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Original paper

Two novel approaches that reduce the effectiveness of the decision maker in decision making under uncertainty environments

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Abstract

Unlike other mathematical models, soft set theory provides a parameterization tool contribution. However, in this theory, since membership degrees are expressed as 0 and 1, for $(0, 1)$, we cannot determine whether any object belongs to a parameter or not. Researchers have tried to overcome this situation by ensuring that the decision maker expresses these values. However, we cannot know the accuracy of the data provided to us by the decision maker. Therefore, in this study, we introduced the concepts of relational membership function, relational non-membership function, inverse relational membership function and inverse relational non-membership function and examined the related properties of these concepts. Then, we propose two new approaches so that uncertainty can be expressed in an ideal way and can be used in decision-making. Finally, the approaches given and some of the important approaches in the literature are compared and analyzed.

Keywords: Soft set, inverse soft set, algorithm, decision making.

1 Introduction

One of the most important properties to be addressed during a data analysis is uncertainty. To decompose the uncertainty of data, which is a very important issue, is not generally an easy task. For this reason, many researchers have introduced a wide variety of mathematical models to the literature in order to overcome this problem. Especially these days, mathematical modeling developed in this field has become an increasingly important issue in various research areas [23, 28, 31, 32, 35, 40, 44, 45].

The first mathematical model proposed to express uncertain situations was the fuzzy set (briefly FS) theory put forward by Zadeh [43]. With this set theory, we can express the membership degree of an element in $[0, 1]$. However, the intuitionistic fuzzy set (briefly IFS) theory, which enables us to express the non-membership degree and thus obtain the indeterminacy degree, was brought to the literature by Atanassov [4, 5]. This mathematical model is a generalization of FSs, and we can understand that it is a very successful model in expressing uncertainty environments from its presence in many studies today [21, 30, 36, 42]. However, the soft set (briefly SS) theory proposed by Molodsov [37] in 1999 introduced a very useful parameterization tool to the literature for evaluating uncertainty environments, thus eliminating a major deficiency of the mathematical models proposed before this theory. Molodtsov initiated a novel concept of SS theory which is a completely new approach for modeling uncertainities and succesfully applied it into several directions such as smoothness of functions, game theory, Riemann Integration, theory of measurement , and so on. The application area and diversity of the SS theory brought to the literature by Molodsov is rapidly increasing due to its success in expressing uncertainty. For example, Maji et al. [33, 34] first defined concepts of equality, subset and complement for SSs, and gave definitions of concepts of null SS and absolute SS. They also gave an application of SSs to the decision making problem. Chen et al. [6] discussed a parameter reduction method based on SSs. Kong et al. [29] introduced the notion of normal parameter reduction of SSs. Ali et al. [1] defined some new operations between

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SSs. Qin and Hong [39] defined concept of soft equality and derived some related properties. Agman and Enginoglu [8] redefined operations between two SSs and studied products of SSs. They also presented a decision making method known as uni-int decision-making method. Besides these, the SS theory expressed as a mapping from the parameter set to the universe set has been redesigned as a mapping from the universe set to the parameter set and the inverse soft set (ISS) theory [11] has been brought to the literature.

Today, it is seen that the mathematical modeling built with the help of SS theory continues to develop rapidly with each passing day [12, 13, 14, 18, 19, 20, 22, 26]. However, this theory also has some shortcomings. One of them is that it limits the membership degree of an element to 0 and 1, which prevents the uncertainty problems encountered from being expressed correctly. In order to overcome this problem, the researchers have considered to handle some set models such as FSs and IFSs together with the SS and introduced many hybrid set models originating from SSs [7, 9, 10, 15, 17, 27]. Then, they stated that the decision maker should express the membership degree and non-membership degree in all these hybrid set models. However, it is a very difficult task for the decision maker to determine a value in $[0, 1]$. On the other hand, even if a solution for uncertainty is obtained by accepting the determined value as correct, it is certain that a highly suspicious approach will be shown regarding the accuracy of this solution. We cannot ignore the influence of the decision maker in order to express the uncertainty in the most accurate way, i.e., it is a very important problem to minimize an error caused by the decision maker. The purpose of this study is that how to determine the membership degree and non-membership degree of an element by keeping the decision maker quite passive. For this purpose, in this study, we introduced the concepts of relational membership function, relational non-membership function,inverse relational membership function and inverse relational non-membership function. Thanks to these concepts, we can determine the membership degree and non-membership degree of the elements of the given SS or ISS. Then, some properties related to these concepts have been studied.

The presentation of the rest of this paper is structured as follows. In the second section, the framework of FS, IFS, SS and ISS are introduced. In the third section, some new technical formulations for decision-making under uncertainty environments are given and some related properties are analyzed. In the fourth section, using the formulations given in the third chapter, two different approaches are proposed to overcome uncertainty and how these approaches can be applied in an uncertainty environment are exemplified. In the fifth chapter, a detailed analysis has been made by comparing the approaches given in our study with some important related approaches in the literature. The final section consists of the conclusion of the study.

2 Preliminaries

This section was created to examine and remind some of the set definitions required for this study. Detailed explanations on the set theories stated can be found in [5, 11, 24, 34, 37, 43].

Throughout this paper, $\mathcal{U} = \{u_1, u_2, ..., u_n\}$ is an initial universe, $\mathcal{P} = \{p_1, p_2, ..., p_m\}$ is a set of parameters and $2^{\mathcal{U}[\mathcal{P}]}$ is the power set of $\mathcal{U}[\mathcal{P}]$.

Definition 2.1. [43] *A FS $\mathcal{F}$ over $\mathcal{U}$ is a set defined by $\mu_{\mathcal{F}} : \mathcal{U} \to [0, 1]$. $\mu_{\mathcal{F}}$ is called the membership function of $\mathcal{F}$, and the value $\mu_{\mathcal{F}}(u)$ is called the grade of membership of $u \in \mathcal{U}$. The value represents the degree of u belonging to the FS $\mathcal{F}$. Thus, an FS $\mathcal{F}$ over $\mathcal{U}$ can be represented by, $\mathcal{F} = \{< \mu_{\mathcal{F}}(u)/u >: u \in \mathcal{U}\}$.*

Definition 2.2. [5] *A IFS $\mathcal{X}$ on $\mathcal{U}$ can be defined by $\mu_{\mathcal{X}} : \mathcal{U} \to [0, 1]$ and $\nu_{\mathcal{X}} : \mathcal{U} \to [0, 1]$ such that $0 \leq \mu_{\mathcal{X}} + \nu_{\mathcal{X}} \leq 1$; $\forall u \in \mathcal{U}$. Here $\mu_{\mathcal{X}}(u)$ and $\nu_{\mathcal{X}}(u)$ are the membership degree and non-membership degree of the element u, respectively. Thus, an IFS $\mathcal{X}$ over $\mathcal{U}$ can be represented by, $\mathcal{X} = \{< u, \mu_{\mathcal{X}}(u), \nu_{\mathcal{X}}(u) >: u \in U\}$. Clearly, when $\nu_{\mathcal{X}}(u) = 1 - \mu_{\mathcal{X}}(u)$; $u \in \mathcal{U}$, the $\mathcal{X}$ becomes an FS.*

Unlike the case of FSs, an indeterminacy degree of u to $\mathcal{X}$ can be defined. It consists of the difference $\pi_{\mathcal{X}}(u) = 1 - (\mu_{\mathcal{X}}(u) + \nu_{\mathcal{X}}(u))$, and if $\pi_{\mathcal{X}}(u) = 0$, i.e. $\mu_{\mathcal{X}}(u) + \nu_{\mathcal{X}}(u) = 1$ for all $u \in \mathcal{U}$, then $\mathcal{X}$ can be identified with a FS.

State that the set of all the IFSs over $\mathcal{U}$ will be denoted by $IFS(\mathcal{U})$.

Example 2.3. *Let $\mathcal{U} = \{u_1 : black, u_2 : orange, u_3 : grey, u_4 : white, u_5 : dark\ blue\}$ be a set of colors. Then the IFS $\mathcal{X}$ on $\mathcal{U}$ can be written as:*

$$\mathcal{X} = \{< u_1, 0.5, 0.3 >, < u_2, 0.25, 0.5 >, < u_3, 0.7, 0.3 >, < u_4, 0, 0 >, < u_5, 0.45, 0.3 >\},$$

or

$$\mathcal{X} = \{< u_1, 0.5, 0.3 >, < u_2, 0.25, 0.5 >, < u_3, 0.7, 0.3 >, < u_5, 0.45, 0.3 >\},$$

To give an example in order to understand the situation here better; the membership degree of the color orange to $\mathcal{X}$ is 0.25, while the non-membership degree is 0.5. Similarly, for white color, since $\mu_{\mathcal{X}}(u_4) + \nu_{\mathcal{X}}(u_4) = 0 + 0 = 0$, then $\pi_{\mathcal{X}}(u_4) = 1$.

Definition 2.4. [11, 37] *(i) A pair* $(\Phi, \mathcal{P})$ *is called an SS over* $\mathcal{U}$ *if and only if* Φ *is a mapping given by* $\Phi : \mathcal{P} \to 2^{\mathcal{U}}$*. Thus an SS* $(\Phi, \mathcal{P})$ *over* $\mathcal{U}$ *can be represented by the set of ordered pairs,*

$$(\Phi, \mathcal{P}) = \{(p, \Phi(p)) : p \in \mathcal{P}, \Phi(p) \in 2^{\mathcal{U}}\}.$$

(ii) A pair $(\psi,\mathcal{U})$ *is called an ISS over* $\mathcal{P}$*, where* ψ *is a mapping given by* $\psi : \mathcal{U} \to 2^{\mathcal{P}}$*.Thus an ISS* $(\psi,\mathcal{U})$ *over* $\mathcal{P}$ *can be represented by the set of ordered pairs,*

$$(\psi,\mathcal{U}) = \{(u, \psi(u)) : u \in \mathcal{U}, \psi(u) \in 2^{\mathcal{P}}\}.$$

State that the set of all SSs [ISSs] over $\mathcal{U}$ [$\mathcal{P}$] will be denoted by $SS(\mathcal{U})$ [$ISS(\mathcal{P})$].

Remark 2.5. [11] *Each SS can be uniquely represented as an ISS.*

Example 2.6. *Let* $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5\}$ *be the set of five students under consideration and* $\mathcal{P} = \{p_1 :$ *successful,* $p_2 :$ *honest,* $p_3 :$ *determined,* $p_4 :$ *willing,* $p_5 :$ *hardworking*$\}$ *be the set of parameters. Then*
(i) By Definition 2.4-(i) we can describe SSs as $\Phi(p_1) = \{u_1, u_3, u_4\}$*,* $\Phi(p_2) = \{u_2, u_3, u_5\}$*,* $\Phi(p_3) = \{u_2, u_3, u_4, u_5\}$*,* $\Phi(p_4) = \{u_1, u_3, u_4\}$ *and* $\Phi(p_5) = \{u_2, u_4, u_5\}$*. Therefore, we obtain an SS*

$$(\Phi, \mathcal{P}) = \left\{ \begin{array}{c} (p_1, \{\langle u_1, 1, 0\rangle, \langle u_3, 1, 0\rangle, \langle u_4, 1, 0\rangle\}), \\ (p_2, \{\langle u_2, 1, 0\rangle, \langle u_3, 1, 0\rangle, \langle u_5, 1, 0\rangle\}), \\ (p_3, \{\langle u_2, 1, 0\rangle, \langle u_3, 1, 0\rangle, \langle u_4, 1, 0\rangle, \langle u_5, 1, 0\rangle\}) \\ (p_4, \{\langle u_1, 1, 0\rangle, \langle u_3, 1, 0\rangle, \langle u_4, 1, 0\rangle, \}), \\ (p_5, \{\langle u_2, 1, 0\rangle, \langle u_4, 1, 0\rangle, \langle u_5, 1, 0\rangle\}) \end{array} \right\},$$

or

$$(\Phi, \mathcal{P}) = \left\{ \begin{array}{c} (p_1, \{u_1, u_3, u_4\}), (p_2, \{u_2, u_3, u_5\}), (p_3, \{u_2, u_3, u_4, u_5\}) \\ (p_4, \{u_1, u_3, u_4\}), (p_5, \{u_2, u_4, u_5\}) \end{array} \right\},$$

which describes the "characteristics of the students".
(ii) By Definition 2.4-(ii) we can describe the ISSs as $\psi(u_1) = \{p_1, p_4\}$*,* $\psi(u_2) = \{p_2, p_3, p_5\}$*,* $\psi(u_3) = \{p_1, p_2, p_3, u_4\}$*,* $\psi(u_4) = \{p_1, p_3, p_4, p_5\}$ *and* $\psi(u_5) = \{p_2, p_3, p_5\}$*. Therefore, we obtain an ISS*

$$(\psi,\mathcal{U}) = \left\{ \begin{array}{c} (u_1, \{\langle p_1, 1, 0\rangle, \langle p_4, 1, 0\rangle\}), \\ (u_2, \{\langle p_2, 1, 0\rangle, \langle p_3, 1, 0\rangle, \langle p_5, 1, 0\rangle\}), \\ (u_3, \{\langle p_1, 1, 0\rangle, \langle p_2, 1, 0\rangle, \langle p_3, 1, 0\rangle, \langle p_4, 1, 0\rangle\}) \\ (u_4, \{\langle p_1, 1, 0\rangle, \langle p_3, 1, 0\rangle, \langle p_4, 1, 0\rangle, \langle p_5, 1, 0\rangle\}), \\ (u_5, \{\langle p_2, 1, 0\rangle, \langle p_3, 1, 0\rangle, \langle p_5, 1, 0\rangle\}) \end{array} \right\},$$

or

$$(\psi,\mathcal{U}) = \left\{ \begin{array}{c} (u_1, \{p_1, p_4\}), (u_2, \{p_2, p_3, p_5\}), (u_3, \{p_1, p_2, p_3, u_4\}) \\ (u_4, \{p_1, p_3, p_4, p_5\}), (u_5, \{p_2, p_3, p_5\}) \end{array} \right\},$$

which describes the "characteristics of the students".

Remark 2.7. *Let* $\mathcal{X}$ *be an IFS and take* $\mathcal{P} = [0, 1]$ *as the parameter set. By using the mapping* $\mu_{\mathcal{X}} : \mathcal{U} \to [0, 1]$ *and* $\nu_{\mathcal{X}} : \mathcal{U} \to [0, 1]$ *we construct a unique ISS* $\psi : \mathcal{U} \to 2^{\mathcal{P}}$*, by* $\psi(u) = [0, \mu_{\mathcal{X}}(u)) \cap [0, \nu_{\mathcal{X}}(u))$*. Thus, each IFS generates an ISS.*

3 New technical formulations for decision-making under uncertainty environments

Thanks to the formulations given in this section, we can express whether the elements that the decision maker cannot clearly determine in the range $(0, 1)$ belong to the set or not by making use of data that expresses whether only one element belongs to a set or not. Also some important properties are given for the proposed new technical formulations.

Unlike other theories, soft set theory makes use of a very useful parameterization tool in expressing uncertainty environments. With this tool, elements in the universe set corresponding to a parameter can be mapping. However, the fact that the objects corresponding to the parameter belong to the parameter with the membership degree 1 and the objects that do not correspond to that parameter belong to that parameter with a membership degree 0 makes it difficult to express the uncertainty in a way close to the ideal. In addition to these, these cases can be considered similarly for the inverse soft set theory, which is proposed by replacing the universe set and the parameter set. Different theories have been proposed to overcome this problem [7, 9, 19, 25, 27, 38, 41]. However, all of these theories are based on the assumption that the membership degrees of the elements are determined or given. In other words, all of these theories focused on the decision maker and the decision maker expected to express these membership values in the range $[0,1]$. In addition, it is difficult for the decision maker to express a membership or non-membership degree in this range. Moreover, there is a problem of trust in these values expressed by the decision maker. If we want to overcome uncertainty in the most ideal way, we need to minimize the mistakes of the decision maker.

In this section, some new formulations are introduced to overcome this problem. First, for SSs, the concepts of Relational Membership Function and Relational non-Membership Function are defined as follows:

Definition 3.1. *Let $(\Phi,\mathcal{P})\in SS(\mathcal{U})$. For $u_k\in\mathcal{U}\backslash\Phi(p_i)$ and $u_j\in\Phi(p_i)$, the relational membership degree and relational non-membership degree of u_k to $\Phi(p_i)$ is expressed with the help of mapping given by $\overset{(\Phi,\mathcal{P})}{\Theta_\mu},\ \overset{(\Phi,\mathcal{P})}{\Theta_\nu}:\mathcal{P}^{\mathcal{U}\backslash\Phi(p_i)}\to[0,1)$, respectively. For $1\le k,j\le n$, $1\le i\le m$ and $n,m\ge 2$; firstly, the Relational Membership Function is given as follows:*

$$\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_i^{u_k})=\frac{1}{(n-1)(m-1)}\sum_{u_j\in\Phi(p_i)}\sum_{p\in\mathcal{P}}\overset{(\Phi,\mathcal{P})}{[\Upsilon_p]}_\mu(u_k,u_j),\tag{1}$$

where

$$\overset{(\Phi,\mathcal{P})}{[\Upsilon_p]}_\mu(u_k,u_j)=\begin{cases}1, & \mu_{\Phi(p)}(u_k)+\mu_{\Phi(p)}(u_j)=2\\ 0, & otherwise\end{cases},\quad\forall p\in\mathcal{P},$$

is a mapping given by $\overset{(\Phi,\mathcal{P})}{[\Upsilon_p]}_\mu:[\mathcal{U}\setminus\Phi(p_i)]\times\Phi(p_i)\to\{0,1\}$. Next, the Relational non-Membership Function is given as follows:

$$\overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_i^{u_k})=\frac{1}{(n-1)(m-1)}\sum_{u_j\in\Phi(p_i)}\sum_{p\in\mathcal{P}}\overset{(\Phi,\mathcal{P})}{[\Upsilon_p]}_\nu(u_k,u_j),\tag{2}$$

where

$$\overset{(\Phi,\mathcal{P})}{[\Upsilon_p]}_\nu(u_k,u_j)=\begin{cases}1, & \mu_{\Phi(p)}(u_k)=0\ and\ \mu_{\Phi(p)}(u_j)=1\\ 0, & otherwise\end{cases},\quad\forall p\in\mathcal{P}-\{p_i\},$$

is a mapping given by $\overset{(\Phi,\mathcal{P})}{[\Upsilon_p]}_\nu:[\mathcal{U}\setminus\Phi(p_i)]\times\Phi(p_i)\to\{0,1\}$. Here $\mu_{\Phi(p)}$ is the membership function of Φ.

Example 3.2. *Let $\mathcal{U}=\{u_1,u_2,u_3,u_{4=n}\}$ and $\mathcal{P}=\{p_1,p_2,p_3,p_4,p_{5=m}\}$. If $\Phi(p_1)=\{u_1,u_2,u_4\}$, $\Phi(p_2)=\{u_1,u_3,u_4\}$, $\Phi(p_3)=\{u_2,u_3\}$, $\Phi(p_4)=\{u_1,u_4\}$, $\Phi(p_5)=\{u_3,u_4\}$, then the SS $(\Phi,\mathcal{P})$ is written by*

$$(\Phi,\mathcal{P})=\left\{\begin{array}{c}(p_1,\{u_1,u_2,u_4\}),(p_2,\{u_1,u_3,u_4\})\\(p_3,\{u_2,u_3\}),(p_4,\{u_1,u_4\}),(p_5,\{u_3,u_4\})\end{array}\right\}.$$

Now let's calculate all relational membership degrees and relational non-membership degrees for SS $(\Phi,\mathcal{P})$,

For** $p_{1=i}$**:

Since $u_{3=k}\in[\mathcal{U}\setminus\Phi(p_{1=i})]$, $\overset{(\Phi,\mathcal{P})}{[\Upsilon_{p_1}]}_\mu(u_3,u_j)=0$ and $\overset{(\Phi,\mathcal{P})}{[\Upsilon_{p_1}]}_\nu(u_3,u_j)=0$ for $u_j\in\Phi(p_1)$, $p\in\mathcal{P}-\{p_i\}$. Therefore we have

$$\begin{aligned}\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_1^{u_3}) &= \frac{1}{3.4}\left[\begin{array}{c}\overset{(\Phi,\mathcal{P})}{[\Upsilon_{p_2}]}_\mu(u_3,u_j)+\overset{(\Phi,\mathcal{P})}{[\Upsilon_{p_3}]}_\mu(u_3,u_j)+\\ \overset{(\Phi,\mathcal{P})}{[\Upsilon_{p_4}]}_\mu(u_3,u_j)+\overset{(\Phi,\mathcal{P})}{[\Upsilon_{p_5}]}_\mu(u_3,u_j)\end{array}\right],\quad\forall u_j\in\Phi(p_1)\\ &= \frac{[(1+0+1)+(0+1+0)+(0+0+0)+(0+0+1)]}{12}=1/3,\end{aligned}$$

$$\begin{aligned}\overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_1^{u_3}) &= \frac{1}{3.4}\left[\begin{array}{c}\overset{(\Phi,\mathcal{P})}{[\Upsilon_{p_2}]}_\nu(u_3,u_j)+\overset{(\Phi,\mathcal{P})}{[\Upsilon_{p_3}]}_\nu(u_3,u_j)+\\ \overset{(\Phi,\mathcal{P})}{[\Upsilon_{p_4}]}_\nu(u_3,u_j)+\overset{(\Phi,\mathcal{P})}{[\Upsilon_{p_5}]}_\nu(u_3,u_j)\end{array}\right],\quad\forall u_j\in\Phi(p_1)\\ &= \frac{[(0+0+0)+(0+0+0)+(1+0+1)+(0+0+0)]}{12}=1/6.\end{aligned}$$

Hence; since $\pi_{\Phi(p_1)}(u_3) = 1 - \big(\mu_{\Phi(p_1)}(u_3) + \nu_{\Phi(p_1)}(u_3)\big) = 1 - (1/3 + 1/6) = 1/2$*, then the indeterminancy degree of* u_3 *to* $\Phi(p_1)$ *in* $(\Phi, \mathcal{P})$ *is* $1/2$*. Here;* $\pi_{\Phi(p)}$*,* $\mu_{\Phi(p)}$ *and* $\nu_{\Phi(p)}$ *are the intedeterminancy degree, membership degree and non-membership degree of* $\Phi(p)$ *for* $p \in \mathcal{P}$*, respecitvely. Similarly,*

For p_2***:*** *Since* $u_2 \in [\mathcal{U} \setminus \Phi(p_2)]$*, then* $\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_2^{u_2}) = 1/4$*,* $\overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_2^{u_2}) = 1/3$*. Thus* $\pi_{\Phi(p_2)}(u_2) = 5/12$*.*

For p_3***:*** *Since* $u_1, u_4 \in [\mathcal{U} \setminus \Phi(p_3)]$*, then* $\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_3^{u_1}) = 1/6$*,* $\overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_3^{u_1}) = 1/12$ *and* $\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_3^{u_4}) = 1/4$*,* $\overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_3^{u_4}) = 0$*. Thus* $\pi_{\Phi(p_3)}(u_1) = 3/4$ *and* $\pi_{\Phi(p_3)}(u_4) = 3/4$*.*

For p_4***:*** *Since* $u_2, u_3 \in [\mathcal{U} \setminus \Phi(p_4)]$*, then* $\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_4^{u_2}) = 1/6$*,* $\overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_4^{u_2}) = 1/4$ *and* $\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_4^{u_3}) = 1/4$*,* $\overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_4^{u_3}) = 1/6$*. Thus* $\pi_{\Phi(p_4)}(u_2) = 7/12$ *and* $\pi_{\Phi(p_4)}(u_3) = 7/12$*.*

For p_5***:*** *Since* $u_1, u_2 \in [\mathcal{U} \setminus \Phi(p_5)]$*, then* $\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_5^{u_1}) = 1/3$*,* $\overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_5^{u_1}) = 1/12$ *and* $\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_5^{u_2}) = 1/6$*,* $\overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_5^{u_2}) = 1/4$*. Thus* $\pi_{\Phi(p_5)}(u_1) = 7/12$ *and* $\pi_{\Phi(p_5)}(u_2) = 7/12$*.*

Additionally, we can write by matrix form as

$$(\Phi, \mathcal{P}) = \left\{ \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline p_1 & \langle 1,0\rangle & \langle 1,0\rangle & \langle 1/3,1/6\rangle & \langle 1,0\rangle \\ p_2 & \langle 1,0\rangle & \langle 1/4,1/3\rangle & \langle 1,0\rangle & \langle 1,0\rangle \\ p_3 & \langle 1/6,1/12\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 1/4,0\rangle \\ p_4 & \langle 1,0\rangle & \langle 1/6,1/4\rangle & \langle 1/4,1/6\rangle & \langle 1,0\rangle \\ p_5 & \langle 1/3,1/12\rangle & \langle 1/6,1/4\rangle & \langle 1,0\rangle & \langle 1,0\rangle \end{array} \right\}.$$

Now, for ISSs, the concepts of Inverse Relational Membership Function and Inverse Relational non-Membership Function are defined as follows:

Definition 3.3. *Let* $(\psi, \mathcal{U}) \in ISS(\mathcal{P})$*. For* $p_t \in \mathcal{P} \setminus \psi(u_j)$ *and* $p_i \in \psi(u_j)$*, the inverse relational membership degree and inverse relational non-membership degree of* p_t *to* $\psi(u_j)$ *is expressed by the help of mapping given by* $\overset{(\psi,\mathcal{U})}{\Xi_\mu}, \overset{(\psi,\mathcal{U})}{\Xi_\nu} : \mathcal{U}^{\mathcal{P}\setminus\psi(u_j)} \to [0,1)$*, respectively. For* $1 \le j \le n$*,* $1 \le i, t \le m$ *and* $n, m \ge 2$*; firstly, the Inverse Relational Membership Function is given as follows:*

$$\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_j^{p_t}) = \frac{1}{(m-1)(n-1)} \sum_{p_i \in \psi(u_j)} \sum_{u \in \mathcal{U}} \overset{(\psi,\mathcal{U})}{[\Upsilon_u]_\mu}(p_t, p_i), \tag{3}$$

where

$$\overset{(\psi,\mathcal{U})}{[\Upsilon_u]_\mu}(p_t, p_i) = \left\{ \begin{array}{ll} 1, & \mu_{\psi(u)}(p_t) + \mu_{\psi(u)}(p_i) = 2 \\ 0, & \textit{otherwise} \end{array} \right. , \quad \forall u \in \mathcal{U},$$

is a mapping given by $\overset{(\psi,\mathcal{U})}{[\Upsilon_u]_\mu} : [\mathcal{P} \setminus \psi(u_j)] \times \psi(u_j) \to \{0,1\}$*. Next, the Inverse Relational non-Membership Function is given as follows:*

$$\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_j^{p_t}) = \frac{1}{(m-1)(n-1)} \sum_{p_i \in \psi(u_j)} \sum_{u \in \mathcal{U}} \overset{(\psi,\mathcal{U})}{[\Upsilon_u]_\nu}(p_t, p_i), \tag{4}$$

where

$$\overset{(\psi,\mathcal{U})}{[\Upsilon_u]_\nu}(p_t, p_i) = \left\{ \begin{array}{ll} 1, & \mu_{\psi(u)}(p_t) = 0 \textit{ and } \mu_{\psi(u)}(p_i) = 1 \\ 0, & \textit{otherwise} \end{array} \right. , \quad \forall u \in \mathcal{U} - \{u_j\},$$

is a mapping given by $\overset{(\psi,\mathcal{U})}{[\Upsilon_u]_\nu} : [\mathcal{P} \setminus \psi(u_j)] \times \psi(u_j) \to \{0,1\}$*. Here* $\mu_{\psi(u)}$ *is the membership function of* ψ*.*

Proposition 3.4. *Let* $(\Phi, \mathcal{P}) \in SS(\mathcal{U})$ *and* $(\psi, \mathcal{U}) \in ISS(\mathcal{P})$*. Then, we have* $\overset{(\Phi,\mathcal{P})}{[\Upsilon_p]_\mu}(u_k, u_j) = \overset{(\Phi,\mathcal{P})}{[\Upsilon_p]_\mu}(u_j, u_k)$ *and* $\overset{(\psi,\mathcal{U})}{[\Upsilon_u]_\mu}(p_t, p_i) = \overset{(\psi,\mathcal{U})}{[\Upsilon_u]_\mu}(p_i, p_t)$*;* $\forall p \in \mathcal{P}, u \in \mathcal{U}$*. However, the similar situation may not be valid for functions* $\overset{(\Phi,\mathcal{P})}{[\Upsilon_p]_\nu}$ *and* $\overset{(\psi,\mathcal{U})}{[\Upsilon_u]_\nu}$*.*

Proof. Straightforward. □

Example 3.5. *Consider Example 3.2; we have* $\psi(u_1)=\{p_1,p_2,p_4\}$, $\psi(u_2)=\{p_1,p_3\}$, $\psi(u_3)=\{p_2,p_3,p_5\}$, $\psi(u_4)=\{p_1,p_2,p_4,p_5\}$. *Therefore ISS* $(\psi,\mathcal{U})$ *is written by*

$$(\psi,\mathcal{U})=\left\{\begin{array}{c}(u_1,\{p_1,p_2,p_4\}),(u_2,\{p_1,p_3\})\\(u_3,\{p_2,p_3,p_5\}),(u_4,\{p_1,p_2,p_4,p_5\})\end{array}\right\}.$$

Now let's calculate all inverse relational membership degrees and inverse relational non-membership degrees for ISS $(\psi,\mathcal{U})$,
For $u_{1=j}$***:***
Since $p_{3=t}\in[\mathcal{P}\setminus\psi(u_{1=j})]$, $\overset{(\psi,\mathcal{U})}{[\Upsilon_{u_1}]_\mu}(p_3,p_i)=0$ *and* $\overset{(\psi,\mathcal{U})}{[\Upsilon_{u_1}]_\nu}(p_3,p_i)=0$ *for* $p_i\in\psi(u_1)$, $u\in\mathcal{U}-\{u_j\}$. *Therefore, we have*

$$\begin{aligned}\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_1^{p_3}) &= \frac{1}{4.3}\left[\overset{(\psi,\mathcal{U})}{[\Upsilon_{u_2}]_\mu}(p_3,p_i)+\overset{(\psi,\mathcal{U})}{[\Upsilon_{u_3}]_\mu}(p_3,p_i)+\overset{(\psi,\mathcal{U})}{[\Upsilon_{u_4}]_\mu}(p_3,p_i)\right],\quad \forall p_i\in\psi(u_1)\\ &= \frac{[(1+0+0+0)+(0+1+0+0)+(0+0+0+0)]}{12}=1/6,\end{aligned}$$

and

$$\begin{aligned}\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_1^{p_3}) &= \frac{1}{4.3}\left[\overset{(\psi,\mathcal{U})}{[\Upsilon_{u_2}]_\nu}(p_3,p_i)+\overset{(\psi,\mathcal{U})}{[\Upsilon_{u_3}]_\nu}(p_3,p_i)+\overset{(\psi,\mathcal{U})}{[\Upsilon_{u_4}]_\nu}(p_3,p_i)\right],\quad \forall p_i\in\psi(u_1)\\ &= \frac{[(0+0+0+0)+(0+0+0+0)+(1+1+1+0)]}{12}=1/4.\end{aligned}$$

Hence; since $\pi_{\psi(u_1)}(p_3)=1-\big(\mu_{\Phi(u_1)}(p_3)+\nu_{\Phi(u_1)}(p_3)\big)=1-(1/6+1/4)=7/12$, *then the indeterminancy degree of* p_3 *to* $\psi(u_1)$ *in* $(\psi,\mathcal{U})$ *is* $7/12$. *Here;* $\pi_{\psi(u)}$, $\mu_{\psi(u)}$ *and* $\nu_{\psi(u)}$ *are the intederminancy degree, membership degree and non-membership degree of* $\psi(u)$ *for* $u\in\mathcal{U}$, *respecitvely. Similarly,*
For u_1***:*** *Since* $p_5\in[\mathcal{P}\setminus\psi(u_1)]$, *then* $\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_1^{p_5})=1/3$, $\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_1^{p_5})=1/12$. *Thus* $\pi_{\psi(u_1)}(p_5)=7/12$.
For u_2***:*** *Since* $p_2,p_4,p_5\in[\mathcal{P}\setminus\psi(u_2)]$, *then* "$\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_2^{p_2})=1/4$, $\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_2^{p_2})=0$", $\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_2^{p_4})=1/6$, $\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_2^{p_4})=1/12$ *and* "$\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_2^{p_5})=1/6$, $\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_2^{p_5})=1/12$". *Thus* $\pi_{\psi(u_2)}(p_2)=3/4$, $\pi_{\psi(u_2)}(p_4)=3/4$ *and* $\pi_{\psi(u_2)}(p_5)=3/4$.
For u_3***:*** *Since* $p_1,p_4\in[\mathcal{P}\setminus\psi(u_3)]$, *then* $\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_3^{p_1})=1/3$, $\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_3^{p_1})=0$ *and* $\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_3^{p_4})=1/4$, $\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_3^{p_4})=1/12$. *Thus* $\pi_{\psi(u_3)}(p_1)=2/3$ *and* $\pi_{\psi(u_3)}(p_4)=1/3$.
For u_4***:*** *Since* $p_3\in[\mathcal{P}\setminus\psi(u_4)]$, *then* $\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_4^{p_3})=1/4$, $\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_4^{p_3})=1/4$. *Thus* $\pi_{\psi(u_4)}(p_3)=1/2$.
Additionally, we can write by matrix form as

$$(\psi,\mathcal{U})=\left\{\begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5\\\hline u_1 & \langle 1,0\rangle & \langle 1,0\rangle & \langle 1/6,1/4\rangle & \langle 1,0\rangle & \langle 1/3,1/12\rangle\\ u_2 & \langle 1,0\rangle & \langle 1/4,0\rangle & \langle 1,0\rangle & \langle 1/6,1/12\rangle & \langle 1/6,1/12\rangle\\ u_3 & \langle 1/3,0\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 1/4,1/12\rangle & \langle 1,0\rangle\\ u_4 & \langle 1,0\rangle & \langle 1,0\rangle & \langle 1/4,1/4\rangle & \langle 1,0\rangle & \langle 1,0\rangle\end{array}\right\}.$$

Proposition 3.6. *Let* $(\Phi,\mathcal{P})\in SS(\mathcal{U})$, $(\psi,\mathcal{U})\in ISS(\mathcal{P})$. *Then, we have* $\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_a^{u_b})=\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_b^{p_a})$ *for* $1\le a\le m$, $1\le b\le n$ *and* $m,n\ge 2$. *However, similar situation may not be valid for relational non-membership function and inverse relational non-membership function.*

Proof. This proof is easily obtained from Definitions 3.1 and 3.3. □

Remark 3.7. *Consider Examples 3.2 and 3.5, the following results are obtained,*

$$\begin{aligned}\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_1^{u_3})=1/3 &= 1/3=\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_3^{p_1}),\quad \overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_1^{u_3})=1/6\neq 0=\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_3^{p_1}),\\ \overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_2^{u_2})=1/4 &= 1/4=\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_2^{p_2}),\quad \overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_2^{u_2})=1/3\neq 0=\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_2^{p_2}),\\ \overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_3^{u_1})=1/6 &= 1/6=\overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_1^{p_3}),\quad \overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_3^{u_1})=1/12\neq 1/4=\overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_1^{p_3}),\end{aligned}$$

$$
\begin{aligned}
\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_3^{u_4}) = 1/4 &= 1/4 = \overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_4^{p_3}), \quad \overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_3^{u_4}) = 1/0 \neq 1/4 = \overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_4^{p_3}),\\
\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_4^{u_2}) = 1/6 &= 1/6 = \overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_2^{p_4}), \quad \overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_4^{u_2}) = 1/4 \neq 1/12 = \overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_2^{p_4}),\\
\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_4^{u_3}) = 1/4 &= 1/4 = \overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_3^{p_4}), \quad \overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_4^{u_3}) = 1/6 \neq 1/12 = \overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_3^{p_4}),\\
\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_5^{u_1}) = 1/3 &= 1/3 = \overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_1^{p_5}), \quad \boldsymbol{\overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_5^{u_1}) = 1/12 = 1/12 = \overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_1^{p_5})},\\
\overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_5^{u_2}) = 1/6 &= 1/6 = \overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_2^{p_5}), \quad \overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_5^{u_2}) = 1/4 \neq 1/12 = \overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_2^{p_5}).
\end{aligned}
$$

This situation is not accidental and is a requirement of Proposition 3.6.

4 The proposed decision-making approaches

In this section, we propose two new approaches that we think can be useful in expressing decision-making processes in uncertainty environments based on membership functions introduced in the previous section.

Algorithm 1 Determine the optimal decision based on a SS.

Require: $\mathcal{U} = \{u_1, u_2, ..., u_n\}$, $\mathcal{P} = \{p_1, p_2, ..., p_m\}$, $1 \leq i \leq m$, $1 \leq r \leq n$, $m, n \geq 2$; $i, m, r, n \in \mathbb{N}$

Step 1: Input the SS $(\Phi, \mathcal{P}) \in SS(\mathcal{U})$ as follows:

$$(\Phi, \mathcal{P}) = \{(p_i, \Phi(p_i)) : p_i \in \mathcal{P}, \Phi(p_i) \in 2^{\mathcal{U}}\}.$$

Step 2: Calculate all relational membership degrees and relational non-membership degrees using the selected SS.

Step 3: Input all membership degrees in matrix form:

$$\mathcal{M}_{m\times n}^{(\Phi,\mathcal{P})} = \begin{pmatrix} \overset{(\Phi,\mathcal{P})}{\Omega}(p_1,u_1) & \overset{(\Phi,\mathcal{P})}{\Omega}(p_1,u_2) & \dots & \overset{(\Phi,\mathcal{P})}{\Omega}(p_1,u_n) \\ \overset{(\Phi,\mathcal{P})}{\Omega}(p_2,u_1) & \overset{(\Phi,\mathcal{P})}{\Omega}(p_2,u_2) & \dots & \overset{(\Phi,\mathcal{P})}{\Omega}(p_2,u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \overset{(\Phi,\mathcal{P})}{\Omega}(p_m,u_1) & \overset{(\Phi,\mathcal{P})}{\Omega}(p_m,u_2) & \dots & \overset{(\Phi,\mathcal{P})}{\Omega}(p_m,u_n) \end{pmatrix},$$

where $\overset{(\Phi,\mathcal{P})}{\Omega}(p_i,u_r) = \left\langle \overset{(\Phi,\mathcal{P})}{\Omega_\mu}(p_i,u_r), \overset{(\Phi,\mathcal{P})}{\Omega_\nu}(p_i,u_r) \right\rangle$ such that

$$\overset{(\Phi,\mathcal{P})}{\Omega_\mu}(p_i,u_r) = \begin{cases} 1, & u_r \in \Phi(p_i) \\ \overset{(\Phi,\mathcal{P})}{\Theta_\mu}(p_i^{u_r}), & u_r \in \mathcal{U} \setminus \Phi(p_i) \end{cases}$$

and

$$\overset{(\Phi,\mathcal{P})}{\Omega_\nu}(p_i,u_r) = \begin{cases} 1, & u_r \in \Phi(p_i) \\ \overset{(\Phi,\mathcal{P})}{\Theta_\nu}(p_i^{u_r}), & u_r \in \mathcal{U} \setminus \Phi(p_i) \end{cases}$$

are mappings given by $\overset{(\Phi,\mathcal{P})}{\Omega_\mu}, \overset{(\Phi,\mathcal{P})}{\Omega_\nu} : \mathcal{P} \times \mathcal{U} \to [0,1]$, respectively.

Step 4: Calculate the $\overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_r)$ of elements u_r:

$$\overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_r) = \sum_{i=1}^{m} \left[\overset{(\Phi,\mathcal{P})}{\Omega_\mu}(p_i,u_r) - \overset{(\Phi,\mathcal{P})}{\Omega_\nu}(p_i,u_r) \right].$$

Step 5: Find l, for which $\overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_l) = \max \left\{ \overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_r) \right\}$.

Firstly, we construct Algorithm 1 for decision-making under uncertainty environments (i.e., the first application with the help of membership functions):

Now we use the following example to show how the Algorithm 1 can be applied in an uncertainty environments.

Example 4.1. *Suppose that a married couple going to a real estate agent expresses the characteristics of each parameter they want and do not want to be at home. Taking advantage of these data, the real estate agent struggles to make a choice as to which of the existing homes he/she has is best suited for this couple. We recommend that the real estate agent use the algorithm given above to solve this problem:*

Let $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the set of houses owned by the real estate agent and $\mathcal{P} = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ be the set of parameters that express the characteristics of the house the couple wants from the real estate agent. For $i = 1, 2, ..., 6$, the parameters p_i stand for "gardened", "comfortable", "with a sea view", "cheap", "earthquake resistant" and "easy access", respectively. The real estate agent expressed the characteristics of the existing houses corresponding to each parameter by creating the SS $(\Phi, \mathcal{P}) \in SS(\mathcal{U})$ as follows:

$$(\Phi, \mathcal{P}) = \left\{ \begin{array}{c} (p_1, \{u_2, u_4, u_6\}), (p_2, \{u_1, u_3, u_5\})(p_3, \{u_1, u_2, u_3, u_5\}), \\ (p_4, \{u_2, u_3, u_5\})(p_5, \{u_1, u_3, u_4, u_6\}), (p_6, \{u_3, u_4, u_5\}) \end{array} \right\}.$$

As can be seen, the real estate agent had difficulty in reporting the membership degree in detail about the houses. Being able to express these values is often a difficult task for the decision maker. In order to overcome this problem, lets calculate all relational membership degrees and relational non-membership degrees for the given SS and by applying the instructions in the steps of the given Algorithm 1, the matrix $\mathcal{M}_{m\times n}^{(\Phi,\mathcal{P})}$ is easily obtained as follows:

$$\mathcal{M}_{m\times n}^{(\Phi,\mathcal{P})} = \begin{pmatrix} \langle 0.12, 0.08\rangle & \langle 1, 0\rangle & \langle 0.2, 0\rangle & \langle 1, 0\rangle & \langle 0.12, 0.08\rangle & \langle 1, 0\rangle \\ \langle 1, 0\rangle & \langle 0.2, 0.16\rangle & \langle 1, 0\rangle & \langle 0.16, 0.25\rangle & \langle 1, 0\rangle & \langle 0.08, 0.28\rangle \\ \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.2, 0.24\rangle & \langle 1, 0\rangle & \langle 0.12, 0.32\rangle \\ \langle 0.24, 0.12\rangle & \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.16, 0.2\rangle & \langle 1, 0\rangle & \langle 0.08, 0.28\rangle \\ \langle 1, 0\rangle & \langle 0.2, 0.16\rangle & \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.28, 0.08\rangle & \langle 1, 0\rangle \\ \langle 0.24, 0.12\rangle & \langle 0.2, 0.16\rangle & \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 1, 0\rangle & \langle 0.12, 0.24\rangle \end{pmatrix}.$$

By calculating, we get $\overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_r)$ of elements u_r $(r = 1, 2, 3, 4, 5, 6)$:

$$\overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_1) = 3.28, \quad \overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_2) = 3.12, \quad \overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_3) = 5.2,$$

$$\overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_5) = 2.83, \quad \overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_6) = 4.24, \quad \overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_7) = 1.28.$$

Finally, since $\overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_{l=3}) = \max\left\{\overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_r)\right\}$, then u_3 is determined to be the best decision.

Now, we present Algorithm 2 as a second type for a decision-making in uncertainty environment as follows:

Proposition 4.2. *Let $(\Phi, \mathcal{P}) \in SS(\mathcal{U})$, $(\psi, \mathcal{U}) \in ISS(\mathcal{P})$. Then; for $1 \leq a \leq m$, $1 \leq b \leq n$ and $m, n \geq 2$,*

(i) If $\overset{(\Phi,\mathcal{P})}{\Omega_\nu}(p_a, u_b) = \overset{(\psi,\mathcal{U})}{\Lambda_\nu}(u_b, p_a)$, then $\left[\mathcal{M}_{n\times m}^{(\Phi,\mathcal{P})}\right]^T = \mathcal{M}_{n\times m}^{(\psi,\mathcal{U})}$.

(ii) $\overset{(\Phi,\mathcal{P})}{\Omega_\mu}(p_a, u_b) = \overset{(\psi,\mathcal{U})}{\Lambda_\mu}(u_b, p_a)$.

(iii) If $\overset{(\Phi,\mathcal{P})}{\Omega_\nu}(p_a, u_b) = \overset{(\psi,\mathcal{U})}{\Lambda_\nu}(u_b, p_a)$, then $\overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_j) = \overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_j)$; $\forall u_j \in \mathcal{U}$.

Proof. Straightforward. □

Now, we show the principle and steps of Algorithm 2 using the following example.

Algorithm 2 Determine the optimal decision based on a ISS.

Require: $\mathcal{U}=\{u_1,u_2,...,u_n\}$, $\mathcal{P}=\{p_1,p_2,...,p_m\}$, $1\le s\le m$, $1\le j\le n$, $m,n\ge 2$; $s,m,j,n\in\mathbb{N}$

Step 1: Input the ISS $(\psi,\mathcal{U})\in ISS(\mathcal{P})$ as follows:

$$(\psi,\mathcal{U})=\{(u_j,\psi(u_j)):u_j\in\mathcal{U},\psi(u_j)\in 2^{\mathcal{P}}\}.$$

Step 2: Calculate all inverse relational membership degrees and inverse relational non-membership degrees using the selected ISS.

Step 3: Input all membership degrees in matrix form:

$$\mathcal{M}_{n\times m}^{(\psi,\mathcal{U})}=\begin{pmatrix} \overset{(\psi,\mathcal{U})}{\Lambda}(u_1,p_1) & \overset{(\psi,\mathcal{U})}{\Lambda}(u_1,p_2) & \dots & \overset{(\psi,\mathcal{U})}{\Lambda}(u_1,p_m)\\ \overset{(\psi,\mathcal{U})}{\Lambda}(u_2,p_1) & \overset{(\psi,\mathcal{U})}{\Lambda}(u_2,p_2) & \dots & \overset{(\psi,\mathcal{U})}{\Lambda}(u_2,p_m)\\ \vdots & \vdots & \ddots & \vdots\\ \overset{(\psi,\mathcal{U})}{\Lambda}(u_n,p_1) & \overset{(\psi,\mathcal{U})}{\Lambda}(u_n,p_2) & \dots & \overset{(\psi,\mathcal{U})}{\Lambda}(u_n,p_m)\end{pmatrix},$$

where $\overset{(\psi,\mathcal{U})}{\Lambda}(u_j,p_s)=\left\langle \overset{(\psi,\mathcal{U})}{\Lambda_\mu}(u_j,p_s),\ \overset{(\psi,\mathcal{U})}{\Lambda_\nu}(u_j,p_s)\right\rangle$ such that

$$\overset{(\psi,\mathcal{U})}{\Lambda_\mu}(u_j,p_s)=\begin{cases} 1, & p_s\in\psi(u_j)\\ \overset{(\psi,\mathcal{U})}{\Xi_\mu}(u_j^{p_s}), & p_s\in\mathcal{P}\setminus\psi(u_j)\end{cases}$$

and

$$\overset{(\psi,\mathcal{U})}{\Lambda_\nu}(u_j,p_s)=\begin{cases} 0, & p_s\in\psi(u_j)\\ \overset{(\psi,\mathcal{U})}{\Xi_\nu}(u_j^{p_s}), & p_s\in\mathcal{P}\setminus\psi(u_j)\end{cases}$$

are mappings given by $\overset{(\psi,\mathcal{U})}{\Lambda_\mu},\ \overset{(\psi,\mathcal{U})}{\Lambda_\nu}:\mathcal{U}\times\mathcal{P}\to[0,1]$, respectively.

Step 4: Calculate the $\overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_j)$ of elements u_j:

$$\overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_j)=\sum_{s=1}^{m}\left[\overset{(\psi,\mathcal{U})}{\Lambda_\mu}(u_j,p_s)-\overset{(\psi,\mathcal{U})}{\Lambda_\nu}(u_j,p_s)\right].$$

Step 5: Find l, for which $\overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_l)=\max\left\{\overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_j)\right\}$.

Example 4.3. *Consider Example 4.1. In this case, ISS $(\psi,\mathcal{U})\in ISS(\mathcal{P})$ is given by*

$$(\psi,\mathcal{U})=\left\{\begin{array}{c}(u_1,\{p_2,p_3,p_5\}),(u_2,\{p_1,p_3,p_4\}),(u_3,\{p_2,p_3,p_4,p_5,p_6\}),\\ (u_4,\{p_1,p_5,p_6\}),(u_5,\{p_2,p_3,p_4,p_6\}),(u_6,\{p_1,p_5\})\end{array}\right\}.$$

Next, lets calculate all inverse relational membership degrees and inverse relational nonmembership degrees for the given ISS and by applying the instructions in the steps of the given Algorithm 2, the matrix $\mathcal{M}_{n\times m}^{(\psi,\mathcal{U})}$ is easily obtained as follows:

$$\mathcal{M}_{n\times m}^{(\psi,\mathcal{U})}=\begin{pmatrix} \langle 0.12,0.2\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.24,0.08\rangle & \langle 1,0\rangle & \langle 0.24,0.08\rangle\\ \langle 1,0\rangle & \langle 0.2,0.08\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.2,0.08\rangle & \langle 0.2,0.08\rangle\\ \langle 0.2,0.28\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 1,0\rangle\\ \langle 1,0\rangle & \langle 0.16,0.12\rangle & \langle 0.2,0.08\rangle & \langle 0.16,0.12\rangle & \langle 1,0\rangle & \langle 1,0\rangle\\ \langle 0.12,0.24\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.28,0.08\rangle & \langle 1,0\rangle\\ \langle 1,0\rangle & \langle 0.08,0.12\rangle & \langle 0.12,0.08\rangle & \langle 0.08,0.12\rangle & \langle 1,0\rangle & \langle 0.12,0.08\rangle\end{pmatrix}.$$

By calculating, we get $\overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_j)$ *of elements* u_j $(j = 1,2,3,4,5,6)$*:*

$$\overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_1) = 2.72, \quad \overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_2) = 3.12, \quad \overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_3) = 4.16,$$

$$\overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_5) = 3.16, \quad \overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_6) = 4.32, \quad \overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_7) = 3.32.$$

Finally, since $\overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_{l=5}) = \max\left\{\overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_j)\right\}$*, then* u_5 *is determined to be the best decision.*

5 Comparison analysis

This section has been created for a comparison and analysis between the algorithms we propose in this study and some important related algorithms.

First, lets make a comparison between the algorithms we recommend. When Examples 4.1 and 4.3 given for the uncertainty environments are examined, Algorithm 1 and Algorithm 2 have determined the order of the objects being the best decision as follows:

$$u_3 > u_5 > u_1 > u_2 > u_4 > u_6 \quad \text{and} \quad u_5 > u_3 > u_6 > u_4 > u_2 > u_1$$

,respectively. As can be seen, the ordering between the objects has been obtained differently. This is expected and the reason is explained in Proposition 4.2. For this reason, algorithms should be handled together in order to manage the decision making processes in uncertainty environments in the most accurate way. So; For $\sum(u_l) = \max\left\{\overset{(\Phi,\mathcal{P})}{\mathcal{S}}(u_j), \overset{(\psi,\mathcal{U})}{\mathcal{S}}(u_j)\right\}$ the object u_l should be expressed as the best decision. Thus, the order between objects is obtained as follows:

$$u_3 > u_5 > u_2 > u_1 > u_4 > u_6$$

Here, for Algorithms 1 and 2, the following points should be noted:

(i) There is no superiority between Algorithm 1 and Algorithm 2. Both approaches are equally important to overcome uncertainty and should be considered together.

(ii) The proposed approaches aim to minimize the margin of error of the decision maker since they only receive data in the form of 0 and 1 from the decision maker. This is the most important difference between the algorithms we propose and other algorithms in the literature.

(iii) Algorithms 1 and 2 that we have elaborated here arrive at their decisions by considering the theories of SS and ISS. Hence, we can apply Algorithm 1 (or 2) to mathematical models such as virtual fuzzy parametrized soft sets [14], fuzzy soft sets [9], intuitionistic fuzzy soft set [10], bipolar fuzzy soft sets [3], bipolar neutrosophic soft set [2], fuzzy parametrized fuzzy soft sets [7], bipolar fuzzy parametrized soft sets [16], intuitionistic fuzzy parametrized soft sets [15] and interval valued intuitionistic fuzzy parametrized soft sets [17]. Further, Algorithm 2 can be applied to inverse fuzzy soft sets [27] and inverse soft rough sets [19]. Therefore, it is clear that the proposed algorithms can be considered in many areas where these set theories find application areas.

Now lets compare some of the related approaches in the literature with the results we have obtained by taking the two algorithms proposed in this study together. The first approach we consider is the selection algorithm for a house given by Maji and Roy in [33]. If we apply this algorithm to Example 4.1, the results are as follows:

$$u_3 > u_5 > u_1 = u_2 = u_4 > u_6$$

It can be easily seen here that some houses could not be separated, i.e., the same values were obtained. However, our approach has achieved a different value for each house. In addition, both approaches stated the most suitable house as u_3. Another impressive approach we have considered is the algorithm given by Agman et al. in [9]. If we use this algorithm to solve the uncertainty problem given in Example 4.1, the order between existing houses is as follows:

$$u_3 > u_5 > u_1 > u_2 = u_4 > u_6$$

However, the results of this approach were insufficient to find a different value for each house. In addition, the fact that both approaches focus membership degrees on the decision maker, which is very important in expressing uncertainty, poses a serious problem for these approaches. Because the decision maker may not always be able to express clear results, that is, it is necessary to eliminate the margin of error of the decision maker in order to eliminate uncertainty. The approaches we propose in this study have achieved very successful results in constructing uncertainty independently from the decision maker. Therefore, we recommend the use of our approaches given in our study in order to make the ordering between elements more clearly and to reduce the workload of the decision maker.

6 Conclusion

Since its first introduction to the literature, soft set theory continues to attract the attention of researchers in combating uncertainty environments. However, this theory has two major problems. The first problem is the distinction between objects belonging to the parameter and objects that do not belong to the parameter, i.e., this theory does not talk about choosing a value in the range $(0, 1)$. Various mathematical models have been proposed to overcome this problem. The second problem is that all these proposed mathematical models require the decision maker to choose the value in the range $(0, 1)$. However, to what extent would it be correct for the decision maker to find the most ideal values in this range?

In this study, it was emphasized that it may be wrong to trust a membership degree or non-membership degree that a decision maker can express in the range $(0, 1)$. Therefore, the concepts of relational membership function, relational non-membership function,inverse relational membership function and inverse relational non-membership function were defined and some related properties were studied. In addition, using these concepts, two different decision-making approaches for uncertainty are proposed. Moreover, the given approaches are compared with some related approaches in the literature and a detailed analysis is given. Finally, thanks to the proposed approaches, we show that we minimize the possibility of the decision maker entering incorrect data for the mathematical model used. It is important to notice that the approachs differs from the previously introduced approaches which only aim to select the single best alternative. Therefore, we propose to use these algorithms that can overcome the problems encountered with a different approach in uncertainty environments.

References

[1] M. I. Ali, F. Feng, X. Liu, W. K. Min, M. Shabir, *On some new operations in soft set theory*, Computers and Mathematics with Applications, **57** (2007), 1547-1553.

[2] M. I. Ali, L. H. Son, I. Deli, N. D. Tien, *Bipolar neutrosophic soft sets and applications in decision making*, Journal of Intelligent and Fuzzy Systems, **33**(6) (2017), 4077-4087.

[3] M. Aslam, S. Abdullah, K. Ullah, *Bipolar fuzzy soft sets and its applications in decision making problem*, Journal of Intelligent and Fuzzy Systems, **27**(2) (2014), 729-742.

[4] K. Atanassov, *Intuitionistic fuzzy sets: Theory and applications*, Heidelberg, Germany: Physica-Verlag, 1999.

[5] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87-96.

[6] D. G. Chen, E. C. C. Tsang, D. S. Yeung, W. Xizhao, *The parameterization reduction of soft sets and its applications*, Computers and Mathematics with Applications, **49** (2005), 757-763.

[7] N. Çagman, F. Çıtak, S. Enginoglu, *Fuzzy parameterized fuzzy soft set theory and its applications*, Turkish Journal of Fuzzy Systems, **1**(1) (2010), 21-35.

[8] N. Çagman, S. Enginoglu, *Soft set theory and uni-int decision making*, European Journal of Operational Research, **207** (2010), 848-855.

[9] N. Çagman, S. Enginoglu, F. Çıtak, *Fuzzy soft set theory and its applications*, Iranian Journal of Fuzzy Systems, **8**(3) (2011), 137-147.

[10] N. Çagman, S. Karatas, *Intuitionistic fuzzy soft set theory and its decision making*, Journal of Intelligent and Fuzzy Systems, **24**(4) (2013), 829-836.

[11] V. Çetkin, A. Aygunoglu, H. Aygun, *A new approach in handling soft decision making problems*, Journal of Nonlinear Sciences and Applications, **9** (2016), 231-239.

[12] O. Dalkılıç, *An application of VFPFSSs in decision making problems*, Journal of Polytechnic, (2021), Doi: 10.2339/politeknik.758474.

[13] O. Dalkılıç, N. Demirtas, *Bipolar soft filter*, Journal of Universal Mathematics, **3**(1) (2020), 21-27.

[14] O. Dalkılıç, N. Demirtas, *VFP-soft sets and its application on decision making problems*, Journal of Polytechnic, (2021), Doi: 10.2339/politeknik.685634.

[15] I. Deli, N. Çagman, *Intuitionistic fuzzy parameterized soft set theory and its decision making*, Applied Soft Computing, **28** (2015), 109-113.

[16] I. Deli, F. Karaaslan, *Bipolar FPSS-theory with applications in decision making*, Afrika Matematika, **31**(3) (2019), 493-505.

[17] I. Deli, S. Karatas, *Interval valued intuitionistic fuzzy parameterized soft set theory and its decision making*, Journal of Intelligent and Fuzzy Systems, **30**(3) (2016), 2073-2082.

[18] N. Demirtas, O. Dalkılıç, *Decompositions of soft α-continuity and soft A-continuity*, Journal of New Theory, **31** (2020), 86-94.

[19] N. Demirtas, S. Hussan, O. Dalkılıç, *New approaches of inverse soft rough sets and their applications in a decision making problem*, Journal of Applied Mathematics and Informatics, **38**(3-4) (2020), 335-349.

[20] H. Garg, R. Arora, *Maclaurin symmetric mean aggregation operators based on t-norm operations for the dual hesitant fuzzy soft set*, Journal of Ambient Intelligence and Humanized Computing, **11**(1) (2020), 375-410.

[21] H. Garg, K. Kumar, *A novel possibility measure to interval-valued intuitionistic fuzzy set using connection number of set pair analysis and its applications*, Neural Computing and Applications, **32**(8) (2020), 3337-3348.

[22] K. Hayat, M. I. Ali, F. Karaaslan, B. Y. Cao, M. H. Shah, *Design concept evaluation using soft sets based on acceptable and satisfactory levels: An integrated TOPSIS and Shannon entropy*, Soft Computing, **24**(3) (2020), 2229-2263.

[23] C. Huang, M. Lin, Z. Xu, *Pythagorean fuzzy MULTIMOORA method based on distance measure and score function: Its application in multicriteria decision making process*, Knowledge and Information Systems, **62**(11) (2020), 4373-4406.

[24] A. M. Irfan, F. Feng, X. Liu, W. K. Minc, M. Shabir, *On some new operations in soft set theory*, Computers and Mathematics with Applications, **57** (2009), 1547-1553.

[25] Y. Jiang, Y. Tang, Q. Chen, *An adjustable approach to intuitionistic fuzzy soft sets based decision making*, Applied Mathematical Modelling, **35** (2011), 824-836.

[26] A. M. Khalil, D. Cao, A. A. Azzam, F. Smarandache, W. Alharbi, *Combination of the single-valued neutrosophic fuzzy set and the soft set with applications in decisionmaking*, Symmetry, **12**(8) (2020), 1361.

[27] A. M. Khalil, N. Hassan, *Inverse fuzzy soft set and its application in decision making*, International Journal of Information and Decision Sciences, **11**(1) (2019), 73-92.

[28] M. Kirişçi, *Medical decision making with respect to the fuzzy soft sets*, Journal of Interdisciplinary Mathematics, **23**(4) (2020), 767-776.

[29] Z. Kong, L. Q. Gao, L. F. Wang, S. Li, *The normal parameter reduction of soft sets and its algorithm*, Computers and Mathematics with Applications, **56** (2008), 3029-3037.

[30] P. S. Kumar, *Algorithms for solving the optimization problems using fuzzy and intuitionistic fuzzy set*, International Journal of System Assurance Engineering and Management, **11**(1) (2020), 189-222.

[31] M. Lin, C. Huang, R. Chen, H. Fujita, X. Wang, *Directional correlation coefficient measures for Pythagorean fuzzy sets: Their applications to medical diagnosis and cluster analysis*, Complex and Intelligent Systems, **7**(2) (2021), 1025-1043.

[32] M. Lin, X. Li, R. Chen, H. Fujita, J. Lin, *Picture fuzzy interactional partitioned Heronian mean aggregation operators: An application to MADM process*, Artificial Intelligence Review, (2021), 1-38.

[33] P. K. Maji, A. R. Roy, R. Biswas, *An application of soft sets in a decision making problem*, Computers and Mathematics with Applications, **44** (2002), 1077-1083.

[34] P. K. Maji, A. R. Roy, R. Biswas, *Soft set theory*, Computers and Mathematics with Applications, **24** (2003), 555-562.

[35] S. Manna, T. M. Basu, S. K. Mondal, *A soft set based VIKOR approach for some decision-making problems under complex neutrosophic environment*, Engineering Applications of Artificial Intelligence, **89** (2020), 103432, Doi: 10.1016/j.engappai.2019.103432.

[36] A. R. Mishra, A. Mardani, P. Rani, E. K. Zavadskas, *A novel EDAS approach on intuitionistic fuzzy set for assessment of health-care waste disposal technology using new parametric divergence measures*, Journal of Cleaner Production, **272** (2020), 122807.

[37] D. Molodtsov, *Soft set theory-first results*, Computers and Mathematics with Applications, **37** (1999), 19-31.

[38] X. Peng, Y. Yang, *Algorithms for interval-valued fuzzy soft sets in stochastic multi-criteria decision making based on regret theory and prospect theory with combined weight*, Applied Soft Computing, **54** (2017), 415-430.

[39] K. Qin, Z. Hong, *On soft equality*, Journal of Computational and Applied Mathematics, **234**(5) (2010), 1347-1355.

[40] M. Saeed, M. Saqlain, A. Mehmood, K. Naseer, S. Yaqoob, *Multi-polar neutrosophic soft sets with application in medical diagnosis and decision-making*, Neutrosophic Sets and Systems, **33**(1) (2020), 183-207.

[41] P. Suebsan, *Inverse int-fuzzy soft bi-ideals over semigroups*, Annals of Fuzzy Mathematics and Informatics, **18**(1) (2019), 15-30.

[42] N. R. Thi, L. H. Son, M. Ali, D. Tamir, *Representing complex intuitionistic fuzzy set by quaternion numbers and applications to decision making*, Applied Soft Computing, **87**(4) (2020), 105961, Doi:10.1016/j.asoc.2019.105961.

[43] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.

[44] X. Zhang, B. Sun, X. Chen, X. Chu, J. Yang, *An approach to evaluating sustainable supply chain risk management based on BWM and linguistic value soft set theory*, Journal of Intelligent and Fuzzy Systems, **39**(3) (2020), 4369-4382.

[45] R. M. Zulqarnain, X. L. Xin, M. Saeed, N. Ahmed, *Application of interval valued fuzzy soft max-min decision making method in medical diagnosis*, International Journal of Pharmaceutical Sciences Review and Research, **62**(1) (2020), 56-60.