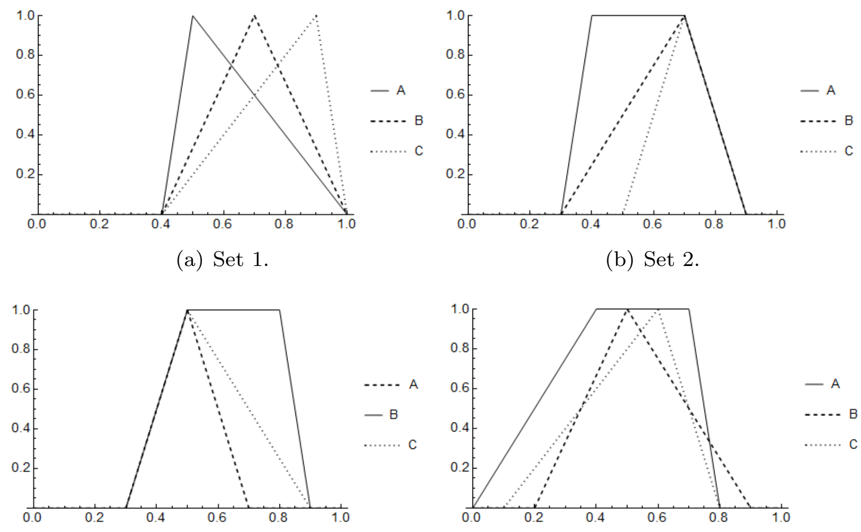
Example 4.5. *Consider the fuzzy numbers* $A = (0.4, 0.5, 1)$ *and the following generalized LR fuzzy number as shown in Figure 8:*

$$\mu_B(x) = \begin{cases} \sqrt{1-(x-2)^2}, & \text{if } 1 \leq x \leq 2, \\ \sqrt{1-\frac{1}{4}(x-2)^2}, & \text{if } 2 < x \leq 4, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Table 2, depicts a comparison of the proposed method and other existing method.

5 Conclusion

In this paper, an approach for ranking a crisp real number and a GFN was proposed with respect to the membership function or based on the accuracy of comparison on accuracy at a larger (smaller) number. Then, this method is generalised to rank two GFNs with comparable accuracy on a smaller or larger basis. In fact, to compare two GFNs A and B, all values in $supp(A)$ with the membership degree were compared with B and vice versa. In this method, a



(a) Set 1. (b) Set 2.

(c) Set 3. (d) Set 4.

Figure 7: Fuzzy numbers of the Example 4.4.

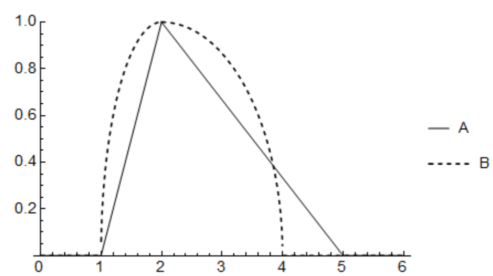


Figure 8: Figure of the fuzzy number $A = (1, 2, 5)$ and the generalized LR fuzzy number in Example 4.5.

Table 2: A comparison of the method and other existing methods of the Example 4.5.

Methods	Ranking
Asady & Zendehnam [9]	$B \prec A$
Cheng distance [14]	$B \prec A$
Chu & Tsao [17]	$B \prec A$
Deng et al. [22]	$B \prec A$
Nejad & Mashinchi [40]	$B \prec A$
Wang et al. [56]	$B \prec A$
de Hierro et al. [34]	$B \prec A$
Proposed method	$B \prec A$

continuous function with accuracy of comparison has been utilized. Moreover, some useful properties enjoyed by the proposed method has been discussed thoroughly. Further, some numerical examples has been presented in order to compare the proposed method of ranking with some existing approaches.

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