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Original paper

A new stability criterion for high-order dynamic fuzzy systems

Z. Zeighami[1], M. R. Jahed-Motlagh[2], A. Moarefianpour[3] and G. Heydari[4]

[1,3] *Department of Electrical Engineering, Science and Research Branch,Islamic Azad University,Tehran,Iran*
[2] *Department of Computer Engineering, Iran University of Science and Technology, Tehran, Iran*
[4] *Department of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman*

zohreh_zeighami@yahoo.com, jahedmr@iust.ac.ir, moarefian@srbiau.ac.ir, gholamali.heydari@math.uk.ac.ir

Abstract

Fuzzy modeling is a well-known solution for simplified modeling and predicting nonlinear systems behavior. Dynamic TSK Fuzzy Systems are an important branch in fuzzy modeling and are used for complex nonlinear dynamic systems modeling vastly. High order fuzzy systems have been developed recently in the fuzzy modeling field, aiming to reduce number of the fuzzy model rules compared to zero and first order systems, not in cost of a larger modeling error. Employing high order TSK in dynamic TSK fuzzy systems, motivates finding a better model for nonlinear dynamic systems. Closed loop control system design is an important usage of dynamic TSK models, including the stability analysis as the first step.

While stability investigation is a main part of any Controller design process, in this paper, a criterion has been investigated based on Lyapunov second method, for High Order Dynamic TSK Fuzzy System stability. Although the controller design process is not totally discussed in this paper, however some examples are provided to verify the proposed stability criterion.

Keywords: Dynamic fuzzy systems, high-order TSK fuzzy systems, stability, nonlinear state feedback controller.

1 Introduction

Stability of dynamic systems is an important property in control engineering. In recent years, fuzzy modeling has been addressed in many researches as a solution for predicting or simulating behavior of complex nonlinear dynamic systems [3–5, 11, 23, 29, 30]. Also TSK fuzzy modeling has been used for stability investigation of the modeled system [28]. Zero and first order TSK modeling fuzzy systems have displayed significant results [8], but when modeling complex nonlinear systems, a larger fuzzy rule base is needed for a keeping modeling error small enough. This leads to a longer simulation and calculation of the obtained model [13, 16]. High order TSK systems deal much better with this issue [10, 12, 17, 19, 24, 31].

Recently, high-order fuzzy systems have been used in various fields including, predictive control of photovoltaic electricity generation plant [14, 15], discrete-time fuzzy predictive controller for Continuous Stirred Tank Reactor [7]. Recent researches on high order dynamic TSK system include [9, 18, 25, 26].

Stability analysis of nonlinear dynamic systems has been always considered as a challenge, due to the special complexities of these systems and the existence of different behaviors in their various operating points. In [2, 27], the stabilization problem has been investigated for fuzzy nonlinear systems. A general idea for this purpose is to model the dynamic nonlinear system first, using a TSK fuzzy model. Next, the stability analysis of the dynamic nonlinear systems is performed based on the stability analysis of its fuzzy model [28]. For example, in [2], the nonlinear system has been modeled with a fuzzy polynomial model and the polynomial fuzzy controller is designed based on it for tracking the reference model in the discrete-time domain. The designed controller ensures the system stability based on the Lyapunov stability theorem in the velocity variation range.

Corresponding Author: M. R. Jahed-Motlagh

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In this paper, the stability of high-order dynamic fuzzy systems is investigated and a criterion is obtained. The proposed approach is totally new and is motivated by Wang [28] and Heydari [14] works. The obtained criterion is also verified through few examples.

2 Mathematical prerequisites

2.1 Linear dynamic TSK systems, definition and stability

If the consequent of rules of a first-order fuzzy system is a set of discrete-time state-space equations like $x(k+1) = \hat{A}x(k) + \hat{B}u(k)$, it is called a dynamic fuzzy system [28]. The general form of the lth rule in these systems is as follows [28]:

$$\text{If } x_1 \in M_1^l \ \& \ x_2 \in M_2^l \ \& \ ... \ \& \ x_n \in M_n^l, \text{ then } x^l(k+1) = \hat{A}^l x(k) + \hat{B}^l u(k). \tag{1}$$

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}, \quad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_m(k) \end{bmatrix}. \tag{2}$$

Where the "l" superscript is the rule number and "n" is the number of state variables,

$x(k)$ is called the state variables vector and *u(k)* is referred to as the state equations' inputs vector where u_i is the equations input and m is the number of inputs, $\widehat{\mathrm{A}}, \widehat{\mathrm{B}}$ respectively are $\mathrm{n} \times \mathrm{n}$ and $\mathrm{n} \times \mathrm{m}$ matrix that n is the number of state variables and m is the number of inputs. The method used for calculating matrixes $\hat{A}^l$ and $\hat{B}^l$ is provided in the appendix.

Now, if the system is closed-loop and the inputs are excited by a state-space fuzzy controller, the closed-loop fuzzy system's rule could be written as follows [28]:

$$\text{if } x_1 \in \ D_1^l \ \& \ x_2 \in \ D_2^l \& \ldots . \& \ x_n \in \ D_n^l, \text{ then } x^l(k+1) = A^l x(k). \tag{3}$$

It should be noticed that D_i^l and A^l in this rule depend on both open-loop system and controller parameters. *x(k+1)* may be written by inferencing fuzzy rules as follows:

$$x(k+1) = \frac{\sum_{l=1}^{N} x^l(k+1)v^l}{\sum_{l=1}^{N} v^l} = \frac{\sum_{l=1}^{N} A^l x(k)v^l}{\sum_{l=1}^{N} v^l}, \tag{4}$$

where N is the total number of fuzzy rules and v^l is the weight or impact of the l^{th} rule which is calculated regarding the membership degree of state variables (x_i) in the fuzzy sets of the l^{th} rule : (D_1^l , D_2^l , ... D_n^l).

$$v^l = \prod_{i=1}^{n} \mu_{D_i^l}(x_i), \tag{5}$$

where $\mu_{D_i^l}$ is the membership function of the fuzzy set D_i^l.

2.2 High order dynamic TSK systems (dynamic HTSK systems)

The main idea behind the current work is inspired form [14, 16]. High-order fuzzy systems are referred to as a TSK fuzzy system that the consequent section of its fuzzy rules is a polynomial in the input variables, with the order higher than one. For simplicity, these fuzzy systems are called HTSK from now on.

Definition 2.1. *The general form of the l^{th} fuzzy rule from an HTSK system of order n with m input variable is as follows* [14]*:*

$$\textit{If } x_1 \textit{ is } A_1^l \ \& \ x_2 \textit{ is } A_2^l \ \& \cdots \& \ x_m \textit{ is } A_m^l, \textit{ then } y^l = \sum_{\substack{j_1 + j_2 + \cdots + j_m \le n \\ j_1, j_2, \cdots, j_m \ge 0}} \left(a^l_{j_1, j_2, \cdots, j_m}\right) x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}. \tag{6}$$

In this relation, the n^{th} degree polynomial is the rule's consequent and l is the number of the rule. As shown in [16], the above system may be represented with a number of zero or first order fuzzy systems, however this is not crucial but is used later in simulations for controller implementation. *Dynamic* HTSK systems may be introduced based on the above definition.

Definition 2.2. *Suppose M_i follows:*

$$M_i\,(x) = x_1^{j1} x_2^{j2} \ldots x_m^{jm}. \tag{7}$$

$M_i(x)$ is called a monomial with the maximum degree of R if, $R \geq j_1 + j_2 + \ldots j_n$.

Now, the l^{th} rule of a dynamic HTSK system is defined as follows:

$$\text{if } x_1 \in\ D_1^l \ \&\ x_2 \in\ D_2^l \& \ldots \& \ x_n \in\ D_n^l, \text{ then } x^l\,(k+1) = \sum_{i=1}^{Q} M_i\,(x)\,A_i^l x(k), \tag{8}$$

where Q is the number of the terms of the consequent of fuzzy rules as described in [16] and $M_i(x)$ is a monomial of degree *n-1*. A_i^l matrixes are *n-by-n* and may be determined by many various identification methods, however in the current paper these matrixes are calculated by the Taylor expansion of the nonlinear system state equations as it will be discussed later in the simulation and the appendix sections. This expansion is done around points whose coordinates mostly belong to $D_1^l, D_2^l, \cdots, D_n^l$ fuzzy sets. The result of the fuzzy inference in such system will be as follows:

$$x\,(k+1) = \frac{\sum_{l=1}^{N} \sum_{i=1}^{Q} M_i\,(x)\,{A_i}^l x(k) v^l}{\sum_{l=1}^{N} v^l}. \tag{9}$$

As it is obvious the proposed representation is actually a decomposition of nonlinear state space equations in to ${A_i}^l$ matrixes and as it will be shown next, these matrices are key factors in determining the stability criterion for the proposed dynamic HTSK system. Numerical examples provided in section 4 and the appendix contents, explain the applications of 8 in details.

2.3 Stability analysis perquisites

The main theoretical approach for the stability analysis of the proposed system is based on the Lyapunov direct method for dynamic discrete-time systems. The stability analysis is performed first by selecting a Lyapunov candidate function and then the stability criterion is found by using the Lyapunov theorem.

Theorem 2.3. *Consider the dynamic discrete-time system:*

$$x\,((k+1)\,T) = f(x\,(kT)), \tag{10}$$

where x is an n-dimensional vector defined in a closed convex subset of $\mathbb{R}^n$ like $X \subset \mathbb{R}^n$, $x \in X$, f(0)=0 and T is the sampling period.

Suppose there exists a continuous scalar function V(x) so that

1- for $x \neq 0$ $V\,(x) > 0$.

2- $\Delta V\,(\mathrm{x}) < 0$ for $x \neq 0$ where

$$\Delta V\,(x\,(kT)) = V\,(x\,(k+1)\,T) - V\,(x\,(kT)) = V(f\,(x\,(kT))) - V(x\,(kT)).$$

3- $V\,(0) = 0$.

4- If $\|x\| \to \infty$, then $V(x) \to \infty$.

In this case, the equilibrium point $x = 0$ is the globally asymptomatically stable and $V\,(x)$ is a Lyapunov function [21].

Lemma 2.4. *If matrix P is positive definite, such that $A^T PA - P$ and $B^T PB - P$ are negative definite and $A, B, P \in R^{n\times n}$, then $A^T PB - B^T PA - 2P$ is negative definite* [28].

Now, it is possible to investigate the *stability criterion of dynamic HTSK systems* which is the main contribution of this paper, presented in the following theorem.

3 Stability analysis of dynamic HTSK system

Theorem 3.1. *The dynamic HTSK system as defined in 8 is globally asymptomatically stable at* $x^T = \begin{bmatrix} 0 & 0 \ldots & 0 \end{bmatrix}$ *(the state space origin), if there exists a positive bound like* x_{imax} *for each state variable* x_i *such that*

$$|x_i| \le x_{imax} \quad , i = 1, 2, \ldots, n$$

and a positive definite matrix like P could be founded, such that:

$$\left\{ \begin{array}{c} \forall\, l = 1, 2, \ldots, N \\ \forall\, x \in X, x \neq 0 \end{array} \right. \implies \left[\sum_{i=1}^{Q} M_i\left(x_{max}\right) {A_i}^l \right]^T P \left[\sum_{i=1}^{Q} M_i\left(x_{max}\right) {A_i}^l \right] - P < 0. \tag{11}$$

Proof. For simplicity, define

$$C_l(x\,(k)) \triangleq \sum_{i=1}^{Q} M_i\left(x(k)\right) {A_i}^l. \tag{12}$$

Assume P is the proposed positive definite matrix and $V(x\,(k))$ is defined as $V(x\,(k)) = x^T(k)Px\,(k)$, then $V(x\,(k))$ is positive definite and $\Delta V(x\,(k))$ is as follows:

$$\Delta V(x\,(k)) = V(x\,(k+1)) - V(x\,(k)) = x^T\,(k+1)\,Px\,(k+1) - x^T(k)Px\,(k)\,.$$

Merging 9 & 12 gives $x\,(k+1) = \frac{\sum_{l=1}^{N} \sum_{i=1}^{Q} C_l(x(k))x(k)v^l}{\sum_{l=1}^{N} v^l}$ Thus,

$$\Delta V(x\,(k)) = x^T\,(k) \left[\left(\left(\frac{\sum_{l=1}^{N} {C_l}^T\,(x\,(k))\,v^l}{\sum_{l=1}^{N} v^l} \right)^T P \frac{\sum_{l=1}^{N} C_l\,(x\,(k))\,v^l}{\sum_{l=1}^{N} v^l} \right) - P \right] x(k) \Rightarrow$$

$$\Delta V(x\,(k)) = \frac{\sum_{l=1}^{N} \left\{ \sum_{m=1}^{N} \left[v^l v^m x^T(k)({C_l}^T\,(x\,(k))\,PC_m(x\,(k)) - P)x(k) \right] \right\}}{\sum_{l=1}^{N} \sum_{m=1}^{N} v^l v^m}. \tag{13}$$

As it was assumed, there exists a positive bound like x_{imax} for state variable x_i such that:

$$|x_i| \le x_{imax} \quad , i = 1, 2, \ldots, n$$

It is possible to show that the function $x^T(k)({C_l}^T\,(x_{max})\,PC_l(x_{max}) - P)x(k)$ is negative definite where, $C_l(x_{max}) = \sum_{i=1}^{Q} M_i\,(x_{max})\,{A_i}^l$. It is trivial that any element of matrix $C_l\,(x\,(k))$ is less than or equal to the corresponding one in $C_l\,(x_{max})$. Since P is assumed positive definite, then ${C_l}^T\,(x_{max})\,PC_l(x_{max})$ and ${C_l}^T\,(x\,(k))\,PC_l x\,(k)$ matrixes are also positive definite and for any vector of the state variables like:
$x^T\,(k) = \begin{bmatrix} x_1(k) & x_2(k) & \ldots x_n(k) \end{bmatrix}$, where, $i = 1, 2, \ldots, n,\ |x_i| \le x_{imax}$,
the following inequality holds:

$$x^T\,(k) \left({C_l}^T\,(x_{max})\,PC_l\,(x_{max}) \right) x\,(k) \ge x^T(k)({C_l}^T\,(x\,(k))\,PC_l\,(x\,(k)))x(k) \Longrightarrow$$

$$\Delta V(x\,(k)) \le \frac{\sum_{l=1}^{N} \left\{ \sum_{m=1}^{N} \left[v^l v^m x^T(k)({C_l}^T\,(x_{max}\,(k))\,PC_m(x_{max}\,(k)) - P)x(k) \right] \right\}}{\sum_{l=1}^{N} \sum_{m=1}^{N} v^l v^m}. \tag{14}$$

Calling the right side of the above term $\Delta V_{max}(x\,(k))$ gives:

$$\Delta V(x\,(k)) \le \Delta V_{max}(x\,(k))\,. \tag{15}$$

Thus, to show that $\Delta V(x\,(k))$ is negative definite, it is enough to show that $\Delta V_{max}(x\,(k))$ is negative definite. For this purpose, it may be written:

$$\Delta V_{max}(x\,(k)) = \frac{\sum_{l=1}^{N} \left(v^l\right)^2 x^T\,(k) \left[{C_l}^T\,(x_{max})\,PC_l\,(x_{max}) - P \right] x\,(k)}{\sum_{l=1}^{N} \sum_{m=1}^{N} v^l v^m} + \ldots\ldots +$$

$$\frac{\sum_{l<m}^{N} v^l v^m x^T\,(k) \left[{C_l}^T\,(x_{max})\,PC_m\,(x_{max}) + {C_m}^T\,(x_{max})\,PC_l\,(x_{max}) - 2P \right] x\,(k)}{\sum_{l=1}^{N} \sum_{m=1}^{N} v^l v^m}. \tag{16}$$

While $v^l \geq 0$, the first term of the above relation is negative definite according to the current theorem assumptions. It is obvious that the second part is also negative definite according to the 2.4. So, $\Delta V_{max}(x(k))$ is negative definite which leads to negative definiteness of $\Delta V(x(k))$ and according to the Lyapunov second method for discrete systems, the considered fuzzy system is globally asymptomatically stable at $x^T = \left[\begin{array}{ccc} 0 & 0\ldots & 0 \end{array}\right]$.

□

4 Numerical simulations

Two examples have been provided here for a better understanding of 8 and to illustrate Theorem 3.1 validity.

4.1 Example 1

Suppose a pendulum connected by a rod to the shaft of an electric motor H whose torque can be adjusted directly through a drive circuit (Figure 1) [6].

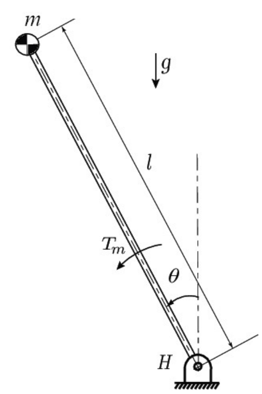

Figure 1: Inverted Pendulum

The rod inertia is supposed to be negligible and all measurements are by SI units: *l= 1m , m=0.1 Kg, g=9.8 m/s*^{2}. Motor torque (T_m) direction is assumed to increase the pendulum angle (θ).

The dynamics of this system is described by these equations:

$$ml^2\ddot{\theta} = mgl\sin\theta \; + T_m. \tag{17}$$

Defining $x_1 = \theta, \quad x_2 = \dot{\theta}$ as state variable gives following state space equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g}{l}\sin x_1 \; + \frac{T_m}{ml^2}. \end{aligned} \tag{18}$$

Employing multivariable Taylor expansion (as in 47) around (θ_0 , 0) leads to:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &\simeq \frac{g}{l}\sin\theta_0 \; + (x_1 - \theta_0)\frac{g}{l}\cos\theta_0 \; - \frac{1}{2!}(x_1 - \theta_0)^2\frac{g}{l}\sin\theta_0 \; + \frac{T_m}{ml^2}. \end{aligned} \tag{19}$$

Or,

$$\dot{\Delta x} = A_0(\theta_0)\Delta x + x_1 A_2(\theta_0)\Delta x + B_0\Delta u. \tag{20}$$

Where,

$$A_0 = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\left(\cos\theta_0 \ + \theta_0 \sin\theta_0\ \right) & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ -\frac{g}{2l}\sin\theta_0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix},\ \Delta u = \Delta T_m, \Delta x = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} x_1 - \theta_0 \\ x_2 - 0 \end{bmatrix},$$
$$\dot{x} = \begin{bmatrix} \dot{\Delta x_1} \\ \dot{\Delta x_2} \end{bmatrix} \tag{21}$$

It must be noted that 19 is an estimation determined by the Taylor expansion and certainly has an error. For the case of using Taylor expansion, the error bound can be calculated as discussed in [20]. It is crucial to keep the error band small enough to ensure the validity of the fuzzy model and stability analysis results based on this model.

While discrete-time difference equations are needed (as in 8), a first order discrete derivative estimation is used:

$$\dot{\Delta x} \simeq \frac{\Delta x\left(k+1\right) - \Delta x\left(k\right)}{t_s} \Longrightarrow \Delta x\left(k+1\right) = t_s \dot{\Delta x}(k) + \Delta x(k), \tag{22}$$

where $x(k)$ is defined in 2-2 and t_S is the sampling period and it is obvious that it has to be short. Here $t_s = 0.01s$ is chosen which is short enough for a reasonable estimation. Merging 19 and 20 gives a set of estimated difference state space equation for the proposed plant:

$$\Delta x\left(k+1\right) = \left(t_s A_0 + I\right) \Delta x(k) + t_s x_1(k) A_1 x(k) + t_s B \Delta u. \tag{23}$$

Finally the equations may be written into 8 matrix form:

$$\Delta x\left(k+1\right) = \begin{bmatrix} 1 & t_s \\ t_s \frac{g}{l}\left(\cos\theta_0 \ + \theta_0 \sin\theta_0\ \right) & 1 \end{bmatrix} \Delta x\left(k\right) + \Delta x_1\left(k\right) \begin{bmatrix} 0 & 0 \\ \frac{-t_s g}{2l}\sin\theta_0 & 0 \end{bmatrix} \Delta x\left(k\right) + \begin{bmatrix} 0 \\ \frac{t_s}{ml^2} \end{bmatrix} \Delta u. \tag{24}$$

Fuzzy modeling is done around three values $\theta_0 = 0$, $\pi/3$ and $-\pi/3$ and three membership functions (named *Z, P* and *N*) are considered for these values respectively (Figure 4-2). There will be 3 rules for each value of θ_0 respectively:

$$\text{If } \theta \in Z,\ \text{then } \Delta x\left(k+1\right) = \begin{bmatrix} 1 & 0.01 \\ 0.098 & 1 \end{bmatrix} \Delta x\left(k\right) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \Delta u,$$
$$\text{If } \theta \in P,\ \text{then } \Delta x\left(k+1\right) = \begin{bmatrix} 1 & 0.01 \\ 0.1379 & 1 \end{bmatrix} \Delta x\left(k\right) + \Delta x_1\left(k\right) \begin{bmatrix} 0 & 0 \\ -0.0424 & 0 \end{bmatrix} \Delta x\left(k\right) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \Delta u,$$
$$\text{If } \theta \in N,\ \text{then } \Delta x\left(k+1\right) = \begin{bmatrix} 1 & 0.01 \\ 0.1379 & 1 \end{bmatrix} \Delta x\left(k\right) + \Delta x_1\left(k\right) \begin{bmatrix} 0 & 0 \\ 0.0424 & 0 \end{bmatrix} \Delta x\left(k\right) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \Delta u. \tag{25}$$

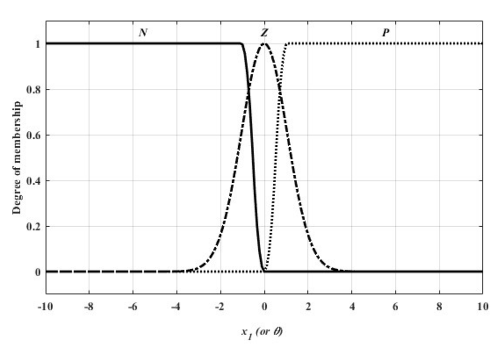

Figure 2: Input Membership Functions

For investigating closed-loop stability a feedback control law has to be determined for each rule. This can be done by various methods to achieve an optimal control but while this is not the aim of this paper, a simple control law has been provided *just for stabilizing* the closed loop system. A feedback vector is defined for each rule:

$$\begin{array}{l} \text{If } \theta \in Z, \text{ then } u = -\left(\begin{bmatrix} 4.5 & 1.5 \end{bmatrix} + \Delta x_1(k)\begin{bmatrix} 1 & 2 \end{bmatrix}\right)\Delta x(k), \\ \text{If } \theta \in P, \text{ then } u = -\left(\begin{bmatrix} 6 & 1.5 \end{bmatrix} + \Delta x_1(k)\begin{bmatrix} 1 & 2 \end{bmatrix}\right)\Delta x(k), \\ \text{If } \theta \in N, \text{ then } u = -\left(\begin{bmatrix} 6 & 1.5 \end{bmatrix} + \Delta x_1(k)\begin{bmatrix} 2 & 1 \end{bmatrix}\right)\Delta x(k). \end{array} \tag{26}$$

While antecedent membership functions of the plant model 25 and the controller 26 coincide, combining 25 and 26 gives the closed loop equation as follows [28]:

$$\begin{array}{l} \text{If } \theta \in Z, \text{ then } \Delta x(k+1) = \begin{bmatrix} 1 & 0.01 \\ -0.3520 & 0.85 \end{bmatrix}\Delta x(k) + \Delta x_1(k)\begin{bmatrix} 0 & 0 \\ -0.1000 & -0.2 \end{bmatrix}\Delta x(k), \\ \text{If } \theta \in P, \text{ then } \Delta x(k+1) = \begin{bmatrix} 1 & 0.01 \\ -0.4621 & 0.85 \end{bmatrix}\Delta x(k) + \Delta x_1(k)\begin{bmatrix} 0 & 0 \\ -0.2425 & -0.1 \end{bmatrix}\Delta x(k), \\ \text{If } \theta \in N, \text{ then } \Delta x(k+1) = \begin{bmatrix} 1 & 0.01 \\ -0.4621 & 0.85 \end{bmatrix}\Delta x(k) + \Delta x_1(k)\begin{bmatrix} 0 & 0 \\ -0.0576 & -0.2 \end{bmatrix}\Delta x(k). \end{array} \tag{27}$$

Regarding state variable bounds (Theorem 3-1) $x_{max}^T = \begin{bmatrix} 1.5 & 1.5 \end{bmatrix}$ and the positive definite matrix $P = \begin{bmatrix} 2 & 10 \\ 0 & 1 \end{bmatrix}$ (found with some effort by trial and error such that 3-1 holds true), gives:

$$\begin{array}{l} C_1(x_{max}) = \begin{bmatrix} 1 & 0.01 \\ -0.3520 & 0.85 \end{bmatrix} + 1.5\begin{bmatrix} 0 & 0 \\ -0.1000 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0.01 \\ -0.5020 & 0.55 \end{bmatrix}, \\ C_2(x_{max}) = \begin{bmatrix} 1 & 0.01 \\ -0.4621 & 0.85 \end{bmatrix} + 1.5\begin{bmatrix} 0 & 0 \\ -0.2425 & -0.1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -0.8258 & 0.7 \end{bmatrix}, \\ C_3(x_{max}) = \begin{bmatrix} 1 & 0.01 \\ -0.4621 & 0.85 \end{bmatrix} + 1.5\begin{bmatrix} 0 & 0 \\ -0.0576 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0.01 \\ -0.5485 & 0.55 \end{bmatrix}. \end{array} \tag{28}$$

According to 3-1, $C_l(x_{max})^T P C_l(x_{max}) - P$ has to be negative definite for *l=1,2,3*:

$$\begin{array}{l} C_1(x_{max})^T P C_1(x_{max}) - P = \begin{bmatrix} -4.7680 & -4.7561 \\ -0.3063 & -0.6423 \end{bmatrix} \Longrightarrow Eigen\ Values: -5.0952, \quad -0.3151, \\ C_2(x_{max})^T P C_2(x_{max}) - P = \begin{bmatrix} -7.5759 & -3.5580 \\ -0.6406 & -0.4398 \end{bmatrix} \Longrightarrow Eigen\ Values: -7.8821, \quad -0.1335, \\ C_3(x_{max})^T P C_3(x_{max}) - P = \begin{bmatrix} -5.1839 & -4.7817 \\ -0.3365 & -0.6423 \end{bmatrix} \Longrightarrow Eigen\ Values: -5.5142, \quad -0.3120. \end{array} \tag{29}$$

While all Eigen values are negative, 11 holds and the proposed feedback control vector stabilizes the system in the current situation at the origin. The response of the system to an initial deviation equal to +45° is illustrated in Figure 3. As it is clear the closed loop system is asymptotically stable.

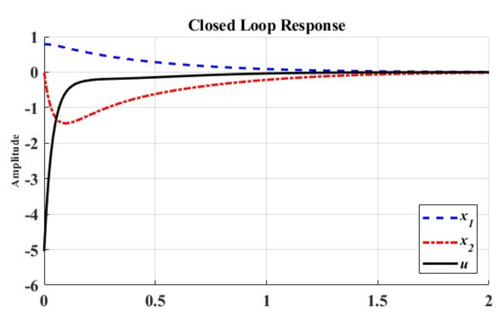

Figure 3: Closed Loop Resp. (x_1=θ)

4.2 Example 2

The traditional inverted pendulum on a moving cart [22] is stabilized by the proposed Theorem 3.1 in this example.

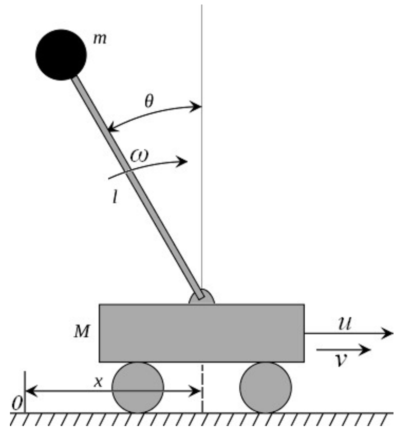

Figure 4: Inverted Pendulum on a Moving Cart

The rod inertia is supposed to be negligible and all measurements are by SI units: *l= 1m , m=2 Kg, M=1 Kg, g=9.8* m/s^2.

The dynamics of this system is described by *cart position* (x), *cart velocity* (v), *pendulum angle* (θ) and *pendulum angular velocity* (ω) as follows:

$$x_1 = x,\ x_2 = v,\ \ x_3 = \theta,\ \ x_4 = \omega. \tag{30}$$

State space equations:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{u}{m} + \frac{l{x_4}^2 sinx_3}{\frac{M}{m} + sin^2 x_3} - \frac{g\ sinx_3\ cosx_3}{\frac{M}{m} + sin^2 x_3} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{\left(-\frac{u\ cosx_3}{m} - l{x_4}^2 sinx_3\ cosx_3 + \frac{(M+m)g\ sinx_3}{m}\right)}{l(\frac{M}{m} + sin^2 x_3)}
\end{aligned} \tag{31}$$

For Taylor expansion, suppose x=0, v=0, $\theta \in \left\{ -\frac{\pi}{4},\ \ 0,\ \ \frac{\pi}{4} \right\}$, $\omega \in \left\{ -10,\ \ 0,\ \ 10 \right\}$. Combination of these values gives 9 points in the state space. A_0^l, A_i^l, A_u^l, B_0^l, B_i^l and B_u^l matrixes (described in 48 and l=1,...,9), are calculated in details in the appendix 6.2:

$$\dot{\Delta X} = A_0 \Delta X + B_0 \Delta u + \frac{1}{2}\left[\sum_{i=1}^{n} \Delta x_i \left(A_i \Delta X + B_i \Delta u\right) + \Delta u \left(A_u \Delta X + B_u \Delta u\right)\right]. \tag{32}$$

Similar to Example 1, the set of estimated difference state space equation for each point in the state space of the proposed plant (regarding t_s = *0.01s*) is calculated and the general form of the l^{th} rule will be:

$$\begin{aligned}
&\text{If } \theta \in S_1^l \ and\ \omega \in S_2^l, \text{ then } \Delta X(k+1) = \\
&\left(t_s A_0^l + I\right) \Delta X(k) + t_s B_0^l \Delta u(k) + \frac{t_s}{2}\left[\sum_{i=1}^{4} \Delta x_i(k) \left(A_i^l \Delta X(k) + B_i^l \Delta u(k)\right) + \Delta u(k) \left(A_u^l \Delta X(k) + B_u^l \Delta u(k)\right)\right].
\end{aligned} \tag{33}$$

Where, l=1,...,9, membership functions S_1^l and S_2^l are defined around values $\theta = 0$, $\pi/4$ and $-\pi/4$ and $\omega = 0$, 10 and -10 (called *med1, high* and *low*) as shown in Figure 5. There will be 9 rules one for each pair of (θ,ω) respectively.

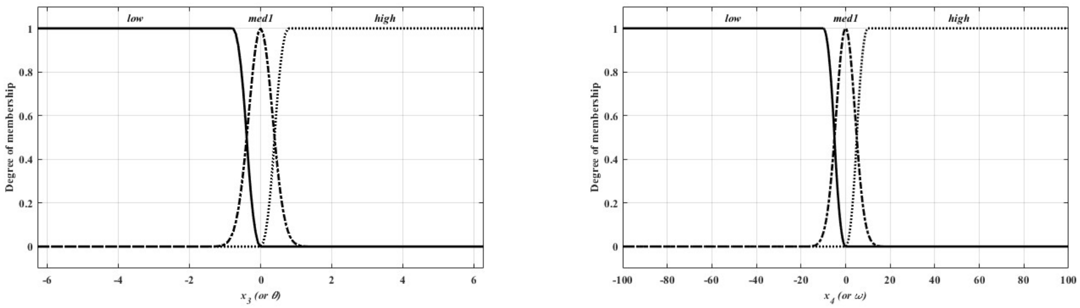

Figure 5: Input Membership Functions

Table 1: Values of Feedback Matrixes for Inverted Pendulum on Moving Cart (K_0^l and K_i^l)

l	θ	ω	K_0^l				K_1^l		K_2^l			K_3^l			K_4^l		
1	$-\pi/4$	-10	-3.387	-4.643	-319.7	-15.05	0	0	-0.0133	0.0124	-0.0055	-0.0004	0.0004	-0.0002	-0.0133	0.0125	-0.0055
2	$-\pi/4$	0	-52.21	-71.59	-247.3	-138	0	0	0.0054	-0.0037	0	0.0002	-0.0001	0	0.0055	-0.0037	0
3	$-\pi/4$	10	-3.387	-4.643	-319.7	-71.62	0	0	-0.0115	0.0101	-0.0037	-0.0004	0.0003	-0.0001	-0.0115	0.0101	-0.0034
4	0	-10	-9.230	-12.65	99.99	-25.66	0	0	-0.0154	0.0143	-0.0061	-0.0005	0.0004	-0.0002	-0.0154	0.0143	-0.0062
5	0	0	-9.230	-12.56	-100.0	-25.66	0	0	0.0024	-0.0013	-0.0006	0	0	0	0.0024	-0.0013	-0.0006
6	0	10	-9.230	-12.65	99.99	-25.65	0	0	0.0032	-0.0022	0	0.0001	0	0	0.0032	-0.0022	0
7	$\pi/4$	-10	-3.387	-4.644	-319.7	-71.62	0	0	-0.0017	0.0008	0.0005	0	00	0	-0.0017	0.0008	0.0005
8	$\pi/4$	0	-52.21	-71.59	-247.4	-138.0	0	0	-0.0021	0.0011	0.0005	0	0	0	-0.0021	0.0011	0.0005
9	$\pi/4$	10	-3.387	-4.644	-319.7	-15.05	0	0	-0.0025	0.0015	0.0003	0	0	0	-0.0025	0.0015	0.0003

Values for θ, ω, A_0^l, A_i^l, A_u^l, B_0^l, B_i^l and B_u^l matrixes corresponding to the l^{th} rule are the same as values of the l^{th} row of Table 6.2.

For investigating closed-loop stability a feedback control law has to be determined for each rule. As discussed in previous example, a simple control law has been provided by trial and error *just for stabilizing* the closed loop system. Supposing an HTSK dynamic system as the controller, the Feedback control signal variation for the l^{th} rule (Δu^l) is defined as follows:

$$If\ \theta \in S_1^l\ and\ \omega \in S_2^l\ Then\ \Delta u^l = -\left(K_0^l + \sum_{i=1}^{4} \Delta x_i(k) K_i^l\right) \Delta X(k). \tag{34}$$

As discussed in previous example, combining 33 and 34, the l^{th} rule of the closed loop control system fuzzy model will be:

$$\begin{aligned}
&\text{If } \theta \in S_1^l\ and\ \omega \in S_2^l,\ \text{then } \Delta X^l(k+1) = \left(t_s A_0^l + I - t_s B_0^l \left(K_0^l + \sum_{i=1}^{4} \Delta x_i(k) K_i^l\right)\right) \Delta X(k) \\
&+ \frac{t_s}{2}\left[\sum_{i=1}^{4} \Delta x_i(k) \left(A_i^l - B_i^l \left(K_0^l + \sum_{i=1}^{4} \Delta x_i(k)\, K_i^l\right)\right)\right] \Delta X(k) \\
&- \frac{t_s}{2}\left[\left(K_0^l + \sum_{i=1}^{4} \Delta x_i(k) K_i^l\right) \Delta X(k) \left(A_u^l - B_u^l \left(K_0^l + \sum_{i=1}^{4} \Delta x_i(k) K_i^l\right)\right)\right] \Delta X(k).
\end{aligned} \tag{35}$$

Doing some simplification, it is possible to show that 35 is in the form of 8 and Theorem 3.1 may be applied to it, so to stabilize the closed loop system it is enough to determine K_0^l and K_i^l (for i=1,…,4) such that there exists a positive definite matrix P as described in Theorem 3-1. For this reason, some values for K_0^l and K_i^l are determined by trial and error (Table 4.2).

Regarding state variable bounds (Theorem 3.1) $x_{max}^T = \begin{bmatrix} 10 & 50 & \pi & 50 \end{bmatrix}$ there exists a positive definite matrix:

$$P = \begin{bmatrix} 0.0029965 & 0.0010841 & -0.8224392 & 1.2072510 \\ -0.1134833 & -0.0002033 & 0.6296882 & -0.822439 \\ -0.0009793 & 0.0000183 & -0.0002033 & 0.0010841 \\ 0.1809570 & -0.0009793 & -0.1134833 & 0.0029965 \end{bmatrix}. \tag{36}$$

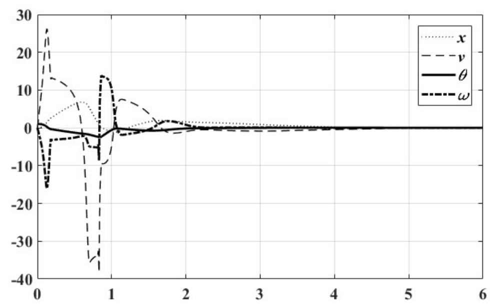

Figure 6: Closed Loop Response of Inverted Pendulum on Moving Cart

such that 3-1 holds true. The response of the system to an initial deviation equal to θ=60° is illustrated in Figure 6.

As it is clear the closed loop system is asymptotically stable.

5 Conclusions

In the current paper a stability criterion for high-order dynamic fuzzy systems was proved as a theorem. The proposed theorem is based on a representation of high-order dynamic fuzzy systems that leads to a matrix decomposition of nonlinear state space equations. Matrixes found by this decomposition, play a key role in determining the stability condition discussed in the proposed theorem. The validity of this theorem was investigated by 2 examples in the last section. For future works it is recommended to provide optimization over the proposed control system. Moreover it is recommended to use nonlinear system identification methods for estimating A_0^l, A_i^l, A_u^l, B_0^l, B_i^l and B_u^l matrixes.

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6 Appendixes

6.1 Nonlinear dynamic systems and Taylor expansion

A non-linear dynamic system can usually be modeled by a set of fist order differential equations (i.e. state space equations) as follows [22]:

$$\begin{cases} \dot{x} = F(x(t), u(t)) \\ y(t) = G(x(t), u(t)) \end{cases} \tag{37}$$

Where t is the time and *x(t)* is the state variables' vector and *u(t)* is the input value (i.e. independent variable) of the proposed system at the moment *t*. F & G are C^{∞} (i.e. continuous, infinitely differentiable), vector functions and dependent to *x(t)* and *u(t)*, and *y(t)* is the system's output value. In this case, Taylor's expansion of functions F & G could be considered.

If $z \in \mathbb{R}^n$, $F(z) : \mathbb{R}^n \to \mathbb{R}$ is a C^{∞} real valued function and $a \in \mathbb{R}^n$ is a point where *F(a)* is defined, in this case, the Taylor expansion of function F from the m^{th} degree around $a \in \mathbb{R}^n$ could be shown as following [1]:

$$F(z) \cong \sum_{|\alpha|<m} \frac{D^{\alpha}(F(a))}{\alpha!}(z-a)^{\alpha}. \tag{38}$$

In the above relation, the following definitions can be considered:

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_n \\ \alpha! &= \alpha_1!\alpha_2!\alpha_3!\ldots\alpha_n!. \end{aligned} \tag{39}$$

And supposing that "z" and "a" are vectors as follows:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}. \tag{40}$$

In this case, $(z-a)^{\alpha} = (z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\ldots(z-a_n)^{\alpha_n}$ and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are integers whose sum should be less than or equal to "m" (i.e.$|\alpha| < m$). The operator D is actually a partial differentiation operator and is defined in details in [1].

For simplicity, suppose a continuous-time nonlinear dynamic system is described by a set of state space equations as follows:

$$\dot{X} = F(X,u) = \begin{bmatrix} f_1(X,u) \\ f_2(X,u) \\ \vdots \\ f_n(X,u) \end{bmatrix} \quad , \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{41}$$

Where, $X \in \mathbb{R}^n$ is the vector of state variables, $u \in \mathbb{R}$ is the input value of the proposed system, $\dot{X}$ is the state variables time-derivation vector and f_i is C^{∞}. To simplify the notation more, following augmented vector Z is defined:

$$Z = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ u \end{bmatrix}. \tag{42}$$

Now, letting $Z_0 \in R^n$ a point where $f_1, f_2, \ldots, f_n$ are defined, and, regarding 38, gives:

$$f_i\left(Z_0+\Delta Z\right) \cong f_i\left(Z_0\right)+\nabla f_i\left(Z_0\right) \Delta Z+\frac{1}{2}\ \Delta Z^T H_i \Delta Z. \tag{43}$$

Where,

$$\Delta Z^T = \left[\Delta x_1 \quad \Delta x_2 \ \ldots \ \Delta x_n \quad \Delta u\right], \tag{44}$$

is vector of small enough deviations around Z_0 (i.e. ΔZ in the near neighborhood of Z_0), and,

$$\nabla f_i(Z)=\left[\begin{array}{cccccc} \frac{\partial f_i}{\partial x_1} & \frac{\partial f_i}{\partial x_2} & \cdots & \frac{\partial f_i}{\partial x_n} & \frac{\partial f_i}{\partial u} \end{array}\right], \tag{45}$$

$$H_i=\left[\begin{array}{cccc} \frac{\partial^2 f_i}{\partial x_1{}^2} & \cdots & \frac{\partial^2 f_i}{\partial x_1 \partial x_n} & \frac{\partial^2 f_i}{\partial x_1 \partial u} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 f_i}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f_i}{\partial x_n{}^2} & \frac{\partial^2 f_i}{\partial x_n \partial u} \\ \frac{\partial^2 f_i}{\partial u \partial x_1} & \cdots & \frac{\partial^2 f_i}{\partial u \partial x_n} & \frac{\partial^2 f_i}{\partial u^2} \end{array}\right]. \tag{46}$$

are gradient and hessian of each f_i respectively. Relation 43 is the 2^{nd} degree approximation of 41 based on Taylor expansion. This approximation helps to form rules for 2^{nd} order dynamic TSK systems (i.e. the simplest form of dynamic HTSK systems) as follows.

Theorem 6.1. *State-space equations of a dynamic non-linear system similar to 41 may be approximated by the Taylor expansion of degree 2 in the following matrix form:*

$$F(X, u) \cong F_0+A_0 \Delta X+B_0 \Delta u+\frac{1}{2}\left[\sum_{i=1}^n \Delta x_i\left(A_i \Delta X+B_i \Delta u\right)+\Delta u\left(A_u \Delta X+B_u \Delta u\right)\right]. \tag{47}$$

Where,

$$F_0=F\left(X_0, u_0\right), \quad \Delta X=\left[\begin{array}{c} \Delta x_1 \\ \vdots \\ \Delta x_n \end{array}\right], J_A(F)=\left[\begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{array}\right], \quad J_B(F)=\left[\begin{array}{c} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{array}\right]$$

$$A_i=\left.\frac{\partial J_A(F)}{\partial x_i}\right|_{\substack{X=X_0 \\ u=u_0}}, \quad A_u=\left.\frac{\partial J_A(F)}{\partial u}\right|_{\substack{X=X_0 \\ u=u_0}} \quad A_0=\left.J_A(F)\right|_{\substack{X=X_0 \\ u=u_0}},$$

$$B_i=\left.\frac{\partial J_B(F)}{\partial x_i}\right|_{\substack{X=X_0 \\ u=u_0}}, \quad B_u=\left.\frac{\partial J_B(F)}{\partial u}\right|_{\substack{X=X_0 \\ u=u_0}} \quad B_0=\left.J_B(F)\right|_{\substack{X=X_0 \\ u=u_0}} \tag{48}$$

Proof. Main idea here is inspired from 43, let,

$$F^T=\left[f_1 \quad f_2 \quad f_3 \ \ldots \ f_n\right]. \tag{49}$$

43 gives,

$$\left[\begin{array}{c} f_1\left(Z_0+\Delta Z\right) \\ \vdots \\ f_n\left(Z_0+\Delta Z\right) \end{array}\right] \cong\left[\begin{array}{c} f_1\left(Z_0\right) \\ \vdots \\ f_n\left(Z_0\right) \end{array}\right]+\left[\begin{array}{c} \nabla f_1\left(Z_0\right) \Delta Z \\ \vdots \\ \nabla f_n\left(Z_0\right) \Delta Z \end{array}\right]+\frac{1}{2}\left[\begin{array}{c} \Delta Z^T H_1 \Delta Z \\ \vdots \\ \Delta Z^T H_n \Delta Z \end{array}\right]. \tag{50}$$

Where Z_0 and ΔZ are as defined in 42 and 44. It is clear that,

$$F_0 = F\left(X_0, u_0\right) = \begin{bmatrix} f_1\left(Z_0\right) \\ \vdots \\ f_n\left(Z_0\right) \end{bmatrix}. \tag{51}$$

Regarding 45 and the supposition it is possible to define and rewrite following statement,

$$\begin{bmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial u} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial u} \end{bmatrix} \triangleq J\left(F\right) = \begin{bmatrix} J_A\left(F\right) & J_B\left(F\right) \end{bmatrix}. \tag{52}$$

And by considering 44,48 clearly shows,

$$\begin{bmatrix} \nabla f_1\left(Z_0\right)\Delta Z \\ \vdots \\ \nabla f_n\left(Z_0\right)\Delta Z \end{bmatrix} = \begin{bmatrix} J_A\left(F\right) & J_B\left(F\right) \end{bmatrix} \Delta Z = A_0 \Delta X + B_0 \Delta u. \tag{53}$$

Now, considering 46, it may be written,

$$Z^T H_i Z = \begin{bmatrix} x_1 & x_2 & \ldots & x_n & u \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f_i}{\partial {x_1}^2} & \frac{\partial^2 f_i}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial x_1 \partial x_n} & \frac{\partial^2 f_i}{\partial x_1 \partial u} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\partial^2 f_i}{\partial x_n \partial x_1} & \frac{\partial^2 f_i}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial {x_n}^2} & \frac{\partial^2 f_i}{\partial x_n \partial u} \\ \frac{\partial^2 f_i}{\partial u \partial x_1} & \frac{\partial^2 f_i}{\partial u \partial x_2} & \cdots & \frac{\partial^2 f_i}{\partial u \partial x_n} & \frac{\partial^2 f_i}{\partial u^2} \end{bmatrix} Z. \tag{54}$$

Or equally,

$$Z^T H_i Z = x_1 \left[\frac{\partial \nabla f_i}{\partial x_1}\right] Z + \cdots + x_n \left[\frac{\partial \nabla f_i}{\partial x_n}\right] Z + u \left[\frac{\partial \nabla f_i}{\partial u}\right] Z. \tag{55}$$

Considering $Z^T H_1 Z$, $Z^T H_2 Z$, $\ldots$, $Z^T H_n Z$ as elements of a vector,

$$\begin{bmatrix} Z^T H_1 Z \\ \vdots \\ Z^T H_n Z \end{bmatrix} = \begin{bmatrix} x_1 \left[\frac{\partial \nabla f_1}{\partial x_1}\right] Z + \cdots + x_n \left[\frac{\partial \nabla f_1}{\partial x_n}\right] Z + u \left[\frac{\partial \nabla f_1}{\partial u}\right] \\ x_1 \left[\frac{\partial \nabla f_2}{\partial x_1}\right] Z + \cdots + x_n \left[\frac{\partial \nabla f_2}{\partial x_n}\right] Z + u \left[\frac{\partial \nabla f_2}{\partial u}\right] \\ \vdots \\ x_1 \left[\frac{\partial \nabla f_n}{\partial x_1}\right] Z + \cdots + x_n \left[\frac{\partial \nabla f_n}{\partial x_n}\right] Z + u \left[\frac{\partial \nabla f_n}{\partial u}\right] \end{bmatrix}. \tag{56}$$

So it leads to,

$$\begin{bmatrix} Z^T H_1 Z \\ \vdots \\ Z^T H_n Z \end{bmatrix} = \left[x_1 \frac{\partial J\left(F\right)}{\partial x_1} + \ldots x_n \frac{\partial J\left(F\right)}{\partial x_n} + u \frac{\partial J\left(F\right)}{\partial u}\right] Z \tag{57}$$

$$= x_1 \frac{\partial}{\partial x_1} \begin{bmatrix} J_A\left(F\right) & J_B\left(F\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ u \end{bmatrix} + \ldots x_n \frac{\partial}{\partial x_n} \begin{bmatrix} J_A\left(F\right) & J_B\left(F\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ u \end{bmatrix} + u \frac{\partial}{\partial u} \begin{bmatrix} J_A\left(F\right) & J_B\left(F\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ u \end{bmatrix}.$$

Regarding 41, recent equation can easily be reformed as follows,

$$\begin{bmatrix} Z^T H_1 Z \\ \vdots \\ Z^T H_n Z \end{bmatrix} = x_1 \left[\left(\frac{\partial}{\partial x_1} J_A(F) \right) X + \left(\frac{\partial}{\partial x_1} J_B(F) \right) u \right] + \dots$$

$$+ x_n \left[\left(\frac{\partial}{\partial x_n} J_A(F) \right) X + \left(\frac{\partial}{\partial x_n} J_B(F) \right) u \right] + u \left[\left(\frac{\partial}{\partial u} J_A(F) \right) X + \left(\frac{\partial}{\partial u} J_B(F) \right) u \right]. \tag{58}$$

As it is clear, it is possible to replace Z by ΔZ, X by ΔX and u by Δu, without loss of generality and regarding 48 gives,

$$\begin{bmatrix} \Delta Z^T H_1 \Delta Z \\ \vdots \\ \Delta Z^T H_n \Delta Z \end{bmatrix} = \Delta x_1 \left[A_1 \Delta X + B_1 \Delta u \right] + \dots \Delta x_n \left[A_n \Delta X + B_n \Delta u \right] + \Delta u \left[A_u \Delta X + B_u \Delta u \right]. \tag{59}$$

Placing 51,53 and 59 in 50 completes the proof. □

Although it is not a result of this paper, but it has to be noted that recent theorem may be extended to 3^{rd} and higher degree statements almost easily by using tensor representation.

6.2 Taylor expansion of the inverted pendulum on cart

State space equations are described in 31. Employing multivariable Taylor expansion (as in 47 and 48 around these points leads to following matrices:

$$J_A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\omega^2 cos\theta - 9.81 cos^2\theta + 9.81 sin^2\theta}{sin^2\theta + \frac{1}{2}} - \frac{2cos\theta sin\theta \left(\omega^2 sin\theta + \frac{u}{2} - 9.81 cos\theta sin\theta \right)}{\left(sin^2\theta + \frac{1}{2} \right)^2} & \frac{2\omega sin\theta}{sin^2\theta + \frac{1}{2}} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-\omega^2 cos^2\theta + \omega^2 sin^2\ \theta + 14.715 cos\theta + \frac{u}{2} sin\theta}{sin^2\theta + \frac{1}{2}} + \frac{2cos\theta sin\theta \left(\omega^2 cos\theta sin\theta - 14.715 sin\theta + \frac{u}{2} cos\theta \right)}{\left(sin^2\theta + \frac{1}{2} \right)^2} & -\frac{2\omega cos\theta sin\theta}{sin^2\theta + \frac{1}{2}} \end{bmatrix}, \tag{60}$$

$$J_B = \begin{bmatrix} 0 \\ \frac{1}{2\left(sin^2\theta + \frac{1}{2} \right)} \\ 0 \\ \frac{-cos\theta}{2\left(sin^2\theta + \frac{1}{2} \right)} \end{bmatrix}, \tag{61}$$

$$\frac{\partial J_A(F)}{\partial x} = \frac{\partial J_A(F)}{\partial v} = 0_{4\times 4} \quad , \quad \frac{\partial J_B(F)}{\partial x} = \frac{\partial J_B(F)}{\partial v} = 0_{4\times 1}, \tag{62}$$

$$\frac{\partial J_A(F)}{\partial \theta} =$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2\sigma_3 sin^2\theta}{\sigma_1{}^2} - \frac{2\sigma_3 cos^2\theta}{\sigma_1{}^2} - \frac{\omega^2 sin\theta - \frac{\sigma_6}{25}}{\sigma_1} + \frac{8cos^2\theta sin^2\theta \sigma_3}{\sigma_1{}^3} - \frac{\Omega_0}{\sigma_1{}^2} & \frac{2\omega cos\theta}{\sigma_1} - \frac{4\omega cos\theta sin^2\theta}{\sigma_1{}^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4\omega^2 cos\theta sin\theta - \sigma_4 + \sigma_5}{\sigma_1} + \frac{2\sigma_2 cos^2\theta}{\sigma_1{}^2} - \frac{2\sigma_2 sin^2\theta}{\sigma_1{}^2} - \frac{8\sigma_2 cos^2\theta sin^2\theta}{\sigma_1{}^3} - \frac{\Omega_1}{\sigma_1{}^2} & \frac{2\omega sin^2\theta}{\sigma_1} - \frac{2\omega cos^2\theta}{\sigma_1} + \frac{4\omega cos^2\theta sin^2\theta}{\sigma_1{}^2} \end{bmatrix}, \tag{63}$$

$$\Omega_0 = 4cos\theta sin\theta(\omega^2 cos\theta - 9.81cos^2\theta + 9.81sin^2\theta$$

$$\Omega_1 = 4cos\theta sin\theta(-\omega^2 cos^2\theta + \omega^2 sin^2\theta + 14.715cos\theta + \frac{usin\theta}{2})$$

$$\frac{\partial J_B(F)}{\partial\theta} = \begin{bmatrix} 0 \\ \frac{-cos\theta sin\theta}{\left(sin^2\theta+\frac{1}{2}\right)^2} \\ 0 \\ \frac{sin\theta}{2\left(sin^2\theta+\frac{1}{2}\right)} + \frac{cos^2\theta sin\theta}{\left(sin^2\theta+\frac{1}{2}\right)^2} \end{bmatrix}, \tag{64}$$

$$\frac{\partial J_A}{\partial\omega} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2\omega cos\theta}{sin^2\theta+\frac{1}{2}} - \frac{4\omega sin^2\theta cos\theta}{\left(sin^2\theta+\frac{1}{2}\right)^2} & \frac{2sin\theta}{sin^2\theta+\frac{1}{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4\omega cos^2\theta sin^2\theta}{\left(sin^2\theta+\frac{1}{2}\right)^2} - \frac{2\omega cos^2\theta - 2\omega sin^2\theta}{sin^2\theta+\frac{1}{2}} & \frac{-2cos\theta sin\theta}{sin^2\theta+\frac{1}{2}} \end{bmatrix}, \ \frac{\partial J_B}{\partial\omega} = 0_{4\times1}, \tag{65}$$

$$\frac{\partial J_A}{\partial u} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-cos\theta sin\theta}{(sin^2\theta+\frac{1}{2})^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{sin\theta}{2(sin^2\theta+\frac{1}{2})} + \frac{cos^2\theta sin\theta}{(sin^2\theta+\frac{1}{2})^2} & 0 \end{bmatrix}, \ \frac{\partial J_B}{\partial u} = 0_{4\times1}. \tag{66}$$

Where,

$$\sigma_1 = sin^2\theta + \frac{1}{2},\ \sigma_2 = \omega^2 cos\theta sin\theta - \sigma_4 + \sigma_5,\ \sigma_3 = \omega^2 sin\theta + \frac{u}{2} - \frac{\sigma_6}{100},\ \sigma_4 = \frac{2943}{200}sin\theta,\ \sigma_5 = \frac{ucos\theta}{2},\ \sigma_6 = 981cos\theta sin\theta$$

Replacing θ and ω with numeric value considering all possible combinations ($\theta \in \{\ -\frac{\pi}{4},\quad 0,\quad \frac{\pi}{4}\ \}, \omega \in \{\ -10,\quad 0,\quad 10\ \}$), gives 9 sets of A_0, A_i, A_u, B_0, B_i and B_u matrixes, described in 48. Table 6.2 shows these matrixes.

Table 2: Matrixes of Inverted Pendulum on Moving Cart (A_0^l, A_i^l, A_u^l, B_0^l, B_i^l and B_u^l)

l	θ	ω	A_0^l				B_0^l	A_1^l				B_1^l	A_2^l				B_2^l	A_3^l				B_3^l	A_4^l				B_4^l	A_u^l				B_u^l
1	$-\frac{\pi}{4}$	-10	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	4.905	14.14	0.5	0	0	0	0	0	0	0	0	0	0	0	0	60.90	0	0.5	0	0	0	-1.414	0	0	0	0.5	0	0
			0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	50	-10	-0.3535	0	0	0	0	0	0	0	0	0	0	0	0	-89.59	-10	-0.7071	0	0	-10	1	0	0	0	-0.7071	0	0
2	$-\frac{\pi}{4}$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	4.905	0	0.5	0	0	0	0	0	0	0	0	0	0	0	0	-9.81	0	0.5	0	0	0	-1.414	0	0	0	0.5	0	0
			0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	0	0	-0.3535	0	0	0	0	0	0	0	0	0	0	0	0	10.40	0	-0.7071	0	0	0	1	0	0	0	-0.7071	0	0
3	$-\frac{\pi}{4}$	10	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	4.905	-14.14	0.5	0	0	0	0	0	0	0	0	0	0	0	0	60.90	0	0.5	0	0	0	-1.414	0	0	0	0.5	0	0
			0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	50	10	-0.3535	0	0	0	0	0	0	0	0	0	0	0	0	-89.59	10	-0.7071	0	0	10	1	0	0	0	-0.7071	0	0
4	0	-10	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	180.3	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	-40	0	0	0	-40	0	0	0	0	0	0	0
			0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	-170.6	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	40	0	0	0	40	0	0	0	0	0	0	0
5	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	-19.62	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	29.43	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	10	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	180.4	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	40	0	0	0	40	0	0	0	0	0	0	0
			0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	-170.6	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	-40	0	0	0	-40	0	0	0	0	0	0	0
7	$\frac{\pi}{4}$	-10	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	4.905	-14.14	0.5	0	0	0	0	0	0	0	0	0	0	0	0	-60.90	0	-0.5	0	0	0	1.414	0	0	0	-0.5	0	0
			0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	50	10	-0.3535	0	0	0	0	0	0	0	0	0	0	0	0	89.59	-10	0.7071	0	0	-10	-1	0	0	0	0.7071	0	0
8	$\frac{\pi}{4}$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	4.905	0	0.5	0	0	0	0	0	0	0	0	0	0	0	0	9.81	0	-0.5	0	0	0	1.414	0	0	0	-0.5	0	0
			0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	0	0	-0.3535	0	0	0	0	0	0	0	0	0	0	0	0	-10.40	0	0.7071	0	0	0	-1	0	0	0	0.7071	0	0
9	$\frac{\pi}{4}$	10	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	4.905	14.14	0.5	0	0	0	0	0	0	0	0	0	0	0	0	-60.90	0	-0.5	0	0	0	1.414	0	0	0	-0.5	0	0
			0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
			0	0	50	-10	-0.3535	0	0	0	0	0	0	0	0	0	0	0	0	89.59	10	0.7071	0	0	10	-1	0	0	0	0.7071	0	0