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Original paper

Riemann integrability based optimality criteria for fractional optimization problems with fuzzy parameters

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Abstract

This paper aims to establish the Karush-Kuhn-Tucker type optimality criteria for linear fractional optimization problems with fuzzy parameters. To evolve the desired criteria first, the fractional optimization problem is transformed into the non-fractional optimization problem with fuzzy parameters. Then Hukuhara differentiability for the differentiation of functions with fuzzy parameters and Hausdorff metric to expound the distance between the fuzzy numbers is invoked. Optimality criteria are then elicited for the non-fractional optimization problems by introducing Lagrange multipliers and Riemann integration theory. In order to validate the developed theory, two numerical optimization problems are also verified.

Keywords: Hukuhara difference, Hausdorff metric, H-differentiability, Riemann integration, Lagrange multipliers.

1 Introduction

In the physical world, most of the optimization problems are not deterministic in nature because of the involvement of imprecise data. This means that there are some types of uncertainty/fuzziness available in the data due to the lack of information, measurement errors, prediction errors, and major emergency like COVID-19. These problems occur in different areas of applications such as economic, social, management and industrial etc. Since the last five decades, the fuzzy-set theory has been used as an indispensable tool to tackle these problems and widely applied to model the real world optimization problems where the uncertainty exists.

Zadeh [36] initiated the fuzzy set theory in the year 1965. After that, the development of fuzzy-set theory has been increased exponentially with its applications in the field of optimization. To combine the fuzzy targets and fuzzy decision space, Bellman and Zadeh [5] provided the concept of aggregation operators. Later the developed aggregation operators of Bellman and Zadeh [5] were regenerated by Zimmermann [37]. Zimmermann [38] noticed two kinds of uncertainties like fuzziness and stochastic. To optimize the ratio of one or more functions, the term fractional programming came into existence. The vague data in fractional optimization leads to introduce a new problem called fuzzy fractional optimization problem. Most of the researchers developed many algorithms to obtain an optimal solution for fuzzy optimization problems, including fuzzy fractional optimization problems. Also, the KKT conditions are developed for fuzzy linear optimization problems as well as for fractional optimization problems. But less attention is given to evolve the optimality criteria for the fractional optimization problems with fuzzy parameters. So, we have considered this problem and developed KKT optimality conditions by taking reference of the previous research done by different authors.

The KKT optimality criteria for linear optimization problem with fuzzy parameters have been established by Wu [28] using the saddle-point criteria. After that, the optimality criteria for single-objective optimization problem with fuzzy parameters have also been developed by Wu [29, 32]. The problem for establishing the optimality criteria was extended

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to the fuzzy multi-objective optimization problems by Wu [33, 34] and then to the interval-valued programming problems by Wu [30, 31, 35]. Necessary and sufficient conditions were developed for non-linear fuzzy optimization problems by Pathak and Pirzada [19]. Hosseinzade and Hassanpour [16] developed optimality criteria for multi-objective programming problems with interval-valued constraints and objective functions. Singh, Dar and Kim [23] also obtained the KKT(Karush-Kuhn-Tucker) optimality criteria for multi-objective optimization problems with interval-valued functions using the concept of generalized differentiability and convexity. Agarwal et al. [2] established the KKT optimality criteria for multi-objective fractional optimization problems with fuzzy parameters. Agarwal et al. [1] developed a branch-bound method for solving sum of linear fractional multi-objective optimization problems. Singh et al. [22] also developed a new branch and bound cut technique for solving multi-objective fractional optimization problems involving non-linear variables.
Recently, new research has been done by many author on current subject. Chen [7] characterized interval-valued functions in the Hadamard manifolds context and generated KKT optimality conditions for the same. Chen et al. [9] considered a convex multi-objective optimization problem with fuzzy data in both constraints and objective functions. They discussed robust optimality criteria by using image space analysis (ISA) and ε-constraints scalarization method. Su & Luu [25] studied higher order strong and weak KKT type optimal conditions in order to obtain the efficient solution of semi-infinite multi-objective optimization problem. Here, the constraints involved in the problem are in terms of higher order Studniarski derivatives. Tung [26] considered multi-objective semi-infinite programming problem (MSIP)and obtained KKT optimality conditions by using suitable generalized qualifications & tangential subdifferentials. Wei et al. [27] considered scalar robust optimization problem and derived robust necessary optimality conditions. Hejazi & Nobakhtian [14] considered multi-objective fractional optimization and used the idea of convexificators to obtain KKT necessary conditions at weak efficient solution for the problem. Gadhi et al. [13] worked on necessary and sufficient optimality conditions for set value optimization problem. For this they used a modified generalized convexity, which is related to directional convexificators. Chen et al. [8] considered a complex fractional optimization problem and established KKT type and Fritz-John type robust necessary conditions. Dar et al. [10] derived KKT type & Fritz John type conditions under the interval uncertainty for interval multi-objective fractional problem. Fathy [12] developed an iterative technique to solve for fully rough multi-objective multi-level linear fractional optimization problem. To deal with the roughness of the given problem, he has presented an extension of interval method. Borza and Rambely [6] developed an effective method to compute linear fractional programming problems with fuzzy parameters by converting into equivalent bi-objective linear programming problems.
Motivation: This current research work is motivated by the work of Wu [32]. Wu [32] developed optimality conditions for linear fuzzy optimization problem with classical constraints using Riemann integration theory and Hukuhara differentiability for the differentiation of functions with fuzzy parameters. In this paper, the work of Wu [32] is extended for fractional optimization problems with fuzzy parameters under the non-fuzzy constraints space.
The fractional optimization problem with fuzzy parameters (FOPFP) is converted into a non-fractional optimization problem with fuzzy parameters (OPFP) using Dinkelbach [11] and Jagannathan [17, 18] approaches, then the optimality criteria are developed with the help of the non-fractional optimization problem. The proposed optimality criteria are verified on two numerical optimization problems. For the verification, the problem is converted into a deterministic problem by ranking functional approach. After getting the approximate deterministic solution of the problem, the obtained value is represented by a fuzzy number. Then the proposed optimality criteria are verified. Remaining part of this paper is arranged in sections. In Section 2 preliminaries are discussed. Section 3 describes the problem formulation. The proposed optimality criteria are established in Section 4 and it also provides brief details of optimality criteria for the conventional optimization problem. The developed optimality criteria are verified in Section 5 and conclusions are presented in Section 6.

2 Preliminaries

In this section, basic concepts and definitions of the fuzzy numbers, fuzzy-sets, canonical-fuzzy numbers, triangular fuzzy numbers, function with fuzzy parameters and their operations are presented. The different operations for the fuzzy numbers are also defined below. This section also includes Hukuhara difference and Hausdorff metric for the difference and distance between two fuzzy numbers.

2.1 Fuzzy sets and fuzzy numbers

1. *Fuzzy set:* A fuzzy-set $\widetilde{A}$ can be defined as $\{(\mathbf{u}, \xi_{\widetilde{A}}(\mathbf{u}))|\ \mathbf{u} \in \mathbb{R}\}$ for the membership function $\xi_{\widetilde{A}} : \mathbb{R} \to [0, 1]$. Here set of real-numbers is denoted by a universal set $\mathbb{R}$.

2. *α-cut of a fuzzy set:* A crisp set is called α-level or α-cut set which is denoted and defined as $\widetilde{A}_\alpha = \{\mathbf{u} \in \mathbb{R} : \xi_{\widetilde{A}}(\mathbf{u}) \geq \alpha\}$. The closure set $\widetilde{A}_0 = \{\mathbf{u} \in \mathbb{R} : \xi_{\widetilde{A}}(\mathbf{u}) > 0\} = cl(\{\mathbf{u} \in \mathbb{R} : \xi_{\widetilde{A}}(\mathbf{u}) > 0\})$ is also called 0-level set.

3. *Normal-fuzzy set:* $\widetilde{A}$ is said to be normal if $\exists \ \mathbf{u} \in \mathbb{R}$ s.t., $\xi_{\widetilde{A}} = 1$.

4. *Convex fuzzy-set:* A fuzzy set $\tilde{A}$ is said to be convex on $\mathbb{R}$ if and only if for all $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}$ and $\forall \ \lambda \in [0, 1]$ we have

$$\xi_{\tilde{A}}(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) \geq \min(\xi_{\tilde{A}}(\mathbf{u}), \xi_{\tilde{A}}(\mathbf{v})).$$

Here, min sign indicates the minimum operator.

5. *Fuzzy number:* A normal and convex fuzzy set with its bounded support is called fuzzy number $\widetilde{a}$. Also we write $\widetilde{a}_\alpha = [\widetilde{a}^L_\alpha, \widetilde{a}^U_\alpha]$ and $\mathbb{F}(\mathbb{R})$ denotes the set of all fuzzy numbers.

6. *Non-negative fuzzy number:* If $\widetilde{a}^L_\alpha, \widetilde{a}^U_\alpha \geq 0$ for all $\alpha \in [0, 1]$, then the fuzzy number $\widetilde{a}$ is called the non-negative fuzzy number and if $\widetilde{a}^L_\alpha, \widetilde{a}^U_\alpha > 0$ for all $\alpha \in [0, 1]$, then the fuzzy number $\widetilde{a}$ is called positive fuzzy number.

7. *Canonical-fuzzy number:* Let $\mathbb{F}_c(\mathbb{R})$ be the canonical-fuzzy number's set. If the functions $\widetilde{a}^L_\alpha, \widetilde{a}^U_\alpha$ are continuous on [0, 1] where $\widetilde{a}_\alpha = [\widetilde{a}^L_\alpha, \widetilde{a}^U_\alpha]$, then $\widetilde{a}$ is called a canonical-fuzzy number.

8. *Triangular fuzzy number:* The membership function for a triangular fuzzy number $\widetilde{a} = (a^L, a, a^U)$ is defined as:

$$\gamma_{\widetilde{a}}(s) = \begin{cases} (s - a^L)/(a - a^L) & \text{if } a^L \leq s \leq a \\ (a^U - s)/(a^U - a) & \text{if } a < s \leq a^U \\ 0 & \text{otherwise} \end{cases}$$

The α-cut of $\widetilde{a}$ is given below:

$$\widetilde{a}_\alpha = [(1-\alpha)a^L + \alpha a, (1-\alpha)a^U + \alpha a];$$

i.e.,

$$\widetilde{a}^L_\alpha = (1-\alpha)a^L + \alpha a \text{ and } \widetilde{a}^U_\alpha = (1-\alpha)a^U + \alpha a. \tag{1}$$

2.2 Arithmetic operations for fuzzy numbers

Since $\widetilde{a}$ and $\widetilde{b}$ are two canonical fuzzy numbers, let us define "$\boxdot$" the binary operations "$\boxtimes$" and "$\boxplus$" corresponding to "$\times$" and "$+$". By the extension principle (Zadeh), the membership function for $\widetilde{a} \boxdot \widetilde{b}$ is given as:

$$\xi_{\widetilde{a}\boxdot\widetilde{b}}(\mathbf{w}) = \text{Sup}_{\mathbf{u}\boxdot\mathbf{v}=\mathbf{w}} \min\{\xi_{\widetilde{a}}(\mathbf{u}), \xi_{\widetilde{b}}(\mathbf{v})\}.$$

We have some results for the fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$ as below:

1. $\left(\widetilde{a} \boxplus \widetilde{b}\right)_\alpha = \left[\widetilde{a}^L_\alpha + \widetilde{b}^L_\alpha,\ \widetilde{a}^U_\alpha + \widetilde{b}^U_\alpha\right]$, where $\left(\widetilde{a} \boxplus \widetilde{b}\right) \in \mathbb{F}(\mathbb{R})$.

2. $\left(\widetilde{k} \boxtimes \widetilde{b}\right)_\alpha = \left[\min\left(\widetilde{k}^L_\alpha\widetilde{b}^L_\alpha,\ \widetilde{k}^L_\alpha\widetilde{b}^U_\alpha,\ \widetilde{k}^U_\alpha\widetilde{b}^L_\alpha,\ \widetilde{k}^U_\alpha\widetilde{b}^U_\alpha\right),\ \max\left(\widetilde{k}^L_\alpha\widetilde{b}^L_\alpha,\ \widetilde{k}^L_\alpha\widetilde{b}^U_\alpha,\ \widetilde{k}^U_\alpha\widetilde{b}^L_\alpha,\ \widetilde{k}^U_\alpha\widetilde{b}^U_\alpha\right)\right]$, where $\left(\widetilde{k} \boxtimes \widetilde{b}\right) \in \mathbb{F}(\mathbb{R})$.
Also for the non-negative fuzzy numbers $\widetilde{v}$ and $\widetilde{b}$, we have $\left(\widetilde{k} \boxtimes \widetilde{b}\right)_\alpha = \left[\widetilde{k}^L_\alpha\widetilde{b}^L_\alpha,\ \widetilde{k}^U_\alpha\widetilde{b}^U_\alpha\right]$.

3. *Hukuhara difference:* (from Puri and Ralescu [20]) The Hukuhara difference between two non-negative fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$ is given by the fuzzy number $\widetilde{c}$ s.t., $\widetilde{c} = \widetilde{a} \boxminus_H \widetilde{b}$. Also if $\widetilde{c} = \widetilde{a} \boxminus_H \widetilde{b}$ exists, then $\forall \ \ \alpha \in [0, 1]$ we have:

$$\widetilde{c}^L_\alpha = \widetilde{a}^L_\alpha - \widetilde{b}^L_\alpha \quad \text{and} \quad \widetilde{c}^U_\alpha = \widetilde{a}^U_\alpha - \widetilde{b}^U_\alpha.$$

Here, $\boxminus_H$ indicate the Hukuhara-difference between the two fuzzy numbers.

4. For the positive fuzzy number $\tilde{k}$, the hukuhara difference between the fuzzy numbers $\tilde{a}$ and $\left(\widetilde{k} \boxtimes \widetilde{b}\right)$ is defined as:
$\left(\widetilde{a} - \left(\widetilde{k} \boxtimes \widetilde{b}\right)\right)_\alpha = \left[\widetilde{a}^L_\alpha - \widetilde{k}^L_\alpha\widetilde{b}^L_\alpha,\ \widetilde{a}^U_\alpha - \widetilde{k}^U_\alpha\widetilde{b}^U_\alpha\right]$, where $\widetilde{a} - \left(\widetilde{k} \boxtimes \widetilde{b}\right) \in \mathbb{F}(\mathbb{R})$.

5. *Hausdorff metric:* Suppose $A, B \subseteq \mathbb{R}^n$, so we can define the Hausdorff metric as:

$$d_H(A,\, B) = \max\{\sup_{b\in B}\inf_{a\in A} \parallel a-b \parallel,\, \sup_{a\in A}\inf_{b\in B} \parallel a-b \parallel\}.$$

Also, for the fuzzy numbers $\widetilde{a},\, \widetilde{b} \in \mathbb{F}(\mathbb{R})$, the Hausdorff metric $d_{\mathbb{F}}$ is given by Wu [32] as:

$$d_{\mathbb{F}}(\widetilde{a},\, \widetilde{b}) = \sup_{0\le\alpha\le 1} d_H(\widetilde{a}_\alpha,\, \widetilde{b}_\alpha) = \max\left(|\widetilde{a}^L_\alpha - \widetilde{b}^L_\alpha|,\, |\widetilde{a}^U_\alpha - \widetilde{b}^U_\alpha|\right) \quad \forall \quad \alpha \in [0,\, 1].$$

6. For the canonical fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$, if $d_{\mathbb{F}}(\widetilde{a},\, \widetilde{b}) < \epsilon$, then for all $\alpha \in [0,\, 1]$, we have $|\widetilde{a}^L_\alpha - \widetilde{b}^L_\alpha| < \epsilon$ and $|\widetilde{a}^U_\alpha - \widetilde{b}^U_\alpha| < \epsilon$.

2.3 Properties for function with fuzzy parameters

Let a function with fuzzy parameters, $\widetilde{N} : \mathbb{R}^n \to \mathbb{F}(\mathbb{R})$ be defined on $\mathbb{R}^n$ and $\widetilde{N}(u)$ is a canonical-fuzzy number for $u \in \mathbb{R}^n$. We now define two real-valued functions i.e., for any fixed $\alpha \in [0,\, 1]$ we have, $\widetilde{N}^L_\alpha(u) = (\widetilde{N}(u))^L_\alpha$ and $\widetilde{N}^U_\alpha(u) = (\widetilde{N}(u))^U_\alpha$.

1. *Limits and Continuity:* (from Wu [32]) We say the function with fuzzy parameters $\widetilde{N}$ have a limit $\widetilde{a}$ (canonical-fuzzy number) i.e., for $\mathbf{d} \in \mathbb{R}^{\mathbf{n}}$ we have

$$\lim_{u\to\mathbf{d}} \widetilde{N}(u) = \widetilde{a},$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ s.t., we have $d_{\mathbb{F}}(\widetilde{N}(u),\, \widetilde{a}) < \epsilon$ for $\parallel \mathbf{u} - \mathbf{d} \parallel < \delta$.
Also the function $\widetilde{N}$ is continuous at $\mathbf{d}$ if and only if we have

$$\lim_{u\to\mathbf{d}} \widetilde{N}(u) = \widetilde{N}(\mathbf{d}).$$

A function with fuzzy parameters, $\widetilde{N}$ is said to be level-wise continuous at $\mathbf{d}$, if we have the real-valued functions $\widetilde{N}^L_\alpha$ and $\widetilde{N}^U_\alpha$ for all $\alpha \in [0,\, 1]$, which are continuous at $\mathbf{d}$.

2. *Differentiability:* Hukuhara-difference (by Puri and Ralescu [20]) of two fuzzy numbers develops a new concept of Hukuhara differentiability. For an open subset V of $\mathbb{R}^n$, the function $\widetilde{N} : V \to \mathbb{F}_c(\mathbb{R})$ is called Hukuhara-differentiable at $\bar{u}$ if $\exists$ a canonical fuzzy number $\widetilde{a}(\bar{u})$ s.t., the limits exist and equal to $\widetilde{a}(\bar{u})$ i.e.,

$$\lim_{h\to 0^+} \widetilde{1}_{\{\frac{1}{h}\}} \boxtimes [\widetilde{N}(\bar{u}+h) \ \boxminus_{\mathrm{H}} \ \widetilde{N}(\bar{u})] \quad \text{and} \quad \lim_{h\to 0^-} \widetilde{1}_{\{\frac{1}{h}\}} \boxtimes [\widetilde{N}(\bar{u}) \ \boxminus_{\mathrm{H}} \ \widetilde{N}(\bar{u}-h)].$$

Here $\widetilde{1}_{\frac{1}{h}}$ is a crisp number with its value $\frac{1}{h}$. The fuzzy number $\widetilde{a}(\bar{u})$ is called Hukuhara-derivative of the function $\widetilde{N}$ at $\bar{u}$.

3 Formulation of the problem

Consider a single objective fractional optimization problem with fuzzy parameters as follows:

$$\begin{aligned} (FOPFP) \qquad & \min \ \frac{\widetilde{N}(u)}{\widetilde{D}(u)} \\ & \text{subject to,} \\ & l_h(u) \le 0, \quad h = 1, 2, 3, \ldots, m, \\ & u \in V \subseteq \mathbb{R}^n. \end{aligned}$$

Here $\widetilde{N}(u),\ \widetilde{D}(u) > 0$ are the continuous functions with fuzzy parameters on the feasible set $V \subseteq \mathbb{R}^n$. The functions $\widetilde{N}(u)$ and $\widetilde{D}(u)$ are considered as:

$$\widetilde{N}(u) = (\widetilde{a}_{(1)} \boxtimes \widetilde{1}_{(u_1)}) \ \boxplus (\widetilde{a}_{(2)} \boxtimes \widetilde{1}_{(u_2)}) \ \boxplus \ldots \boxplus (\widetilde{a}_{(n)} \boxtimes \widetilde{1}_{(u_n)}),$$

and

$$\widetilde{D}(u) = (\widetilde{b}_{(1)} \boxtimes \widetilde{1}_{(u_1)}) \boxplus (\widetilde{b}_{(2)} \boxtimes \widetilde{1}_{(u_2)}) \boxplus \ldots \boxplus (\widetilde{b}_{(n)} \boxtimes \widetilde{1}_{(u_n)}),$$

where $\widetilde{a}_i$ and $\widetilde{b}_i$ are the canonical fuzzy numbers. Here each $\widetilde{1}_(u_i) = u_i$ is a crisp number for $i = 1, 2, 3, \ldots, n$.
The problem $(FOPFP)$ is now transformed into non-fractional programming problem using the approaches of Dinkelbach [11] and Jagannathan [18]. The transformed problem can be defined as:

$$(OPFP) \qquad \min \left(\widetilde{N}(u) - \widetilde{k} \boxtimes \widetilde{D}(u)\right)$$
$$\text{subject to,}$$
$$l_h(u) \leq 0, \quad h = 1, 2, 3, \ldots, m,$$
$$u \in V \subseteq \mathbb{R}^n.$$

Here $\widetilde{k}$ is a canonical fuzzy vector number.
Suppose we have two fuzzy numbers $\tilde{c}$ and $\tilde{d}$ such that for all $\alpha \in [0, 1]$, $\tilde{c}_\alpha = [\tilde{c}^L_\alpha, \tilde{c}^U_\alpha]$ and $\tilde{d}_\alpha = [\tilde{d}^L_\alpha, \tilde{d}^U_\alpha]$ are closed intervals in $\mathbb{R}$. We say $\tilde{c} \preceq \tilde{d}$ if and only if for all $\alpha \in [0, 1]$, $\tilde{c}_\alpha \leq \tilde{d}_\alpha$. Also for all $\alpha \in [0, 1]$ we can write $\tilde{c}^L_\alpha \leq \tilde{d}^L_\alpha$ and $\tilde{c}^U_\alpha \leq \tilde{d}^U_\alpha$. $\preceq$ is a partial ordering on $\mathbb{F}(\mathbb{R})$.
$\tilde{c} \prec \tilde{d}$ iff for all $\alpha \in [0, 1]$ we have

$$\begin{cases} \tilde{c}^L_\alpha \leq \tilde{d}^L_\alpha \\ \tilde{c}^U_\alpha < \tilde{d}^U_\alpha \end{cases} \text{or} \begin{cases} \tilde{c}^L_\alpha < \tilde{d}^L_\alpha \\ \tilde{c}^U_\alpha \leq \tilde{d}^U_\alpha \end{cases} \text{or} \begin{cases} \tilde{c}^L_\alpha < \tilde{d}^L_\alpha \\ \tilde{c}^U_\alpha < \tilde{d}^U_\alpha \end{cases} \tag{2}$$

3.1 Fuzzy-optimal solution

Definition 3.1. *Suppose $u^o \in V$ be a feasible solution then, we can say*
(i) u^o is non-dominated solution for problem (FOPFP) if $\nexists\ \bar{u}(\neq u^o)$ s.t., $\frac{\widetilde{N}(\bar{u})}{\widetilde{D}(\bar{u})} \prec \frac{\widetilde{N}(u^o)}{\widetilde{D}(u^o)}$.
(ii) u^o is strongly non-dominated solution for problem (FOPFP) if $\nexists\ \bar{u}(\neq u^o)$ s.t., $\frac{\widetilde{N}(\bar{u})}{\widetilde{D}(\bar{u})} \preceq \frac{\widetilde{N}(u^o)}{\widetilde{D}(u^o)}$.

Definition 3.2. *Suppose that $u^o \in V$ is a feasible solution then, we can say*
(i) u^o is non-dominated solution for problem (OPFP) if $\nexists\ \bar{u}(\neq u^o)$ s.t., $\widetilde{N}(\bar{u}) - \widetilde{k} \boxtimes \widetilde{D}(\bar{u}) \prec \widetilde{N}(u^o) - \widetilde{k} \boxtimes \widetilde{D}(u^o)$.
(ii) u^o is strongly non-dominated solution for problem (OPFP) if $\nexists\ \bar{u}(\neq u^o)$ s.t., $\widetilde{N}(\bar{u}) - \widetilde{k} \boxtimes \widetilde{D}(\bar{u}) \preceq \widetilde{N}(u^o) - \widetilde{k} \boxtimes \widetilde{D}(u^o)$.
(It can be noted that u^o is a weakly-non-dominated solution for the problem (OPFP) if $\nexists\ \bar{u}(\neq u^o)$ s.t., $\widetilde{N}(\bar{u}) - \widetilde{k} \boxtimes \widetilde{D}(\bar{u})$ strongly-dominates $\widetilde{N}(u^o) - \widetilde{k} \boxtimes \widetilde{D}(u^o)$.)

u^o is a non-dominated solution for problem (OPFP) if $\nexists$ any $\bar{u}(\neq u^o) \in V$ such that $(\widetilde{N} - \widetilde{k} \cdot \widetilde{D})(u) \prec (\widetilde{N} - \widetilde{k} \cdot \widetilde{D})(u^o)$, then from (2)

$$\begin{cases} \widetilde{N}^L_\alpha(\bar{u}) - \widetilde{k}^L_\alpha \cdot \widetilde{D}^L_\alpha(\bar{u}) < \widetilde{N}^L_\alpha(u^o) - \widetilde{k}^L_\alpha \cdot \widetilde{D}^L_\alpha(u^o) \\ \widetilde{N}^U_\alpha(\bar{u}) - \widetilde{k}^U_\alpha \cdot \widetilde{D}^U_\alpha(\bar{u}) \leq \widetilde{N}^U_\alpha(u^o) - \widetilde{k}^U_\alpha \cdot \widetilde{D}^U_\alpha(u^o) \end{cases} \text{or}$$
$$\begin{cases} \widetilde{N}^L_\alpha(\bar{u}) - \widetilde{k}^L_\alpha \cdot \widetilde{D}^L_\alpha(\bar{u}) \leq \widetilde{N}^L_\alpha(u^o) - \widetilde{k}^L_\alpha \cdot \widetilde{D}^L_\alpha(u^o) \\ \widetilde{N}^U_\alpha(\bar{u}) - \widetilde{k}^U_\alpha \cdot \widetilde{D}^U_\alpha(\bar{u}) < \widetilde{N}^U_\alpha(u^o) - \widetilde{k}^U_\alpha \cdot \widetilde{D}^U_\alpha(u^o) \end{cases} \text{or}$$
$$\begin{cases} \widetilde{N}^L_\alpha(\bar{u}) - \widetilde{k}^L_\alpha \cdot \widetilde{D}^L_\alpha(\bar{u}) < \widetilde{N}^L_\alpha(u^o) - \widetilde{k}^L_\alpha \cdot \widetilde{D}^L_\alpha(u^o) \\ \widetilde{N}^U_\alpha(\bar{u}) - \widetilde{k}^U_\alpha \cdot \widetilde{D}^U_\alpha(\bar{u}) < \widetilde{N}^U_\alpha(u^o) - \widetilde{k}^U_\alpha \cdot \widetilde{D}^U_\alpha(u^o) \end{cases} \tag{3}$$

for all $\alpha \in [0, 1]$. Also, if u^o is a non-dominated solution for problem (OPFP) then u^o is also a weakly non-dominated solution for problem (OPFP).

The results are provided in the following lemma for the problem (FOPFP) and (OPFP).

Lemma 3.3. *u^o is said to be non-dominated solution for problem (FOPFP) if and only if for a fuzzy number $\widetilde{k}$ s.t., $\widetilde{k} = \frac{\widetilde{N}(u^o)}{\widetilde{D}(u^o)} \in \mathbb{F}_c(\mathbb{R})$, u^o is the non-dominated solution for the problem (OPFP).*

Proof. Let $\widetilde{k} = \frac{\widetilde{N}(u^o)}{\widetilde{D}(u^o)} \in \mathbb{F}_c(\mathbb{R})$.
For $\widetilde{k} = [\widetilde{k}^L_\alpha,\ \widetilde{k}^U_\alpha]$, where $(\widetilde{k})^L_\alpha = \frac{\widetilde{N}^L_\alpha(u^o)}{\widetilde{D}^L_\alpha(u^o)}$ and $(\widetilde{k})^U_\alpha = \frac{\widetilde{N}^U_\alpha(u^o)}{\widetilde{D}^U_\alpha(u^o)} \in \mathbb{F}_c(\mathbb{R})$.
The division of the fuzzy numbers are given according to Stefanini [24] for a particular case. However, this is a restrictive case.
Let u^o is not a non-dominated solution, therefore $\exists\ \bar{u} \neq u^o \in V$ s.t.,
$(\widetilde{N})^L_\alpha(\bar{u}) - (\widetilde{k})^L_\alpha \cdot (\widetilde{D})^L_\alpha(\bar{u}) \prec (\widetilde{N})^L_\alpha(u^o) - (\widetilde{k})^L_\alpha \cdot (\widetilde{D})^L_\alpha(u^o)$ and $(\widetilde{N})^U_\alpha(\bar{u}) - (\widetilde{k})^U_\alpha \cdot (\widetilde{D})^U_\alpha(\bar{u}) \prec (\widetilde{N})^U_\alpha(u^o) - (\widetilde{k})^U_\alpha \cdot (\widetilde{D})^U_\alpha(u^o)$
$\Rightarrow (\widetilde{N})^L_\alpha(\bar{u}) - (\widetilde{k})^L_\alpha \cdot (\widetilde{D})^L_\alpha(\bar{u}) \prec (\widetilde{N})^L_\alpha(u^o) - \frac{\widetilde{N}^L_\alpha(u^o)}{\widetilde{D}^L_\alpha(u^o)} \cdot (\widetilde{D})^L_\alpha(u^o)$ and $(\widetilde{N})^L_\alpha(\bar{u}) - (\widetilde{k})^L_\alpha \cdot (\widetilde{D})^L_\alpha(\bar{u}) \prec (\widetilde{N})^L_\alpha(u^o) - \frac{\widetilde{N}^L_\alpha(u^o)}{\widetilde{D}^L_\alpha(u^o)} \cdot (\widetilde{D})^L_\alpha(u^o)$
$\Rightarrow (\widetilde{N})^L_\alpha(\bar{u}) \prec \frac{\widetilde{N}^L_\alpha(u^o)}{\widetilde{D}^L_\alpha(u^o)} \cdot (\widetilde{D})^L_\alpha(\bar{u})$ and $(\widetilde{N})^U_\alpha(\bar{u}) \prec \frac{\widetilde{N}^U_\alpha(u^o)}{\widetilde{D}^U_\alpha(u^o)} \cdot (\widetilde{D})^U_\alpha(\bar{u}) \Rightarrow \frac{(\widetilde{N})^L_\alpha(\bar{u})}{(\widetilde{D})^L_\alpha(\bar{u})} \prec \frac{\widetilde{N}^L_\alpha(u^o)}{\widetilde{D}^L_\alpha(u^o)}$ and $\frac{(\widetilde{N})^U_\alpha(\bar{u})}{(\widetilde{D})^U_\alpha(\bar{u})} \prec \frac{\widetilde{N}^U_\alpha(u^o)}{\widetilde{D}^U_\alpha(u^o)}$
This contradicts the assumption in (FOPFP).
Conversely, suppose $\exists$ a feasible point $\bar{u}$ s.t.,
$\Rightarrow \frac{(\widetilde{N})^L_\alpha(\bar{u})}{(\widetilde{D})^L_\alpha(\bar{u})} \prec \frac{\widetilde{N}^L_\alpha(u^o)}{\widetilde{D}^L_\alpha(u^o)} = (\widetilde{k})^L_\alpha$ and $\frac{(\widetilde{N})^U_\alpha(\bar{u})}{(\widetilde{D})^U_\alpha(\bar{u})} \prec \frac{\widetilde{N}^U_\alpha(u^o)}{\widetilde{D}^U_\alpha(u^o)} = (\widetilde{k})^U_\alpha \Rightarrow (\widetilde{N})^L_\alpha(\bar{u}) - (\widetilde{k})^L_\alpha \cdot (\widetilde{D})^L_\alpha(\bar{u}) \prec 0 = (\widetilde{N})^L_\alpha(u^o) - (\widetilde{k})^L_\alpha \cdot (\widetilde{D})^L_\alpha(u^o)$
and

$$(\widetilde{N})^U_\alpha(\bar{u}) - (\widetilde{k})^U_\alpha \cdot (\widetilde{D})^U_\alpha(\bar{u}) \prec 0 = (\widetilde{N})^U_\alpha(u^o) - (\widetilde{k})^U_\alpha \cdot (\widetilde{D})^U_\alpha(u^o).$$

Which is a contradiction. □

3.2 Continuity and differentiability of fuzzy-valued functions ($\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$):

The continuity and differentiability of the function with fuzzy parameters ($\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$) have been proved below using section (2.3).
Let for any $u \in \mathbb{R}^n$, $\widetilde{N}(u)$ and $\widetilde{D}(u)$ are canonical functions with fuzzy parameters defined on $\mathbb{R}^n$. For a fuzzy number $\widetilde{k}$, $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is also a canonical function with fuzzy parameters. Then for any fixed $\alpha \in [0,\ 1]$, the two real-valued functions can be defined as follows:

$$(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha(u) = ((\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u))^L_\alpha \quad \text{and} \quad (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha(u) = ((\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u))^U_\alpha.$$

For $\mathbf{c} \in \mathbb{R}^n$, $\widetilde{N}$ and $\widetilde{D}$ are continuous at $\mathbf{c}$. Therefore $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is also continuous at $\mathbf{c}$ if

$$\lim_{u \to \mathbf{c}} (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(\mathbf{c}),$$

and if $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is continuous at $\mathbf{c}$ i.e., $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha$ and $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha$ are also continuous at $\mathbf{c}$ for all $\alpha \in [0,\ 1]$.
Now the right hand limit of a function with fuzzy parameters $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}) : V \to \mathbb{F}_c(\mathbb{R})$ can be defined as:

$$\lim_{u \to \mathbf{c}^+} (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u) \quad \text{and} \quad \lim_{u \to \mathbf{c}^+} (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u) = \widetilde{a} - \widetilde{k} \boxtimes \widetilde{b}.$$

Since $\widetilde{a}, \widetilde{b}$ and $\widetilde{k}$ are fuzzy numbers i.e., $\widetilde{a} - \widetilde{k} \boxtimes \widetilde{b}$ is also a fuzzy number. Therefore we have:

$$\lim_{u \to \mathbf{c}^+} (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha(u) = \widetilde{a}^L_\alpha - \widetilde{k}^L_\alpha \cdot \widetilde{b}^L_\alpha \quad \text{and} \quad \lim_{u \to \mathbf{c}^+} (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha(u) = \widetilde{a}^U_\alpha - \widetilde{k}^L_\alpha \cdot \widetilde{b}^U_\alpha,$$

for all $\alpha \in [0,\ 1]$.

Now the differentiation of the function with fuzzy parameters, $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is defined in the following lemma using the difference of two fuzzy-numbers. In this lemma it is assumed that, $\widetilde{N},\ \widetilde{D} : V \to \mathbb{F}_c(\mathbb{R})$, the functions with fuzzy parameters are Hukuhara-differentiable at $\bar{u}$ with the Hukuhara-derivatives $\widetilde{a}(\bar{u})$ and $\widetilde{b}(\bar{u})$, respectively. Where $\widetilde{a}$ and $\widetilde{b}$ are canonical fuzzy numbers.

Lemma 3.4. *Let $\widetilde{b}$ and $\widetilde{k}$ are two fuzzy numbers and the function with fuzzy parameters, $\widetilde{D} : V \to \mathbb{F}_c(\mathbb{R})$ is Hukuhara-differentiable at $\bar{u}$ with the derivative $(\widetilde{D})'(\bar{u}) = \widetilde{b} \geq 0$ then $\widetilde{k} \boxtimes \widetilde{D}$ is also Hukuhara-differentiable at $\bar{u}$ with its derivative $(\widetilde{k} \boxtimes \widetilde{D})'(\bar{u}) = \widetilde{k} \boxtimes \widetilde{b}$.*

Proof. A function $\widetilde{D} : V \to \mathbb{F}_c(\mathbb{R})$ is said to be Hukuhara-differentiable at $\bar{u}$ iff $\exists$ a fuzzy number $\widetilde{b}$ s.t.,$(\widetilde{D})'(\bar{u}) = \widetilde{b}$ and limit exists which is equal to $\widetilde{b}$ i.e.,

$$\lim_{h \to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [\widetilde{D}(\bar{u} + h) \boxminus_H \widetilde{D}(\bar{u})] \quad \text{and} \quad \lim_{h \to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [\widetilde{D}(\bar{u}) \boxminus_H \widetilde{D}(\bar{u} - h)].$$

To prove that $\widetilde{k} \boxtimes \widetilde{D}$ is Hukuhara-differentiable at $\bar{u}$, we now consider-

$$\begin{aligned}(\widetilde{k} \boxtimes \widetilde{D})'(\bar{u}) &= \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [(\widetilde{k} \boxtimes \widetilde{D})(\bar{u}+h) \boxminus_H (\widetilde{k} \boxtimes \widetilde{D})(\bar{u})] \\ &= \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes \widetilde{k} \boxtimes [\widetilde{D}(\bar{u}+h) \boxminus_H \widetilde{D}(\bar{u})] \\ &= \widetilde{k} \boxtimes \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [\widetilde{D}(\bar{u}+h) \boxminus_H \widetilde{D}(\bar{u})] \\ &= \widetilde{k} \boxtimes (\widetilde{D})'(\bar{u}) \\ &= \widetilde{k} \boxtimes \widetilde{b},\end{aligned}$$

i.e., it is equal to a fuzzy-number $\widetilde{k} \boxtimes \widetilde{b}$ and also the limits exists. Again,

$$\begin{aligned}(\widetilde{k} \boxtimes \widetilde{D})'(\bar{u}) &= \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [(\widetilde{k} \boxtimes \widetilde{D})(\bar{u}) \boxminus_H (\widetilde{k} \boxtimes \widetilde{D})(\bar{u}-h)] \\ &= \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes \widetilde{k} \boxtimes [\widetilde{D}(\bar{u}) \boxminus_H \widetilde{D}(\bar{u}-h)] \\ &= \widetilde{k} \boxtimes \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [\widetilde{D}(\bar{u}) \boxminus_H \widetilde{D}(\bar{u}-h)] \\ &= \widetilde{k} \boxtimes (\widetilde{D})'(\bar{u}) \\ &= \widetilde{k} \boxtimes \widetilde{b}.\end{aligned}$$

Here both the limits exist and are equal to $\widetilde{k} \boxtimes \widetilde{b}$. Therefore $\widetilde{k} \boxtimes \widetilde{b}$ is Hukuhara-differentiable at $\bar{u}$. □

Lemma 3.5. *Suppose $\widetilde{N}$ and $\widetilde{D}$ are Hukuhara-differentiable at $\bar{u}$ with the limits equal to the canonical fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$ respectively, then for a canonical fuzzy vector number $\widetilde{k}$, $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is also Hukuhara-differentiable at $\bar{u}$.*

Proof. Consider that

$$\begin{aligned}(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})'(\bar{u}) &= \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(\bar{u}+h) \boxminus_H (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(\bar{u})] \\ &= \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [\Big(\widetilde{N}(\bar{u}+h) - (\widetilde{k} \boxtimes \widetilde{D})(\bar{u}+h)\Big) \boxminus_H \Big(\widetilde{N}(\bar{u}) - (\widetilde{k} \boxtimes \widetilde{D})(\bar{u})\Big)] \\ &= \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [\Big(\widetilde{N}(\bar{u}+h) \boxminus_H \widetilde{N}(\bar{u})\Big) - \Big((\widetilde{k} \boxtimes \widetilde{D})(\bar{u}+h) \boxminus_H (\widetilde{k} \boxtimes \widetilde{D})(\bar{u})\Big)] \\ &= \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [\widetilde{N}(\bar{u}+h) \boxminus_H \widetilde{N}(\bar{u})] - \lim_{h\to 0+} \widetilde{1}_{\frac{1}{h}} \boxtimes [(\widetilde{k} \boxtimes \widetilde{D})(\bar{u}+h) \boxminus_H (\widetilde{k} \boxtimes \widetilde{D})(\bar{u})].\end{aligned}$$

Since the functions $\widetilde{N}$ and $\widetilde{D}$ are Hukuhara-differentiable at $\bar{u}$ and also $\widetilde{k} \boxtimes \widetilde{D}$ is Hukuhara-differentiable at $\bar{u}$ by Lemma 3.4. Therefore we have

$$(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})'(\bar{u}) = (\widetilde{N})'(\bar{u}) - (\widetilde{k} \boxtimes \widetilde{D})'(\bar{u}) = \widetilde{a} - \widetilde{k} \boxtimes \widetilde{b}.$$

It is a fuzzy number. For other case, similar results can be obtained. Hence $\widetilde{N} - (\widetilde{k} \boxtimes \widetilde{D})$ is Hukuhara-differentiable at $\bar{u}$. □

3.3 H-differentiability for $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})$

Let $V \subseteq \mathbb{R}$ be an open subset. If a function with fuzzy parameters, $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}) : V \to \mathbb{F}_c(\mathbb{R})$ is Hukuhara-differentiable with Hukuhara-derivative $(\widetilde{a} - \widetilde{k} \boxtimes \widetilde{b})$ at $\bar{u}$ then for all $\alpha \in [0,\, 1]$, $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha$ and $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha$ are differentiable at $\bar{u}$. Also, we have

$$((\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha)'(\bar{u}) = (\widetilde{a} - \widetilde{k} \boxtimes \widetilde{b})^L_\alpha(\bar{u}) \quad \text{and} \quad ((\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha)'(\bar{u}) = (\widetilde{a} - \widetilde{k} \boxtimes \widetilde{b})^U_\alpha(\bar{u}). \tag{4}$$

This can be verified from Lemma 3.5.

Now consider $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$, the function with fuzzy parameters defined on V, an open subset of $\mathbb{R}^n$. That is $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u_1, u_2, \ldots, u_n)$ is a fuzzy number for each $u = (u_1, u_2, \ldots, u_n) \in V$. Also the real valued functions can be defined on V as:

$$\begin{aligned}&(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha(u) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha(u_1, u_2, \ldots, u_n) = ((\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u_1, u_2, \ldots, u_n))^L_\alpha \quad \text{and} \\ &(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha(u) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha(u_1, u_2, \ldots, u_n) = ((\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u_1, u_2, \ldots, u_n))^U_\alpha,\end{aligned}$$

$\forall\ \alpha \in [0,\, 1]$.

Proposition 3.6. *Since $\widetilde{N}$ and $\widetilde{D}$ are the functions with fuzzy parameters defined on $\mathbb{R}^n$ i.e., $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is also a function with fuzzy parameters on $\mathbb{R}^n$. (By Theorem 12.11 from Apostol [3]) One of the partial derivatives $\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})/\partial u_1, \ldots, \partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})/\partial u_n$ of the function with fuzzy parameters $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ exists at $\bar{u}$ and rest of $(n-1)$ partial derivatives exists in the neighbourhood of $\bar{u}$ and are continuous at $\bar{u}$. Then $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is said to be differentiable at $\bar{u}$.*

From the above proposition, the following definitions can be proposed.

Definition 3.7. *Consider a function with fuzzy parameters, $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ defined on $V \subseteq \mathbb{R}^n$, where V is an open subset of $\mathbb{R}^n$. Let $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n) \in V$ be fixed.*
(i) A function $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is said to have the i^{th} partial Hukuhara-derivative $\widetilde{a} - \widetilde{k} \boxtimes \widetilde{b}$ at $\bar{u}$ if the function with fuzzy parameters, $\widetilde{N}(u_i) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(\bar{u}_1, \ldots, \bar{u}_{i-1}, \bar{u}_i, \bar{u}_{i+1}, \ldots, \bar{u}_n)$ is Hukuhara-differentiable with the Hukuhara-derivative $\widetilde{a} - \widetilde{k} \boxtimes \widetilde{b}$ at $\bar{u}_i$. Also, $(\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})/\partial u_i)(\bar{u}) = (\widetilde{a} - \widetilde{k} \boxtimes \widetilde{b})_i(\bar{u})$.
(ii) A function $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is said to be Hukuhara-differentiable at $\bar{u}$ if one of the partial Hukuhara-derivative $\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})/\partial u_1, \ldots, \partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})/\partial u_n$ exists at $\bar{u}$ and $(n-1)$ partial derivatives exist in the neighbourhood of $\bar{u}$ and are continuous at $\bar{u}$.
(iii) A function is said to be continuously Hukuhara-differentiable at $\bar{u}$ if all of the partial Hukuhara-derivatives $\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})/\partial u_i$ for $i = 1, 2, 3, \ldots, n$, exist and continuous on some neighbourhood of $\bar{u}$.

For $n = 1$, in Definition 3.7(ii) the function $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is Hukuhara-differentiable with partial Hukuhara-derivative at $\bar{u}$. From equation (4) and the above definition, we conclude that:

$$\frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha}{\partial u_i} = \left(\frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})}{\partial u_i}\right)^L_\alpha \quad \text{and} \quad \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha}{\partial u_i} = \left(\frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})}{\partial u_i}\right)^U_\alpha. \tag{5}$$

Next the properties of function with fuzzy parameters are presented to develop the optimality criteria.
Let ψ be a real-valued functions with two variables. Following are the sufficient conditions for the equality provided by Rudin [21].

$$\frac{d}{du}\int_c^d \psi(u,v)dv = \int_c^d \frac{\partial \psi}{\partial u}(u,v)dv.$$

The more details can be presented below:

Proposition 3.8. *Let us consider a two variables function ψ with real values, which are defined on $U \times [a, b]$, where U is an interval in $\mathbb{R}$. Now assume the conditions given below are satisfied:*
(i) For every $u \in U$ the function with real values, $g(v) = \psi(u,v)$ is Riemann-integrable on $[a, b]$. So we can write $(N - kD)(u) = \int_a^b \psi(u,v)dv$;
(ii) Let a fixed point $\bar{u} \in int(U)$, where U is an open interval and $int(U)$ is the interior of U. For every $\epsilon > 0$, there exists a $\delta > 0$ s.t.,

$$\left|\frac{\partial \psi}{\partial u}(u,v) - \frac{\partial \psi}{\partial u}(\bar{u},v)\right| < \epsilon,$$

for all $v \in [a, b]$ and all $u \in (\bar{u} - \delta, \bar{u} + \delta)$.
Therefore $(\partial\psi/\partial u)(\bar{u}, v)$ is Riemann integrable on the interval $[a, b]$ i.e., $(N - kD)'(\bar{u})$ exists, and

$$(N - kD)'(\bar{u}) = \int_a^b \frac{\partial \psi}{\partial u}(\bar{u},v)dv.$$

Next the properties of Riemann integrable are defined below for function with fuzzy parameters:
Let $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}) : V \to \mathbb{F}_c(\mathbb{R})$ is a function with fuzzy parameters defined on $V \subset \mathbb{R}^n$. Then $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha$ and $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha$ are the functions with real values defined on V. The corresponding functions with real values s.t., $\varsigma^L(\alpha) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha(u)$ and $\varsigma^U(\alpha) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha(u)$ are defined on [0, 1]. As the value of the function $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})$ is a canonical-fuzzy number so the functions with real values, $\varsigma^L(\alpha) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha(u)$ and $\varsigma^U(\alpha) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha(u)$ are continuous on [0, 1]. That is from the definition both the functions are also Riemann integrable on [0, 1]. Therefore for every $u \in V$, we can write

$$H^L(u) = \int_0^1 (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha(u) \quad \text{and} \quad H^U(u) = \int_0^1 (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha(u). \tag{6}$$

From the above results the following proposition can be given:

Proposition 3.9. *Let a function with fuzzy parameters, $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}) : V \to \mathbb{F}_c(\mathbb{R})$ is defined on $V \subseteq \mathbb{R}^n$. Assume that the function $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})$ is continuously Hukuhara-differentiable on the neighbourhood of $\bar{u}$. Then the functions defined in* (6) *i.e., H^L and H^U are real-valued continuously differentiable function at $\bar{u}$ and*

$$\frac{\partial H^L}{\partial u_i}(\bar{u}) = \int_0^1 \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha}{\partial u_i}(\bar{u}) d\alpha \quad \text{and} \quad \frac{\partial H^U}{\partial u_i}(\bar{u}) = \int_0^1 \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha}{\partial u_i}(\bar{u}) d\alpha, \tag{7}$$

for all $i = 1, 2, 3, \ldots, n$.

Proof. In this proposition, our aim is only to prove that the partial derivatives $\partial H^L/\partial u_i$ and $\partial H^U/\partial u_i$ exists on some neighbourhood of $\bar{u}$ for all $i = 1, 2, 3, \ldots, n$, and are continuous at $\bar{u}$. Since $V \subseteq \mathbb{R}^n$ and V is convex, so the one dimensional projection of V is an interval. From equation (6), it is clear that the condition (i) in Proposition 3.8 is satisfied. Now our aim is to prove the condition (ii) in Proposition 3.8 . Since $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is continuous Hukuhara-differentiable on the neighbourhoods i.e., $Q(\bar{u})$ of $\bar{u}$. Now suppose for any $\hat{u} \in Q(\bar{u})$, the function $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is continuous Hukuhara-differentiable at $\hat{u}$, i.e., all the partial Hukuhara-derivatives $\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})/\partial u_i$ for all $i = 1, 2, 3, \ldots, n$ are continuous and exist on some neighbourhood of $\hat{u}$. As we know if $\widetilde{a}$ and $\widetilde{b}$ are two canonical-fuzzy numbers then $d_{\mathbb{F}}(\widetilde{a}, \widetilde{b}) < \epsilon$ implies $|\widetilde{a}^L_\alpha - \widetilde{b}^L_\alpha|$ and also $|\widetilde{a}^U_\alpha - \widetilde{b}^U_\alpha|$. Therefore using the same concept and equation (5) we have, for every $\epsilon > 0$ there exists a $\delta > 0$ s.t., for all $\alpha \in [0, 1]$ we have

$$\|u - \hat{u}\| < \delta \Rightarrow \left| \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha}{\partial u_i}(u) - \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha}{\partial u_i}(\hat{u}) \right|, \tag{8}$$

where $\hat{u} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n)$ is fixed. Let $\psi(u_i, \alpha) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha(\hat{u}_1, \ldots, \hat{u}_{i-1}, u_i, \hat{u}_{i+1}, \ldots, \hat{u}_n)$. Assume that $|u_i - \hat{u}_i| < \delta$. Also if $u = (\hat{u}_1, \ldots, \hat{u}_{i-1}, u_i, \hat{u}_{i+1}, \ldots, \hat{u}_n)$, then $\|u - \hat{u}\| = |u_i - \hat{u}_i| < \delta$. Therefore for every $\epsilon > 0$ $\exists$ a $\delta > 0$ and from equation (8) we have

$$\left| \frac{\partial \psi}{\partial u_i}(u_i, \alpha) - \frac{\partial \psi}{\partial u_i}(\hat{u}_i, \alpha) \right| < \epsilon,$$

for all $\alpha \in [0, 1]$ and all $u_i \in (\hat{u}_i - \delta, \hat{u}_i + \delta)$. So for any $\hat{u} \in Q(\bar{u})$ and by Proposition 3.8 we have:

$$\frac{\partial H^L}{\partial u_i}(\hat{u}) = \int_0^1 \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha}{\partial u_i}(\hat{u}) d\alpha.$$

Again from equation (8), since

$$\begin{aligned} \left| \frac{\partial H^L}{\partial u_i}(u) - \frac{\partial H^L}{\partial u_i}(\bar{u}) \right| &= \left| \int_0^1 \left[\frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha}{\partial u_i}(u) - \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha}{\partial u_i}(\bar{u}) \right] d\alpha \right| \\ &\leq \int_0^1 \left| \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha}{\partial u_i}(u) - \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha}{\partial u_i}(\bar{u}) \right| d\alpha < \epsilon, \end{aligned}$$

i.e., we say the partial derivatives $\partial H^L/\partial u_i$ for all $i = 1, 2, 3, \ldots, n$, are continuous at $\bar{u}$. Similarly, the other case of $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha$ and $\partial H^U/\partial u_i$ can be discuss. This concludes that the upper and lower partial derivatives i.e., $\partial H^L/\partial u_i$ and $\partial H^U/\partial u_i$ exist on the $Q(\bar{u})$ $\forall$ $i = 1, 2, 3, \ldots, n$ and also continuous at $\bar{u}$. Thus the proof is complete. □

3.4 Convexity for $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$

This subsection renders the concept of convexity for the function with fuzzy parameters $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$. Let $V \in \mathbb{R}^n$ and $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is defined on V, then the function $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is said to be convex at u^o if for each $u \in V$ and for each $\lambda \in (0, 1)$, we have

$$(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(\lambda u^o + (1 - \lambda)u) \preceq \left(\widetilde{1}_\lambda \boxtimes (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u^o)\right) \boxplus \left(\widetilde{1}_{1-\lambda} \boxtimes (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u)\right),$$

where $\widetilde{1}_\lambda$ and $\widetilde{1}_{1-\lambda}$ are the crisp numbers with the values λ and $1 - \lambda$ respectively.
Since $\widetilde{N}$ and $\widetilde{D}$ are convex functions with fuzzy parameters at u^o. Therefore, we have

$$\widetilde{N}(\lambda u^o + (1 - \lambda)u) \preceq \left(\widetilde{1}_\lambda \boxtimes \widetilde{N}(u^o)\right) \boxplus \left(\widetilde{1}_{1-\lambda} \boxtimes \widetilde{N}(u)\right) \quad \text{and} \tag{9}$$

$$\widetilde{D}(\lambda u^o + (1-\lambda)u) \preceq \left(\widetilde{1}_\lambda \boxtimes \widetilde{D}(u^o)\right) \boxplus \left(\widetilde{1}_{1-\lambda} \boxtimes \widetilde{D}(u)\right). \tag{10}$$

For $\widetilde{N} \succeq \widetilde{D}$ and $\widetilde{k} \succ 0$ we have

$$(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(\lambda u^o + (1-\lambda)u) = (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})\lambda u^o + (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(1-\lambda)u = \widetilde{N}(\lambda u^o + (1-\lambda)u) - \widetilde{k} \boxtimes \widetilde{D}(\lambda u^o + (1-\lambda)u).$$

From (9) and (10), we have

$$\begin{aligned}
&\preceq \left[\left(\widetilde{1}_\lambda \boxtimes \widetilde{N}(u^o)\right) \boxplus \left(\widetilde{1}_{1-\lambda} \boxtimes \widetilde{N}(u)\right)\right] - \widetilde{k} \boxtimes \left[\left(\widetilde{1}_\lambda \boxtimes \widetilde{D}(u^o)\right) \boxplus \left(\widetilde{1}_{1-\lambda} \boxtimes \widetilde{D}(u)\right)\right] \\
&\preceq \left[\left(\widetilde{1}_\lambda \boxtimes \widetilde{N}(u^o)\right) - \widetilde{k} \boxtimes \left(\widetilde{1}_\lambda \boxtimes \widetilde{D}(u^o)\right)\right] \boxplus \left[\left(\widetilde{1}_{1-\lambda} \boxtimes \widetilde{N}(u)\right) - \widetilde{k} \boxtimes \left(\widetilde{1}_{1-\lambda} \boxtimes \widetilde{D}(u)\right)\right] \\
&\preceq \left(\widetilde{1}_\lambda \boxtimes (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u^o)\right) \boxplus \left(\widetilde{1}_{1-\lambda} \boxtimes (\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u)\right).
\end{aligned}$$

So $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is convex at u^o.
Hence a function $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is said to be convex at u^o if and only if the real-valued functions $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha$ and $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha$ are convex at u^o for all $\alpha \in [0, 1]$.

4 Optimality criteria

4.1 Optimality conditions for conventional optimization problem

Let N, D and l_h, be the functions with real values, which are defined on $\mathbb{R}^n$. Then we have the following optimization problem (OP)

$$\begin{aligned}
(OP) \qquad & \min \ (N - k \cdot D) \\
& \text{subject to,} \\
& l_h(u) \leq 0, \quad h = 1,2,3,\ldots,m, \\
& u \in V \subseteq \mathbb{R}^n.
\end{aligned}$$

Since $(N - kD)$ is a real-valued function and the real-valued functions l_h are convex on $\mathbb{R}^n$ for each $h = 1,2,3,\ldots,m$, where $V = \{u \in \mathbb{R}^n : l_h(u) \leq 0,\ h = 1,2,3,\ldots,m\}$. Then the optimality criteria for the problem (OP) (by Bazaraa et al. [4] and Horst et al. [15]) are given below.

Theorem 4.1. *Let $u^o \in V$ and the objective function $(N - kD) : V \to \mathbb{R}$ be convex at u^o. Assume that for $h = 1,2,3,\ldots,m$, the real-valued constraints functions l_h are convex at u^o and the functions $(N - kD)$, l_h are continuously differentiable at u^o. If there exist multipliers (Lagrange) $\mu_h \in \mathbb{R}$ and $\mu_h \geq 0$ for $h = 1,2,3,\ldots,m$ s.t.,*

$$\begin{aligned}
&(i) \quad \triangledown (N - k \cdot D)(u^o) + \sum_{h=1}^{m} \mu_h \cdot \triangledown l_h(u^o) = 0; \\
&(ii) \quad \mu_h \cdot l_h(u^o) = 0, \quad \textit{for all } h = 1,2,3,\ldots,m,
\end{aligned}$$

then u^o is said to be an optimal solution for the problem (OP).

4.2 Optimality criteria for optimization problem with fuzzy parameters

In this subsection, the considered fractional optimization problem with fuzzy parameters (FPFOP) is converted into non-fractional optimization problem with fuzzy parameters (OPFP) by using Dinkelbach [11] and Jagannathan [17] technique. In this problem, the objective functions are fuzzy-valued and the constraints are real-valued functions, also $l_h : \mathbb{R}^n \to \mathbb{R}$ for $h = 1,2,3,\ldots,m$ are convex on $\mathbb{R}^n$. The problem can be formulated below as:

$$\begin{aligned}
(OPFP) \qquad & \min \ \left(\widetilde{N}(u) - \widetilde{k} \boxtimes \widetilde{D}(u)\right) \\
& \text{subject to,} \\
& l_h(u) \leq 0, \quad h = 1,2,3,\ldots,m, \\
& u \in V \subseteq \mathbb{R}^n.
\end{aligned}$$

The gradients of $(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha$ and $(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha$ for the function with fuzzy parameters $\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D}$, where $\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D}$ is continuously Hukuhara-differentiable at $\bar{u}$ can be defined as follows:

$$\bigtriangledown(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha(\bar{u}) = \left(\frac{\partial(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha}{\partial u_1}(\bar{u}),\ldots,\frac{\partial(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha}{\partial u_n}(\bar{u})\right)^T \quad \text{and}$$

$$\bigtriangledown(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha(\bar{u}) = \left(\frac{\partial(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha}{\partial u_1}(\bar{u}),\ldots,\frac{\partial(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha}{\partial u_n}(\bar{u})\right)^T,$$

for all $\alpha \in [0,\, 1]$. Also we have

$$\int_0^1 \bigtriangledown(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha(\bar{u})d\alpha = \left(\int_0^1 \frac{\partial(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha}{\partial u_1}(\bar{u})d\alpha,\ldots,\int_0^1 \frac{\partial(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha}{\partial u_n}(\bar{u})d\alpha\right)^T,$$

and

$$\int_0^1 \bigtriangledown(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha(\bar{u})d\alpha = \left(\int_0^1 \frac{\partial(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha}{\partial u_1}(\bar{u})d\alpha,\ldots,\int_0^1 \frac{\partial(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha}{\partial u_n}(\bar{u})d\alpha\right)^T.$$

If the function with fuzzy parameters $\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D}$ is continuously Hukuhara-differentiable in some neighbourhood of $\bar{u}$, then by Proposition 3.9 H^L and H^U are also differentiable at $\bar{u}$. Therefore we have

$$\bigtriangledown H^L(\bar{u}) = \bigtriangledown \int_0^1 (\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha(\bar{u})d\alpha = \int_0^1 \bigtriangledown(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha(\bar{u})d\alpha = \int_0^1 \left(\bigtriangledown\widetilde{N}^L_\alpha(\bar{u}) - \widetilde{k}^L_\alpha\cdot\bigtriangledown\widetilde{D}^L_\alpha(\bar{u})\right)d\alpha, \quad \text{and}$$

$$\bigtriangledown H^U(\bar{u}) = \bigtriangledown \int_0^1 (\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha(\bar{u})d\alpha = \int_0^1 \bigtriangledown(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha(\bar{u})d\alpha = \int_0^1 \left(\bigtriangledown\widetilde{N}^U_\alpha(\bar{u}) - \widetilde{k}^U_\alpha\cdot\bigtriangledown\widetilde{D}^U_\alpha(\bar{u})\right)d\alpha.$$

Now we are able to present optimality criteria for problem (OPFP).

Theorem 4.2. *Let a point $u^o \in V$ where, $V = \{u \in \mathbb{R}^n : l_h(u) \le 0, h = 1,2,3,\ldots,m\}$ be a set of feasible solutions for problem (OPFP). Considering the constraint functions with real-values, $l_h : \mathbb{R}^n \to \mathbb{R}$ for $h = 1,2,3,\ldots,m$ which are continuously differentiable at u^o and also convex on $\mathbb{R}^n$. The objective function with fuzzy parameters, $(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D}) : V \to \mathbb{F}_c(\mathbb{R})$ is convex at u^o and continuously Hukuhara-differentiable in the neighbourhood of u^o. u^o is said to have a non-dominated solution for problem (OPFP) if for $h = 1,2,3,\ldots,m$, there exist the multipliers $\mu_h \ge 0$ where $\mu_h \in \mathbb{R}$ and $0 < \lambda^L,\ \lambda^U \in \mathbb{R}$ s.t.,*

$$(i)\quad \lambda^L\cdot\int_0^1 \left(\bigtriangledown\widetilde{N}^L_\alpha(u^o) - \widetilde{k}^L_\alpha\cdot\bigtriangledown\widetilde{D}^L_\alpha(u^o)\right)d\alpha + \lambda^U\cdot\int_0^1 \left(\bigtriangledown\widetilde{N}^U_\alpha(u^o) - \widetilde{k}^U_\alpha\cdot\bigtriangledown\widetilde{D}^U_\alpha(u^o)\right)d\alpha + \sum_{h=1}^m \mu_h\cdot\bigtriangledown l_h(u^o) = 0;$$

(ii) $\mu_h \cdot l_h(u^o) = 0$ *for all* $h = 1,2,3,\ldots,m$.

Proof. Since the functions $(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha$ and $(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha$ are Riemann integrable on [0, 1] with respect to a fixed variable u then by equation (6), a real-valued function can be defined as follows:

$$H(u) = \lambda^L\int_0^1 (\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha(u)d\alpha + \lambda^U\int_0^1 (\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha(u)d\alpha. \tag{11}$$

Where,

$$(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^L_\alpha(u) = \widetilde{N}^L_\alpha(u) - \widetilde{k}^L_\alpha\cdot\widetilde{D}^L_\alpha(u) \quad \text{and} \quad (\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})^U_\alpha(u) = \widetilde{N}^U_\alpha(u) - \widetilde{k}^U_\alpha\cdot\widetilde{D}^U_\alpha(u).$$

Since $(\widetilde{N}-\widetilde{k}\boxtimes\widetilde{D})$ is continuously Hukuhara-differentiable on some neighbourhoods of u^o then from Proposition 3.9, H is continuously differentiable at u^o and by equation (11) we have

$$\bigtriangledown H(u^o) = \lambda^L\cdot\int_0^1 \left(\bigtriangledown\widetilde{N}^L_\alpha(u^o) - \widetilde{k}^L_\alpha\cdot\bigtriangledown\widetilde{D}^L_\alpha(u^o)\right)d\alpha + \lambda^U\cdot\int_0^1 \left(\bigtriangledown\widetilde{N}^U_\alpha(u^o) - \widetilde{k}^U_\alpha\cdot\bigtriangledown\widetilde{D}^U_\alpha(u^o)\right)d\alpha, \tag{12}$$

i.e., the condition (i) in the present theorem is well defined. Since the function $\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}$ is convex at u^o then the function H is also convex at u^o, this statement can be proved from section (3.3). The result will be proved by contradiction, by assuming u^o is not a non-dominated solution. Then $\exists$ a $\bar{u}(\neq u^o) \in V$ such that $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(\bar{u})$ dominates $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})(u^o)$ i.e., from equation (3) we have

$$\lambda^L \cdot \left(\widetilde{N}^L_\alpha(\bar{u}) - \widetilde{k}^L_\alpha \cdot \widetilde{D}^L_\alpha(\bar{u})\right) + \lambda^U \cdot \left(\widetilde{N}^U_\alpha(\bar{u}) - \widetilde{k}^U_\alpha \cdot \widetilde{D}^U_\alpha(\bar{u})\right) < \lambda^L \cdot \left(\widetilde{N}^L_\alpha(u^o) - \widetilde{k}^L_\alpha \cdot \widetilde{D}^L_\alpha(u^o)\right) + \lambda^U \cdot \left(\widetilde{N}^U_\alpha(u^o) - \widetilde{k}^U_\alpha \cdot \widetilde{D}^U_\alpha(u^o)\right),$$

$\forall\ \alpha \in [0,\ 1]$. Also $\lambda^L,\ \lambda^U > 0$, from equation (11) we get

$$H(\bar{u}) < H(u^o). \tag{13}$$

Since H is continuously differentiable convex function at u^o, therefore using equation (12) we obtain two new conditions as below:

$$(a)\ \ \triangledown H(u^o) + \sum_{h=1}^{m} \mu_h \cdot \triangledown l_h(u^o) = 0;$$
$$(b)\ \mu_h \cdot l_h(u^o) = 0\ \forall\ h = 1, 2, 3, \ldots, m.$$

By using Theorem 4.1 we observe that for the real-valued function H with respect to the constraints defined in problem (OPFP), u^o is an optimal solution i.e., we have

$$H(u^o) \leq H(\bar{u})\ \ \forall\ \ \bar{u}(\neq u^o) \in V.$$

This contradicts equation (13). Hence the proof is complete. □

The function H in equation (12) can be defined from the above proof as follows:

$$\triangledown H(u^o) = \int_0^1 \left(\triangledown \widetilde{N}^L_\alpha(u^o) - \widetilde{k}^L_\alpha \cdot \triangledown \widetilde{D}^L_\alpha(u^o)\right) d\alpha + \int_0^1 \left(\triangledown \widetilde{N}^U_\alpha(u^o) - \widetilde{k}^U_\alpha \cdot \triangledown \widetilde{D}^U_\alpha(u^o)\right) d\alpha.$$

If we replace the condition (i) of the above theorem by the condition given below, then the Theorem 4.2 is still true i.e.,

$$\int_0^1 \left(\triangledown \widetilde{N}^L_\alpha(u^o) - \widetilde{k}^L_\alpha \cdot \triangledown \widetilde{D}^L_\alpha(u^o)\right) d\alpha\ + \int_0^1 \left(\triangledown \widetilde{N}^U_\alpha(u^o) - \widetilde{k}^U_\alpha \cdot \triangledown \widetilde{D}^U_\alpha(u^o)\right) d\alpha + \sum_{h=1}^{m} \mu_h \cdot \triangledown l_h(u^o)\ = 0.$$

Definition 4.3. ***Pseudo-convexity:*** *Let V be an open convex subset of $\mathbb{R}^n$ and $(N - k \cdot D)$ be a real-valued function. We say $(N-k\cdot D)$ is convex at $\bar{u}$ iff $(N-k\cdot D)(u)-(N-k\cdot D)(\bar{u}) \geq \triangledown(N-k\cdot D)(\bar{u})^T(u-\bar{u})$ for $u \in V$ (from Theorem 3.3.3 of Bazaraa et al. [4]). Now we say the function $(N - k \cdot D)$ is pseudo-convex at $\bar{u}$ if $(N-k\cdot D)(u) < (N-k\cdot D)(\bar{u})$ then $\triangledown(N - k \cdot D)(\bar{u})^T(u - \bar{u}) < 0$ for $u \in V$. By Section (3.3) the definition of pseudo-convexity for function with fuzzy parameters can be proposed.*

Let V be a non-empty convex subset of $\mathbb{R}^n$ and the function with fuzzy parameters $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})$ is defined on V. The function $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})$ is said to be pseudo-convex at $\bar{u}$ if and only if $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^L_\alpha$ and $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})^U_\alpha$ are pseudo-convex at $\bar{u}$ for all $\alpha \in [0,\ 1]$.
Let us denote R, as the cone of feasible direction of the non-empty feasible set V at $u^o \in cl\ V$ (the closure of V) which is defined as:

$$R = \{\mathbf{r} \in \mathbb{R}^n : \mathbf{r} \neq 0,\ \exists \text{ a } \delta > 0 \text{ such that } u^o + \zeta \mathbf{r} \in V \quad \forall \quad \zeta \in (0,\ \delta)\}.$$

Here each $\mathbf{r}$ of R is called feasible direction of V. The proposition can be defined as follows (by Bazaraa et al. [4]).

Proposition 4.4. *(Form the Lemma 4.2.4 Bazaraa et al. [4]) Let $u^o \in V$ and $V = \{u \in \mathbb{R}^n : l_h(u) \leq 0, h = 1, 2, 3, \ldots, m\}$ be a feasible set. Suppose, l_h for $h = 1, 2, 3, \ldots, m$ are differentiable at u^o. Let the set I be an index set (for active constraints) which is defined as $I = \{h : l_h(u^o) = 0\}$. Then*

$$R \subseteq \{\boldsymbol{r} \in \mathbb{R}^n : \triangledown l_h(u^o)^T \boldsymbol{r} \leq 0 \text{ for each } h \in I\}.$$

(The above proposition is still valid if we assume that the constraints functions $l_h,$ for $h \in I$ are continuous (instead of differentiable) at u^o.)

The alternative statement of Tucker's theorem for any matrices P and S is given as,

$$\text{System } (a) : Pu \leq 0,\ Pu \neq 0,\ Su \leq 0 \text{ for some } u \in \mathbb{R}^n;$$
$$\text{System } (b) : P^T\lambda + S^T\mu = 0 \text{ for some } (\lambda, \mu),\ \lambda > 0,\ \mu \geq 0.$$

After imposing some conditions on the objective function with fuzzy parameters, the refine conditions can be obtain using the above alternative statements.

Theorem 4.5. *Let a point $u^o \in V$ where, $V = \{u \in \mathbb{R}^n : l_h(u) \leq 0, h = 1, 2, 3, \ldots, m\}$ is a feasible convex subset of $\mathbb{R}^n$ for problem (OPFP). Suppose the constraint functions with real values $l_h : \mathbb{R}^n \to \mathbb{R}$ for $h = 1, 2, 3, \ldots, m$ are differentiable at u^o and the objective function with fuzzy parameters, $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D}) : V \to \mathbb{F}_c(\mathbb{R})$ is continuously Hukuhara-differentiable in the neighbourhood of u^o which is also pseudo-convex at u^o. Then u^o is said to be weakly-non-dominated solution of problem (OPFP) if for $h = 1, 2, 3, \ldots, m$, there exist the multipliers $\mu_h \geq 0$ where $0 \leq \mu_h^L,\ \mu_h^U \in \mathbb{R}$ s.t.,*

(i) $\int_0^1 \left(\bigtriangledown \widetilde{N}_\alpha^L(u^o) - \widetilde{k}_\alpha^L \cdot \bigtriangledown \widetilde{D}_\alpha^L(u^o)\right) d\alpha\ + \sum_{h=1}^{m} \mu_h \cdot \bigtriangledown l_h(u^o)\ = 0;$

(ii) $\int_0^1 \left(\bigtriangledown \widetilde{N}_\alpha^U(u^o) - \widetilde{k}_\alpha^U \cdot \bigtriangledown \widetilde{D}_\alpha^U(u^o)\right) d\alpha + \sum_{h=1}^{m} \mu_h \cdot \bigtriangledown l_h(u^o)\ = 0;$

(iii) $\mu_h^L \cdot l_h(u^o) = 0 = \mu_h^U \cdot l_h(u^o)$ *for all* $h = 1, 2, 3, \ldots, m.$

Proof. The result will be prove by contradiction, by assuming that u^o is not a weakly-non-dominated solution. That is $\exists$ an $\bar{u}(\neq u^o)$ such that $\widetilde{N}(\bar{u}) - \widetilde{k} \boxtimes \widetilde{D}(\bar{u})$ strongly-dominates $\widetilde{N}(u^o) - \widetilde{k} \boxtimes \widetilde{D}(u^o)$ i.e., from equation (3) and for all $\alpha \in [0,\ 1]$ we have $\widetilde{N}_\alpha^L(\bar{u}) - \widetilde{k}_\alpha^L \cdot \widetilde{N}_\alpha^L(\bar{u}) < \widetilde{N}_\alpha^L(u^o) - \widetilde{k}_\alpha^L \cdot \widetilde{N}_\alpha^L(u^o)$ or $\widetilde{N}_\alpha^U(\bar{u}) - \widetilde{k}_\alpha^U \cdot \widetilde{N}_\alpha^U(\bar{u}) < \widetilde{N}_\alpha^U(u^o) - \widetilde{k}_\alpha^U \cdot \widetilde{N}_\alpha^U(u^o)$. Since the function $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})$ is pseudo-convex at $\widetilde{u}^o$, so the real-valued functions $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})_\alpha^L$ and $(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})_\alpha^U$ are also pseudo-convex at $u^o\ \forall\ \alpha \in [0,\ 1]$. Therefore we can write $\bigtriangledown(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})_\alpha^L(u^o)^T(\bar{u} - u^o) < 0$ and $\bigtriangledown(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})_\alpha^U(u^o)^T(\bar{u} - u^o) < 0$ for all $\alpha \in [0,\ 1]$, i.e.,

$$\sum_{i=1}^{n}(\bar{u}_i - u_i^o) \cdot \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})_\alpha^L}{\partial u_i}(u^o) < 0 \quad \text{and} \quad \sum_{i=1}^{n}(\bar{u}_i - u_i^o) \cdot \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})_\alpha^U}{\partial u_i}(u^o) < 0,$$

for all $\alpha \in [0,\ 1]$. This also implies that

$$\sum_{i=1}^{n}(\bar{u}_i - u_i^o) \cdot \int_0^1 \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})_\alpha^L}{\partial u_i}(u^o)d\alpha < 0 \quad \text{and} \quad \sum_{i=1}^{n}(\bar{u}_i - u_i^o) \cdot \int_0^1 \frac{\partial(\widetilde{N} - \widetilde{k} \boxtimes \widetilde{D})_\alpha^U}{\partial u_i}(u^o)d\alpha < 0. \tag{14}$$

By Proposition 3.9 we see that, H^L and H^U are differentiable functions at u^0. So by using equations (7) and (14), we have

$$\sum_{i=1}^{n}(\bar{u}_i - u_i^o) \cdot \frac{\partial H_\alpha^L}{\partial u_i}(u^o) < 0 \quad \text{and} \quad \sum_{i=1}^{n}(\bar{u}_i - u_i^o) \cdot \frac{\partial H_\alpha^U}{\partial u_i}(u^o) < 0,$$

also

$$\bigtriangledown H^L(u^o)^T(\bar{u} - u^o) < 0 \quad \text{and} \quad \bigtriangledown H^U(u^o)^T(\bar{u} - u^o) < 0. \tag{15}$$

First we are going to consider the case of $\bigtriangledown H^L(u^o)^T(\bar{u} - u^o) < 0$. Now let $\mathbf{r} = \bar{u} - u^o$, then $u = u^o + \zeta\mathbf{r} = \zeta\bar{u} + (1-\zeta)u^o \in V$ for $\zeta \in (0,\ 1)$. Since $\bar{u},\ u^o \in V$, where V is the convex set which shows $\mathbf{r} \in R$. Therefore from Proposition 4.4, we have

$$\bigtriangledown l_h(u^o)^T\mathbf{r} \leq 0 \text{ for each } h \in I, \tag{16}$$

here I is the set of active constraints. consider a matrix $P = \bigtriangledown H^L(u^o)^T$ and a matrix S with the rows $\bigtriangledown l_h(u^o)^T$ for $h \in I$. From the alternative statements of Tucker's theorem we have system (a), which has a solution $\mathbf{r} = \bar{u} - u^o$. So by equations (15) and (16), $\nexists$ multipliers $0 \leq \bar{u}_h \in \mathbb{R}$ for $h \in I$ and $0 < \lambda \in \mathbb{R}$, s.t.,

$$\lambda \cdot \bigtriangledown H^L(u^o) + \sum_{h \in I} \bar{\mu}_h \cdot \bigtriangledown l_h(u^o) = 0;$$

or $\nexists$ multipliers $0 \leq \mu_h \in \mathbb{R}$ for $h \in I$, such that

$$\lambda \cdot \bigtriangledown H^L(u^o) + \sum_{h\in I} \mu_h \cdot \bigtriangledown l_h(u^o) = 0.$$

Here $\mu_h = \bar{\mu}_h/\lambda$, which is a contradiction to condition (i) and condition (ii) of the present theorem, since $\sum_{h\in I} \mu_h \bigtriangledown l_h(u^o) = \sum_{h=1}^{m} \mu_h \cdot \bigtriangledown l_h(u^o)$ with $\mu_h \cdot l_h(u^o)$ for all $h = 1, 2, 3, \ldots, m$ (i.e., $l_h(u^o) \neq 0$ for $h \notin I$). Similarly for the other case $\bigtriangledown H^U(u^o)^T(\bar{u} - u^o) < 0$, the conditions (ii) and (iii) of the present theorem can be violated. Hence proof is done. □

5 Numerical problems

Next, we are going to consider numerical example as follows.

Example 5.1. *Considering the optimization problem with fuzzy parameters given below-*

$$min\ \widetilde{Z} = \frac{(-2,\ -1,\ 0)u_1 \boxplus (0,\ 1,\ 2)u_2 \boxplus (2,\ 3,\ 4)}{(0,\ 1,\ 2)u_1 \boxplus (0,\ 1,\ 2)u_2 \boxplus (1,\ 2,\ 3)}$$

subject to, $u_1 + u_2 \leq 2, \quad u_1 - u_2 \leq 1,$ *and* $u_1,\ u_2 \geq 0,$
where,
$(0,\ 1,\ 2), (-2,\ -1,\ 0),\ (2,\ 3,\ 4)$ *and* $(1, 2, 3)$ *are triangular fuzzy numbers.*
After the Defuzzification of considered problem and solving it by Dinkelbach Algorithm, we obtained $k = 0.5$. *Again fuzzify the obtained number* $\tilde{k}$, *we get* $\widetilde{k} = (0,\ 0.5,\ 1)$. *So that* $(\widetilde{k})_\alpha = [\widetilde{k}^L_\alpha, \widetilde{k}^U_\alpha] = [0.5\alpha, 1 - 0.5\alpha]$

$$\begin{aligned}
\widetilde{N}(u_1,\ u_2) &= (-2,\ -1,\ 0)u_1 \boxplus (0,\ 1,\ 2)u_2 \boxplus (2,\ 3,\ 4),\\
\widetilde{D}(u_1,\ u_2) &= (0,\ 1, 2)u_1 \boxplus (0,\ 1,\ 2)u_2 \boxplus (1,\ 2,\ 3)\\
l_1(u_1,\ u_2) &= u_1 + u_2 - 2\\
l_2(u_1,\ u_2) &= u_1 - u_2 - 1\\
l_3(u_1,\ u_2) &= -u_1\\
l_4(u_1,\ u_2) &= -u_2
\end{aligned}$$

By arithmetic operations on fuzzy numbers in subsection (2.2) and equation (1), *we get*

$$\begin{aligned}
\widetilde{N}^L_\alpha(u_1, u_2) &= u_1(-2+\alpha) + u_2(\alpha) + (\alpha+2)\\
\widetilde{N}^U_\alpha(u_1, u_2) &= u_1(-\alpha) + u_2(2 - \alpha) + (4-\alpha)\\
\widetilde{D}^L_\alpha(u_1, u_2) &= u_1(\alpha) + u_2(\alpha) + (\alpha+1)\\
\widetilde{D}^U_\alpha(u_1, u_2) &= u_1(2-\alpha) + u_2(2-\alpha) + (3-\alpha)
\end{aligned}$$

$\forall\ \alpha \in [0, 1]$. *We can also obtain*

$$\bigtriangledown \widetilde{N}^L_\alpha(u) = \begin{bmatrix} -2+\alpha \\ \alpha \end{bmatrix}, \quad \bigtriangledown \widetilde{N}^U_\alpha(u) = \begin{bmatrix} -\alpha \\ -\alpha+2 \end{bmatrix}, \quad \bigtriangledown \widetilde{D}^L_\alpha(u) = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}, \quad \bigtriangledown \widetilde{D}^U_\alpha(u) = \begin{bmatrix} 2-\alpha \\ 2-\alpha \end{bmatrix},$$

$$\bigtriangledown l_1(u) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } \bigtriangledown l_2(u) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore we have

$$\int_0^1 \bigtriangledown \widetilde{N}^L_\alpha(u)d\alpha = \begin{bmatrix} -1.5 \\ 0.5 \end{bmatrix}, \quad \int_0^1 \bigtriangledown \widetilde{N}^U_\alpha(u)d\alpha = \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}, \quad \int_0^1 \widetilde{k}^L_\alpha \cdot \bigtriangledown \widetilde{D}^L_\alpha(u)d\alpha = \begin{bmatrix} 0.167 \\ 0.167 \end{bmatrix},$$

$$\int_0^1 \widetilde{k}^U_\alpha \cdot \bigtriangledown \widetilde{D}^U_\alpha(u)d\alpha = \begin{bmatrix} 1.167 \\ 1.167 \end{bmatrix},$$

After solving the equations $l_1(u) = 0 = l_2(u)$, *we get*

$$u^o = (u^o_1, u^o_2) = (1.5, 0.5).$$

By the (ii) condition of Theorem 4.2 we obtain $\mu_3(\alpha) = 0 = \mu_4(\alpha)$, *since* $l_3(u^o) = -1.5$ *and* $l_4(u^o) = -0.5$. *Now applying the condition (i) of Theorem 4.2 to* u^o, *we get*

$$\lambda^L \cdot \int_0^1 \left(\triangledown \widetilde{N}^L_\alpha(u^o) - \widetilde{k}^L_\alpha \cdot \triangledown \widetilde{D}^L_\alpha(u^o)\right) d\alpha \; + \; \lambda^U \cdot \int_0^1 \left(\triangledown \widetilde{N}^U_\alpha(u^o) - \widetilde{k}^U_\alpha \cdot \triangledown \widetilde{D}^U_\alpha(u^o)\right) d\alpha + \sum_{h=1}^{4} \mu_h \cdot \triangledown l_h(u^o)$$

$$= \begin{bmatrix} -1.667\lambda^L - 1.667\lambda^U + \mu_1 + \mu_2 \\ 0.333\lambda^L + 0.333\lambda^U + \mu_1 - \mu_2 \end{bmatrix} = 0.$$

After numerical evaluation we get the non-negative real-valued-functions,
$\mu_1 = 1.334$, $\mu_2 = 2$ *and* $\mu_h = 0$ *for* $h = 3, 4$.
Therefore by Theorem 4.2, $u^o = (1.5, 0.5)$ *is efficient solution.*

6 Applications of fractional optimization problems in production planning

Example 6.1. *Suppose a company gets profit in the production of a product A, while the company faces loss in the production of a product B. The profit and loss are around 3 \$ and 2 \$ per unit respectively. The company faces some additional loss of amount around 12 \$ during the manufacturing of products. The production cost of each product A and B is around 1 \$ per unit respectively with a fixed charge of around 3 \$ for storage purpose.*
The management company has given a task to minimize the loss in the business with respect to some conditions. Material needed per unit for A and B are around 2 pounds and around 3 pounds respectively and restricted to around 15 pounds. Also the production of A is more than production of B at most by around 1 unit. Here u_1 *and* u_2 *are the production units produced by the company.*

This production planning problem can be formulated below and solved by the above described method.

$$min\ \widetilde{Z} \;=\; \frac{(-4,\,-3,\,-2)u_1 \boxplus (1,\,2,\,3)u_2 \boxplus (11,\,12,\,13)}{(0,\,1,\,2)u_1 \boxplus (0,\,1,\,2)u_2 \boxplus (2,\,3,\,4)},$$

subject to $\quad u_1 - u_2 \leq 1, \quad 2u_1 + 3u_2 \leq 15, \quad u_1,\, u_2 \geq 0,$
where,
$(-4,\,-3,\,-2), (1,\,2,\,3), (11,\,12,\,13), (0,\,1,\,2)$ *and* $(2,\,3,\,4)$ *are triangular fuzzy numbers.*
After the Defuzzification of considered problem and solving it by Dinkelbach Algorithm, we obtained $k = 0.7$. *Again fuzzify the obtained number* $\tilde{k}$, *we get* $\tilde{k} = (0.6,\, 0.7,\, 0.8)$. *So that* $(\tilde{k})_\alpha = [\tilde{k}^L_\alpha,\, \tilde{k}^U_\alpha] = [0.1\alpha + 0.6,\, 0.8 - 0.1\alpha]$

$$\begin{aligned}
\widetilde{N}(u_1,\, u_2) &= (-4,\, -3, -2)u_1 \boxplus (1,\, 2,\, 3)u_2 \boxplus (11,\, 12,\, 13),\\
\widetilde{D}(u_1,\, u_2) &= (0,\, 1,\, 2)u_1 \boxplus (0,\, 1,\, 2)u_2 \boxplus (2,\, 3,\, 4)\\
l_1(u_1,\, u_2) &= u_1 - u_2 - 1\\
l_2(u_1,\, u_2) &= 2u_1 + 3u_2 - 15\\
l_3(u_1,\, u_2) &= -u_1\\
l_4(u_1,\, u_2) &= -u_2
\end{aligned}$$

By arithmetic operations on fuzzy numbers in subsection (2.2) and equation (1), *we get*

$$\begin{aligned}
\widetilde{N}^L_\alpha(u_1,\, u_2) &= u_1(-4 + \alpha) + u_2(1 + \alpha) + (11 + \alpha)\\
\widetilde{N}^U_\alpha(u_1,\, u_2) &= u_1(-\alpha - 2) + u_2(3 - \alpha) + (13 - \alpha)\\
\widetilde{D}^L_\alpha(u_1,\, u_2) &= u_1(\alpha) + u_2(\alpha) + (\alpha + 2)\\
\widetilde{D}^U_\alpha(u_1,\, u_2) &= u_1(2 - \alpha) + u_2(2 - \alpha) + (4 - \alpha)
\end{aligned}$$

for all $\alpha \in [0, 1]$. *We can also obtain*

$$\triangledown \widetilde{N}^L_\alpha(u) = \begin{bmatrix} -4 + \alpha \\ 1 + \alpha \end{bmatrix}, \quad \triangledown \widetilde{N}^U_\alpha(u) = \begin{bmatrix} -\alpha - 2 \\ 3 - \alpha \end{bmatrix}, \quad \triangledown \widetilde{D}^L_\alpha(u) = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}, \quad \triangledown \widetilde{D}^U_\alpha(u) = \begin{bmatrix} 2 - \alpha \\ 2 - \alpha \end{bmatrix},$$

$$\triangledown l_1(u) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad and \quad \triangledown\, l_2(u) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Therefore we have

$$\int_0^1 \bigtriangledown \widetilde{N}_\alpha^L(u)d\alpha = \begin{bmatrix} -3.5 \\ 1.5 \end{bmatrix}, \quad \int_0^1 \bigtriangledown \widetilde{N}_\alpha^U(u)d\alpha = \begin{bmatrix} -2.5 \\ 2.5 \end{bmatrix}, \quad \int_0^1 \widetilde{k}_\alpha^L \cdot \bigtriangledown \widetilde{D}_\alpha^L(u)d\alpha = \begin{bmatrix} 0.333 \\ 0.333 \end{bmatrix},$$
$$\int_0^1 \widetilde{k}_\alpha^U \cdot \bigtriangledown \widetilde{D}_\alpha^U(u)d\alpha = \begin{bmatrix} 1.133 \\ 1.133 \end{bmatrix},$$

After solving the equations $l_1(u) = 0 = l_2(u)$, *we get*

$$u^o = (u_1^o, u_2^o) = (3.60, 2.60).$$

By condition (ii) of the Theorem 4.2 we see that $\mu_3(\alpha) = 0 = \mu_4(\alpha)$, *since* $l_3(u^o) = -3.60$ *and* $l_4(u^o) = -2.60$. *Now applying the condition (i) of the Theorem 4.2 to* u^o, *we obtain*

$$\lambda^L \cdot \int_0^1 \left(\bigtriangledown \widetilde{N}_\alpha^L(u^o) - \widetilde{k}_\alpha^L \cdot \bigtriangledown \widetilde{D}_\alpha^L(u^o) \right) d\alpha + \lambda^U \cdot \int_0^1 \left(\bigtriangledown \widetilde{N}_\alpha^U(u^o) - \widetilde{k}_\alpha^U \cdot \bigtriangledown \widetilde{D}_\alpha^U(u^o) \right) d\alpha + \sum_{h=1}^4 \mu_h \cdot \bigtriangledown l_h(u^o)$$

$$= \begin{bmatrix} -3.83\lambda^L - 3.633\lambda^U + \mu_1 + 2\mu_2 \\ 1.17\lambda^L + 1.367\lambda^U - \mu_1 + 3\mu_2 \end{bmatrix} = 0$$

After numerical evaluation we get the non-negative real-valued-functions
$\mu_1 = 5.4926,\ \mu_2 = 0.9852$ *and* $\mu_h = 0$ *for* $h = 3, 4$.
Therefore by Theorem 4.2, $u^o = (3.60, 2.60)$ *is efficient solution.*

7 Limitations and advantage

The proposed work have some advantage as well as limitations. Some of the methods are proposed randomly to solve fuzzy fractional optimization problems by many authors. However our proposed method have advantage that it justify the optimal solution after checking KKT optimality conditions. This optimality conditions are general and applicable for more than two variables, it worked for large scale problem also. The limitation of the work is that, in our problem we have considered our constraints classical only not fuzzy.
For the validation of above result, we have considered two numerical examples. In which one of the example is considered from production planning. The developed optimality conditions are verified by checking conditions on optimal solution, this validates our result.

8 Conclusion

This article inspired by the work of Wu [32] and delineates the KKT optimality criteria for linear fractional-optimization problems with fuzzy parameters. For the differentiation of function with fuzzy parameters, Hausdorff metric and Hukuhara-difference are invoked. Using the Riemann integration theory, α-cut theory and Lagrange multipliers, the optimality criteria are successfully derived. Numerical optimization problems are presented to verify the effectiveness of the developed optimality criteria. There are numerous applications of the considered problem in real life, that can be solved with the proposed work. Some of them are from different areas like transportations theory, production planning, inventory problems, management science, and economics. For the validation of the result we have considered a real-life problem from production planning and verified developed KKT conditions for the optimal solution.
Future work: In future this work can be extended for multi-objective fractional optimization problem with fuzzy parameters for classical as well as fuzzy constraints. Also we can try to obtain KKT conditions for interval-valued fractional optimization problems.

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