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Original paper

⊤-uniform convergence spaces

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Abstract

We show, for a commutative and integral quantale, that the recently introduced category of ⊤-uniform convergence spaces is a topological category which possesses natural function spaces, making it Cartesian closed. Furthermore, as two important examples for ⊤-uniform convergence spaces, we study probabilistic uniform spaces and quantale-valued metric spaces. The underlying ⊤-convergence spaces are also described and it is shown that in the case of a probabilistic uniform space this ⊤-convergence is the convergence of a fuzzy topology with conical neighbourhood filters. Finally it is shown that the category of ⊤-uniform convergence spaces can be embedded into the category of stratified lattice-valued uniform convergence spaces as a reflective subcategory.

Keywords: Topology, top-filter, uniform convergence space, lattice-valued uniform convergence space, probabilistic uniform space, quantale-valued metric space.

1 Introduction

It is known that the category of uniform spaces with uniformly continuous mappings as morphisms is not Cartesian closed. This deficiency can be overcome by a supercategory, the category of uniform convergence spaces, defined by Cook and Fischer [3], which has more convenient categorical properties, i.e. which is topological (i.e. it allows initial constructions) and – in a modification by Wyler [26] – Cartesian closed (i.e. it has canonical function spaces) [18]. Cook and Fischer's category, as improved by Wyler, was generalised to a category of lattice-valued uniform convergence spaces in [15] for a complete Heyting algebra as a lattice and in [4] for an enriched cl-premonoid, in order to capture more examples. However, Cartesian closedness of this category is only guaranteed in the complete Heyting algebra case.

Recently, lattice-valued convergence spaces based on so-called ⊤-filters, were considered [6, 23, 27, 29]. In order to study completeness and completion, related Cauchy spaces were studied and also ⊤-uniform convergence spaces were defined and used [24]. In this paper, we study these ⊤-uniform convergence spaces in more detail. We show that the category of these spaces has nice categorical properties like topologicalness and Cartesian closedness, the latter also in the more general case of a commutative and integral quantale, which is divisible [12] or a value quantale [7], as underlying lattice structure. In this way, natural function spaces are available also in the case of a non-idempotent quantale operation like e.g. Lawvere's quantale or in the probabilistic case. We also show that probabilistic uniform spaces [11, 28] are captured and that quantale-valued metric spaces, such as metric spaces and probabilistic metric spaces [25], possess natural ⊤-uniform (convergence) structures. Furthermore, we study suitable underlying ⊤-convergence structures.

The paper is organised as follows. In Section 2, we fix the lattice background and the notation. Then, in Sections 2 and 3, the basic theory of ⊤-filters is reviewed and ⊤-filters on $X \times X$ are defined and studied in detail. Section 5 defines ⊤-uniform convergence spaces and studies the resulting category. In Section 6 the underlying ⊤-convergence spaces are studied. As examples, probabilistic uniform spaces and quantale-valued metric spaces are given in Sections 7

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and 8 and finally, in Section 9, the category of $\top$-uniform convergence spaces is embedded into the category of stratified L-uniform convergence spaces studied previously.

2 Preliminaries

Let L be a complete lattice with distinct top and bottom elements $\top \neq \bot$. In any complete lattice L we can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. A complete lattice is completely distributive if and only if we have $\alpha = \bigvee\{\beta \,:\, \beta \lhd \alpha\}$ for any $\alpha \in L$, [22]. For more details and results on lattices we refer to [8].

We consider in this paper *commutative and integral quantales*, i.e. triples $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice with order relation $\leq$, and $(L, *)$ is a commutative semigroup for which the top element of L acts as the unit, i.e. $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, i.e. $(\bigvee_{i\in J} \alpha_i) * \beta = \bigvee_{i\in J}(\alpha_i * \beta)$, see e.g. [14]. For simplicity, we often simply speak of a quantale, always including that it is commutative and integral. Typical examples of such quantales are e.g. the unit interval $[0, 1]$ with a left-continuous *t-norm* [25]. Another important example is given by *Lawvere's quantale*, the interval $[0, \infty]$ with the opposite order and addition $\alpha * \beta = \alpha + \beta$, extended by $\alpha + \infty = \infty + a = \infty$, see e.g. [7]. A further important example is the quantale of distance distribution functions. A *distance distribution function* $\varphi : [0, \infty] \longrightarrow [0, 1]$, satisfies $\varphi(x) = \sup\{\varphi(y) : y < x\}$ for all $x \in [0, \infty]$. The set of all distance distribution functions is denoted by Δ^+ and with the pointwise order Δ^+ becomes a completely distributive lattice [7]. A quantale operation $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ is also called a *sup-continuous triangle function* [25].

In a commutative and integral quantale, we can define an *implication* by $\alpha \to \beta = \bigvee\{\delta \in L \,:\, \alpha * \delta \leq \beta\}$. Then $\delta \leq \alpha \to \beta \iff \delta * \alpha \leq \beta$. A commutative and integral quantale is an *MV-algebra* [12] if $(\alpha \to \beta) \to \beta = \alpha \vee \beta$ for all $\alpha, \beta \in L$.

At some places we will need that the lattice $(L, \leq)$ of the quantale $\mathsf{L} = (L, \leq, *)$ is *distributive*, $\alpha \wedge (\beta \vee \gamma) = (\alpha\wedge\gamma)\vee(\beta\wedge\gamma)$ for all $\alpha, \beta, \gamma \in L$, or even stronger, that the lattice is a *frame*, i.e. satisfies the *frame law* $(\bigvee_{i\in J} \alpha_i)\wedge\beta = \bigvee_{i\in J}(\alpha_i \wedge \beta)$ for all $\alpha_i, \beta \in L$. This can be ensured if the quantale is *divisible*, i.e. if for all $\alpha, \beta \in L$ with $\alpha \leq \beta$ there is $\gamma \in L$ such that $\alpha = \beta * \gamma$ [12]. We also note that $(\Delta^+, \leq)$ is a frame, as it is completely distributive [7], however $(\Delta^+, \leq, *)$ is in general not divisible, see [10].

We call a commutative and integral quantale $\mathsf{L} = (L, \leq, *)$ a *value quantale* [7] if $(L, \leq)$ is completely distributive and if $\alpha, \beta \lhd \top$ implies $\alpha \vee \beta \lhd \top$ for all $\alpha, \beta \in L$. We note that for a sup-continuous triangle function, $(\Delta^+, \leq, *)$ is a value quantale [7].

Lemma 2.1. *Let $\mathsf{L} = (L, \leq, *)$ be a commutative and integral quantale. If $\mathsf{L} = (L, \leq, *)$ is divisible or if $\mathsf{L} = (L, \leq, *)$ is a value quantale, then for $A \subseteq L$ with $\bigvee A = \top$ we have $\bigvee_{\alpha\in A}(\alpha * \alpha) = \top$.*

Proof. We first assume that $\mathsf{L} = (L, \leq, *)$ is divisible. From [12] we know that $\alpha * \beta \leq (\alpha * \alpha) \vee (\beta * \beta)$. We conclude $\bigvee A * \bigvee A = \bigvee_{\alpha\in A}(\alpha * \alpha)$, see [21], from which the claim immediately follows.

Let now $\mathsf{L} = (L, \leq, *)$ be a value quantale. From $\top = \bigvee_{\beta\lhd\top} \beta$ we conclude

$$\top = \top * \top = \bigvee_{\beta\lhd\top} \beta * \bigvee_{\gamma\lhd\top} \gamma \leq \bigvee_{\beta\vee\gamma\lhd\top} ((\beta \vee \gamma) * (\beta \vee \gamma)) \leq \bigvee_{\epsilon\lhd\top} (\epsilon * \epsilon).$$

Let now $\epsilon \lhd \top = \bigvee A$. Then there is $\beta \in A$ such that $\epsilon \leq \beta$ and hence $\epsilon * \epsilon \leq \beta * \beta \leq \bigvee_{\alpha\in A}(\alpha * \alpha)$. This is true for all $\epsilon \lhd \top$ and we conclude $\top = \bigvee_{\epsilon\lhd\top} \epsilon * \epsilon \leq \bigvee_{\alpha\in A}(\alpha * \alpha)$. □

We denote the set of L-sets $a, b, c, ...$ by $L^X = \{a : X \longrightarrow L\}$. A constant L-set with value $\alpha \in L$ is denoted by α_X. For a function $\varphi : X \longrightarrow Y$ and $a \in L^X$ and $b \in L^Y$ the *image of* a, $\varphi(a) \in L^Y$, is defined by $\varphi(a)(y) = \bigvee_{\varphi(x)=y} a(x)$ for $y \in Y$, and the *inverse image of* b is defined by $\varphi^{\leftarrow}(b) = b \circ \varphi$. The lattice operations are extended pointwisely from L to L^X. For $a \in L^X$ and $b \in L^Y$ we define the *Cartesian product* $a \times b$ by $(a \times b)(x, y) = a(x) \wedge b(y)$ for all $(x, y) \in X \times Y$.

For $b, d \in L^X$ we denote $[b, d] = \bigwedge_{x\in X}(b(x) \to d(x))$. $[\cdot, \cdot]$ is sometimes called the *fuzzy inclusion order* [2]. We collect some of the properties that we will need later.

Lemma 2.2. *Let $a, a', b, b', c \in L^X$, $d \in L^Y$ and $\varphi : X \longrightarrow Y$ be a mapping. Then*

(i) $a \leq b$ *if and only if* $[a, b] = \top$;

(ii) $a \leq a'$ *implies* $[a', b] \leq [a, b]$ *and* $b \leq b'$ *implies* $[a, b] \leq [a, b']$;

(iii) $[a,c] \wedge [b,c] = [a \vee b, c]$;

(iv) $[\varphi(a), d] = [a, \varphi^{\leftarrow}(d)]$.

If $a, c \in L^X$ *and* $b, d \in L^Y$ *then*

(v) $[a \times b, c \times d] \geq [a,c] \wedge [b,d]$.

Proof. Most properties are simple consequences of the properties of the implication. We show (iv) and (v).

(iv) For $a \in L^X$ we have

$$\begin{aligned}[\varphi(a), d] &= \bigwedge_{y\in Y} (\varphi(a)(y) \to d(y)) = \bigwedge_{y \in Y} ((\bigvee_{x\in X:\varphi(x)=y} a(x)) \to d(y)) \\ &= \bigwedge_{y\in Y} (\bigwedge_{x:\varphi(x)=y} (a(x) \to d(y))) = \bigwedge_{x\in X} (a(x) \to d(\varphi(x))) \\ &= \bigwedge_{x \in X} (a(x) \to \varphi^{\leftarrow}(d)(x)) = [a, \varphi^{\leftarrow}(d)].\end{aligned}$$

(v) If $a, c \in L^X, b, d \in L^Y$, then

$$\begin{aligned}[a \times b, c \times d] &= \bigwedge_{(x,y)\in X\times Y} ((a(x) \wedge b(y)) \to (c(x) \wedge d(y)) \\ &= \bigwedge_{(x,y)\in X\times Y} ((a(x) \wedge b(y)) \to c(x)) \wedge (a(x) \wedge b(y)) \to d(y)) \\ &\geq \bigwedge_{(x,y)\in X\times Y} (a(x) \to c(x)) \wedge (b(y) \to d(y)) \ \geq \ [a,c] \wedge [b,d].\end{aligned}$$

□

3 $\top$-filters

Definition 3.1. [9, 11, 27] *A subset* $\mathbb{F} \subseteq L^X$ *is called a* $\top$-filter *if*

($\top$-F1) $\bigvee_{x\in X} b(x) = \top$ *for all* $b \in \mathbb{F}$*;*

($\top$-F2) $a, b \in \mathbb{F}$ *implies* $a \wedge b \in \mathbb{F}$*;*

($\top$-F3) $\bigvee_{b \in \mathbb{F}} [b, d] = \top$ *implies* $d \in \mathbb{F}$.

We denote the set of all $\top$-filters on X by $\mathsf{F}_{\mathsf{L}}^{\top}(X)$.

Example 3.2. *For* $x \in X$, $[x] = \{a \in L^X \ : \ a(x) = \top\}$ *is a* $\top$*-filter. More general, if* $a(x) = \top$*, then* $[a] = \{b \in L^X \ : \ a \leq b\}$ *is a* $\top$*-filter.*

Definition 3.3. [9, 11, 27] *A subset* $\mathbb{B} \subseteq L^X$ *is called a* $\top$-filter base *if*

($\top$-B1) $\bigvee_{x\in X} b(x) = \top$ *for all* $b \in \mathbb{B}$*;*

($\top$-B2) $a, b \in \mathbb{B}$ *implies* $\bigvee_{c\in\mathbb{B}} [c, a \wedge b] = \top$.

For a $\top$-filter base $\mathbb{B}$, $[\mathbb{B}] = \{a \in L^X \ : \ \bigvee_{b\in\mathbb{B}} [b, a] = \top\}$ is a $\top$-filter, the $\top$-filter generated by $\mathbb{B}$.

The set $\mathsf{F}_{\mathsf{L}}^{\top}(X)$ is ordered by $\mathbb{F} \leq \mathbb{G}$ if $\mathbb{F} \subseteq \mathbb{G}$. The meet of a non-empty family $(\mathbb{F}_j)_{j\in J}$ of $\top$-filters on X is given by $\bigwedge_{j\in J} \mathbb{F}_j = \bigcap_{j \in J} \mathbb{F}_j$ and a $\top$-filter base for $\mathbb{F} \wedge \mathbb{G}$ is given by $\{f \vee g \ : \ f \in \mathbb{F}, g \in \mathbb{G}\}$.

Proposition 3.4 (Join of two $\top$-filters, [27])**.** *Let* $\mathbb{F}, \mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$*. Then* $\mathbb{B} = \{f \wedge g \ : \ f \in \mathbb{F}, g \in \mathbb{G}\}$ *is a* $\top$*-filter base if and only if* $\bigvee_{x\in X} f \wedge g(x) = \top$ *for all* $f \in \mathbb{F}, g \in \mathbb{G}$.

Proof. If $\mathbb{B}$ is a $\top$-filter base, then the condition follows from ($\top$-B1). Conversely, the condition ensures ($\top$-B1) for $\mathbb{B}$. For ($\top$-B2), let $a, b \in \mathbb{B}$. Then $a = f_1 \wedge g_1$ and $b = f_2 \wedge g_2$ with $f_1, f_2 \in \mathbb{F}$ and $g_1, g_2 \in \mathbb{G}$. Being $\top$-filters, then $f_1 \wedge f_2 \in \mathbb{F}$ and $g_1 \wedge g_2 \in \mathbb{G}$ and hence

$$\bigvee_{c\in\mathbb{B}} [c, a \wedge b] = \bigvee_{f\in\mathbb{F}, g\in\mathbb{G}} [f \wedge g, f_1 \wedge g_1 \wedge f_2 \wedge g_2] \geq [f_1 \wedge f_2 \wedge g_1 \wedge g_2, f_1 \wedge f_2 \wedge g_1 \wedge g_2] = \top.$$

□

If the condition of Proposition 3.4 is satisfied, then we denote the generated $\top$-filter by $\mathbb{F} \vee \mathbb{G}$ and say that $\mathbb{F} \vee \mathbb{G}$ exists. The notation is justified. To this end, we note that $\top_X \in \mathbb{G}$ as $\bigvee_{g\in\mathbb{G}}[g, \top_X] = \top$. Hence $f = f \wedge \top_X \in \mathbb{B}$ for all $f \in \mathbb{F}$ and therefore $\bigvee_{f'\in\mathbb{F},g'\in\mathbb{G}}[f' \wedge g', f] \geq [f \wedge \top_X, f] = \top$, i.e. we have $f \in \mathbb{F} \vee \mathbb{G}$ for all $f \in \mathbb{F}$, i.e. $\mathbb{F} \leq \mathbb{F} \vee \mathbb{G}$. In the same way, we can see $\mathbb{G} \leq \mathbb{F} \vee \mathbb{G}$. Let now $\mathbb{H} \geq \mathbb{F}, \mathbb{G}$ be a $\top$-filter. For $a \in \mathbb{F} \vee \mathbb{G}$ then $\bigvee_{h\in\mathbb{H}}[h, a] \geq \bigvee_{f\in\mathbb{F},g\in\mathbb{G}}[f \wedge g, a] = \top$ and we have $a \in \mathbb{H}$. Therefore $\mathbb{F} \vee \mathbb{G}$ is the join of $\mathbb{F}$ and $\mathbb{G}$ in $\mathsf{F}_L^{\top}(X)$ whenever it exists.

It is well-known, that for a $\top$-filter $\mathbb{F} \in \mathsf{F}_L^{\top}(X)$ and a mapping $\varphi : X \longrightarrow Y$, the set $\mathbb{B} = \{\varphi(a) : a \in \mathbb{F}\}$ is a $\top$-filter base on Y and we denote $\varphi(\mathbb{F})$ the generated $\top$-filter on Y, the *image of* $\mathbb{F}$ *under* φ, see [11].

Lemma 3.5. *Let $\mathbb{F} \in \mathsf{F}_\mathsf{L}^{\top}(X)$ and $\varphi : X \longrightarrow Y$ be a mapping. Then $b \in \varphi(\mathbb{F})$ if and only if $\varphi^{\leftarrow}(b) \in \mathbb{F}$.*

Proof. We have that $b \in \varphi(\mathbb{F})$ is equivalent to $\top = \bigvee_{f\in\mathbb{F}}[\varphi(f), b] = \bigvee_{f\in\mathbb{F}}[f, \varphi^{\leftarrow}(b)]$ which is equivalent to $\varphi^{\leftarrow}(b) \in \mathbb{F}$. □

We conclude that $\varphi([x]) = [\varphi(x)]$: We have $b \in \varphi([x])$ iff $\varphi^{\leftarrow}(b) \in [x]$ iff $\varphi^{\leftarrow}(b)(x) = b(\varphi(x)) = \top$ iff $b \in [\varphi(x)]$.

Proposition 3.6. *Let $\mathbb{F}, \mathbb{G} \in \mathsf{F}_\mathsf{L}^{\top}(X)$ and $\varphi : X \longrightarrow Y$ be a mapping. Then $\varphi(\mathbb{F}) \wedge \varphi(\mathbb{G}) = \varphi(\mathbb{F} \wedge \mathbb{G})$.*

Proof. We apply Lemma 3.5. We have $a \in \varphi(\mathbb{F}) \wedge \varphi(\mathbb{G})$ if and only if $\varphi^{\leftarrow}(a) \in \mathbb{F} \wedge \mathbb{G}$ if and only if $a \in \varphi(\mathbb{F} \wedge \mathbb{G})$. □

Proposition 3.7. [11, 27] *Let $\varphi : X \longrightarrow Y$ and $\mathbb{F} \in \mathsf{F}_\mathsf{L}^{\top}(Y)$. Then $\mathbb{B} = \{\varphi^{\leftarrow}(b) : b \in \mathbb{F}\}$ is a $\top$-filter base if and only if $\bigvee_{y\in\varphi(X)} b(y) = \top$ for all $b \in \mathbb{F}$.*

Proof. Let first $\mathbb{B}$ be a $\top$-filter base. Then $\top = \bigvee_{x\in X} \varphi^{\leftarrow}(b)(x) = \bigvee_{x\in X} b(\varphi(x)) = \bigvee_{y\in\varphi(X)} b(y)$ by ($\top$-B1).

Let now the stated condition be satisfied. Then ($\top$-B1) is true. For ($\top$-B2), let $b_1, b_2 \in \mathbb{B}$. Then there are $f_1, f_2 \in \mathbb{F}$ with $b_1 = \varphi^{\leftarrow}(f_1)$, $b_2 = \varphi^{\leftarrow}(f_2)$ and we have

$$\bigvee_{c\in\mathbb{B}}[c, b_1 \wedge b_2] = \bigvee_{b\in\mathbb{F}}[\varphi^{\leftarrow}(b), \varphi^{\leftarrow}(f_1) \wedge \varphi^{\leftarrow}(f_2)] = \bigvee_{b\in\mathbb{F}}[\varphi^{\leftarrow}(b), \varphi^{\leftarrow}(f_1 \wedge f_2)] \geq \bigvee_{b\in\mathbb{F}}[b, f_1 \wedge f_2] \geq [f_1 \wedge f_2, f_1 \wedge f_2] = \top,$$

as $\mathbb{F}$ is a $\top$-filter. □

We now turn to products of $\top$-filters.

Proposition 3.8. [27] *Let $\mathbb{F} \in \mathsf{F}_\mathsf{L}^{\top}(X)$ and $\mathbb{G} \in \mathsf{F}_\mathsf{L}^{\top}(Y)$. Then $\mathbb{B} = \{f \times g : f \in \mathbb{F}, g \in \mathbb{G}\}$ is a $\top$-filter base on $X \times Y$.*

Proof. ($\top$-B1) We have $\bigvee_{(x,y)\in X\times Y}(f(x) \wedge g(y)) = \bigvee_{x\in X} \bigvee_{y\in Y}(f(x) \wedge g(y)) \geq \bigvee_{x\in X} \bigvee_{y\in Y}(f(x) * g(y)) = \bigvee_{x\in X} f(x) * \bigvee_{y\in Y} g(y) = \top * \top = \top$, as $\alpha \wedge \beta \geq \alpha * \beta$ for an integral quantale.

($\top$-B2) Let $f \times g, f' \times g' \in \mathbb{B}$. Then

$$\begin{aligned}\bigvee_{f''\times g''\in\mathbb{B}}[f'' \times g'', (f \times g) \wedge (f' \times g')] &= \bigvee_{f''\times g''\in\mathbb{B}}[f'' \times g'', (f \wedge f') \times (g \wedge g')] \\ &= \bigvee_{f''\in\mathbb{F},g''\in\mathbb{G}}[f'', f \wedge f'] \wedge [g'', g \wedge g'] \geq \top\end{aligned}$$

because $f \wedge f' \in \mathbb{F}$ and $g \wedge g' \in \mathbb{G}$. □

We denote the generated $\top$-filter by $\mathbb{F} \times \mathbb{G}$. Note that for $f \in \mathbb{F}$ and $g \in \mathbb{G}$ we have $pr_X^{\leftarrow}(f) \wedge pr_Y^{\leftarrow}(g)(x, y) = f(x) \wedge g(y) = f \times g(x, y)$. Hence a $\top$-filter base for $\mathbb{F} \times \mathbb{G}$ is given by $\{pr_X^{\leftarrow}(f) \wedge pr_Y^{\leftarrow}(g) : f \in \mathbb{F}, g \in \mathbb{G}\}$ and this is also a $\top$-filter base of $pr_X^{\leftarrow}(\mathbb{F}) \vee pr_Y^{\leftarrow}(\mathbb{G})$, i.e. we have $\mathbb{F} \times \mathbb{G} = pr_X^{\leftarrow}(\mathbb{F}) \vee pr_Y^{\leftarrow}(\mathbb{G})$.

Proposition 3.9. [27] *Let the underlying lattice of L satisfy the frame law. Let $\mathbb{F} \in \mathsf{F}_\mathsf{L}^{\top}(X)$ and $\mathbb{G} \in \mathsf{F}_\mathsf{L}^{\top}(Y)$ and $\varphi : X \longrightarrow U, \psi : Y \longrightarrow V$ be mappings. Then $(\varphi \times \psi)(\mathbb{F} \times \mathbb{G}) = \varphi(\mathbb{F}) \times \psi(\mathbb{G})$.*

Proof. We note that for a frame L and $a \in L^X, b \in L^Y$ we have that $\varphi \times \psi(a \times b) = \varphi(a) \times \psi(b)$. Hence we have $d \in (\varphi \times \psi)(\mathbb{F} \times \mathbb{G})$ if and only if

$$\top = \bigvee_{f\in\mathbb{F},g\in\mathbb{G}}[(\varphi \times \psi)(f \times g), d] = \bigvee_{f\in\mathbb{F},g\in\mathbb{G}}[\varphi(f) \times \psi(g), d].$$

As $\{\varphi(f) \times \varphi(g) : f \in \mathbb{F}, g \in \mathbb{G}\}$ is a $\top$-filter base of $\varphi(\mathbb{F}) \times \psi(\mathbb{G})$, the latter is equivalent to $d \in \varphi(\mathbb{F}) \times \psi(\mathbb{G})$. □

Proposition 3.10. *Let $\mathbb{F}, \mathbb{G} \in \mathsf{F}_\mathsf{L}^{\top}(X)$ and $\mathbb{H}, \mathbb{K} \in \mathsf{F}_\mathsf{L}^{\top}(Y)$.*

(i) If $\mathbb{F} \leq \mathbb{G}$ *and* $\mathbb{H} \leq \mathbb{K}$*, then* $\mathbb{F} \times \mathbb{H} \leq \mathbb{G} \times \mathbb{K}$.

(ii) If the lattice L *is distributive, then* $(\mathbb{F} \wedge \mathbb{G}) \times \mathbb{H} = (\mathbb{F} \times \mathbb{H}) \wedge (\mathbb{G} \times \mathbb{H})$.

Proof. (i) Let $a \in \mathbb{F} \times \mathbb{H}$. Then $\top = \bigvee_{f\in\mathbb{F},h\in\mathbb{H}}[f \times h, a] \leq \bigvee_{f\in\mathbb{G},h\in\mathbb{K}}[f \times h, a]$ and hence $a \in \mathbb{G} \times \mathbb{K}$.

(ii) One inequality follows directly from (i). Let now $a \in (\mathbb{F} \times \mathbb{H}) \wedge (\mathbb{G} \times \mathbb{H})$. Then $\top = \bigvee_{f\in\mathbb{F},h\in\mathbb{H}}[f \times h, a]$ and $\top = \bigvee_{g\in\mathbb{G},k\in\mathbb{H}}[g \times k, a]$. We conclude

$$\begin{aligned}
\top &= \bigvee_{f\in\mathbb{F},h\in\mathbb{H}}[f \times h, a] * \bigvee_{g\in\mathbb{G},k\in\mathbb{H}}[g \times k, a] = \bigvee_{f\in\mathbb{F},g\in\mathbb{G},h,k\in\mathbb{H}}[f \times h, a] * [g \times k, a] \\
&\leq \bigvee_{f\in\mathbb{F},g\in\mathbb{G},h,k\in\mathbb{H}}[f \times h, a] \wedge [g \times k, a] \leq \bigvee_{f\in\mathbb{F},g\in\mathbb{G},h,k\in\mathbb{H}}[(f \times h) \vee (g \times k), a] \\
&\leq \bigvee_{f\in\mathbb{F},g\in\mathbb{G},h,k\in\mathbb{H}}[(f \times (h \wedge k)) \vee (g \times (h \wedge k)), a] \leq \bigvee_{f\in\mathbb{F},g\in\mathbb{G},l\in\mathbb{H}}[(f \times l) \vee (g \times l), a] \\
&= \bigvee_{f\in\mathbb{F},g\in\mathbb{G},l\in\mathbb{H}}[(f \vee g) \times l, a]
\end{aligned}$$

and hence $a \in (\mathbb{F} \wedge \mathbb{G}) \times \mathbb{H}$. The distributivity of L was used in the last step. □

Proposition 3.11. *Let* $\Phi \in \mathsf{F}_\mathsf{L}^\top(X \times Y)$*. Then* $pr_X(\Phi) \times pr_Y(\Phi) \leq \Phi$.

Proof. Let $a \in pr_X(\Phi) \times pr_Y(\Phi)$. Then $\top = \bigvee_{k_1\in pr_X(\Phi),k_2\in pr_Y(\Phi)}[k_1 \times k_2, a]$. For $k_1 \in pr_X(\Phi)$ we have $\top = \bigvee_{\phi\in\Phi}[pr_X(\phi), k_1] = \bigvee_{\phi\in\Phi}[\phi, pr_X^{\leftarrow}(k_1)]$ and similarly, for $k_2 \in pr_Y(\Phi)$, $\top = \bigvee_{\phi\in\Phi}[\phi, pr_Y^{\leftarrow}(k_2)]$. Hence

$$\begin{aligned}
\top &= \bigvee_{\phi\in\Phi,\psi\in\Phi}([\phi, pr_X^{\leftarrow}(k_1)] \wedge [\psi, pr_Y^{\leftarrow}(k_2)]) \leq \bigvee_{\phi\in\Phi,\psi\in\Phi}([\phi \wedge \psi, pr_X^{\leftarrow}(k_1)] \wedge [\phi \wedge \psi, pr_Y^{\leftarrow}(k_2)]) \\
&= \bigvee_{\phi\in\Phi}([\phi, pr_X^{\leftarrow}(k_1) \wedge pr_Y^{\leftarrow}(k_2)]) = \bigvee_{\phi\in\Phi}[\phi, k_1 \times k_2].
\end{aligned}$$

Hence $k_1 \times k_2 \in \Phi$ and we conclude $\bigvee_{\phi\in\Phi}[\phi, a] \geq \bigvee_{k_1\in pr_X(\Phi),k_2\in pr_Y(\Phi)}[k_1 \times k_2, a] = \top$ and we have $a \in \Phi$. □

Proposition 3.12. [27] *Let* $\mathbb{F} \in \mathsf{F}_\mathsf{L}^\top(X)$*,* $\mathbb{G} \in \mathsf{F}_\mathsf{L}^\top(Y)$*. Then* $pr_X(\mathbb{F} \times \mathbb{G}) = \mathbb{F}$ *and* $pr_Y(\mathbb{F} \times \mathbb{G}) = \mathbb{G}$.

Proof. We have $pr_X(\mathbb{F} \times \mathbb{G}) = pr_X(pr_X^{\leftarrow}(\mathbb{F}) \vee pr_Y^{\leftarrow}(\mathbb{G})) \geq pr_X(pr_X^{\leftarrow}(\mathbb{F})) = \mathbb{F}$ as pr_X is a surjection. To show the other inequality, let $a \in pr_X(\mathbb{F}\times\mathbb{G})$. Then $pr_X^{\leftarrow}(a) \in \mathbb{F}\times\mathbb{G}$, i.e. we have $\top = \bigvee_{f\in\mathbb{F},g\in\mathbb{G}}[f\times g, pr_X^{\leftarrow}(a)] = \bigvee_{f\in\mathbb{F},g\in\mathbb{G}}[pr_X(f\times g), a]$. As $pr_X(f \times g)(x) = \bigvee_{y\in Y} f(x) \wedge g(y) \geq \bigvee_{y\in Y} f(x) * g(y) = f(x) * \bigvee_{y\in Y} g(y) = f(x)$ whenever $\bigvee_{y\in Y} g(y) = \top$, we conclude $\top \leq \bigvee_{f\in\mathbb{F}}[f, a]$ and hence $a \in \mathbb{F}$ because $\mathbb{F}$ is a $\top$-filter. The second projection can be dealt with analogously. □

Let $\mathbb{F} \in \mathsf{F}_\mathsf{L}^\top(X)$ and let $x \in X$. We define $\mathbb{F}_x = \{d \in L^{X\times X} \ : \ d(\cdot, x) \in \mathbb{F}\}$.

Proposition 3.13. *Let* $\mathbb{F} \in \mathsf{F}_\mathsf{L}^\top(X)$ *and* $x \in X$*. Then* $\mathbb{F}_x = \mathbb{F} \times [x]$.

Proof. Let $d \in \mathbb{F}_x$. Then $d(\cdot, x) \in \mathbb{F}$. We have $d(\cdot, x) \times \top_{\{x\}}(\cdot)(u, v) = d(u, x) \wedge \top_{\{x\}}(v) \leq d(u, v)$. Hence $\top = [d(\cdot, x) \times \top_{\{x\}}, d] \leq \bigvee_{f\in\mathbb{F},b\in[x]}[f \times b, d]$ and consequently $d \in \mathbb{F} \times [x]$.

On the other hand, let $d \in \mathbb{F} \times [x]$. Then

$$\begin{aligned}
\top &= \bigvee_{f\in\mathbb{F},b\in[x]}[f \times b, d] = \bigvee_{f\in\mathbb{F},b(x)=\top}\ \bigwedge_{(u,v)\in X\times X}((f(u) \wedge b(v)) \to d(u, v)) \\
&\leq \bigvee_{f\in\mathbb{F},b(x)=\top}\ \bigwedge_{u\in X}((f(u) \wedge b(x)) \to d(u, x)) = \bigvee_{f\in\mathbb{F}}[f, d(\cdot, x)].
\end{aligned}$$

As $\mathbb{F}$ is a $\top$-filter, we conclude $d(\cdot, x) \in \mathbb{F}$. □

As clearly $(\mathbb{F} \wedge \mathbb{G})_x = \mathbb{F}_x \wedge \mathbb{G}_x$, it follows that $(\mathbb{F} \times [x]) \wedge (\mathbb{G} \times [x]) = (\mathbb{F} \wedge \mathbb{G}) \times [x]$. This is a special case of Lemma 3.10(ii), however the distributivity of L is not required.

Proposition 3.14. *Let* $\mathbb{F} \in \mathsf{F}_\mathsf{L}^\top(X)$ *and let* $x \in X$ *and let* $\varphi, \psi : X \longrightarrow Y$*. Then* $(\varphi \times \psi)(\mathbb{F}_x) = \varphi(\mathbb{F})_{\psi(x)}$.

Proof. We have $a \in \varphi(\mathbb{F})_{\psi(x)}$ if and only if $\varphi^{\leftarrow}(a(\cdot, \psi(x))) \in \mathbb{F}$. We now note that for $a \in L^{Y\times Y}$ we have

$$\varphi^{\leftarrow}(a(\cdot, \psi(x)))(x') = a(\cdot, \psi(x))(\varphi(x')) = a(\varphi(x'), \psi(x)) = (\varphi \times \psi)^{\leftarrow}(a)(x', x),$$

i.e. we have $\varphi^{\leftarrow}(a(\cdot, \psi(x))) = (\varphi \times \psi)^{\leftarrow}(a)(\cdot, x)$. Hence $a \in \varphi(\mathbb{F})_{\psi(x)}$ is equivalent to $a(\cdot, \psi(x)) \in \varphi(\mathbb{F})$. This is in turn equivalent to $\varphi^{\leftarrow}(a(\cdot, \psi(x))) \in \mathbb{F}$, i.e. to $(\varphi \times \psi)^{\leftarrow}(a)(\cdot, x) \in \mathbb{F}$. This is again equivalent to $(\varphi \times \psi)^{\leftarrow}(a) \in \mathbb{F}_x$, i.e. to $a \in (\varphi \times \psi)(\mathbb{F})$. □

As a consequence we have $(\varphi \times \psi)(\mathbb{F} \times [x]) = \varphi(\mathbb{F}) \times [\psi(x)]$, without the assumption of the underlying lattice being a frame.

4 $\top$-filters on $X \times X$

We consider the *swapping mapping* $\sigma_X : X \times X \longrightarrow X \times X$, $\sigma_X(x, x') = (x', x)$ for all $x, x' \in X$ and we denote, for $a \in L^{X\times X}$ the *inverse* $a^{-1} = \sigma_X^{\leftarrow}(a)$, i.e. we have for $x, x' \in X$ that $a^{-1}(x, x') = a(\sigma_X(x, x')) = a(x', x)$. Further, we denote for $\Phi \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X)$ the $\top$-filter $\Phi^{-1} = \sigma_X(\Phi)$, see [24].

Proposition 4.1. *Let $\Phi \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X)$. Then $\Phi^{-1} = \{a^{-1} \,:\, a \in \Phi\}$.*

Proof. With Lemma 3.5 we obtain $a \in \Phi^{-1} = \sigma_X(\Phi)$ if and only if $a^{-1} = \sigma_X^{\leftarrow}(a) \in \Phi$. □

Proposition 4.2. *Let $\Phi, \Psi \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$. Then $(\Phi^{-1})^{-1} = \Phi$ and $\Phi \le \Psi$ implies $\Phi^{-1} \le \Psi^{-1}$.*

Proof. The first assertion follows from $\sigma_X \circ \sigma_X = id_X$, while the second assertion is clear. □

Proposition 4.3. *Let $\varphi : X \longrightarrow Y$ and $\Phi \in \mathsf{F}_{\mathsf{L}}^{\top} L(X \times X)$. Then $(\varphi \times \varphi)(\Phi^{-1}) = ((\varphi \times \varphi)(\Phi))^{-1}$.*

Proof. This follows from $(\varphi \times \varphi) \circ \sigma_X = \sigma_Y \circ (\varphi \times \varphi)$. □

Proposition 4.4. *Let $\mathbb{F}, \mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$. Then $(\mathbb{F} \times \mathbb{G})^{-1} = \mathbb{G} \times \mathbb{F}$.*

Proof. We first note that for $f, g \in L^X$ we have $(f \times g)^{-1}(x, y) = f \times g(y, x) = f(y) \wedge g(x) = g(x) \wedge f(y) = (g \times f)(x, y)$. Hence we have $d \in (\mathbb{F} \times \mathbb{G})^{-1}$ if and only if $\top = \bigvee_{f\in\mathbb{F}, g\in\mathbb{G}}[(f \times g)^{-1}, d] = \bigvee_{f\in\mathbb{F}, g\in\mathbb{G}}[g \times f, d]$ if and only if $d \in \mathbb{G} \times \mathbb{F}$. □

Proposition 4.5. [24] *Let $\Phi, \Psi \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X)$. Then $\mathbb{B} = \{\phi \circ \psi \,:\, \phi \in \Phi, \psi \in \Psi\}$ is a $\top$-filter base if and only if $\bigvee_{(x,y)\in X\times X} \phi \circ \psi(x, y) = \top$ for all $\phi \in \Phi, \psi \in \Psi$. Here, $\phi \circ \psi(x, y) = \bigvee_{z\in X} \phi(x, z) * \psi(z, y)$ for all $(x, y) \in X \times X$.*

Proof. If $\mathbb{B}$ is a $\top$-filter base then the condition is satisfied by ($\top$-B1). Conversely, let the condition be true. Then ($\top$-B1) is satisfied and we need to check ($\top$-B2). Let $a, b \in \mathbb{B}$. Then $a = \phi \circ \psi$ and $b = \phi' \circ \psi'$ with $\phi, \phi' \in \Phi, \psi, \psi' \in \Psi$. Then $(\phi \wedge \phi') \circ (\psi \wedge \psi') \in \mathbb{B}$ and we have $\bigvee_{c\in\mathbb{B}}[c, a \wedge b] \ge [(\phi \wedge \phi') \circ (\psi \wedge \psi'), a \wedge b]$. It is not difficult to show that $(\phi \wedge \phi') \circ (\psi \wedge \psi') \le (\phi \circ \psi) \wedge (\phi' \circ \psi') = a \wedge b$ and hence $[(\phi \wedge \phi') \circ (\psi \wedge \psi'), a \wedge b] = \top$. □

If the condition in Proposition 4.5 is satisfied, then we denote the generated $\top$-filter by $\Phi \circ \Psi$ and say that $\Phi \circ \Psi$ exists.

Proposition 4.6. *Let $\Phi, \Psi, \Gamma, \Theta \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X)$.*

(i) If $\Phi \le \Gamma, \Psi \le \Theta$ and $\Gamma \circ \Theta$ exists, then $\Phi \circ \Psi$ exists and is $\le \Gamma \circ \Theta$.

(ii) If $\Phi \circ \Psi$ exists, then $\Psi^{-1} \circ \Phi^{-1}$ exists and $(\Phi \circ \Psi)^{-1} = \Psi^{-1} \circ \Phi^{-1}$.

Proof. (i) Let $\phi \in \Phi$ and $\psi \in \Psi$. Then $\phi \in \Gamma$ and $\psi \in \Theta$ and because $\Gamma \circ \Theta$ exists we have $\bigvee_{(x,y)\in X\times X} \phi \circ \psi(x, y) = \top$. Hence $\Phi \circ \Psi$ exists.

(ii) Let $\theta \in \Psi^{-1}, \Upsilon \in \Phi^{-1}$. Then $\theta^{-1} \in \Psi, \Upsilon^{-1} \in \Phi$ and we have $\bigvee_{(x,y)\in X\times X} \theta \circ \Upsilon(x, y) = \bigvee_{(x,y)\in X\times X} \Upsilon^{-1} \circ \theta^{-1}(x, y) = \top$ as $\Phi \circ \Psi$ exists. Hence $\Psi^{-1} \circ \Phi^{-1}$ exists. Moreover, we have $\eta \in (\Phi \circ \Psi)^{-1}$ if and only if $\eta^{-1} \in \Phi \circ \Psi$. This is equivalent to

$$\top = \bigvee_{\phi\in\Phi,\psi\in\Psi} [\phi \circ \psi, \eta^{-1}] = \bigvee_{\phi\in\Phi,\psi\in\Psi} [(\phi \circ \psi)^{-1}, \eta] = \bigvee_{\phi\in\Phi,\psi\in\Psi} [\psi^{-1} \circ \phi^{-1}, \eta] = \bigvee_{\zeta\in\Phi^{-1},\rho\in\Psi^{-1}} [\zeta \circ \rho, \eta],$$

i.e. equivalent to $\eta \in \Psi^{-1} \circ \Phi^{-1}$. □

Proposition 4.7. *Let $\mathsf{L} = (L, \leq, *)$ be a commutative and integral quantale which is divisible or is a value quantale. and let $\Phi \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X)$. Then $\Phi^{-1} \circ \Phi$ exists.*

Proof. Let $\phi \in \Phi$. Then

$$\begin{aligned}
\bigvee_{x,y \in X} \phi^{-1} \circ \phi(x,y) &= \bigvee_{x,y,z \in X} \phi^{-1}(x,z) * \phi(z,y) \;=\; \bigvee_{x,y,z \in X} \phi(z,x) * \phi(z,y) \\
&= \bigvee_{x,z \in X} \phi(z,x) * \bigvee_{y \in X} \phi(z,y) \;\geq\; \bigvee_{x,z \in X} \phi(z,x) * \phi(z,x) \\
&= \bigvee_{x,z \in X} \phi(z,x) \;=\; \top,
\end{aligned}$$

the last step because Φ is a $\top$-filter and Proposition 2.1. $\square$

Proposition 4.8. *Let $\Phi, \Psi, \Gamma, \Theta \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X)$ and $\varphi : X \longrightarrow Y$ be a mapping. If $\Phi \circ \Psi$ exists, then $(\varphi \times \varphi)(\Phi) \circ (\varphi \times \varphi)(\Psi)$ exists and $(\varphi \times \varphi)(\Phi) \circ (\varphi \times \varphi)(\Psi) \leq (\varphi \times \varphi)(\Phi \circ \Psi)$.*

Proof. Let $\eta \in (\varphi \times \varphi)(\Phi)$ and $\rho \in (\varphi \times \varphi)(\Psi)$. Then $(\varphi \times \varphi)^{\leftarrow}(\eta) \in \Phi$ and $(\varphi \times \varphi)^{\leftarrow}(\rho) \in \Psi$. Hence $\top = \bigvee_{(x,y) \in X \times X} (\varphi \times \varphi)^{\leftarrow}(\eta) \circ (\varphi \times \varphi)^{\leftarrow}(\rho)(x,y) = \bigvee_{x,y,z \in X} (\varphi \times \varphi)^{\leftarrow}(\eta)(x,z) * (\varphi \times \varphi)^{\leftarrow}(\rho)(z,y) = \bigvee_{x,y,z \in X} \eta(\varphi(x), \varphi(z)) * \rho(\varphi(z), \varphi(y)) \leq \bigvee_{x,y \in X} \eta \circ \rho(\varphi(x), \varphi(y)) \leq \bigvee_{(u,v) \in Y \times Y} \eta \circ \rho(u,v)$. Hence $(\varphi \times \varphi)(\Phi) \circ (\varphi \times \varphi)(\Psi)$ exists.

To show the inequality, we first note that for $\phi, \psi \in L^{X \times X}$ we have

$$\begin{aligned}
(\varphi \times \varphi)(\phi) \circ (\varphi \times \varphi)(\psi)(u,v) &= \bigvee_{z \in X} (\varphi \times \varphi)(\phi)(u,z) * (\varphi \times \varphi)(\psi)(z,v) \\
&= \bigvee_{z \in X} \Big(\bigvee_{\varphi(x) = u, \varphi(y) = z} \phi(x,y) \Big) * \Big(\bigvee_{\varphi(x') = z, \varphi(y') = v} \psi(x', y') \Big) \\
&\geq \bigvee_{z \in X} \bigvee_{\varphi(x) = u, \varphi(x') = z, \varphi(y') = v} \phi(x, x') * \psi(x', y') \\
&= \bigvee_{\varphi(x) = u, \varphi(y') = v} \bigvee_{x' \in X} \phi(x, x') * \psi(x', y') \\
&= \bigvee_{\varphi(x) = u, \varphi(y') = v} \phi \circ \psi(x, y') \\
&= (\varphi \times \varphi)(\phi \circ \psi)(u,v).
\end{aligned}$$

Hence, if $\rho \in (\varphi \times \varphi)(\Phi) \circ (\varphi \times \varphi)(\Psi)$, then $\top = \bigvee_{\phi \in \Phi, \psi \in \Psi} [(\varphi \times \varphi)(\phi) \circ (\varphi \times \varphi)(\psi), \rho] \leq \bigvee_{\phi \in \Phi, \psi \in \Psi} [(\varphi \times \varphi)(\phi \circ \psi), \rho]$ and this implies $\rho \in (\varphi \times \varphi)(\Phi \circ \Psi)$. $\square$

We will later need the following bijection.

$$S : \begin{cases} (X \times X) \times (Y \times Y) & \longrightarrow \quad (X \times Y) \times (X \times Y) \\ ((x, x'), (y, y')) & \longmapsto \quad ((x, y), (x', y')) \end{cases}$$

Proposition 4.9. *Let $\Phi \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X)$ and $\Psi \in \mathsf{F}_{\mathsf{L}}^{\top}(Y \times Y)$. Then $S(\Phi^{-1} \times \Psi) = (S(\Phi \times \Psi^{-1}))^{-1}$.*

Proof. We first note that for $\phi \in L^{X \times X}$ and $\psi \in L^{Y \times Y}$ we have

$$\begin{aligned}
(S(\phi \times \psi))^{-1}((x_1, y_1), (x_2, y_2)) &= S(\phi \times \psi)(((x_2, y_2), (x_1, y_1)) \\
&= \phi \times \psi((x_2, x_1), (y_2, y_1)) \\
&= \phi(x_2, x_1) \wedge \psi(y_2, y_1) \\
&= \phi^{-1}(x_1, x_2) \wedge \psi^{-1}(y_1, y_2)) \\
&= \phi^{-1} \times \psi^{-1}((x_1, x_2), (y_1, y_2)) \\
&= S(\phi^{-1} \times \psi^{-1})((x_1, y_1), (x_2, y_2)).
\end{aligned}$$

A $\top$-filter base for $S(\Phi^{-1} \times \Psi)$ is given by

$$\{ S(\phi^{-1} \times \psi) \;:\; \phi \in \Phi, \psi \in \Psi \} = \{ (S(\phi \times \psi^{-1}))^{-1} \;:\; \phi \in \Phi, \psi \in \Psi \},$$

which is also a $\top$-filter base of $(S(\Phi \times \Psi^{-1}))^{-1}$. $\square$

Proposition 4.10. *Let $\Phi \in \mathsf{F}_\mathsf{L}^\top(X \times X)$ and $\Psi \in \mathsf{F}_\mathsf{L}^\top(Y \times Y)$ and denote $pr_X : X \times Y \longrightarrow X$ and $pr_Y : X \times Y \longrightarrow Y$ the projections. Then $\Phi \leq (pr_X \times pr_X)(S(\Phi \times \Psi))$ and $\Psi \leq (pr_Y \times pr_Y)(S(\Phi \times \Psi))$.*

Proof. We only show the first inequality. Let $\phi \in \Phi$. Then $S(\phi \times \top_{Y \times Y})$ is in the $\top$-filter base of $S(\Phi \times \Psi)$ and because

$$\begin{aligned}(pr_X \times pr_X)(S(\phi \times \top_{Y\times Y}))(x_1, x_2) &= \bigvee_{y_1,y_2 \in Y} S(\phi \times \top_{Y\times Y})((x_1, y_1), (x_2, y_2)) \\ &= \bigvee_{y_1,y_2 \in Y} \phi \times \top_{Y\times Y}((x_1, x_2), (y_1, y_2)) \\ &= \bigvee_{y_1,y_2 \in Y} \phi(x_1, x_2) \wedge \top_{Y\times Y}(y_1, y_2) \\ &= \phi(x_1, x_2),\end{aligned}$$

also ϕ is in this $\top$-filter base and hence in $(pr_X \times pr_X)(S(\Phi \times \Psi))$. ☐

Lemma 4.11. [15, Lemma 3.10] *Let $a, b \in L^{(X\times Y)\times(X\times Y)}$. Then $S((pr_X \times pr_X)(a) \times ((pr_Y \times pr_Y)(b)) \geq a \wedge b$.*

Proposition 4.12. *Let $\mathbb{H} \in \mathsf{F}_\mathsf{L}^\top((X \times Y) \times (X \times Y))$. Then $S((pr_X \times pr_X)(\mathbb{H}) \times (pr_Y \times pr_Y)(\mathbb{H})) \leq \mathbb{H}$.*

Proof. For an element of the $\top$-filter base of $S((pr_X \times pr_X)(\mathbb{H}) \times (pr_Y \times pr_Y)(\mathbb{H}))$, we have

$$S((pr_X \times pr_X)(h_1) \times ((pr_Y \times pr_Y)(h_2)) \geq h_1 \wedge h_2,$$

with $h_1, h_2 \in \mathbb{H}$. Both $\mathbb{H}$ and $S((pr_X \times pr_X)(\mathbb{H}) \times (pr_Y \times pr_Y)(\mathbb{H}))$ being $\top$-filters, this shows that all members of the $\top$-filter base are in $\mathbb{H}$, from which the claim follows. ☐

5 $\top$-uniform convergence spaces

Definition 5.1. [24] *A pair (X, Λ) is called a* $\top$-uniform convergence space *if $\Lambda \subseteq \mathsf{F}_\mathsf{L}^\top(X \times X)$ satisfies*
($\top$-UCS1) *$[(x, x)] \in \Lambda$ for all $x \in X$;*
($\top$-UCS2) *If $\Phi \leq \Psi$ and $\Phi \in \Lambda$, then $\Psi \in \Lambda$;*
($\top$-UCS3) *$\Phi \wedge \Psi \in \Lambda$ whenever $\Phi, \Psi \in \Lambda$;*
($\top$-UCS4) *$\Phi \circ \Psi \in \Lambda$ whenever $\Phi, \Psi \in \Lambda$ and $\Phi \circ \Psi$ exists;*
($\top$-UCS5) *$\Phi^{-1} \in \Lambda$ whenever $\Phi \in \Lambda$.*
A mapping $\varphi : (X, \Lambda) \longrightarrow (X', \Lambda')$ is called uniformly continuous *if $(\varphi \times \varphi)(\Phi) \in \Lambda'$ whenever $\Phi \in \Lambda$.*

It is clear that the identity mapping $id_X : (X, \Lambda) \longrightarrow (X, \Lambda)$ is uniformly continuous and that for two uniformly continuous mappings $\varphi : (X, \Lambda) \longrightarrow (X', \Lambda')$, $\psi : (X', \Lambda') \longrightarrow (X'', \Lambda'')$ the composition $\psi \circ \varphi : (X, \Lambda) \longrightarrow (X'', \Lambda'')$ is uniformly continuous. We denote the category which has as objects the $\top$-uniform convergence spaces and as morphisms the uniformly continuous mappings by $\top$-UCTS.

For a constant mapping $\varphi_0 : (X, \Lambda) \longrightarrow (X', \Lambda')$, $\varphi_0(x) = y_0$ for all $x \in X$, we have for $a \in L^{X\times X}$ that $(\varphi_0 \times \varphi_0)(a)(y_1, y_2) = \bigvee_{u,v\in X} a(u, v)$ if $y_1 = y_2 = y_0$ and $= \bot$ otherwise. For $b \in [(y_0, y_0)]$ we have $b(y_0, y_0) = \top$ and hence for any $\Phi \in \Lambda$ and $a \in \Phi$ we have $b \geq (\varphi_0 \times \varphi_0)(a)$ and hence $b \in (\varphi_0 \times \varphi_0)(\Phi)$, i.e. we have $[(y_0, y_0)] \leq (\varphi_0 \times \varphi_0)(\Phi)$. Therefore, by ($\top$-UCS1) and ($\top$-UCS2), the constant mapping φ_0 is uniformly continuous.

Proposition 5.2. *The category $\top$-UCTS is well-fibred and topological over SET.*

Proof. There is only one $\top$-filter $\Phi = [(x, x)]$ in $\mathsf{F}_L^\top(\{x\} \times \{x\})$ and the set of all $\top$-uniform convergence structures on a set X is a subset of $\{0, 1\}^{(\{0,1\}^{\mathsf{F}_\mathsf{L}^\top(X\times X)})}$, i.e. is a set. So it is sufficient to describe the initial structures. For a family of mappings $(\varphi_j : X \longrightarrow (X_j, \Lambda_j))_{j\in J}$ we define $\Lambda \subseteq \mathsf{F}_\mathsf{L}^\top(X \times X)$ by

$$\Phi \in \Lambda \iff (\varphi_j \times \varphi_j)(\Phi) \in \Lambda_j \ \forall j \in J.$$

Then $(X, \Lambda) \in |\top\text{-}\mathsf{UCS}|$. The axiom ($\top$-UCS4) follows from Proposition 4.8 and ($\top$-UCS5) follows from Proposition 4.3. The other axioms are easy. Let now $(X', \Lambda') \in |\top\text{-}\mathsf{UCS}|$ and consider a mapping $\psi : X' \longrightarrow X$ such that $\varphi_j \circ \psi : (X', \Lambda') \longrightarrow (X_j, \Lambda_j)$ is uniformly continuous for all $j \in J$. For $\Phi' \in \Lambda'$ then $(\varphi_j \circ \psi) \times (\varphi_j \circ \psi)(\Phi') \in \Lambda_j$ for all $j \in J$ and because $(\varphi_j \circ \psi) \times (\varphi_j \circ \psi) = (\varphi_j \times \varphi_j) \circ (\psi \times \psi)$ we see that $(\psi \times \psi)(\Phi') \in \Lambda$ and ψ is uniformly continuous. ☐

We will later need the product of two $\top$-uniform convergence spaces $(X,\Lambda),(Y,\Sigma)$. This is denoted by $(X\times Y,\Lambda\times\Sigma)$ or by $(X,\Lambda)\times(Y,\Sigma)$ with the product structre $\Lambda\times\Sigma$ as the initial $\top$-uniform convergence structure on $X\times Y$ with respect to the projections $pr_X : X\times Y \longrightarrow X$ and $pr_Y : X\times Y \longrightarrow Y$, i.e. we have $\Phi\in\Lambda\times\Sigma$ if and only if $(pr_X\times pr_X)(\Phi)\in\Lambda$ and $(pr_Y\times pr_Y)(\Phi)\in\Sigma$.

We now turn to function spaces. For $(X,\Lambda),(X',\Lambda')\in|\top\text{-}\mathsf{UCS}|$ we denote

$$UC=\{\varphi:(X,\Lambda)\longrightarrow(X',\Lambda') \text{ uniformly continuous}\}.$$

Further we denote the *evaluation mapping* $\mathrm{ev}: UC\times X\longrightarrow X'$, $\mathrm{ev}(\varphi,x)=\varphi(x)$ and $S:(UC\times UC)\times(X\times X)\longrightarrow (UC\times X)\times(UC\times X)$, $S((\varphi,\psi),(x,x'))=((\varphi,x),(\psi,x'))$, so that $(\mathrm{ev}\times\mathrm{ev})\circ S((\varphi,\psi),(x,x'))=(\varphi(x),\psi(x'))$.

Proposition 5.3. *Let $\mathsf{L}=(L,\le,*)$ be a commutative and integral quantale which is divisible or is a value quantale. Let $\mathbb{H},\mathbb{K}\in\mathsf{F}^\top_\mathsf{L}(UC\times UC)$ and $\Phi\in\mathsf{F}^\top_\mathsf{L}(X\times X)$. If $\mathbb{H}\circ\mathbb{K}$ exists, then $((\mathrm{ev}\times\mathrm{ev})(S(\mathbb{H}\times\Phi)))\circ((\mathrm{ev}\times\mathrm{ev})(S(\mathbb{K}\times(\Phi^{-1}\circ\Phi))))$ exists and is $\le((\mathrm{ev}\times\mathrm{ev})(S((\mathbb{H}\circ\mathbb{K})\times\Phi)))$.*

Proof. It is shown in [15] that for $h,k\in L^{UC\times UC}$ amd $\phi\in L^{X\times X}$ we have

$$(\mathrm{ev}\times\mathrm{ev})(S(h\circ k)\times\phi))\le((\mathrm{ev}\times\mathrm{ev})(S(h\times\phi)))\circ((\mathrm{ev}\times\mathrm{ev})(S(h\times(\phi^{-1}\circ\phi)))).$$

Therefore, if $\eta\in((\mathrm{ev}\times\mathrm{ev})(S(\mathbb{H}\times\Phi)))\circ((\mathrm{ev}\times\mathrm{ev})(S(\mathbb{K}\times(\Phi^{-1}\circ\Phi))))$, then

$$\begin{aligned}\top &= \bigvee_{h\in\mathbb{H},k\in\mathbb{K},\phi\in\Phi}[((\mathrm{ev}\times\mathrm{ev})(S(h\times\phi)))\circ((\mathrm{ev}\times\mathrm{ev})(S(h\times(\phi^{-1}\circ\phi)))),\eta]\\ &\le \bigvee_{h\in\mathbb{H},k\in\mathbb{K},\phi\in\Phi}[(\mathrm{ev}\times\mathrm{ev})(S(h\circ k)\times\phi)),\eta],\end{aligned}$$

which means that $\eta\in((\mathrm{ev}\times\mathrm{ev})(S((\mathbb{H}\circ\mathbb{K})\times\Phi)))$. □

We now define $\Lambda_C\subseteq\mathsf{F}^\top_L(UC\times UC)$ by

$$\mathbb{H}\in\Lambda_c \iff \forall\Phi\in\mathsf{F}^\top_\mathsf{L}(X\times X):\Phi\in\Lambda\Rightarrow(\mathrm{ev}\times\mathrm{ev})(S(\mathbb{H}\times\Phi))\in\Lambda'.$$

Proposition 5.4. *Let $\mathsf{L}=(L,\le,*)$ be a commutative and integral quantale which is divisible or is a value quantale. Let $(X,\Lambda),(X',\Lambda')\in|\top\text{-}\mathsf{UCS}|$. Then $(UC,\Lambda_C)\in|\top\text{-}\mathsf{UCS}|$.*

Proof. ($\top$-UCS1) It is shown in [15] that for $\phi\in L^{X\times X}$ we have $(\varphi\times\varphi)(\phi)=(\mathrm{ev}\times\mathrm{ev})(S(\top_{(\varphi,\varphi)}\times\phi))$. Hence, for $\Phi\in\mathsf{F}^\top_L(X\times X)$ and $a\in(\varphi\times\varphi)(\Phi)$, we have $\top=\bigvee_{\phi\in\Phi}[(\varphi\times\varphi)(\phi),a]=\bigvee_{\phi\in\Phi}[(\mathrm{ev}\times\mathrm{ev})(S(\top_{(\varphi,\varphi)}\times\phi)),a]$ and consequently $a\in(\mathrm{ev}\times\mathrm{ev})(S([(\varphi,\varphi)]\times\Phi))$. Therefore, as $(\varphi\times\varphi)(\Phi)\in\Lambda'$ we conclude $(\mathrm{ev}\times\mathrm{ev})(S([(\varphi,\varphi)]\times\Phi))\in\Lambda'$ and we get $[(\varphi,\varphi)]\in\Lambda_C$.

($\top$-UCS2) and ($\top$-UCS3)are clear.

($\top$-UCS4) Let $\mathbb{H},\mathbb{K}\in\Lambda_C$ and let $\mathbb{H}\circ\mathbb{K}$ exist and let $\Phi\in\Lambda$. We fix $x\in X$. Then, by ($\top$-UCS1), ($\top$-UCS5) and ($\top$-UCS3) also $\Phi\wedge[(x,x)],\Phi^{-1}\wedge[(x,x)]\in\Lambda$. As $[(x,x)]\circ[(x,x)]$ exists we conclude by ($\top$-UCS4) that $(\Phi^{-1}\wedge[(x,x)])\circ(\Phi\wedge[(x,x)])\in\Lambda$ and therefore, by definition of Λ_C, $(\mathrm{ev}\times\mathrm{ev})(S(\mathbb{H}\times(\Phi\wedge[(x,x)])))\in\Lambda'$ and $(\mathrm{ev}\times\mathrm{ev})(S(\mathbb{K}\times((\Phi^{-1}\wedge[(x,x)])\circ(\Phi\wedge[(x,x)]))))\in\Lambda'$. From Proposition 5.3 and ($\top$-UCS2) we then conclude that also $(\mathrm{ev}\times\mathrm{ev})(S((\mathbb{H}\circ\mathbb{K})\times\Phi))\in\Lambda'$ and hence $\mathbb{H}\circ\mathbb{K}\in\Lambda_C$.

($\top$-UCS5) Let $\mathbb{H}\in\Lambda_C$. If $\Phi\in\Lambda$, then $\Phi^{-1}\in\Lambda$ and hence $(\mathrm{ev}\times\mathrm{ev})(S(\mathbb{H}\times\Phi^{-1}))\in\Lambda'$. From Proposition 4.9 we deduce $\Lambda'\ni((\mathrm{ev}\times\mathrm{ev})(S(\mathbb{H}\times\Phi^{-1})))^{-1}=(\mathrm{ev}\times\mathrm{ev})(S(\mathbb{H}^{-1}\times\Phi))$ and therefore $\mathbb{H}^{-1}\in\Lambda_C$. □

Λ_C is called the $\top$-*uniform convergence structure of uniform continuous convergence.*

Proposition 5.5. *Let $\mathsf{L}=(L,\le,*)$ be a commutative and integral quantale which is divisible or is a value quantale. Let $(X,\Lambda),(X',\Lambda')\in|\top\text{-}\mathsf{UCS}|$. Then $\mathrm{ev}:(UC\times X,\Lambda_C\times\Lambda)\longrightarrow(X',\Lambda')$ is uniformly continuous.*

Proof. Let $\mathbb{H}\in\Lambda_C\times\Lambda$. Then $(pr_{UC}\times pr_{UC})(\mathbb{H})\in\Lambda_C$ and $(pr_X\times pr_X)(\mathbb{H})\in\Lambda$. The definition of Λ_C then yields $(\mathrm{ev}\times\mathrm{ev})(S((pr_{UC}\times pr_{UC})(\mathbb{H})\times(pr_X\times pr_X)(\mathbb{H})))\in\Lambda'$ and hence also the finer $\top$-filter $(ev\times ev)(\mathbb{H})\in\Lambda'$. □

We consider now a mapping $\varphi:X\times Y\longrightarrow Z$ and $x\in X$ and we define

$$\varphi_x:\left\{\begin{array}{ccc} Y & \longrightarrow & Z\\ y & \longmapsto & \varphi(x,y)\end{array}\right. \quad\text{and}\quad \varphi^*:\left\{\begin{array}{ccc} X & \longrightarrow & Z^Y\\ x & \longmapsto & \varphi_x\end{array}\right. .$$

Then

$$E:\left\{\begin{array}{ccc} Z^{X\times Y} & \longrightarrow & (Z^Y)^X\\ \varphi & \longmapsto & \varphi^*\end{array}\right.$$

is called the *exponential map.*

Lemma 5.6. *Let $x \in X$ and $\phi \in L^{X\times X}$ with $\phi(x,x) = \top$ and $\psi \in L^{Y\times Y}$ and $\varphi : X \times Y \longrightarrow Z$. Then $(\varphi_x \times \varphi_x)(\psi) \leq (\varphi \times \varphi)(S(\phi \times \psi))$.*

Proof. We have for $z_1, z_2 \in Z$

$$\begin{aligned}(\varphi\times\varphi)(S(\phi\times\psi))(z_1,z_2) &= \bigvee_{\varphi(x_1,y_1)=z_1,\varphi(x_2,y_2)=z_2} \phi(x_1,x_2)\wedge\psi(y_1,y_2)\\ &\geq \bigvee_{\varphi(x,y_1)=z_1,\varphi(x,y_2)=z_2} \underbrace{\phi(x,x)}_{=\top}\wedge\psi(y_1,y_2)\\ &= \bigvee_{\varphi_x(y_1)=z_1,\varphi_x(y_2)=z_2} \psi(y_1,y_2)\\ &= (\varphi_x\times\varphi_x)(\psi)(z_1,z_2).\end{aligned}$$

□

Proposition 5.7. *Let $x \in X$ and $\Psi \in \mathsf{F}_{\mathsf{L}}^{\top}(Y\times Y)$ and $\varphi : X \times Y \longrightarrow Z$. Then $(\varphi\times\varphi)(S([(x,x)]\times\Psi)) \leq (\varphi_x\times\varphi_x)(\Psi)$.*

Proof. Let $\rho \in (\varphi\times\varphi)(S([(x,x)]\times\Psi))$. Then

$$\top = \bigvee_{\phi(x,x)=\top,\psi\in\Psi} [(\varphi\times\varphi)(S(\phi\times\psi)),\rho] \leq \bigvee_{\phi(x,x)=\top,\psi\in\Psi} [(\varphi_x\times\varphi_x)(\psi),\rho],$$

and hence $\rho \in (\varphi_x\times\varphi_x)(\Psi)$. □

Proposition 5.8. *Let $\varphi : (X,\Lambda)\times(Y,\Sigma) \longrightarrow (Z,\Gamma)$ be uniformly continuous and let $x \in X$. Then also $\varphi_x : (Y,\Sigma) \longrightarrow (Z,\Gamma)$ is uniformly continuous.*

Proof. Let $\Psi \in \Sigma$. As $[(x,x)] \in \Lambda$ and $(pr_X \times pr_X)(S([(x,x)]\times\Psi)) \geq [(x,x)]$ and $(pr_Y\times pr_Y)(S([(x,x)]\times\Psi)) \geq \Psi$, then $S([(x,x)]\times\Psi) \in \Lambda\times\Sigma$ and by the uniform continuity of φ we get $(\varphi\times\varphi)(S([(x,x)]\times\Psi)) \in \Gamma$. Proposition 5.7 then yields $(\varphi_x\times\varphi_x)(\Psi) \in \Gamma$. □

Proposition 5.9. *Let $\mathsf{L} = (L,\leq,*)$ be a commutative and integral quantale which is divisible or is a value quantale. Let $\varphi : (X,\Lambda)\times(Y,\Sigma) \longrightarrow (Z,\Gamma)$ be uniformly continuous. Then $E(\varphi) = \varphi^* : (X,\Lambda) \longrightarrow (UC(Y,Z),\Lambda_C)$ is uniformly continuous.*

Proof. By Proposition 5.8, φ^* is well-defined. Let $\Phi \in \Lambda$. We have to show that $(\varphi^*\times\varphi^*)(\Phi) \in \Lambda_C$. Let $\Psi \in \Sigma$. As φ is uniformly continuous we conclude that $(\varphi\times\varphi)(S(\Phi\times\Psi)) \in \Gamma$. Now we note that for $\phi \in L^{X\times X}$ and $\psi \in L^{Y\times Y}$ we have $(\mathrm{ev}\times\mathrm{ev})(S((\varphi^*\times\varphi^*)(\phi)\times\psi)) = (\varphi\times\varphi)(S(\phi\times\psi))$ [15]. Hence $(\varphi\times\varphi)(S(\Phi\times\Psi)) = (\mathrm{ev}\times\mathrm{ev})(S((\varphi^*\times\varphi^*)(\Phi)\times\Psi))$ and by definition of Λ_C we get $(\varphi^*\times\varphi^*)(\Phi) \in \Lambda_C$. □

Putting everything together, we obtain the main theorem of this section.

Theorem 5.10. *Let $\mathsf{L} = (L,\leq,*)$ be a commutative and integral quantale which is divisible or is a value quantale. Then the category $\top$-UCS is Cartesian closed.*

Proof. We have seen above that $\top$-UCS has function spaces in the sense of [1]. As a well-fibred topological category, it is therefore Cartesian closed. □

6 The underlying $\top$-convergence space

Let $(X,\Lambda) \in |\top\text{-}\mathsf{UCS}|$. We define $q^\Lambda : \mathsf{F}_{\mathsf{L}}^{\top}(X) \longrightarrow \mathsf{P}(X)$ by $x \in q^\Lambda(\mathbb{F}) \iff \mathbb{F}\times[x] \in \Lambda$.

Proposition 6.1. *Let $(X,\Lambda) \in |\top\text{-}\mathsf{UCS}|$. Then (X,q^Λ) is a $\top$-limit space [6], i.e. satisfies the axioms*
($\top$-LS1) $x \in q^\Lambda([x])$ *for all* $x \in X$;
($\top$-LS2) $q^\Lambda(\mathbb{F}) \subseteq q^\Lambda(\mathbb{G})$ *whenever* $\mathbb{F} \leq \mathbb{G}$;
($\top$-LS3)$q^\Lambda(\mathbb{F}) \cap q^\Lambda(\mathbb{G}) \subseteq q^\Lambda(\mathbb{F}\wedge\mathbb{G})$.

Proof. ($\top$-LS1) follows as $[x]\times[x] = [(x,x)]$, ($\top$-LS2) follows as $\mathbb{F} \leq \mathbb{G}$ implies $\mathbb{F}\times[x] \leq \mathbb{G}\times[x]$ and ($\top$-LS3) finally follows from $(\mathbb{F}\times[x])\wedge(\mathbb{G}\times[x]) = (\mathbb{F}\wedge\mathbb{G})\times[x]$. □

Proposition 6.2. *Let $\varphi(X,\Lambda) \longrightarrow (X',\Lambda')$ be uniformly continuous. Then $\varphi : (X, q^\Lambda) \longrightarrow (X', q^{\Lambda'})$ is continuous, i.e. $\varphi(x) \in q^{\Lambda'}(\varphi(\mathbb{F}))$ whenever $x \in q^\Lambda(\mathbb{F})$.*

Proof. This follows from $(\varphi \times \varphi)(\mathbb{F} \times [x]) = \varphi(\mathbb{F}) \times \varphi([x]) = \varphi(\mathbb{F}) \times [\varphi(x)]$. □

Denoting the category of $\top$-limit spaces with continuous mappings as morphisms by $\top$-LS, we have a functor

$$\mathsf{H} : \begin{cases} \top\text{-}\mathsf{UCS} & \longrightarrow & \top\text{-}\mathsf{LS} \\ (X,\Lambda) & \longmapsto & (X, q^\Lambda) \\ \varphi & \longmapsto & \varphi \end{cases}$$

The category $\top$-LS is topological and initial constructions are done as follows [6]. For a source $(\varphi_j : X \longrightarrow (X_j, q_j))_{j\in J}$, the initial structure q on X is defined by

$$x \in q(\mathbb{F}) \iff \varphi_j(x) \in q_j(\varphi_j(\mathbb{F})) \text{ for all } j \in J.$$

The functor H preserves initial constructions.

Proposition 6.3. *Let $(\varphi_j : X \longrightarrow (X_j, \Lambda_j))_{j\in J}$ be a source in $\top$-UCS and let Λ be the initial structure on X. Then q^Λ is the initial structure in $\top$-LS for the source $(\varphi_j : X \longrightarrow (X_j, q^{\Lambda_j}))_{j\in J}$.*

Proof. Let q be the initial structure for the source $(\varphi_j : X \longrightarrow (X_j, q^{\Lambda_j}))_{j\in J}$. We then have $x \in q(\mathbb{F})$ if and only if $\varphi_j(x) \in q^{\Lambda_j}(\varphi_j(\mathbb{F}))$ for all $j \in J$ if and only if $(\varphi_j \times \varphi_j)(\mathbb{F} \times [x]) = \varphi_j(\mathbb{F}) \times \varphi_j([x]) \in \Lambda_j$ for all $j \in J$ if and only if $\mathbb{F} \times [x] \in \Lambda$ if and only if $x \in q^\Lambda(\mathbb{F})$. □

7 Example: Probabilistic uniform spaces

Definition 7.1. [11, 28] *A $\top$-filter $\mathcal{U} \in \mathsf{F}^\top_\mathsf{L}(X \times X)$ is called a* probabilistic uniformity *on X if*
($\top$-U1) *$U(x,x) = \top$ for all $U \in \mathcal{U}$, $x \in X$;*
($\top$-U2) *$\bigvee_{V\in\mathcal{U}}[V \circ V, U] = \top$ for all $U \in \mathcal{U}$;*
($\top$-U3) *$U^{-1} \in \mathcal{U}$ for all $U \in \mathcal{U}$.*
The pair $(X,\mathcal{U})$ is called a probabilistic uniform space. *A mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ is called* uniformly continuous *if $(\varphi \times \varphi)^\leftarrow(U') \in \mathcal{U}$ whenever $U' \in \mathcal{U}'$. The category of probabilistic uniform spaces with uniformly continuous mappings is denoted by $\top$-UNIF.*

The axioms can also be written more concisely as follows. ($\top$-U1) $\mathcal{U} \le [(x,x)]$ for all $x \in X$; ($\top$-U2) $\mathcal{U} \le \mathcal{U} \circ \mathcal{U}$ and ($\top$-U3) $\mathcal{U} \le \mathcal{U}^{-1}$, and the uniform continuity of a mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ can be characterized by $(\varphi \times \varphi)(\mathcal{U}) \ge \mathcal{U}'$.

For a probabilistic uniform space $(X,\mathcal{U})$ we define $\Lambda^\mathcal{U} = \{\Phi \in \mathsf{F}^\top_\mathsf{L}(X \times X) \,:\, \mathcal{U} \le \Phi\}$.

Proposition 7.2. *Let $(X,\mathcal{U}) \in |\top\text{-}\mathsf{UNIF}|$. Then $(X,\Lambda^\mathcal{U}) \in |\top\text{-}\mathsf{UCS}|$.*

Proof. The proof is easy. □

Proposition 7.3. *Let $(X,\mathcal{U}), (X',\mathcal{U}') \in |\top\text{-}\mathsf{UNIF}|$ and let $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ be uniformly continuious. Then $\varphi : (X,\Lambda^\mathcal{U}) \longrightarrow (X',\Lambda^{\mathcal{U}'})$ is uniformly continuous.*

Proof. Let $\Phi \in \Lambda^\mathcal{U}$. Then $\Phi \ge \mathcal{U}$ and hence $(\varphi \times \varphi)(\Phi) \ge (\varphi \times \varphi)(\mathcal{U}) \ge \mathcal{U}'$ and therefore $(\varphi \times \varphi)(\Phi) \in \Lambda^{\mathcal{U}'}$. □

As a result, we can define a functor

$$\mathsf{F} : \begin{cases} \top\text{-}\mathsf{UNIF} & \longrightarrow & \top\text{-}\mathsf{UCS} \\ (X,\mathcal{U}) & \longmapsto & (X,\Lambda^\mathcal{U}) \\ \varphi & \longmapsto & \varphi \end{cases} .$$

We consider now $(X,\Lambda) \in |\top\text{-}\mathsf{UCS}|$ and define $\mathcal{U}^\Lambda = \bigwedge_{\Phi\in\Lambda} \Phi$. A $\top$-uniform convergence space (X,Λ) for which $\mathcal{U}^\Lambda \in \Lambda$ is called a *principal $\top$-uniform convergence space.* We denote the subcategory of principal $\top$-uniform convergence spaces by $\top$-PUCS. Clearly, for a probabilistic uniform space $(X,\mathcal{U})$, the space $(X,\Lambda^\mathcal{U})$ is principal and we have $\mathcal{U}^{(\Lambda^\mathcal{U})} = \mathcal{U}$. Hence the functor F actually ranges in $\top$-PUCS.

Proposition 7.4. *For a principal $\top$-uniform convergence space, $(X,\mathcal{U}^\Lambda)$ is a probabilistic uniform space.*

Proof. (⊤-U1) As $[(x,x)] \in \Lambda$ we have $\mathcal{U}^\Lambda \le [(x,x)]$.

(⊤-U2) We have $\mathcal{U}^\Lambda \in \Lambda$. Because $\mathcal{U}^\Lambda \le [(x,x)]$, then $\mathcal{U}^\Lambda \circ \mathcal{U}^\Lambda$ exists and hence $\mathcal{U}^\Lambda \circ \mathcal{U}^\Lambda \in \Lambda$. This implies $\mathcal{U}^\Lambda \le \mathcal{U}^\Lambda \circ \mathcal{U}^\Lambda$.

(⊤-U3) $\mathcal{U}^\Lambda \in \Lambda$ implies $(\mathcal{U}^\Lambda)^{-1} \in \Lambda$ and therefore $\mathcal{U}^\Lambda \le (\mathcal{U}^\Lambda)^{-1}$. □

Proposition 7.5. *Let $(X,\Lambda), (X',\Lambda') \in |\top\text{-}\mathsf{UCS}|$ be principal and let $\varphi : (X,\Lambda) \longrightarrow (X',\Lambda')$ be uniformly continuous. Then $\varphi : (X,\mathcal{U}^\Lambda) \longrightarrow (X',\mathcal{U}^{\Lambda'})$ is uniformly continuous.*

Proof. Let $U' \in \mathcal{U}^{\Lambda'}$. Then $U' \in \Psi$ for all $\Psi \in \Lambda'$ and hence $U' \in (\varphi\times\varphi)(\Phi)$ for all $\Phi \in \Lambda$. We conclude $U' \in \bigwedge_{\Phi\in\Lambda}(\varphi\times\varphi)(\Phi) = (\varphi\times\varphi)(\bigwedge_{\Phi\in\Lambda}\Phi) = (\varphi\times\varphi)(\mathcal{U}^\Lambda)$. Hence $(\varphi\times\varphi)^{\leftarrow}(U') \in (\varphi\times\varphi)^{\leftarrow}((\varphi\times\varphi)(\mathcal{U}^\Lambda)) \le \mathcal{U}^\Lambda$. □

As a result, we have a functor

$$\mathsf{G} : \begin{cases} \top\text{-}\mathsf{PUCS} & \longrightarrow & \top\text{-}\mathsf{UNIF} \\ (X,\Lambda) & \longmapsto & (X,\mathcal{U}^\Lambda) \\ \varphi & \longmapsto & \varphi \end{cases} .$$

Proposition 7.6. *Let $(X,\Lambda) \in \top\text{-}\mathsf{UCS}$. Then $\Lambda^{(\mathcal{U}^\Lambda)} \le \Lambda$ and if (X,Λ) is principal, then $\Lambda = \Lambda^{(\mathcal{U}^\Lambda)}$.*

Proof. If $\Phi \in \Lambda$, then $\Phi \ge \mathcal{U}^\Lambda$ and hence $\Phi \in \Lambda^{(\mathcal{U}^\Lambda)}$. If (X,Λ) is principal, then $\Phi\in\Lambda$ is equivalent to $\Phi \ge \mathcal{U}^\Lambda$ which is equivalent to $\Phi \in \Lambda^{(\mathcal{U}^\Lambda)}$. □

Theorem 7.7. *The category ⊤-UNIF is isomorphic with the category ⊤-PUCS and the latter is a reflective subcategory of ⊤-UCS.*

Remark 7.8 (Convergence in ⊤-UNIF)**.** *For $(X,\mathcal{U}) \in |\top\text{-}\mathsf{UNIF}|$ there is a natural ⊤-limit structure. We define*

$$x \in q^{\mathcal{U}}(\mathbb{F}) \iff x \in q^{\Lambda^{\mathcal{U}}}(\mathbb{F}) \iff \mathbb{F}_x = \mathbb{F}\times[x] \ge \mathcal{U}.$$

Now we note that $\{u(\cdot,x) \;:\; u\in\mathcal{U}\}$ is a ⊤-filter base [29] and we denote $\mathcal{U}(x)$ the generated ⊤-filter. Then we have $\mathcal{U} \le \mathbb{F}_x$ if and only if $\mathcal{U}(x) \le \mathbb{F}$: If $\mathcal{U}\le\mathbb{F}_x$ and $a \in \mathcal{U}(x)$, then $\top = \bigvee_{u\in\mathcal{U}}[u(\cdot,x),a] \le \bigvee_{d(\cdot,x)\in\mathbb{F}}[d(\cdot,x),a] \le \bigvee_{f\in\mathbb{F}}[f,a]$ and hence $a\in\mathbb{F}$. Conversely, if $\mathcal{U}(x)\le\mathbb{F}$ and $u\in\mathcal{U}$, then $u(\cdot,x)\in\mathcal{U}(x)\subseteq\mathbb{F}$ and hence $u\in\mathbb{F}_x$.

So we have $x\in q^{\mathcal{U}}(\mathbb{F})$ if and only if $\mathcal{U}(x)\le\mathbb{F}$. Note that $\mathcal{U}(x)$ is a so-called conical ⊤-neighbourhood filter of x, see [29]*. This means that the convergence in a probabilistic uniform space is the convergence of a fuzzy topological space with conical neighbourhood systems [16], see also [19].*

8 Example: L-metric spaces

An L-*metric space* is a pair (X,d) of a set X and an L-*metric* $d : X\times X \longrightarrow L$ such that

(LM1) $d(x,x) = \top$ for all $x\in X$ (*reflexivity*);

(LM2) $d(x,y) * d(y,z) \le d(x,z)$ for all $x,y,z\in X$ (*transitivity*).

A mapping between two L-metric spaces, $f : (X,d) \longrightarrow (X',d')$ is called an L-*metric morphism* if $d(x_1,x_2) \le d'(f(x_1),f(x_2))$ for all $x_1,x_2\in X$. If the L-metric satisfies $d(x,y)=d(y,x)$ for all $x,y\in X$, it is called *symmetric*. We denote the category of symmetric L-metric spaces with L-metric morphisms by sL-MET.

Other names for L-metric spaces are *continuity spaces* [7], L-*categories* [17, 14], or L-*preordered sets* [30]. In case $L=\{0,1\}$, an L-metric space is a preordered set. If $\mathsf{L} = ([0,\infty],\ge,+)$, an L-metric space is a quasimetric space. If $\mathsf{L} = (\Delta^+,\le,*)$, an L-metric space is a probabilistic quasimetric space, see [7].

Classically, a metric space induces a uniform convergence space by defining $\Phi\in\Lambda^d$ if Φ is finer than the uniformity on X generated by the sets $B^d(\epsilon) = \{(x,y) \;:\; d(x,y)<\epsilon\}$, i.e. if for all $\epsilon>0$ there is $\phi\in\Phi$ such that $\phi\subseteq B^d(\epsilon)$. Using Lawvere's quantale $\mathsf{L} = ([0,\infty],\ge,+)$, this can be "translated" as

$$\Phi\in\Lambda^d \iff \forall\epsilon\lhd\top : \bigvee_{\phi\in\Phi}\bigwedge_{(x,y)\in X\times X}(\phi(x,y)\to d(x,y)) \rhd \epsilon.$$

As $L=[0,\infty]$ is completely distributive, this is equivalent to

$$\Phi\in\Lambda^d \iff \bigvee_{\phi\in\Phi}[\phi,d] = \top,$$

which is, Φ being a $\top$-filter, the same as

$$\Phi \in \Lambda^d \iff d \in \Phi.$$

Hence, for a general symmetric L-metric space, we define Λ^d in this way: $\Phi \in \Lambda^d$ if $d \in \Phi$.

We note that $[d] = \{a \in L^{X\times X} \ : \ d \leq a\}$ is a $\top$-filter and that $d \in \Phi$ is equivalent to $[d] \leq \Phi$. In this way, the L-metric generates a probabilistic uniformity $\mathcal{U}^d = [d]$.

Proposition 8.1. *Let (X, d) be a symmetric L-metric space. Then $\mathcal{U}^d = [d]$ is a probabilistic uniformity.*

Proof. ($\top$-U0) We have $U \in [d]$ if $d \leq U$ and by (LM1) then $U(x, x) \geq d(x, x) = \top$.

($\top$-U1) The transitivity (LM2) implies $d \circ d(x, y) = \bigvee_{z\in X} d(x, z) * d(z, y) \leq d(x, y)$ for all $x, y \in X$, i.e. we have $d \circ d \leq d$. For $U \in [d]$, then $d \circ d \leq d \leq U$ and hence $\bigvee_{V \in[d]}[V \circ V, U] \geq [d \circ d, U] = \top$.

($\top$-U3) If $U \in [d]$, then $d \leq U$. The symmetry yields $d^{-1}(x, y) = d(y, x) = d(x, y)$ and hence $d = d^{-1} \leq U^{-1}$ and $U^{-1} \in [d]$. □

Proposition 8.2. *Let $(X, d), (X', d')$ be symmetric L-metric spaces and let $\varphi : X \longrightarrow X'$ be an L-metric morphism. Then $\varphi : (X, [d]) \longrightarrow (X', [d'])$ is uniformly continuous.*

Proof. Let $U' \in [d']$. Then $d' \leq U'$ and hence $d \leq (\varphi \times \varphi)^{\leftarrow}(d') \leq (\varphi \times \varphi)^{\leftarrow}(U')$ and $(\varphi \times \varphi)^{\leftarrow}(U') \in [d]$. □

Hence we have a functor

$$\mathsf{D} : \begin{cases} \mathsf{sL\text{-}MET} & \longrightarrow & \top\text{-}\mathsf{UNIF} \\ (X, d) & \longmapsto & (X, [d]) \\ \varphi & \longmapsto & \varphi \end{cases} .$$

Clearly, for $d \neq d'$ we have $[d] \neq [d']$, i.e. the functor D is injective on objects.

Remark 8.3 (Convergence for $\mathsf{sL\text{-}MET}$)**.** *We can define a $\top$-limit structure for a symmetric L-metric space (X, d) in a natural way by defining $x \in q^d(\mathbb{F}) \iff \mathbb{F}\times[x] \geq [d]$. This is equivalent to $x \in q^d(\mathbb{F}) \iff d(\cdot, x) \in \mathbb{F} \iff [d(\cdot, x)] \leq \mathbb{F}$.*

9 Embedding $\top$-UCS into sL-UCS

In this section we assume that the underlying lattice $(L, \leq)$ is a frame.

Definition 9.1. [13] *A mapping $\mathcal{F} : L^X \longrightarrow L$ is a* stratified L-filter *on X if $\mathcal{F}$ satisfies:*

(LF0) $\mathcal{F}(\top_X) = \top, \quad \mathcal{F}(\bot_X) = \bot$,

(LF1) $a_1, a_2 \in L^X$, $a_1 \leq a_2 \Longrightarrow \mathcal{F}(a_1) \leq \mathcal{F}(a_2)$,

(LF2) $\mathcal{F}(a_1) \wedge \mathcal{F}(a_2) \leq \mathcal{F}(a_1 \wedge a_2)$ *for all* $a_1, a_2 \in L^X$,

(LFS) *for all* $\alpha \in L$, *for all* $a \in L^X$, $\quad \alpha * \mathcal{F}(a) \leq \mathcal{F}(\alpha_X * a)$.

The point filter $[x]_s : L^X \longrightarrow L, a \longmapsto a(x)$ is a stratified L-filter for every $x \in X$.

The set of all stratified L-filters on X is denoted by $\mathsf{F}^s_{\mathsf{L}}(X)$. On $\mathsf{F}^s_{\mathsf{L}}(X)$, a partial order can be defined by: $\mathcal{F} \leq \mathcal{G} \iff \mathcal{F}(a) \leq \mathcal{G}(a) \quad \forall\, a \in L^X$.

Let X and Y be sets, $\varphi : X \longrightarrow Y$ and $\mathcal{F} \in \mathsf{F}^s_{\mathsf{L}}(X)$. The *image of $\mathcal{F}$ under φ*, $\varphi^{\rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$, is always a stratified L-filter on Y and is defined as follows [13]. For $a \in L^Y$ we define $\varphi^{\rightarrow}(\mathcal{F})(a) = \mathcal{F}\big(\varphi^{\leftarrow}(a)\big) = \mathcal{F}(a \circ \varphi)$. It is straightforward to deduce that $\varphi\big([x]_s\big) = \big[\varphi(x)\big]_s$.

Further, we define for $\Phi, \Psi \in \mathsf{F}^s_L(X \times X)$, the stratified L-filters Φ^{-1} and $\Phi \circ \Psi$ by $\Phi^{-1}(a) = \Phi(a^{-1})$ and $\Phi \circ \Psi(a) = \bigvee_{a_1 \circ a_2 \leq a} \Phi(a_1) * \Psi(a_2)$, [4, 15].

In the sequel, we need to pass from $\top$-filters to stratified L-filters and vice versa. We define, for $\mathbb{F} \in \mathsf{F}^{\top}_{\mathsf{L}}(X)$ the stratified L-filter $\mathcal{F}_{\mathbb{F}}$ by $\mathcal{F}_{\mathbb{F}}(a) = \bigvee_{b\in\mathbb{F}}[b, a]$, see [13].

Proposition 9.2. *Let $\mathbb{F}, \mathbb{G} \in \mathsf{F}^{\top}_{\mathsf{L}}(X)$, $\Phi, \Psi \in \mathsf{F}^{\top}_{\mathsf{L}}(X \times X)$, $x \in X$ and let $\varphi : X \longrightarrow Y$ be a mapping. Then we have*

(*i*) $\mathcal{F}_{[x]} = [x]_s$;

(*ii*) $\mathbb{F} \leq \mathbb{G}$ *implies* $\mathcal{F}_{\mathbb{F}} \leq \mathcal{F}_{\mathbb{G}}$;

(*iii*) $\mathcal{F}_{\mathbb{F}\wedge\mathbb{G}} = \mathcal{F}_{\mathbb{F}} \wedge \mathcal{F}_{\mathbb{G}}$;

(*iv*) $\mathcal{F}_{\Phi} \circ \mathcal{F}_{\Psi} \leq \mathcal{F}_{\Phi\circ\Psi}$;

(*v*) $\mathcal{F}_{\mathbb{F}^{-1}} = (\mathcal{F}_{\mathbb{F}})^{-1}$;

(*vi*) $\varphi(\mathcal{F}_{\mathbb{F}}) = \mathcal{F}_{\varphi(\mathbb{F})}$.

Proof. We prove (iv) and (vi). For (iv), let $\Phi, \Psi \in \mathsf{F}^{\top}_{\mathsf{L}}(X \times X)$ such that $\Phi \circ \Psi$ exists. Then, for $a \in L^{X\times X}$ we have

$$\begin{aligned}\mathcal{F}_{\Phi} \circ \mathcal{F}_{\Psi}(a) &= \bigvee_{\phi\circ\psi\leq a} \mathcal{F}_{\Phi}(\phi) * \mathcal{F}_{\Psi}(\psi) = \bigvee_{\phi\circ\psi\leq a}\bigvee_{\phi'\in\Phi}[\phi',\phi] * \bigvee_{\psi'\in\Psi}[\psi',\psi] \\ &= \bigvee_{\phi\circ\psi\leq a}\bigvee_{\phi'\in\Phi,\psi'\in\Psi}[\phi',\phi]*[\psi',\psi] \leq \bigvee_{\phi\circ\psi\leq a}\bigvee_{\phi'\in\Phi,\psi'\in\Psi}[\phi'\circ\psi',\phi\circ\psi] \\ &\leq \bigvee_{\phi'\in\Phi,\psi'\in\Psi}[\phi'\circ\psi', a] \leq \bigvee_{h\in\Phi\circ\Psi}[h,a] = \mathbb{F}_{\Phi\circ\Psi}(a).\end{aligned}$$

For (vi), let $b \in L^Y$. Then

$$\varphi(\mathcal{F}_{\mathbb{F}})(b) = \mathcal{F}_{\mathbb{F}}(\varphi^{\leftarrow}(b)) = \bigvee_{f\in\mathbb{F}}[f,\varphi^{\leftarrow}(b)] = \bigvee_{f\in\mathbb{F}}[\varphi(f),b] = \bigvee_{h\in\varphi(\mathbb{F})}[h,b] = \mathcal{F}_{\varphi(\mathbb{F})}.$$

□

Definition 9.3. [4, 15] *A mapping* $\mathcal{L} : \mathsf{F}^s_{\mathsf{L}}(X \times X) \longrightarrow L$ *is called a* stratified L-uniform convergence structure *if* $\mathcal{L}$ *satisfies the following:*

(LUC1) *for all* $x \in X$, $\mathcal{L}([(x,x)]_s) = \top$,
(LUC2) $\Phi \leq \Psi \Longrightarrow \mathcal{L}(\Phi) \leq \mathcal{L}(\Psi)$,
(LUC3) $\mathcal{L}(\Phi) \wedge \mathcal{L}(\Psi) \leq \mathcal{L}(\Phi \wedge \Psi)$,
(LUC4) $\mathcal{L}(\Phi) * \mathcal{L}(\Psi) \leq \mathcal{L}(\Phi \circ \Psi)$ *whenever* $\Phi \circ \Psi$ *exists,*
(LUC5) $\mathcal{L}(\Phi) \leq \mathcal{L}(\Phi^{-1})$.

The pair $(X, \mathcal{L})$ *is called a* stratified L-uniform convergence space. *A mapping* $\varphi : (X, \mathcal{L}) \longrightarrow (X', \mathcal{L}')$ *is* uniformly continuous *if for all* $\Phi \in \mathsf{F}^s_{\mathsf{L}}(X\times X)$, *we have* $\mathcal{L}(\Phi) \leq \mathcal{L}'\big((\varphi\times\varphi)^{\rightarrow}(\Phi)\big)$. *The category of stratified* L*-uniform convergence spaces is denoted by* sL-UCS.

For $(X, \mathcal{L}) \in |\mathsf{sL\text{-}UCS}|$ we define $\Lambda^{\mathcal{L}} \subseteq \mathsf{F}^{\top}_{\mathsf{L}}(X \times X)$ by $\Phi \in \Lambda^{\mathcal{L}} \iff \mathcal{L}(\mathcal{F}_{\Phi}) = \top$.

Proposition 9.4. *Let* $(X, \mathcal{L}) \in |\mathsf{sL\text{-}UCS}|$. *Then* $(X, \Lambda^{\mathcal{L}}) \in |\top\text{-}\mathsf{UCS}|$.

Proof. ($\top$-UCS1) From (LUC1) we see $\mathcal{L}(\mathcal{F}_{[(x,x)]}) = \mathcal{L}([(x,x)]_s) = \top$ and hence $[(x,x)] \in \Lambda^{\mathcal{L}}$.

($\top$-UCS2) Let $\Phi \leq \Psi$ and $\Phi \in \Lambda^{\mathcal{L}}$. Then $\mathcal{L}(\mathcal{F}_{\Phi}) = \top$ and (LUC2) yields with $\mathcal{F}_{\Phi} \leq \mathcal{F}_{\Psi}$ that $\mathcal{L}(\mathcal{F}_{\Psi}) = \top$, i.e. $\Psi \in \Lambda^{\mathcal{L}}$.

($\top$-UCS3) If $\Phi, \Psi \in \Lambda^{\mathcal{L}}$, then $\mathcal{L}(\mathbb{F}_{\Phi}) = \top = \mathcal{L}(\mathcal{F}_{\Psi})$. From (LUC3) we conclude $\top = \mathcal{L}(\mathcal{F}_{\Phi} \wedge \mathcal{F}_{\Psi}) = \mathcal{L}(\mathcal{F}_{\Phi\wedge\Psi})$ and we have $\Phi \wedge \Psi \in \Lambda^{\mathcal{L}}$.

($\top$-UCS4) If $\Phi, \Psi \in \Lambda^{\mathcal{L}}$ and $\Phi \circ \Psi$ exists, then $\mathcal{L}(\mathcal{F}_{\Phi}) = \mathcal{L}(\Psi) = \top$. Hence, by (LUC4) then $\mathcal{L}(\mathcal{F}_{\Phi} \circ \mathcal{F}_{\Psi}) \geq \mathcal{L}(\mathcal{F}_{\Phi}) * \mathcal{L}(\mathcal{F}_{\Psi}) = \top * \top = \top$ and as $\mathcal{F}_{\Phi\circ\Psi} \geq \mathcal{F}_{\Phi} \circ \mathcal{F}_{\Psi}$ we see from (LUC2) that $\mathcal{L}(\mathcal{F}_{\Phi\circ\Psi}) = \top$ and we have $\Phi \circ \Psi \in \Lambda^{\mathcal{L}}$.

($\top$-UCS5) Let $\Phi \in \Lambda^{\mathcal{L}}$. Then $\mathcal{L}(\mathcal{F}_{\Phi}) = \top$ and by (LUC5) then also $\mathcal{L}((\mathcal{F}_{\Phi})^{-1})) = \mathcal{L}(\mathcal{F}_{\Phi^{-1}}) = \top$ and $\Phi^{-1} \in \Lambda^{\mathcal{L}}$. □

Proposition 9.5. *Let* $\varphi : (X, \mathcal{L}) \longrightarrow (X', \mathcal{L}')$ *be uniformly continuous. Then* $\varphi : (X, \Lambda^{\mathcal{L}}) \longrightarrow (X', \Lambda^{\mathcal{L}'})$ *is uniformly continuous.*

Proof. Let $\Phi \in \Lambda^{\mathcal{L}}$. Then $\mathcal{L}(\mathcal{F}_{\Phi}) = \top$ and hence $\mathcal{L}'(\mathcal{F}_{(\varphi\times\varphi)(\Phi)}) = \mathcal{L}'((\varphi\times\varphi)(\mathcal{F}_{\Phi})) = \top$ and $(\varphi\times\varphi)(\Phi) \in \Lambda^{\mathcal{L}'}$. □

As a consequence, we have a functor

$$\mathsf{K} : \left\{\begin{array}{ccc} \mathsf{sL\text{-}UCS} & \longrightarrow & \top\text{-}\mathsf{UCS} \\ (X,\mathcal{L}) & \longmapsto & (X,\Lambda^{\mathcal{L}}) \\ \varphi & \longmapsto & \varphi \end{array}\right. .$$

From now on, we assume that $\mathsf{L} = (L, \leq, *)$ is a complete MV-algebra. Then all stratified L-filters are tight, i.e. $\mathcal{F}(\alpha_X) = \alpha$ for all $\alpha \in L$ [13] and for a stratified and tight L-filter $\mathcal{F}$, $\mathbb{F}_{\mathcal{F}} = \{a \in L^X : \mathcal{F}(a) = \top\}$ is a $\top$-filter [9]. The $\top$-filter $\mathbb{F}_{\mathcal{F}}$ will be crucial in the sequel. Note that for a complete MV-algebra, the underlying lattice is always a frame, see [12].

Proposition 9.6. *Let $\mathsf{L} = (L, \le, *)$ be a complete MV-algebra. Let further $\mathcal{F}, \mathcal{G} \in \mathsf{F}^s_{\mathsf{L}}(X)$, $\Phi, \Psi \in \mathsf{F}^s_{\mathsf{L}}(X \times X)$, $x \in X$ and $\varphi : X \longrightarrow Y$ be a mapping. We have*

(i) $\mathbb{F}_{[x]_s} = [x]$;

(ii) $\mathcal{F} \le \mathcal{G}$ *implies* $\mathbb{F}_{\mathcal{F}} \le \mathbb{F}_{\mathcal{G}}$;

(iii) $\mathbb{F}_{\mathcal{F}} \wedge \mathbb{F}_{\mathcal{G}} = \mathbb{F}_{\mathcal{F} \wedge \mathcal{G}}$;

(iv) $\mathbb{F}_{\Phi} \circ \mathbb{F}_{\Psi} \le \mathbb{F}_{\Phi \circ \Psi}$;

(v) $\mathbb{F}_{\Phi^{-1}} = (\mathbb{F}_{\Phi})^{-1}$;

(vi) $\varphi(\mathbb{F}_{\mathcal{F}}) = \mathbb{F}_{\varphi(\mathcal{F})}$.

Proof. We only prove (iv) and (vi). For (iv), let $a \in \mathbb{F}_{\Phi} \circ \mathbb{F}_{\Psi}$. Then $\top = \bigvee_{\Phi(f)=\top, \Psi(g)=\top} [f \circ g, a]$. Now we note that if $\Phi(f) = \top = \Psi(g)$, then $\Phi \circ \Psi(f \circ g) \ge \Phi(f) * \Psi(g) = \top$. Hence we conclude $\top = \bigvee_{\Phi\circ\Psi(f\circ g)=\top} [f \circ g, a] \le \bigvee_{\Phi\circ\Psi(h)=\top} [h, a]$ and this means $a \in \mathbb{F}_{\Phi\circ\Psi}$.

For (vi), we have $b \in \varphi(\mathbb{F}_{\mathcal{F}})$ if and only if $\top = \bigvee_{f \in \mathbb{F}_{\mathcal{F}}} [\varphi(f), b] = \bigvee_{f \in \mathbb{F}_{\mathcal{F}}} [f, \varphi^{\leftarrow}(b)]$ if and only if $\varphi^{\leftarrow}(b) \in \mathbb{F}_{\mathcal{F}}$ if and only if $b \in \varphi(\mathbb{F}_{\mathcal{F}})$. □

We define now, for $(X, \Lambda) \in |\top\text{-}\mathsf{UCS}|$, $\mathcal{L}^{\Lambda} : \mathsf{F}^s_{\mathsf{L}}(X \times X) \longrightarrow L$ by $\mathcal{L}^{\Lambda}(\Phi) = \top$ if $\mathbb{F}_{\Phi} \in \Lambda$ and $\mathcal{L}^{\Lambda}(\Phi) = \bot$ otherwise.

Proposition 9.7. *Let $\mathsf{L} = (L, \le, *)$ be a complete MV-algebra. Let $(X, \Lambda) \in |\top\text{-}\mathsf{UCS}|$. Then $(X, \mathcal{L}^{\Lambda}) \in |\mathsf{sL}\text{-}\mathsf{UCS}|$.*

Proof. (LUC1) As $\mathbb{F}_{[(x,x)]_s} = [(x, x)] \in \Lambda$, we have $\mathcal{L}^{\Lambda}([(x, x)]_s) = \top$.

(LUC2) Let $\Phi \le \Psi$ and let $\mathcal{L}^{\Lambda}(\Phi) = \top$. Then $\mathbb{F}_{\Phi} \le \mathbb{F}_{\Psi}$ and $\mathbb{F}_{\Phi} \in \Lambda$. This implies $\mathbb{F}_{\Psi} \in \Lambda$ and hence $\mathcal{L}^{\Lambda}(\Psi) = \top$.

(LUC3) Let $\top = \mathcal{L}^{\Lambda}(\Phi) \wedge \mathcal{L}^{\Lambda}(\Psi)$. Then $\mathbb{F}_{\Phi}, \mathbb{F}_{\Psi} \in \Lambda$ and hence $\mathbb{F}_{\Phi\wedge\Psi} = \mathbb{F}_{\Phi} \wedge \mathbb{F}_{\Psi} \in \Lambda$ and $\mathcal{L}^{\Lambda}(\Phi \wedge \Psi) = \top$.

(LUC4) Let $\mathcal{L}^{\Lambda}(\Phi) * \mathcal{L}^{\Lambda}(\Psi) = \top$. Then $\mathbb{F}_{\Phi}, \mathbb{F}_{\Psi} \in \Lambda$ and hence $\mathbb{F}_{\Phi} \circ \mathbb{F}_{\Psi} \in \Lambda$. The property (⊤-UCS2) then yields $\mathbb{F}_{\Phi\circ\Psi} \in \Lambda$, i.e. $\mathcal{L}^{\Lambda}(\Phi \circ \Psi) = \top$.

(LUC5) Let $\mathcal{L}^{\Lambda}(\Phi) = \top$. Then $\mathbb{F}_{\Phi} \in \Lambda$ and hence $\mathbb{F}_{\Phi^{-1}} = (\mathbb{F}_{\Phi})^{-1} \in \Lambda$ and we have $\mathcal{L}(\Phi^{-1}) = \top$. □

Proposition 9.8. *Let $\mathsf{L} = (L, \le, *)$ be a complete MV-algebra. Let $\varphi : (X, \Lambda) \longrightarrow (X', \Lambda')$ be uniformly continuous. Then $\varphi : (X, \mathcal{L}^{\Lambda}) \longrightarrow (X', \mathcal{L}^{\Lambda'})$ is uniformly continuous.*

Proof. Let $\mathcal{L}^{\Lambda}(\Phi) = \top$. Then $\mathbb{F}_{\Phi} \in \Lambda$ and hence $\mathbb{F}_{(\varphi\times\varphi)(\Phi)} = (\varphi \times \varphi)(\mathbb{F}_{\Phi}) \in \Lambda'$. Therefore $\mathcal{L}^{\Lambda}((\varphi \times \varphi)(\Phi)) = \top$. □

As a consequence, we have a functor

$$\mathsf{M} : \left\{ \begin{array}{ccc} \top\text{-}\mathsf{UCS} & \longrightarrow & \mathsf{sL}\text{-}\mathsf{UCS} \\ (X, \Lambda) & \longmapsto & (X, \mathcal{L}^{\Lambda}) \\ \varphi & \longmapsto & \varphi \end{array} \right. .$$

Proposition 9.9. *Let $\mathsf{L} = (L, \le, *)$ be a complete MV-algebra.*

(i) *For $\mathbb{F} \in \mathsf{F}^{\top}_{\mathsf{L}}(X)$ we have $\mathbb{F}_{(\mathcal{F}_{\mathbb{F}})} = \mathbb{F}$.*

(ii) *For $\mathcal{F} \in \mathsf{F}^s_{\mathsf{L}}(X)$ we have $\mathcal{F}_{(\mathbb{F}_{\mathcal{F}})} \le \mathcal{F}$.*

Proof. (i) We have $a \in \mathbb{F}_{(\mathcal{F}_{\mathbb{F}})}$ if and only if $\mathcal{F}_{\mathbb{F}}(a) = \top$ if and only if $\bigvee_{f \in \mathbb{F}} [b, a] = \top$. As $\mathbb{F}$ is a ⊤-filter, this is equivalent to $a \in \mathbb{F}$.

(ii) According to [5] we have for a stratified L-filter $\mathcal{F}$ and $a, b \in L^X$, $[b, a] \le \mathcal{F}(b) \to \mathcal{F}(a)$. Hence, if $\mathcal{F}(b) = \top$, we have $[b, a] \le \mathcal{F}(a)$. We conclude from this $\mathcal{F}(a) \ge \bigvee_{\mathcal{F}(b)=\top} [b, a] = \bigvee_{b \in \mathbb{F}_{\mathcal{F}}} [b, a] = \mathcal{F}_{(\mathbb{F}_{\mathcal{F}})}(a)$. □

Proposition 9.10. *Let $\mathsf{L} = (L, \le, *)$ be a complete MV-algebra.*

(i) *Let $(X, \Lambda) \in |\top\text{-}\mathsf{UCS}|$. Then $\Lambda = \Lambda^{(\mathcal{L}^{\Lambda})}$.*

(ii) *Let $(X, \mathcal{L}) \in |\mathsf{sL}\text{-}\mathsf{UCS}|$. Then $\mathcal{L}^{(\Lambda^{\mathcal{L}})}(\Phi) \le \mathcal{L}(\Phi)$ for all $\Phi \in \mathsf{F}^s_{\mathsf{L}}(X \times X)$.*

Proof. (i) We have $\Phi \in \Lambda^{(\mathcal{L}^{\Lambda})}$ if and only if $\mathcal{L}^{\Lambda}(\mathcal{F}_{\Phi}) = \top$ if and only if $\Phi = \mathbb{F}_{(\mathcal{F}_{\Phi})} \in \Lambda$.

(ii) If $\mathcal{L}^{(\Lambda^{\mathcal{L}})}(\Phi) = \top$, then $\mathbb{F}_{\Phi} \in \Lambda^{\mathcal{L}}$, i.e. $\top = \mathcal{L}(\mathcal{F}_{(\mathbb{F}_{\Phi})}) \le \mathcal{L}(\Phi)$. □

Theorem 9.11. *Let $\mathsf{L} = (L, \le, *)$ be a complete MV-algebra. Then ⊤-UCS can be embedded into sL-UCS as a reflective subcategory.*

10 Conclusions

We studied a category of quantale-valued uniform convergence spaces based on $\top$-filters that generalizes the category of probabilistic uniform spaces. Recently, completions were studied for these probabilistic uniform spaces [28] and also for the related $\top$-Cauchy spaces results on completion were obtained [24]. We showed that quantale-valued metric spaces possess a natural probabilistic uniform structure. In this sense, a completion theory for $\top$-uniform convergence spaces or probabilistic uniform spaces naturally applies also to quantale-valued metric spaces. This may lead to further insight into the completion of quantale-valued metric spaces and, in turn, the completion of quantale-valued metric spaces may inspire ideas on the completion of $\top$-uniform convergence spaces.

Classically, relaxing the axioms of a uniform convergence space leads to even better behaved supercategories such as the category of semi-uniform convergence spaces [20]. It is a natural question to extend the theory of this paper into this direction.

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