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Original paper

Aggregation of fuzzy metrics and its application in image segmentation

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Abstract

This paper proposes a novel method for the construction of a fuzzy metrics and demonstrates application in image segmentation. Some new properties of t-norms, t-conorms, aggregation functions, and fuzzy metrics are proved, which provides the procedures for constructing a new fuzzy metric. We prove that by applying some types of t-norms, t-conorms and aggregation functions on the sequence of fuzzy metrics, a new fuzzy metric could be obtained. The application of the fuzzy metric constructed in this way is illustrated in image segmentation by using the FCM algorithm. For the purpose of constructing a new fuzzy metric, an extended aggregation function called generalized quasi-arithmetic mean is considered.

Keywords: Aggregation functions, distance function, fuzzy metric, image segmentation, triangular conorms, triangular norms.

1 Introduction

Distance functions, similarity functions, metrics and various types of fuzzy metrics have a significant role in many fundamental scientific areas, as well as in various engineering fields and other applications, see [3, 5, 7, 8, 16, 28, 33]. In algorithms for image segmentation, distance functions and metrics represent a criterion for grouping pixels into segments. By applying appropriate t-norm, t-conorm and other aggregation functions on a set of initial distances of the mentioned types, a new distance function could be obtained. In this paper, we investigate the construction of a new fuzzy metric in such a way and examine some of its properties that depend on the properties of applied aggregation function and also on the initially used fuzzy metrics.

The theory of t-norm, t-conorm and other aggregation functions is well developed, see [4, 9, 14, 15, 20, 21, 32], and they have many applications in informatics and other scientific fields, see [7, 9, 21, 28, 33]. In this article, we examine and prove some new properties of these operations, from the aspect of construction of a new fuzzy metric. A new fuzzy metric constructed in this way is considered, obtained and applied in the experimental section. The application of a new fuzzy metric is presented through image segmentation, by using FCM clustering algorithm, see [1, 2, 13, 23, 28]. In image segmentation experiments, besides color components of pixels, one LBP-motivated pixel descriptor which carries information about spatial relations between pixels is used, see [7].

The remaining part of the paper is organized as follows. Section 2 deals with the notions of t-norms, t-conorms, aggregation functions and their properties which we need for further analysis. In Section 3, we introduce a notion of aggregation function compatible with t-norms and t-conorms, we consider the aggregation function called *generalized quasi-arithmetic mean* that could be suitable for different applications, and we prove some new properties of t-norms, t-conorms and aggregation functions, which are relevant for the topics observed in this paper. Section 4 is dedicated to new fuzzy metric construction. A few examples of fuzzy metrics are presented and some of their properties are examined. Also, examples of the construction of the fuzzy metric by using aggregation functions *generalized quasi-arithmetic mean* and *product* are presented. Section 5 is dedicated to LBP descriptors and one of the LBP modifications

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which is later used in segmentation process, presented through experiments that are discussed in Section 6. Section 7 contains conclusions about the results presented in this paper and directions for future research.

2 Preliminaries

In Section 4 we will consider distances that are fuzzy T-metric and fuzzy S-metric. For that purpose, in this section, we give the definitions and some of the basic properties of triangular norms (fuzzy intersection), triangular conorms (fuzzy union), fuzzy complements and aggregation functions, see [4, 9, 14, 18, 19, 20, 21, 25, 29, 35].

Through this paper, we will often use the notion of monotone n-ary functions, so we state its formal definition below.

Definition 2.1. *The function* $f : [0,1]^n \to [0,1]$ *is:*

(a) strictly monotone increasing *if*

$$(a_1 \le b_1 \wedge \cdots \wedge a_n \le b_n \wedge (a_1,\dots,a_n) \ne (b_1,\dots,b_n)) \quad \Rightarrow \quad f(a_1,\dots,a_n) < f(b_1,\dots,b_n),$$

(b) jointly strictly monotone increasing *if* $(a_1 < b_1 \wedge \cdots \wedge a_n < b_n) \Rightarrow f(a_1,\dots,a_n) < f(b_1,\dots,b_n)$.

2.1 t-norms and t-conorms

In fuzzy operation theory, t-norms are also known as "fuzzy intersection" operation, see [20, 21].

Definition 2.2. The triangular norm *(shorter* t-norm*) is a binary operation* $\mathsf{T} : [0,1]^2 \to [0,1]$ *which is commutative, associative and for all* $a, b, c \in [0,1]$ *satisfies*

(1) $\mathsf{T}(a,1) = a$ *(boundary condition),*

(2) $b \le c \Rightarrow \mathsf{T}(a,b) \le \mathsf{T}(a,c)$ *(monotonicity).*

Remark 2.3. *From the definition of t-norm it directly follows that* $\mathsf{T}(1,a) = a$ *and* $\mathsf{T}(0,a) = \mathsf{T}(a,0) = 0$ *for all* $a \in [0,1]$. *Also the joint monotonicity in both components holds, i.e., for all* $a, b, c, d \in [0,1]$,

$$(a \le b \wedge c \le d) \Rightarrow \mathsf{T}(a,c) \le \mathsf{T}(b,d).$$

Due to the commutativity and associativity, we can inductively extend the definition of t-norm to symmetric (see the Remark 2.6.(1)) n-ary operation $\mathsf{T}_{[n]} : [0,1]^n \to [0,1]$, see [20], Remark 1.10.

Definition 2.4. *For any* $n \ge 2$, extended t-norm $\mathsf{T}_{[n]} : [0,1]^n \to [0,1]$ *is defined by*

$\mathsf{T}_{[2]}(a_1,a_2) = \mathsf{T}(a_1,a_2)$,

$$\mathsf{T}_{[n+1]}(a_1,\dots,a_n,a_{n+1}) = \mathsf{T}\left(\mathsf{T}_{[n]}(a_1,\dots,a_n), a_{n+1}\right).$$

For $a_1 = \cdots = a_n = a \in [0,1]$ *we shall briefly write* $a_{\mathsf{T}}^{(n)} = \mathsf{T}_{[n]}(a,\dots,a)$, *and for* $n = 0$ *and* $n = 1$ *by convention we put* $a_{\mathsf{T}}^{(0)} = 1$ *and* $a_{\mathsf{T}}^{(1)} = a$.

Additionally, t-norm T can have some of the following important properties, see [20, 21].

(tn1) The t-norm T is an *Archimedean* t-norm if for each $a \in (0,1)$ and $b \in (0,1)$ there exists $n \in \mathbb{N}$ such that $a_{\mathsf{T}}^{(n)} < b$.

(tn2) If in the definition of the t-norm, instead of the axiom of monotonicity, strict monotonicity is satisfied, i.e., for all $a, b, c \in [0,1]$, $a \ne 0$ it holds that $b < c \Rightarrow \mathsf{T}(a,b) < \mathsf{T}(a,c)$, then we say that t-norm T is *strictly monotone.*

Strictly monotone t-norm must be also jointly strictly monotone in both components.

Proposition 2.5. *If the t-norm* T *is strictly monotone, then* T *is jointly strictly monotone, i.e., for all* $a, b, c, d \in [0,1]$, $(a < b \wedge c < d) \Rightarrow \mathsf{T}(a,c) < \mathsf{T}(b,d)$.

Proof. Let T be a strictly monotone t-norm, and let $a, b, c, d \in [0,1]$ such that $a < b$ and $c < d$. There exist $a_0 \ne 0$ and $c_0 \ne 0$ such that $a < a_0 < b$ and $c < c_0 < d$. Using monotonicity of t-norms in both components (see Remark 2.3), strict monotonicity of T and axiom of commutativity of t-norms it follows that

$$\mathsf{T}(a,c) \le \mathsf{T}(a_0,c_0) < \mathsf{T}(a_0,d) = \mathsf{T}(d,a_0) < \mathsf{T}(d,b) = \mathsf{T}(b,d).$$

□

Remark 2.6. *For the extended t-norm $\mathsf{T}_{[n]}$, the following statements are satisfied.*

(1) $\mathsf{T}_{[n]}(a_1,\dots,a_n) = \mathsf{T}_{[n]}(a_{i_1},\dots,a_{i_n})$, *where $a_{i_1},\dots,a_{i_n}$ is any permutation of elements $a_1,\dots,a_n$.*

(2) $\mathsf{T}_{[n]}(a_1,a_2,\dots,a_n) = 1 \quad \Leftrightarrow \quad a_1 = \cdots = a_n = 1.$

(3) *If T is a strictly monotone triangular norm, then $\mathsf{T}_{[n]}$ is a jointly strictly monotone increasing function and* $\mathsf{T}_{[n]}(a_1,\dots,a_n) = 0 \quad \Leftrightarrow \quad (a_1 = 0 \vee \cdots \vee a_n = 0).$

Proof. Statement (1) follows from associativity and commutativity of t-norm. Statement (2) follows inductively from $\mathsf{T}(a,b) = 1 \Leftrightarrow a = b = 1$, because $\mathsf{T}(1,1) = 1$ trivially holds (see Remark 2.3), and implication $\mathsf{T}(a,b) = 1 \Rightarrow a = b = 1$ follows from $\mathsf{T}(a,b) = 1 \le \min(a,b)$ (see Remark 1.5 in [20]).

(3) Let T be a strictly monotone triangular norm. By Proposition 2.5 then $\mathsf{T} = \mathsf{T}_{[2]}$ is a jointly strictly monotone increasing function. For $n = 3$ and $a_i < b_i$, $i \in \{1,2,3\}$ is then $\mathsf{T}(a_1,a_2) < \mathsf{T}(b_1,b_2)$ and $a_3 < b_3$, so that from joint strictness of T it follows that $\mathsf{T}_{[3]}(a_1,a_2,a_3) = \mathsf{T}(\mathsf{T}(a_1,a_2),a_3) < \mathsf{T}(\mathsf{T}(b_1,b_2),b_3) = \mathsf{T}_{[3]}(b_1,b_2,b_3)$. In the same way it inductively follows that $\mathsf{T}_{[n]}(a_1,\dots,a_n) < \mathsf{T}_{[n]}(b_1,\dots,b_n)$ holds for each $n \ge 2$ and all $a_i < b_i$, $i \in \{1,\dots,n\}$.

Implication $(a_1 = 0 \vee \cdots \vee a_n = 0) \Rightarrow \mathsf{T}_{[n]}(a_1,\dots,a_n) = 0$ follows inductively from $\mathsf{T}(0,0) = 0$ (see Remark 2.3). Let $\mathsf{T}_{[n]}(a_1,\dots,a_n) = 0$. If $0 < a_i$ for all $i \in \{1,\dots,n\}$, then by joint strict monotonicity of $\mathsf{T}_{[n]}$ we obtain contradiction $0 = \mathsf{T}_{[n]}(0,\dots,0) < \mathsf{T}_{[n]}(a_1,\dots,a_n) = 0$. Hence, $\mathsf{T}_{[n]}(a_1,\dots,a_n) = 0$ implies $a_1 = \cdots = a_n = 0$. □

Some types of t-norms can be represented by a function of one variable called "generator" of t-norm, see section 3.2 in [20] and section 3.3 in [21].

Definition 2.7. (i) Decreasing generator f *is continuous and strict decreasing function $f : [0,1] \to [0,\infty]$, such that $f(1) = 0$.*

(ii) Pseudo-inverse function *for the decreasing generator f is a function $f^{(-1)} : [0,\infty] \to [0,1]$ defined with the equation below (where f^{-1} is the usual inverse function for f):*

$$f^{(-1)}(a) = \begin{cases} f^{-1}(a) & , \quad a \in [0, f(0)] \\ 0 & , \quad a \in (f(0), +\infty] \end{cases} .$$

For a decreasing generator f and its pseudo-inverse function $f^{(-1)}$, the equality $f^{(-1)}(f(a)) = a$ is satisfied for all $a \in [0,1]$, and the following equation holds (see section 3.3 in [21]):

$$f\left(f^{(-1)}(a)\right) = \begin{cases} a & , \quad a \in [0, f(0)] \\ f(0) & , \quad a \in (f(0), +\infty] \end{cases} .$$

Remark 2.8. *Note that decreasing generator f satisfies $f\left(f^{(-1)}(a)\right) \le a$ for each $a \ge 0$.*

The theorem below is a well-known result about the representation of continuous Archimedean t-norm by decreasing generator function, see Theorem 5.1 in [20] and [21, 22, 34].

Theorem 2.9. *The mapping $\mathsf{T} : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if there exists a decreasing generator f such that $\mathsf{T}(a,b) = f^{(-1)}(f(a) + f(b))$ for all $a, b \in [0,1]$.*

The following theorem presents one method for the construction of a new t-norm using one initial t-norm, see Proposition 2.28 in [20].

Theorem 2.10. *Let $\varphi : [0,1] \to [0,1]$ be a strictly increasing surjective mapping. If $\mathsf{T} : [0,1]^2 \to [0,1]$ is a t-norm, then the function T_φ defined by $\mathsf{T}_\varphi(a,b) = \varphi \circ \mathsf{T}\left(\varphi^{-1}(a), \varphi^{-1}(b)\right)$, $a, b \in [0,1]$ is also a t-norm.*

Triangular conorms (fuzzy unions) differ from triangular norms only in a boundary condition, see [20, 21].

Definition 2.11. The triangular conorm *(shorter t-conorm) is a binary operation $\mathsf{S} : [0,1]^2 \to [0,1]$ which is commutative, associative and for all $a, b, c \in [0,1]$ satisfies:*

1. $\mathsf{S}(a,0) = a$ *(boundary condition),*
2. $b \le c \Rightarrow \mathsf{S}(a,b) \le \mathsf{S}(a,c)$ *(monotonicity).*

Remark 2.12. *From the definition of t-conorm it directly follows that* $\mathsf{S}(0,a) = a$ *and* $\mathsf{S}(1,a) = \mathsf{S}(a,1) = 1$ *for all* $a \in [0,1]$*, also as the joint monotonicity in both components, i.e., for all* $a,b,c,d \in [0,1]$ *is* $(a \le b \ \wedge \ c \le d) \quad \Rightarrow$ $\mathsf{S}(a,c) \le \mathsf{S}(b,d)$.

Just as for t-norm, we can inductively extend the definition of t-conorm to symmetric n-ary operation $\mathsf{S}_{[n]} : [0,1]^n \to [0,1]$, see [20, Remark 1.16].

Definition 2.13. *For any* $n \ge 2$*,* extended t-conorm $\mathsf{S}_{[n]} : [0,1]^n \to [0,1]$ *is defined by*
$\mathsf{S}_{[2]}(a_1,a_2) = \mathsf{S}(a_1,a_2)$,

$$\mathsf{S}_{[n+1]}(a_1,\dots,a_n,a_{n+1}) = \mathsf{S}\left(\mathsf{S}_{[n]}(a_1,\dots,a_n),a_{n+1}\right).$$

For $a_1 = \cdots = a_n = a \in [0,1]$ *we shall briefly write* $a_{\mathsf{S}}^{(n)} = \mathsf{S}_{[n]}(a,\dots,a)$*, and for* $n = 0$ *and* $n = 1$ *by convention we put* $a_{\mathsf{S}}^{(0)} = 0$ *and* $a_{\mathsf{S}}^{(1)} = a$.

Additionally, t-conorm S can have some of the following important properties, see [20, 21].

(sn1) The t-conorm S is an *Archimedean* t-conorm if for each $a \in (0,1)$ and $b \in (0,1)$ there exists $n \in \mathbb{N}$ such that $a_{\mathsf{S}}^{(n)} > b$.

(sn2) If in the definition of the t-conorm, instead of the axiom of monotonicity, strict monotonicity is satisfied, i.e., for all $a,b,c \in [0,1]$, $a \ne 1$ it holds that $b < c \Rightarrow \mathsf{S}(a,b) < \mathsf{S}(a,c)$, then we say that t-conorm S is *strictly monotone.*

As in Proposition 2.5 for t-norm, for t-conorm the following holds:

Proposition 2.14. *If the t-conorm* S *is strictly monotone, then* S *is jointly strictly monotone, i.e., for all* $a,b,c,d \in [0,1]$ *is* $(a < b \ \wedge \ c < d) \ \Rightarrow \ \mathsf{S}(a,c) < \mathsf{S}(b,d)$.

Remark 2.15. *Analogously as for the extended t-norms, for the extended t-conorm* $\mathsf{S}_{[n]}$ *the following statements are obviously satisfied.*

(1) $\mathsf{S}_{[n]}(a_1,\dots,a_n) = \mathsf{S}_{[n]}(a_{i_1},\dots,a_{i_n})$*, where* $a_{i_1},\dots,a_{i_n}$ *is any permutation of elements* $a_1,\dots,a_n$.

(2) $\mathsf{S}_{[n]}(a_1,a_2,\dots,a_n) = 0 \ \Leftrightarrow \ a_1 = \cdots = a_n = 0$.

(3) *If* S *is a strictly monotone triangular conorm, then* $\mathsf{S}_{[n]}$ *is a jointly strictly monotone increasing function and* $\mathsf{S}_{[n]}(a_1,\dots,a_n) = 1 \ \Leftrightarrow \ (a_1 = 1 \wedge \cdots \wedge a_n = 1)$.

Some types of t-conorms can be represented by a function of one variable called "generator" of t-conorm, see [20, Section 3.2] and [21, Section 3.4].

Definition 2.16. (i) Increasing generator g *is a continuous and strictly increasing function* $g : [0,1] \to [0,\infty]$*, such that* $g(0) = 0$.

(ii) Pseudo-inverse function *for an increasing generator* g *is the function* $g^{(-1)} : [0,\infty] \to [0,1]$*, defined by the equation below (where* g^{-1} *is the usual inverse function for* g*):*

$$g^{(-1)}(a) = \begin{cases} g^{-1}(a) & , \quad a \in [0,g(1)] \\ 1 & , \quad a \in (g(1),+\infty] \end{cases}.$$

For an increasing generator g and its pseudo-inverse function $g^{(-1)}$, the equality $g^{(-1)}(g(a)) = a$ is satisfied for all $a \in [0,1]$, and the following equation holds (see [21, Section 3.4]):

$$g\left(g^{(-1)}(a)\right) = \begin{cases} a & , \quad a \in [0,g(1)] \\ g(1) & , \quad a \in (g(1),+\infty] \end{cases}.$$

Remark 2.17. *Note that increasing generator* g *satisfies* $g\left(g^{(-1)}(a)\right) \le a$ *for each* $a \ge 0$.

The theorem below is a well-known result about the representation of continuous Archimedean t-conorm by increasing generator function, see [20, Corollary 5.5] and [21, 22, 34].

Theorem 2.18. *The mapping* $\mathsf{S} : [0,1]^2 \to [0,1]$ *is a continuous Archimedean t-conorm if and only if there exist an increasing generator* g *such that* $\mathsf{S}(a,b) = g^{(-1)}(g(a)+g(b))$ *for all* $a,b \in [0,1]$.

The next theorem, see [20, Proposition 2.28], is analog to Theorem 2.10, except that it relates to t-conorm.

Theorem 2.19. *Let $\varphi : [0,1] \to [0,1]$ be a strictly increasing surjective mapping. If $\mathsf{S} : [0,1]^2 \to [0,1]$ is a t-conorm, then the function S_φ defined by $\mathsf{S}_\varphi(a,b) = \varphi \circ \mathsf{S}\left(\varphi^{-1}(a), \varphi^{-1}(b)\right)$, $a, b \in [0,1]$ is also a t-conorm.*

Fuzzy negation is a unary operation which we interpret as a "fuzzy set complement", see [20, Definition 11.3].

Definition 2.20. *The function $\mathsf{c} : [0,1] \to [0,1]$ is a* fuzzy negation *(or* function of fuzzy complement*), if the following conditions hold:*

[C1] $\mathsf{c}(0) = 1$ *and* $\mathsf{c}(1) = 0$ *(boundary conditions),*

[C2] $\forall a, b \in [0,1],\ a \le b \ \Rightarrow\ \mathsf{c}(a) \ge \mathsf{c}(b)$ *(monotonicity).*

Additionally, fuzzy negation c is called a strict fuzzy negation *if it is continuous and strictly decreasing. A fuzzy negation is called a* strong fuzzy negation *if it is involutive, i.e., $\mathsf{c}(\mathsf{c}(a)) = a$ holds for all $a \in [0,1]$.*

Remark 2.21. *It is known that strong fuzzy negation also must be strict fuzzy negation, see* [20, Section 11.1]*, and* [21, Theorem 3.1]*, where strict monotonicity of fuzzy complement follows from its monotonicity and bijectivity.*

In the following sense, t-norm T and t-conorm S can be compatible, see [20, 21].

Definition 2.22. *For t-norm T, t-conorm S and strong fuzzy negation c, $(\mathsf{T}, \mathsf{S}, \mathsf{c})$ is a* De Morgan triple *if the conditions $\mathsf{c}\left(\mathsf{T}(a,b)\right) = \mathsf{S}\left(\mathsf{c}(a), \mathsf{c}(b)\right)$ and $\mathsf{c}\left(\mathsf{S}(a,b)\right) = \mathsf{T}\left(\mathsf{c}(a), \mathsf{c}(b)\right)$ hold for all $a, b \in [0,1]$.*

2.2 Aggregation functions

Aggregation functions are fuzzy operations with a variable number of arguments. They have a very important role in informatics, engineering sciences, social, economics and other scientific fields, see [4, 7, 9, 14, 15, 21, 28, 32]. The role of aggregation function is to unify multiple information, i.e., the fusion of multiple input values into one output value by some defined criteria. They are often used as information fusion models in decision-making processes and other different applications. n-ary aggregation function and aggregation function are defined with just a two requirements, monotonicity and boundary conditions, see [14].

Definition 2.23. *For fixed $n \in \mathbb{N}$, an n-ary* aggregation function *is a function $\mathsf{A}_{[n]} : [0,1]^n \to [0,1]$ such that the following conditions are satisfied.*

[A1] *Boundary conditions $\mathsf{A}_{[n]}(0, ..., 0) = 0$ and $\mathsf{A}_{[n]}(1, ..., 1) = 1$ hold.*

[A2] *$\mathsf{A}_{[n]}$ is a monotonically nondecreasing function in all its arguments, i.e.,*

$$\forall i \in \{1, ..., n\},\ a_i \le b_i \quad \Rightarrow \quad \mathsf{A}_{[n]}(a_1, \ldots, a_n) \le \mathsf{A}_{[n]}(b_1, \ldots, b_n). \tag{1}$$

[A3] *For $n = 1$, $\mathsf{A}_{[1]}$ is the identity function, i.e., $\mathsf{A}_{[1]}(a) = a$ for all $a \in [0,1]$.*

The mapping $\mathsf{A} : \bigcup\limits_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ is called extended aggregation function *if every restriction $\mathsf{A}_{[n]} : [0,1]^n \to [0,1]$, $n \in \mathbb{N}$ is an n-ary aggregation function.*

Additionally, extended aggregation function A can have other important properties, see [14].

1. A is an *idempotent* function if for each $n \in \mathbb{N}$ hold $\mathsf{A}_{[n]}(a, \ldots, a) = a$ for all $a \in [0,1]$, i.e., each restriction $\mathsf{A}_{[n]}$ is an idempotent function.
2. A is *continuous* if each restriction $\mathsf{A}_{[n]}$, $n \in \mathbb{N}$ is a continuous function.
3. A is *symmetric* if each restriction $\mathsf{A}_{[n]}$, $n \in \mathbb{N}$ is a symmetric function, i.e., for any permutation $(p_1, \ldots, p_n)$ of the set $\{1, \ldots, n\}$, $\mathsf{A}_{[n]}(a_1, \ldots, a_n) = \mathsf{A}_{[n]}(a_{p_1}, \ldots, a_{p_n})$.

Some well known examples of aggregation functions are listed below, see [4, 15].

Example 2.24. Quasi-arithmetic mean $\mathsf{M}_h(a_1, a_2, \ldots, a_n) = h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} h(a_i)\right)$, *where $h : [0,1] \to \mathbb{R}$ is a continuous and strictly monotonic function.*

Example 2.25. *The extended t-norm and the extended t-conorm are aggregation functions.*

Example 2.26. *The root-power mean function $\mathsf{M}_p(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}$ for $p \in (-\infty, 0) \cup (0, +\infty)$ is a special case of quasi-arithmetic mean. Marginal members of these classes are M_0 which is the geometric mean, while $\mathsf{M}_\infty = \max$ and $\mathsf{M}_{-\infty} = \min$ are not in the class of quasi-arithmetic mean.*

3 Some new relevant properties and notions related to t-norms, t-conorms and aggregation functions

In this section we introduce some new terms and results related to t-norms, t-conorms and aggregation functions that will be used in Section 4. Also, we introduce a new aggregation function which is called "generalized quasi-arithmetic mean", and it will be further considered in Section 4. "Generalized quasi-arithmetic mean" is a special case of "generated aggregation operator" introduced by M. Komorníková, see [4].

Definition 3.1. *Let* $h : [0,1] \to [0,1]$ *and* $h_i : [0,1] \to [0,1]$, $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$ *be continuous functions that are satisfying one of the following conditions:*

(i) *for all* i, h *and* h_i *are strictly increasing functions, and* $h(0) = h_i(0) = 0$ *and* $h(1) = h_i(1) = 1$,

(ii) *for all* i, h *and* h_i *are strictly decreasing functions, and* $h(0) = h_i(0) = 1$ *and* $h(1) = h_i(1) = 0$.

The function defined by

$$\mathsf{M}(a_1, a_2, \dots, a_n) = h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} h_i(a_i)\right), \tag{2}$$

is n*-ary aggregation function that we call* generalized quasi-arithmetic mean.

Remark 3.2. *The generalized quasi-arithmetic mean is an aggregation function that can be obtained using the construction method presented in* [14, Proposition 6.13].

Now, we introduce the terms of compatibility of aggregation function with the sequence of t-norms or t-conorms with respect to a given t-norm or t-conorm respectively. These terms of compatibility which we define figure in several theorems in this section and section 4.

Definition 3.3. (i) *An* n*-ary aggregation function* $\mathsf{A}_{[n]}$ *is* (continuously) compatible with t-norms $\mathsf{T}_1, \dots, \mathsf{T}_n$ with respect to the (continuous) t-norm T *if*

$$\mathsf{A}_{[n]}\left(\mathsf{T}_1(a_1, b_1), \dots, \mathsf{T}_n(a_n, b_n)\right) \geq \mathsf{T}\left(\mathsf{A}_{[n]}(a_1, \dots, a_n), \mathsf{A}_{[n]}(b_1, \dots, b_n)\right), \tag{3}$$

for all $a_i, b_i \in [0,1]$, $i \in \{1, \dots, n\}$.

(ii) *An* n*-ary aggregation function* A *is* (continuously) compatible with t-conorms $\mathsf{S}_1, \dots, \mathsf{S}_n$ with respect to the (continuous) t-conorm S *if*

$$\mathsf{A}_{[n]}\left(\mathsf{S}_1(a_1, b_1), \dots, \mathsf{S}_n(a_n, b_n)\right) \leq \mathsf{S}\left(\mathsf{A}_{[n]}(a_1, \dots, a_n), \mathsf{A}_{[n]}(b_1, \dots, b_n)\right), \tag{4}$$

for all $a_i, b_i \in [0,1]$, $i \in \{1, \dots, n\}$.

Next example illustrates the term of compatibility from Definition 3.3.

Example 3.4. *Let us observe the well known binary aggregation function "arithmetic mean"* $\mathsf{A}_{[2]} : [0,1]^2 \to [0,1]$ *defined with* $\mathsf{A}_{[2]}(a,b) = \frac{1}{2}a + \frac{1}{2}b$, $a, b \in [0,1]$, *and "minimum" t-norms* $\mathsf{T}_1 : [0,1]^2 \to [0,1]$ *and* $\mathsf{T}_2 : [0,1]^2 \to [0,1]$ *defined with* $\mathsf{T}_1(a,b) = \mathsf{T}_2(a,b) = \min(a,b)$, $a, b \in [0,1]$, *see* [20, 14].

(a) *Aggregation function* $\mathsf{A}_{[2]}$ *is not compatible with t-norms* T_1 *and* T_2 *with respect to the "product" t-norm* $\mathsf{T}_P : [0,1]^2 \to [0,1]$ *defined with* $\mathsf{T}_P(a,b) = ab$, $a, b \in [0,1]$ *(see* [20]*). For example, for* $a = 1$, $b = 0.2$, $c = 0.1$ *and* $d = 0.6$ *we have* $\mathsf{A}_{[2]}\left(\mathsf{T}_1(a,b), \mathsf{T}_2(c,d)\right) = 0.15 < \mathsf{T}_P\left(\mathsf{A}_{[2]}(a,c), \mathsf{A}_{[2]}(b,d)\right) = 0.22$, *which indicates that inequality* (3) *is not satisfied for* $n = 2$ *and all* $a, b, c, d \in [0,1]$.

(b) *Aggregation function* $\mathsf{A}_{[2]}$ *is compatible with t-norms* T_1 *and* T_2 *with respect to the "drastic product" t-norm* $\mathsf{T}_D : [0,1]^2 \to [0,1]$ *defined with* $\mathsf{T}_D(a,b) = \begin{cases} a & , \quad b = 1 \\ b & , \quad a = 1 \\ 0 & , \quad a \neq 1 \wedge b \neq 1 \end{cases}$, $a, b \in [0,1]$ *(see* [20]*). Namely, for* $\mathsf{A}_{[2]}(a,c) \neq 1$ *and* $\mathsf{A}_{[2]}(b,d) \neq 1$ *is* $\mathsf{T}_D\left(\mathsf{A}_{[2]}(a,c), \mathsf{A}_{[2]}(b,d)\right) = 0$, *and inequality* (3) *is then obviously satisfied. For* $\mathsf{A}_{[2]}(a,c) = 1$ *(proof is analogous for the case* $\mathsf{A}_{[2]}(b,d) = 1$*) is* $\mathsf{T}_D\left(\mathsf{A}_{[2]}(a,c), \mathsf{A}_{[2]}(b,d)\right) = \mathsf{A}_{[2]}(b,d)$. *On the other hand,* $\mathsf{A}_{[2]}(a,c) = \frac{1}{2}a + \frac{1}{2}c = 1$ *implies* $a = c = 1$ *and then is* $\mathsf{A}_{[2]}\left(\mathsf{T}_1(a,b), \mathsf{T}_2(c,d)\right) = \mathsf{A}_{[2]}\left(\min(a,b), \min(c,d)\right) = \mathsf{A}_{[2]}(b,d)$, *so that equality holds in* (3). *"Drastic product" t-norm is not a continuous function, so that* $\mathsf{A}_{[2]}$ *is not continuously compatible with* T_1 *and* T_2 *with respect to* T_D.

In the next theorem, compatibility of generalized quasi-arithmetic mean function, which is defined by Equation (2), with t-norms and t-conorms that are generated by continuous, monotonic generators h_i from (2) is proved.

Theorem 3.5. *Let $h : [0,1] \to [0,1]$ and $h_i : [0,1] \to [0,1]$ be continuous and strictly monotonic functions.*

(i) *If functions h and h_i, $i = 1,\ldots,n$ satisfy $h(0) = h_i(0) = 0$, $i = 1,\ldots,n$ and $h(1) = h_i(1) = 1$, $i = 1,\ldots,n$, then the generalized quasi-arithmetic mean M defined by (2) is an aggregation function which is continuously compatible with continuous Archimedean t-conorms $\mathsf{S}_1,\ldots,\mathsf{S}_n$ generated by increasing generators h_i, $i = 1,\ldots,n$, with respect to the continuous t-conorm S generated by increasing generator h.*

(ii) *If functions h and h_i, $i = 1,\ldots,n$ satisfy $h(0) = h_i(0) = 1$, $i = 1,\ldots,n$ and $h(1) = h_i(1) = 0$, $i = 1,\ldots,n$, then the generalized quasi-arithmetic mean M defined by (2) is an aggregation function which is continuously compatible with continuous Archimedean t-norms $\mathsf{T}_1,\ldots,\mathsf{T}_n$ generated with decreasing generators h_i, $i = 1,\ldots,n$, with respect to the continuous t-norm T generated with decreasing generator h.*

Proof. (i) Let us notice that continuous, monotonic functions h and h_i, $i = 1,\ldots,n$ satisfying $h(0) = h_i(0) = 0$, $i = 1,\ldots,n$ and $h(1) = h_i(1) = 1$, $i = 1,\ldots,n$, are strictly increasing, and exist monotonic increasing, continuous functions h^{-1} and h_i^{-1}, such that conditions $h^{-1}(0) = h_i^{-1}(0) = 0$ and $h^{-1}(1) = h_i^{-1}(1) = 1$ are satisfied.

Let M be the aggregation function from Definition 3.1 defined by Equation (2). Let $\mathsf{S}_1,\ldots,\mathsf{S}_n$ be t-conorms generated by increasing functions h_i, $i = 1,...,n$, i.e., for each $i = 1,\ldots,n$ holds,

$$\forall a,b \in [0,1], \quad \mathsf{S}_i(a,b) = h_i^{(-1)}\left(h_i(a) + h_i(b)\right).$$

Let $L = \mathsf{M}\left(\mathsf{S}_1(a_1,b_1),\ldots,\mathsf{S}_n(a_n,b_n)\right) = h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} h_i\left(\mathsf{S}_i(a_i,b_i)\right)\right)$. Because of Remark 2.17, from $\mathsf{S}_i(a,b) = h_i^{(-1)}\left(h_i(a) + h_i(b)\right)$ we get

$$h_i\left(\mathsf{S}_i(a_i,b_i)\right) = h_i\left(h_i^{(-1)}\left(h_i(a_i) + h_i(b_i)\right)\right) \le h_i(a_i) + h_i(b_i),$$

for all $a_i, b_i \in [0,1]$ and each $i = 1,\ldots,n$, and thus

$$\frac{1}{n}\sum_{i=1}^{n} h_i\left(\mathsf{S}_i(a_i,b_i)\right) \le \frac{1}{n}\sum_{i=1}^{n}\left(h_i(a_i) + h_i(b_i)\right),$$

i.e.,

$$h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} h_i\left(\mathsf{S}_i(a_i,b_i)\right)\right) \le h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\left(h_i(a_i) + h_i(b_i)\right)\right).$$

Now, we have

$$\begin{aligned}
L &\le h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\left(h_i(a_i) + h_i(b_i)\right)\right) = h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} h_i(a_i) + \frac{1}{n}\sum_{i=1}^{n} h_i(b_i)\right) \\
&= h^{-1}\left(h\left(h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} h_i(a_i)\right)\right) + h\left(h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} h_i(b_i)\right)\right)\right) \\
&= h^{-1}\left(h\left(\mathsf{M}(a_1,\ldots,a_n)\right) + h\left(\mathsf{M}(b_1,\ldots,b_n)\right)\right) \\
&= \mathsf{S}\left(\mathsf{M}(a_1,\ldots,a_n), \mathsf{M}(b_1,\ldots,b_n)\right).
\end{aligned}$$

So, with the equations above we prove that

$$\mathsf{M}\left(\mathsf{S}_1(a_1,b_1),\ldots,\mathsf{S}_n(a_n,b_n)\right) \le \mathsf{S}\left(\mathsf{M}(a_1,\ldots,a_n), \mathsf{M}(b_1,\ldots,b_n)\right),$$

where S is a continuous t-conorm generated by function h in the sense of Theorem 2.18.

(ii) Analogously as (i). □

The following theorem states the compatibility of t-norms and t-conorms with aggregation functions generated by the same t-norm or t-conorm, i.e., compatibility with corresponding extended t-norm and extended t-conorm aggregation functions.

Theorem 3.6. *Let* T *be a (continuous) t-norm, and* S *be a (continuous) t-conorm. Then* $\mathsf{T}_{[n]}$ *and* $\mathsf{S}_{[n]}$ *are aggregation functions that are (continuously) compatible with t-norms* $\mathsf{T}_1 = \mathsf{T}_2 = \cdots = \mathsf{T}_n = \mathsf{T}$ *and t-conorms* $\mathsf{S}_1 = \mathsf{S}_2 = \cdots = \mathsf{S}_n = \mathsf{S}$*, respectively, with respect to the same corresponding* T *and* S.

Proof. From associativity and commutativity of t-norms and t-conorms, we obtain
$\mathsf{T}_{[n]}\left(\mathsf{T}(a_1,b_1),\ldots,\mathsf{T}(a_n,b_n)\right) = \mathsf{T}_{[2n]}(a_1,b_1,\ldots,a_n,b_n) = \mathsf{T}\left(\mathsf{T}_{[n]}(a_1,\ldots,a_n),\mathsf{T}_{[n]}(b_1,\ldots,b_n)\right)$,
$\mathsf{S}_{[n]}\left(\mathsf{S}(a_1,b_1),\ldots,\mathsf{S}(a_n,b_n)\right) = \mathsf{S}_{[2n]}(a_1,b_1,\ldots,a_n,b_n) = \mathsf{S}\left(\mathsf{S}_{[n]}(a_1,\ldots,a_n),\mathsf{S}_{[n]}(b_1,\ldots,b_n)\right)$. □

In the following lemma, we prove one more important property of aggregation functions.

Lemma 3.7. *If* $\mathsf{A}_{[n]}$ *is a strictly monotone* n*-ary aggregation function, then*

(a) $\mathsf{A}_{[n]}(a_1,\ldots,a_n) = 0 \;\Leftrightarrow\; a_1 = \cdots = a_n = 0,$

(b) $\mathsf{A}_{[n]}(a_1,\ldots,a_n) = 1 \;\Leftrightarrow\; a_1 = \cdots = a_n = 1.$

Proof. Let us prove (a), and then proof of (b) is analogous.

(a) Let $\mathsf{A}_{[n]}(a_1,\ldots,a_n) = 0$. Suppose the opposite, i.e., that there exists $a_j > 0$ for some $j \in \{1,\ldots,n\}$. Beside that, because A is an aggregation function for each $i \in \{1,\ldots,n\}$, it holds that $a_i \geq 0$, and then strict monotonicity of $\mathsf{A}_{[n]}$ implies contradiction $0 = \mathsf{A}_{[n]}(a_1,\ldots,a_n) > \mathsf{A}_{[n]}(0,\ldots,0) = 0$. The opposite holds by definition of aggregation function. □

4 Fuzzy metrics

There are several types of distances and functions similar to that could represent a measure of the difference between two objects. Depending on the method of their construction and considered properties they are differently treated in the literature and used for many diverse applications, see [3, 8, 36]. Also, we could consider the distances that assign a fuzzy set to each ordered pair of elements. In the literature review, we could find fuzzy metric spaces, see [10, 11, 12, 17], with a different application in image processing tasks, see [16, 26]. In this section, we consider the properties and examples of fuzzy T-metric and fuzzy S-metric, see [33], which will be further used for distance function construction as a measure of the difference between the observed image pixels in FCM image segmentation algorithm.

For the purpose of the research presented in this paper, the definition of distance functions and other similar types of functions are adopted from [8].

Definition 4.1. *Let* X *be an arbitrary non-empty set, and* T *and* S *are t-norm and t-conorm, respectively. Let* $d : X^2 \to [0,\infty)$ *be a function with the following properties.*

(1) $\forall x \in X,\ d(x,x) = 0.$

(2) $\forall x,y \in X,\ d(x,y) = d(y,x).$

(3) $\forall x,y \in X,\ d(x,y) = 0 \;\Rightarrow\; x = y.$

(4) $\forall x,y,z \in X,\ d(x,z) \leq d(x,y) + d(y,z).$

Additionally, if $d : X^2 \to [0,1]$*, the following two properties can be satisfied:*

(4') $\forall x,y,z \in X,\ d(x,z) \geq \mathsf{T}\left(d(x,y),d(y,z)\right);$

(4") $\forall x,y,z \in X,\ d(x,z) \leq \mathsf{S}\left(d(x,y),d(y,z)\right).$

Depending on which of the above characteristics are satisfied, we distinguish several types of distances. If d *satisfies properties* (1) *and* (2)*, it is a* distance function*, and the ordered pair* (X,d) *is a* space with distance*. If properties* (1)-(4) *are valid for* d*, then it is a* metric*, and the ordered par* (X,d) *is a* metric space*. If* (1)*,* (2) *and* (4) *are satisfied,* d *is a* pseudo-metric*. If* (1)*,* (2) *and* (3) *are satisfied,* d *is a* semi-metric.

The definitions of fuzzy metrics are not unanimously accepted in the literature. For the purpose of our investigations, we adopt the following definitions from [33].

Definition 4.2. Fuzzy S-metric space *is an ordered triple* $(X,\boldsymbol{s},\mathsf{S})$ *such that* X *is a non-empty set,* S *is a continuous t-conorm and* $\boldsymbol{s}$ *is a fuzzy set at* $X^2 \times (0,\infty)$ *that satisfies the following conditions for all* $x,y,z \in X$ *and* $\alpha,\beta > 0$.

(1) $\boldsymbol{s}(x,y,\alpha) \in [0,1),$

(2) $\boldsymbol{s}(x,y,\alpha) = 0 \;\Leftrightarrow\; x = y,$

(3) $\boldsymbol{s}(x,y,\alpha)=\boldsymbol{s}(y,x,\alpha)$,

(4) $\mathsf{S}(\boldsymbol{s}(x,y,\alpha),\boldsymbol{s}(y,z,\beta))\geq \boldsymbol{s}(x,z,\alpha+\beta)$,

(5) $\boldsymbol{s}(x,y,\_):(0,\infty)\to[0,1]$ *is a continuous function.*

The fuzzy set $\boldsymbol{s}$ *is called a* fuzzy S-metric*. If instead of condition 1. we have* $\boldsymbol{s}(x,y,\alpha)\in[0,1]$*, for the fuzzy set* $\boldsymbol{s}$ *we say that it is a* fuzzy S-metric in the broader sense*, and* $(X,\boldsymbol{s},\mathsf{S})$ *is a* fuzzy S-metric space in the broader sense*. If instead of the condition 2. the following equality holds:*

(2′) $\boldsymbol{s}(x,x,\alpha)=0$,

we say that $\boldsymbol{s}$ *is a* fuzzy S-pseudo metric*.*

Definition 4.3. Fuzzy T-metric space *is an ordered triple* $(X,\boldsymbol{t},\mathsf{T})$ *such that* X *is a non-empty set,* T *is a continuous t-norm and* $\boldsymbol{t}$ *is a fuzzy set at* $X^2\times(0,\infty)$ *that satisfies the following conditions for all* $x,y,z\in X$ *and* $\alpha,\beta>0$.

(1) $\boldsymbol{t}(x,y,\alpha)\in(0,1]$,

(2) $\boldsymbol{t}(x,y,\alpha)=1 \Leftrightarrow x=y$,

(3) $\boldsymbol{t}(x,y,\alpha)=\boldsymbol{t}(y,x,\alpha)$,

(4) $\mathsf{T}\left(\boldsymbol{t}(x,y,\alpha),\boldsymbol{t}(y,z,\beta)\right)\leq \boldsymbol{t}(x,z,\alpha+\beta)$,

(5) $\boldsymbol{t}(x,y,\_):(0,\infty)\to[0,1]$ *is a continuous function.*

The fuzzy set $\boldsymbol{t}$ *is called* fuzzy T-metric*. If instead of condition 1. we have* $\boldsymbol{t}(x,y,\alpha)\in[0,1]$*, for the fuzzy set* $\boldsymbol{t}$ *we say that it is a* fuzzy T-metric in the broader sense*, and* $(X,\boldsymbol{t},\mathsf{T})$ *is a* fuzzy T-metric space in the broader sense*. If instead of the condition 2. the following equality holds:*

(2′) $\boldsymbol{t}(x,x,\alpha)=1$,

we say that $\boldsymbol{t}$ *is a* fuzzy T-pseudo metric*.*

Special case of fuzzy metrics are functions $\boldsymbol{s}$ and $\boldsymbol{t}$ that are not dependent on $\alpha\in(0,\infty)$.

Definition 4.4. *Fuzzy S-metric* $\boldsymbol{s}$ *(fuzzy T-metric* $\boldsymbol{t}$*) is* stationary *on* X *if* $\boldsymbol{s}$ *(*$\boldsymbol{t}$*) does not depend on* α*, i.e., if for all fixed* $x,y\in X$*, function* $\boldsymbol{s}_{x,y}(\alpha)=\boldsymbol{s}(x,y,\alpha)$ *(*$\boldsymbol{t}_{x,y}(\alpha)=\boldsymbol{t}(x,y,\alpha)$*),* $\alpha\in(0,\infty)$ *is a constant function.*

Remark 4.5. *We can notice that, in the case of stationary fuzzy T-metric in the broader sense* $\boldsymbol{t}$ *and stationary fuzzy S-metric in broader sense* $\boldsymbol{s}$*, functions* $\boldsymbol{t}$ *and* $\boldsymbol{s}$ *are distance functions and semi-metrics.*

In [33], the following theorem (Theorem 3) is proved.

Theorem 4.6.

(a) *Let* T *be a t-norm,* S *be a t-conorm, and let* c *be a strong fuzzy negation such that* $(\mathsf{T},\mathsf{S},\mathsf{c})$ *is a De Morgan triple. If* $(X,\boldsymbol{s},\mathsf{S})$ *is a fuzzy S-metric space, then* $(X,\mathsf{c}\circ\boldsymbol{s},\mathsf{T})$ *is a fuzzy T-metric space.*

(b) *Let* T *be a t-norm,* S *be a t-conorm, and let* c *be a strong fuzzy negation such that* $(\mathsf{T},\mathsf{S},\mathsf{c})$ *is a De Morgan triple. If* $(X,\boldsymbol{t},\mathsf{T})$ *is a fuzzy T-metric space, then* $(X,\mathsf{c}\circ\boldsymbol{t},\mathsf{S})$ *is a fuzzy S-metric space.*

Definition 4.7. *For fuzzy metric spaces* $(X,\boldsymbol{s},\mathsf{S})$ *and* $(X,\boldsymbol{t},\mathsf{T})$ *and the corresponding fuzzy metrics* $\boldsymbol{s}$ *and* $\boldsymbol{t}$ *from the previous theorem, we say that they are* dual *with respect to fuzzy negation* c*.*

Let us list some examples of fuzzy metrics which will be used through experiments in Section 6, see [10, 11, 12, 16, 17, 33].

Example 4.8. *For fixed parameter* $K>0$*, the function* $\boldsymbol{t}:[0,\infty)^2\times(0,\infty)\to(0,1]$ *defined by*

$$\boldsymbol{t}(x,y,K)=\frac{\frac{x+y}{2}+K}{\max\{x,y\}+K},$$

is a fuzzy T-metric with respect to "multiplication" t-norm, and the mapping $\boldsymbol{s}:[0,\infty)^2\times(0,\infty)\to[0,1)$ *defined by*

$$\boldsymbol{s}(x,y,K)=\frac{|x-y|}{2(\max(x,y)+K)},$$

is the fuzzy S-metric with respect to the "algebraic sum" t-conorm.

Example 4.9. *If (X, d) is a metric space, then the mapping $\boldsymbol{t} : X^2 \times (0, \infty) \to (0, 1]$ defined by*

$$\boldsymbol{t}(x, y, t) = \frac{t}{t + d(x, y)},$$

is a fuzzy T-metric with respect to the "multiplication" t-norm. With respect to the standard fuzzy negation $c(x) = 1 - x$, $x \in [0, 1]$, the mentioned fuzzy T-metric is dual to the fuzzy S-metric $\boldsymbol{s} : X^2 \times (0, \infty) \to [0, 1)$ defined by

$$\boldsymbol{s}(x, y, t) = 1 - \boldsymbol{t}(x, y, t) = \frac{d(x, y)}{t + d(x, y)},$$

which is fuzzy S-metric with respect to the "algebraic sum" t-conorm.

In the following theorem, we present a new method for a fuzzy S-pseudo metric construction by an application of aggregation function on the sequence of initial fuzzy S-pseudo metrics.

Theorem 4.10. *Let $\mathsf{A}_{[n]}$ be a continuous n-ary aggregation function which is continuously compatible with continuous t-conorms $\mathsf{S}_1, \ldots, \mathsf{S}_n$ with respect to a continuous t-conorm S. If $\boldsymbol{s}_i : X_i^2 \times (0, +\infty) \to [0, 1)$, $i \in I = \{1, \ldots, n\}$ are fuzzy S-pseudo metrics with respect to the triangular conorms S_i, $i \in \{1, \ldots, n\}$ respectively, then for $X = X_1 \times \cdots \times X_n$, function $\boldsymbol{s} : X^2 \times (0, +\infty) \to [0, 1]$ defined with*

$$\boldsymbol{s}\,(x, y, \alpha) = \mathsf{A}_{[n]}\left(\boldsymbol{s}_1(x_1, y_1, \alpha), \ldots, \boldsymbol{s}_n(x_n, y_n, \alpha)\right), \quad x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in X, \quad \alpha > 0$$

is the fuzzy s-pseudo metric in a broader sense with respect to t-conorm S. If $\mathsf{A}_{[n]}$ is a strictly monotone function, then $\boldsymbol{s}$ is a fuzzy S-pseudo metric. If additionally $\boldsymbol{s}_i$ are fuzzy S-metrics, then $\boldsymbol{s}$ is fuzzy S-metric.

Proof. For function $\boldsymbol{s}$, we will prove the properties listed in Definition 4.2.

1. From $\boldsymbol{s}_i(x_i, y_i, \alpha) \in [0, 1)$, $i \in I$ obviously follows

$$\boldsymbol{s}(x, y, \alpha) = \mathsf{A}_{[n]}\left(\boldsymbol{s}_1(x_1, y_1, \alpha), \ldots, \boldsymbol{s}_n(x_n, y_n, \alpha)\right) \in [0, 1].$$

2'. Using property 2'. of fuzzy S-pseudo metrics $\boldsymbol{s}_i$, from $x = y$ we have

$$\forall i \in I,\ x_i = y_i \ \Rightarrow\ \forall i \in I,\ \boldsymbol{s}_i(x_i, y_i, \alpha) = 0,$$

and then, using boundary condition of aggregation function $\mathsf{A}_{[n]}$ we obtain

$$\boldsymbol{s}(x, y, \alpha) = \mathsf{A}_{[n]}\left(\boldsymbol{s}_1(x_1, y_1, \alpha), \ldots, \boldsymbol{s}_n(x_n, y_n, \alpha)\right) = 0.$$

3. Using property 3. of functions $\boldsymbol{s}_i$, $i \in I$ we obtain

$$\boldsymbol{s}(x, y, \alpha) = \mathsf{A}_{[n]}\left(\boldsymbol{s}_1(x_1, y_1, \alpha), \ldots, \boldsymbol{s}_n(x_n, y_n, \alpha)\right) = \mathsf{A}_{[n]}\left(\boldsymbol{s}_1(y_1, x_1, \alpha), \ldots, \boldsymbol{s}_n(y_n, x_n, \alpha)\right) = \boldsymbol{s}(y, x, \alpha).$$

4. Let $x = (x_1, \ldots, x_n) \in X$, $y = (y_1, \ldots, y_n) \in X$ and $z = (z_1, \ldots, z_n) \in X$. From property 4. of the fuzzy S-pseudo metrics $\boldsymbol{s}_i$ it follows that

$$\boldsymbol{s}_i(x_i, z_i, \alpha + \beta) \leq \mathsf{S}_i\left(\boldsymbol{s}_i(x_i, y_i, \alpha), \boldsymbol{s}_i(y_i, z_i, \beta)\right),$$

for all $i \in I$. Using monotonicity of aggregation function $\mathsf{A}_{[n]}$ we obtain

$$\begin{aligned} \boldsymbol{s}(x, z, \alpha + \beta) &= \mathsf{A}_{[n]}\left(\boldsymbol{s}_1(x_1, z_1, \alpha + \beta), \ldots, \boldsymbol{s}_n(x_n, z_n, \alpha + \beta)\right) \\ &\leq \mathsf{A}_{[n]}\left(\mathsf{S}_1\left(\boldsymbol{s}_1(x_1, y_1, \alpha), \boldsymbol{s}_1(y_1, z_1, \beta)\right), \ldots, \mathsf{S}_n\left(\boldsymbol{s}_n(x_n, y_n, \alpha), \boldsymbol{s}_n(y_n, z_n, \beta)\right)\right). \end{aligned}$$

Because aggregation function $\mathsf{A}_{[n]}$ is continuously compatible with continuous t-conorms $\mathsf{S}_1, \ldots, \mathsf{S}_n$ with respect to the continuous t-conorm S, it is

$$\begin{aligned} &\mathsf{A}_{[n]}\left(\mathsf{S}_1\left(\boldsymbol{s}_1(x_1, y_1, \alpha), \boldsymbol{s}_1(y_1, z_1, \beta)\right), \ldots, \mathsf{S}_n\left(\boldsymbol{s}_n(x_n, y_n, \alpha), \boldsymbol{s}_n(y_n, z_n, \beta)\right)\right) \\ &\leq \mathsf{S}\left(\mathsf{A}_{[n]}\left(\boldsymbol{s}_1(x_1, y_1, \alpha), \ldots, \boldsymbol{s}_n(x_n, y_n, \alpha)\right), \mathsf{A}_{[n]}\left(\boldsymbol{s}_1(y_1, z_1, \beta), \ldots, \boldsymbol{s}_n(y_n, z_n, \beta)\right)\right). \end{aligned}$$

So, we could conclude that for continuous t-conorm S it holds that

$$\begin{aligned}\boldsymbol{s}(x,z,\alpha+\beta) &\leq \mathsf{S}\left(\mathsf{A}_{[n]}\left(\boldsymbol{s}_1(x_1,y_1,\alpha),\ldots,\boldsymbol{s}_n(x_n,y_n,\alpha)\right),\mathsf{A}_{[n]}\left(\boldsymbol{s}_1(y_1,z_1,\beta),\ldots,\boldsymbol{s}_n(y_n,z_n,\beta)\right)\right)\\ &= \mathsf{S}\left(\boldsymbol{s}(x,y,\alpha),\boldsymbol{s}(y,z,\beta)\right).\end{aligned}$$

5. Since the fuzzy S-metrics $\boldsymbol{s}_i$, $i\in I$ are continuous in parameter α and $\mathsf{A}_{[n]}$ is a continuous function, then their composition $\boldsymbol{s}$ is a continuous function.

Above we proved that $\boldsymbol{s}$ is a fuzzy S-pseudo metric in a broader sense with respect to the t-conorm S. If $\mathsf{A}_{[n]}$ is a strictly monotone aggregation function, from Lemma 3.7.(b) and $\boldsymbol{s}_i(x_i,y_i,\alpha)\in[0,1)$, $i\in I$ follows $\boldsymbol{s}(x,y,\alpha)=\mathsf{A}_{[n]}\left(\boldsymbol{s}_1(x_1,y_1,\alpha),\ldots,\boldsymbol{s}_n(x_n,y_n,\alpha)\right)\in[0,1)$, i.e., $\boldsymbol{s}$ is a fuzzy S-pseudo metric. If $\mathsf{A}_{[n]}$ is a strictly monotone function and $\boldsymbol{s}_i$ are fuzzy S-metrics, then from Lemma 3.7.(a) follows the Axiom 2. for $\boldsymbol{s}$, i.e., $\boldsymbol{s}$ is fuzzy S-metrics. □

Remark 4.11. *In the previous theorem, if additionally $\boldsymbol{s}_i$ are stationary metrics, they are continuous functions in the parameter α, which implies that $\boldsymbol{s}$ also has the same property, and because of that the assumption about continuity of the function $\mathsf{A}_{[n]}$ is not necessary in that case.*

From Theorem 3.6 and Theorem 4.10, as a special case holds the following theorem proved in [33], see Theorem 9.

Theorem 4.12. *If $\boldsymbol{s}_i$, $i\in\{1,\ldots,n\}$ are fuzzy S-metrics with respect to a continuous strictly monotone triangular conorm S, then $\boldsymbol{s}$ defined by*

$$\boldsymbol{s}(x,y,\alpha)=\mathsf{S}_{[n]}\left(\boldsymbol{s}_1(x_1,y_1,\alpha),\ldots,\boldsymbol{s}_n(x_n,y_n,,\alpha)\right),\quad x=(x_1,\ldots,x_n),\ y=(y_1,\ldots,y_n)\in X,\quad \alpha>0$$

is a fuzzy S-metric with respect to the same triangular conorm S. If S is not strictly monotone, then $\boldsymbol{s}$ is a fuzzy S-metric in the broader sense.

Remark 4.13. *Considering the generalized quasi-arithmetic mean M and appropriate continuous Archimedean t-conorms S_i from Theorem 3.5, we observe that the conditions from Theorem 4.10 are fulfilled, which implies that*

$$\boldsymbol{s}(x,y,\alpha)=\mathsf{M}\left(\boldsymbol{s}_1(x_1,y_1,\alpha),\ldots,\boldsymbol{s}_n(x_n,y_n,\alpha)\right),\quad x=(x_1,\ldots,x_n),\ y=(y_1,\ldots,y_n)\in X,\quad \alpha>0$$

is a fuzzy S-metric, where $\boldsymbol{s}_i$, $i\in\{1,\ldots,n\}$ are the fuzzy S-pseudo metrics with respect to the triangular conorms S_i, $i\in\{1,\ldots,n\}$ respectively.

The following theorem is analog (with analogous proof) to Theorem 4.10, observing now the construction of fuzzy T-pseudo metrics instead of fuzzy S-pseudo metric.

Theorem 4.14. *Let $\mathsf{A}_{[n]}$ be a continuous n-ary aggregation function which is continuously compatible with continuous t-norms $\mathsf{T}_1,\ldots,\mathsf{T}_n$, with respect to a continuous t-norm T. If $\boldsymbol{t}_i: X_i^2\times(0,\infty)\to(0,1]$, $i\in I=\{1,\ldots,n\}$ are fuzzy T-pseudo metrics with respect to the triangular norms T_i respectively, then for $X=X_1\times\cdots\times X_n$, function $\boldsymbol{t}:X^2\times(0,\infty)\to[0,1]$ defined by*

$$\boldsymbol{t}(x,y,\alpha)=\mathsf{A}_{[n]}\left(\boldsymbol{t}_1(x_1,y_1,\alpha),\ldots,\boldsymbol{t}_n(x_n,y_n,\alpha)\right),\quad x=(x_1,\ldots,x_n),\ y=(y_1,\ldots,y_n)\in X,\quad \alpha>0$$

is a fuzzy T-pseudo metric in a broader sense with respect to the triangular norm T. If $\mathsf{A}_{[n]}$ is a strictly monotone aggregation function, then $\boldsymbol{t}$ is a fuzzy T-pseudo metric. Additionally, if $\boldsymbol{t}_i$ are fuzzy T-metrics, then $\boldsymbol{t}$ is fuzzy T-metric.

Remark 4.15. *In the previous theorem, if additionally $\boldsymbol{t}_i$ are stationary metrics, they are continuous functions by the parameter α, which implies that $\boldsymbol{t}$ also has the same property, and because of that the assumption about continuity of the function $\mathsf{A}_{[n]}$ is not necessary in that case.*

From Theorem 3.6 and Theorem 4.14 as a special case holds the following theorem proved in [33], see Theorem 8.

Theorem 4.16. *If $\boldsymbol{t}_i$, $i\in\{1,\ldots,n\}$ are fuzzy T-metrics with respect to a continuous, strictly monotone triangular norm T, then $\boldsymbol{t}(x,y)$*

$$\boldsymbol{t}(x,y,\alpha)=\mathsf{T}_{[n]}\left(\boldsymbol{t}_1(x_1,y_1,\alpha),\ldots,\boldsymbol{t}_n(x_n,y_n,\alpha)\right),\quad x=(x_1,\ldots,x_n),\ y=(y_1,\ldots,y_n)\in X,\quad \alpha$$

is a fuzzy T-metric with respect to the same triangular norm T. If T is not strictly monotone, then $\boldsymbol{t}$ is a fuzzy T-metric in the broader sense.

Remark 4.17. *Specially, for* $x = (x_1, \dots, x_n) \in X$, $y = (y_1, \dots, y_n) \in X$ *and* $\alpha \in (0, \infty)$

$$\boldsymbol{t}(x, y, \alpha) = \prod_{i=1}^{n} \boldsymbol{t}_i(x_i, y_i, \alpha), \quad x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in X, \quad \alpha$$

is the fuzzy T*-metric with respect to the "product" aggregation function, which will be used in the experimental section.*

Remark 4.18. *Considering the generalized quasi-arithmetic mean* M *and the appropriate continuous Archimedean t-norms* $\boldsymbol{t}_i$ *from Theorem 3.5, it follows that the conditions from Theorem* 4.14 *are fulfilled, which implies that* $\boldsymbol{t}$*,*

$$\boldsymbol{t}(x, y, \alpha) = \mathsf{M}\left(\boldsymbol{t}_1(x_1, y_1, \alpha), \dots, \boldsymbol{t}_n(x_n, y_n, \alpha)\right), \quad x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in X, \quad \alpha > 0$$

is a fuzzy T*-metric.*

The following two theorems are the generalization of the results presented in [33].

Theorem 4.19. *Let* $h : [0,1] \to [0,1]$ *be a continuous strictly increasing surjective mapping, and let* X *be an arbitrary non-empty set. If* $\boldsymbol{s} : X^2 \times (0, +\infty) \to [0,1)$ *is a fuzzy* S*-(pseudo) metric with respect to the continuous triangular conorm* S*, then function* $\boldsymbol{s}_h : X^2 \times (0, +\infty) \to [0,1)$ *defined by*

$$\boldsymbol{s}_h(x, y, \alpha) = h\left(\boldsymbol{s}(x, y, \alpha)\right), \quad x, y \in X, \quad \alpha > 0$$

is also a fuzzy S*-(pseudo) metric with respect to the continuous triangular conorm* S_h*,*

$$\mathsf{S}_h(a, b) = h \circ \mathsf{S}\left(h^{-1}(a), h^{-1}(b)\right), \quad a, b \in [0,1].$$

If $\boldsymbol{s}$ *is a fuzzy* S*-(pseudo) metric in a broader sense, then* $\boldsymbol{s}_h$ *is also a fuzzy* S*-(pseudo) metric in a broader sense.*

Proof. Clearly, h is a bijective function such that $h(0) = 0$ and $h(1) = 1$. Function S_h is a triangular conorm (see Theorem 2.19), and S_h is a continuous function because it is a composition of continuous functions.

1. From axiom 1. in Definition 4.2, for $\boldsymbol{s}$ holds $\boldsymbol{s}(x, y, \alpha) \in [0,1)$, and because the function $h : [0,1] \to [0,1]$ is strictly increasing, we obtain $0 = h(0) \leq h\left(\boldsymbol{s}(x, y, \alpha)\right) < h(1) = 1$, i.e., $\boldsymbol{s}_h(x, y, \alpha) \in [0,1)$. If $\boldsymbol{s}$ is a fuzzy S-(pseudo) metric in a broader sense, then $\boldsymbol{s}(x, y, \alpha) \in [0,1]$, and by similar consideration that we listed above follows $\boldsymbol{s}_h(x, y, \alpha) \in [0,1]$.

2. Because h is a bijective function and $h(0) = 0$, it can be concluded that

   $\boldsymbol{s}_h(x, y, \alpha) = h\left(\boldsymbol{s}(x, y, \alpha)\right) = 0 \ \Leftrightarrow\ \boldsymbol{s}(x, y, \alpha) = 0 \ \Leftrightarrow\ x = y.$

   If $\boldsymbol{s}$ is a fuzzy S-pseudo metric, then implications from right to left still holds, and $\boldsymbol{s}_h$ is a fuzzy S-pseudo metric.

3. $\boldsymbol{s}_h(x, y, \alpha) = h\left(\boldsymbol{s}(x, y, \alpha)\right) = h\left(\boldsymbol{s}(y, x, \alpha)\right) = \boldsymbol{s}_h(y, x, \alpha).$

4. Using axiom 4. from Definition 4.2, monotonicity of function h, and the definition of function S_h we obtain

   $\mathsf{S}\left(\boldsymbol{s}(x, y, \alpha), \boldsymbol{s}(y, z, \beta)\right) \geq \boldsymbol{s}(x, z, \alpha + \beta)$

   $\Rightarrow\ h\left(\mathsf{S}\left(\boldsymbol{s}(x, y, \alpha), \boldsymbol{s}(y, z, \beta)\right)\right) \geq h\left(\boldsymbol{s}(x, z, \alpha + \beta)\right)$

   $\Rightarrow\ h \circ \mathsf{S}\left(h^{-1} \circ h\left(\boldsymbol{s}\left(x, y, \alpha\right)\right), h^{-1} \circ h\left(\boldsymbol{s}\left(y, z, \beta\right)\right)\right) \geq \boldsymbol{s}_h\left(x, z, \alpha + \beta\right)$

   $\Rightarrow\ h \circ \mathsf{S}\left(h^{-1}\left(\boldsymbol{s}_h\left((x, y, \alpha)\right), h^{-1}\left(\boldsymbol{s}_h\left(y, z, \beta\right)\right)\right)\right) \geq \boldsymbol{s}_h(x, z, \alpha + \beta)$

   $\Rightarrow\ \mathsf{S}_h\left(\boldsymbol{s}_h(x, y, \alpha), \boldsymbol{s}_h(y, z, \beta)\right) \geq \boldsymbol{s}_h(x, z, \alpha + \beta).$

   According to Theorem 2.19, S_h is a t-conorm, so we can conclude that axiom 4. from Definition 4.2 holds for $\boldsymbol{s}_h$ regarding to t-conorm S_h. Because of the continuity of functions h and S, follows the continuity of S_h.

5. From the continuity of functions $h : [0,1] \to [0,1]$ and $\boldsymbol{s}(x, y, \_) : (0, +\infty) \to [0,1]$ it follows that the function $\boldsymbol{s}_h(x, y, \_) : (0, +\infty) \to [0,1]$ is also a continuous function. □

The proof of the following theorem is analogous to the proof of Theorem 4.19.

Theorem 4.20. *Let $h : [0,1] \to [0,1]$ be a continuous strictly increasing surjective mapping, and let X be an arbitrary non-empty set. If $\boldsymbol{t} : X^2 \times (0,+\infty) \to (0,1]$ is the fuzzy T-(pseudo) metric with respect to the continuous triangular norm T, then $\boldsymbol{t}_h : X^2 \times (0,+\infty) \to (0,1]$ defined by*

$$\boldsymbol{t}_h(x,y,\alpha) = h\left(\boldsymbol{t}(x,y,\alpha)\right), \quad x,y \in X, \quad \alpha > 0$$

is also the fuzzy T-(pseudo) metric with respect to the continuous triangular norm T_h,

$$\mathsf{T}_h(a,b) = h \circ \mathsf{T}\left(h^{-1}(a), h^{-1}(b)\right), \quad a,b \in [0,1].$$

If $\boldsymbol{t}$ is a fuzzy T-(pseudo) metric in a broader sense, then $\boldsymbol{t}_h$ is also a fuzzy T-(pseudo) metric in a broader sense.

Remark 4.21. *Theorem* 4 *and Theorem* 5 *from* [33] *are the consequences of Theorem* 4.19 *and Theorem* 4.20*. Namely, if T is continuous Archimedean t-norm (i.e., if S is continuous Archimedean t-conorm), if function h is a corresponding decreasing (i.e., increasing) generator, and if $\boldsymbol{t}$ is stationary fuzzy T-metric (i.e., if $\boldsymbol{s}$ is stationary fuzzy S-metric), then $h \circ \boldsymbol{t}$ (i.e., $h \circ \boldsymbol{s}$) is a standard metric.*

5 LBP-motivated pixel descriptor

In the area of computer vision, Local Binary Patterns (LBPs) are well known descriptors, mostly used for texture analysis problems, object recognition and classification. LBPs and their variants, which include a lot of different binary and non-binary modifications, have shown to provide well-performing features that are suitable for many applications, see [30, 31]. The main idea of LBP descriptors is based on labelling the pixels of an image by thresholding the local neighborhood of each pixel. A local binary code is created by assigning values 1 or 0 to each of the neighboring pixels, depending on the used threshold value. The histogram of binary codes created for all pixels in the observed region represents the feature vector of that region.

Our descriptor is motivated by LBP family and it is proposed in [7]. Let a predefined threshold value, which represents the limit for the similarity of gray-level or color pixels, be marked with $\alpha \in \{0,1,\ldots,255\}$. We define the indicator function for α-similarity between central pixel $p_{i,j}$ and its nearest neighbors $n_k, k \in \{1,\ldots,8\}$. Indicator value 1 denotes that the difference in gray-level or color between the observed pixels is greater than the previously selected α-value. The 0 value indicates the opposite situation, which means that the observed pixels are similar to each other to the α-level.

Definition 5.1. *For an arbitrary pixel $p_{i,j}$, the selected threshold value $\alpha \in \{0,\ldots,255\}$, and each of the neighboring pixels n_k, $k \in \{1,\ldots,8\}$,* α-similarity indicator $I_{i,j;\alpha}(k) \in \{0,1\}$ *is defined by the following equation:*

$$I_{i,j;\alpha}(k) = \begin{cases} 0 & , \quad |\pi_{i,j} - \eta_k| \le \alpha \\ 1 & , \quad |\pi_{i,j} - \eta_k| > \alpha \end{cases}, \quad k \in \{1,\ldots,8\}, \tag{5}$$

where $\pi_{i,j}, \eta_k \in \{0,\ldots,255\}$ are gray-levels of pixels $p_{i,j}$ and n_k respectively. For the pixel $p_{i,j}$ that is on the edge or in the corner of the image and because of that does not have some of the neighboring pixels n_k, we define $I_{i,j;\alpha}(k) = 1$.

The number of zero occurrences in the indicator function $I_{i,j;\alpha}(k)$ for $k \in \{1,\ldots,8\}$, is the number of zeros in the observed binary pattern that is formed for 3×3 neighborhood and it represents the neighboring pixels which differ in the gray-scale level from the central one $p_{i,j}$, for at most α. That number is called *α-indicator counter*, and is calculated with the equation below:

$$\mathsf{IC}_\alpha(p_{i,j}) = \sum_{k=1}^{8} \left(1 - I_{i,j;\alpha}(k)\right) = 8 - \sum_{k=1}^{8} I_{i,j;\alpha}(k). \tag{6}$$

The same consideration presented for gray-scale images can be extended to RGB images, observing the each color channel (red, green and blue) individually, for details see [7]. The formation of one unique descriptor IC_α formed by combining all color channels $\mathsf{IC}_\alpha^{(\ell)}$, $\ell \in \{\mathsf{R},\mathsf{G},\mathsf{B}\}$ into one representation could be implemented with a suitable choice of aggregation function $\mathsf{A}_{[3]}$,

$$\mathsf{IC}_\alpha^{\mathsf{A}_{[3]}}(p_{i,j}) = \mathsf{A}_{[3]}\left(\mathsf{IC}_\alpha^{(\mathsf{R})}(p_{i,j}), \mathsf{IC}_\alpha^{(\mathsf{G})}(p_{i,j}), \mathsf{IC}_\alpha^{(\mathsf{B})}(p_{i,j})\right). \tag{7}$$

Because of specific values of indicator counter function, $\mathsf{IC}_\alpha(p_{i,j}) \in \{0,1,\ldots,8\}$, and the definition of aggregation function $\mathsf{A}_{[3]}$ that is a mapping $\mathsf{A}_{[3]} : [0,1]^3 \to [0,1]$, Equation (7) should be adapted to

$$\mathsf{IC}_\alpha^{\mathsf{A}_{[3]}}(p_{i,j}) = \mathsf{A}_{[3]}\left(\frac{1}{8}\mathsf{IC}_\alpha^{(\mathsf{R})}(p_{i,j}), \frac{1}{8}\mathsf{IC}_\alpha^{(\mathsf{G})}(p_{i,j}), \frac{1}{8}\mathsf{IC}_\alpha^{(\mathsf{B})}(p_{i,j})\right). \tag{8}$$

6 Application in image segmentation

In this section, we present one of the possible applications of fuzzy metrics that are constructed as an aggregation of the initial metrics.

Image segmentation is the process by which a digital image is divided into meaningful, compact regions, or related sets of pixels, see [13, 23]. The goal of segmentation is to simplify or change the image representation so that it is suitable for further analysis, such as shape and pattern recognition, classification, etc. Fuzzy segmentation implies that the observed pixels belong to the object to some extent. This avoids the crisp decisions at the very beginning of the image processing tasks, a large amount of data and information is preserved and can be used later in the process. However, at the end of every fuzzy clustering process, a crisp representation of object is needed.

6.1 Construction of metric used in experiments

The role of a distance function or a metrics is to represent the criterion by which pixels are divided into groups. In this paper, the new metric that is a criterion for dividing pixel into groups is formed by applying aggregation operator on two initial metrics. With an appropriate selection of the aggregation function, and the initial metrics, we could model some of the desired properties of the newly-constructed metric. The features of the constructed metric will depend on both the characteristics of applied aggregation operator and the choice of initial metrics and their properties.

The determination of the difference between pixels is modelled with two criteria, with their difference in color components and the difference between pixel descriptor value, which incorporates in computations the spatial relations of pixel neighborhood. For determining both of the mentioned criteria, we use fuzzy metrics that are listed in Section 4.

As we observe color components and pixel descriptor value, each pixel will be represented as a four-dimensional vector. First three coordinates are the quantity of color, respectively red, green, and blue, and the last coordinate corresponds to the pixel descriptor value, which is assigned to each observed pixel by the considering spatial relations of its neighbors. Each pixel of the image will be denoted with $p_i = (F_i, D_i)$, $i \in \{1, \ldots, n\}$, where $F_i = (F_i^{(\mathsf{R})}, F_i^{(\mathsf{G})}, F_i^{(\mathsf{B})})$ are the normalized color components, D_i is the normalized descriptor value assigned to the corresponding pixel (see equation (8)), and n is the number of observed pixels in the image, $p_i = (F_i, D_i) \in \left\{0, \frac{1}{255}, \ldots, \frac{254}{255}, 1\right\}^3 \times [0, 1]$.

Fuzzy T-metric which in our experiments is used to model the similarity in color components among pixels is denoted with τ and defined by the following equation:

$$\tau(F_i, F_j, K) = \prod_{\ell \in \{\mathsf{R},\mathsf{G},\mathsf{B}\}} \frac{\frac{F_i^{(\ell)} + F_j^{(\ell)}}{2} + K}{\max\{F_i^{(\ell)}, F_j^{(\ell)}\} + K}, \tag{9}$$

where $K > 0$ is the parameter chosen by the user. This is indeed a fuzzy T-metric that follows from Example 4.8 and Remark 4.17. Fuzzy T-metric which in our experiments is used to model the similarity between pixels neighborhoods that are described by the descriptor, IC_α^f, is denoted with t and defined by the equation below (see Example 4.9):

$$\boldsymbol{t}(D_i, D_j, k) = \frac{k}{k + |D_i - D_j|}, \tag{10}$$

where $k > 0$ is the parameter chosen by the user.

The most important part of our segmentation proposal is a selection of the function that should model the summary of the difference between the observed pixels:

$$\boldsymbol{c}\left((F_i, D_i), (F_j, D_j), K, k\right) = \mathsf{A}\left(\tau(F_i, F_j, K), \boldsymbol{t}(D_i, D_j, k)\right). \tag{11}$$

This function depends on two initial fuzzy metrics τ and $\boldsymbol{t}$, and the appropriate function A used for their aggregation. The new constructed function $\boldsymbol{c}$ is also a fuzzy T-metric, which follows from Theorem 4.14. Good selection of fuzzy metrics τ and $\boldsymbol{t}$ and aggregation function A is of great importance for the segmentation success.

Besides the color components, α-indicator counter descriptor also participates in the formation of the metric. Beside the formulation of aggregation between the two observed metrics, τ and $\boldsymbol{t}$, we need to formulate an adequate aggregation function f for the pixel descriptor, to aggregate its values obtained for each color component individually, as we indicated in Equation (8). We consider that the good choice for function f will be an arithmetic mean aggregation operator. Therefore, the descriptor IC_α^f is defined with the equation below:

$$\mathsf{IC}_\alpha^{\mathsf{AM}}(p_{i,j}) = \frac{1}{3}\left(\frac{1}{8}\mathsf{IC}_\alpha^{(\mathsf{R})}(p_{i,j}) + \frac{1}{8}\mathsf{IC}_\alpha^{(\mathsf{G})}(p_{i,j}) + \frac{1}{8}\mathsf{IC}_\alpha^{(\mathsf{B})}(p_{i,j})\right). \tag{12}$$

Note that with the equation above we normalize the value of the observed descriptor, i.e., $\mathsf{IC}_{\alpha}^{\mathsf{AM}}(p_{i,j}) \in [0,1]$.

Based on the results from Theorem 4.20, and by selecting a function $h(x) = x^{\omega}$, it follows that functions $h(\tau)$ and $h(\boldsymbol{t})$ are fuzzy T-metrics such as τ and $\boldsymbol{t}$. By using aggregation product operator, based on Theorem 4.14 we claim that

$$\boldsymbol{c}_{\omega} = \tau^{\omega} \cdot \boldsymbol{t}^{\omega}, \tag{13}$$

is a fuzzy T-metric with respect to "product" t-norm, i.e., $\mathsf{T}(x,y) = x \cdot y$. This metric, $\boldsymbol{c}_{\omega}$, is used in FCM algorithm for measuring the differences between two observed image pixels.

The implementation of the experimental results imply that we have to determine the values of the parameters k and K that appear in the metrics, as well as the power degree ω in the aggregation function and α threshold value in pixel descriptor formation. The quality of the performed segmentation is conditioned by a good selection of these parameters. The presentation of segmentation results is done by comparing the results obtained with the newly proposed metric $\boldsymbol{c}_{\omega}$ and the ones achieved by using the normalized Euclidean metric, see [7, 28], $d_E : \overline{P}^2 \to [0,1]$ defined by

$$d_E(p_i, p_j) = \frac{1}{255\sqrt{3}} \sqrt{(R_i - R_j)^2 + (G_i - G_j)^2 + (B_i - B_j)^2}, \tag{14}$$

where pixels are triplets of RGB colors $p_i = (R_i, G_i, B_i), p_j = (R_j, G_j, B_j) \in \overline{P} = \{0, \ldots, 255\}^3$.

6.2 Experiments

There are several algorithms which are suitable for image segmentation tasks, see [2, 13, 23]. Our experiments used Fuzzy c-Means Clustering Algorithm (FCM), see [1, 2, 21, 23], which is widely accepted in image segmentation practice. It does not belong to the class of fast segmentation algorithms but gives very good results regarding the compactness of the obtained segments.

By using FCM algorithm, we tested our proposal for new metric in segmentation tasks and confirmed its power through very good results. Beside the observed image, the input parameters of FCM algorithm are distance function or metric $\boldsymbol{c} : P^2 \to [0,1]$ where P is a set of pixels, number of clusters that are going to be acquired by segmentation ($c = 4$ is selected), and weight coefficient m ($m = 2.0$ is selected).

Beside the processed image, as the output parameter of the segmentation algorithm, we observe the value of performance index PI that represents the main measure of the quality of obtained segmentation. It measures the compactness of the cluster data and the lower value of this parameter indicates the stronger grouping of pixels, i.e., more compact clusters.

Firstly, we observe one color image, see Figure 1, which is 481×321 pixels in size. The image is taken from the *De-*

Figure 1: Original image.

partment of Electrical Engineering and Computer Sciences at University of California, Berkeley, segmentation dataset-training images, see [24] and `https://www\discretionary{-}{}{}2.eecs.berkeley.edu/Research/Projects/CS/vision/bsds/`.

In the first test we used parameter value $\alpha = 20$, suggested in [7]. The values for k and K are intuitively set in a few initial trying. By changing the parameter value $\omega \in \{1, 3, 5, 7\}$ in aggregation function, different metrics are formed, $\boldsymbol{c}_{\omega} = \tau^{\omega} \cdot \boldsymbol{t}^{\omega}$. Looking at the results, we can observe that increasing the power degree ω is beneficial for segmentation results. The results of the performed experiments are shown in Table 1. It can be concluded that most of the results performed with the new metric are much better than those obtained with Euclidean metric d_E. Some of the segmented images are shown in Figure 2.

The second test is done on the same image as the first one, see Figure 1. Through this test we observe the dependence of segmentation success on the parameter appearing in the α-indicator counter descriptor. The dependence of segmentation on different α values is discussed. As the mentioned descriptor describes the relation between pixels from

	$\boldsymbol{c}_1$	$\boldsymbol{c}_3$	$\boldsymbol{c}_5$	$\boldsymbol{c}_7$	d_E
PI	8589.1	802.6006	0.9440	0.0206	1424.5

Table 1: Results of segmentation performed on Figure 1 with $K = 10$, $k = 1$ and $\alpha = 20$ in $\boldsymbol{c}_\omega$.

(a) Segmentation with d_E. (b) Segmentation with $\boldsymbol{c}_3$. (c) Segmentation with $\boldsymbol{c}_7$.

Figure 2: Segmented images.

the neighborhood, we find natural that it should be dependent on the content of the observed image or neighborhood. The results performed in [6] confirmed that rather than setting α parameter as a fixed value, it should be dependent on the standard deviation s of the observed image. We set the parameter α to be the constant multiplied by the sample standard deviation s of the image that will be processed by the FCM segmentation algorithm. This test is performed with fuzzy metric $\boldsymbol{c}_3 = \tau^3 \cdot \boldsymbol{t}^3$ and fixed values for parameters $K = 10$ and $k = 1$ in aggregated metrics τ and $\boldsymbol{t}$ respectively, while segmentation results were observed through the changes of α parameter. The relation between this parameter and performance index is illustrated in Figure 3.

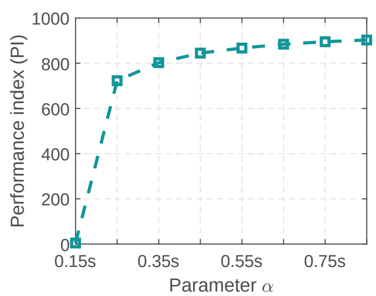

Figure 3: The dependence of segmentation performance on the values of parameter α.

It can be seen in Figure 3 that the best segmentation result PI $= 4.71$, is obtained for $\alpha = 0.15s$, and there is a big gap to the next result. For the threshold value $\alpha = 0.35s = 0.35 \cdot 57.8344 \approx 20$ we reach the PI $= 802.6006$ value from Table 1 indicated for $\boldsymbol{c}_3$. The results which are shown on Figure 3 about the dependence of PI on α value could be discussed through the definition of α-indicator counter descriptor. We observe the closest 3×3 neighborhood, and if we allow the very large value for parameter α this descriptor will not be able to manifest its nature. Because values that are placed in the closest neighborhood are not so different in gray or color levels (except the borders of objects), the large value of parameter α will indicate a lot of codes full of zeros, so a lot of the same descriptor value for different neighborhoods, which means that discriminative power of the descriptor will be lost. With good selection of α, we catch the most of spatial relations within the observed neighborhood. We believe that dependence of parameter on standard deviation of the image is a good choice, because standard deviation shows how much variation from the average value there exist in the image.

The third test confirmed the results performed in the previous two tests. We observed one more color image, 481×321 pixels in size, which is shown in Figure 4. This image is also taken from the *Department of Electrical Engineering and Computer Sciences at University of California, Berkeley, segmentation dataset-training images*, see [24] and `https://www\discretionary{-}{}{}2.eecs.berkeley.edu/Rese\discretionary{-}{}{}arch/Projects/CS/vision/bsds/`. The results from Table 2 confirmed that an increase in degree ω is beneficial for the results, i.e., a better performance index is obtained. Also, we again verify that the dependence of parameter α value on the

Figure 4: Original image.

	$\boldsymbol{c}_1$	$\boldsymbol{c}_3$	$\boldsymbol{c}_5$	$\boldsymbol{c}_7$	d_E	$\boldsymbol{c}_3$	$\boldsymbol{c}_3$	$\boldsymbol{c}_3$
α	20	20	20	20	-	$0.15s$	$0.35s$	$0.55s$
PI	11365	34.6579	0.2843	0.0300	816.0351	26.6366	33.0776	1841.7

Table 2: Results of segmentation performed on Figure 4 with $K = 10$, $k = 1$ and α in $\boldsymbol{c}_\omega$.

standard deviation of the observed image strongly affects segmentation results.

The presented experiments show the results of the application of constructed fuzzy metrics in image segmentation, which are compared with usage of Euclidean metric. Some of the obtained results are also significantly better in comparison to the results of image segmentation obtained in our previous investigation presented in [7, 27, 28], where instead of fuzzy metric constructed in Subsection 6.1 we used distance functions constructed by applying aggregation functions on the sequence of initial distance functions.

7 Conclusions

The results presented in this paper are about the construction of a new fuzzy metrics by applying aggregation function on a set of initial fuzzy metrics, which gives us the opportunity to construct new fuzzy metrics with some desired properties, depending on the application. The results that are presented in this paper leave room for further analysis and investigations about the properties of constructed fuzzy metrics. The contribution of the paper is reflected in several new theoretical results and illustrations of their application in image segmentation tasks, through very good experimental results in comparison with image segmentation using Euclidean metric. We believe that the construction of fuzzy metrics presented in this paper can find many other applications beside the image processing tasks, by highlighting data classification and decision-making processes. In attractive topics of image processing fields, we suggest further investigations of applications in image denoising, processing of image textures, shape recognition, etc. The presented methodology of the construction of new fuzzy metrics is generally applicable wherever is constructing a fuzzy metric joint decision-making criteria based on aggregated particular criterion described by initial fuzzy metrics.

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