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Original paper

Fuzzy betweenness spaces on continuous lattices

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Abstract

The notion of betweenness spaces is very important in convex structure theory. In this paper, it is generalized to continuous lattices. Then its some characterizations and properties are investigated. Finally the categorical relationship between fuzzy generalized convex spaces and fuzzy betweenness spaces is discussed by using a special antitone Galois connection.

Keywords: Fuzzy convex space, fuzzy betweenness space, generalized convex space, Galois connection, continuous lattice

1 Introduction

Convex sets widely exist in various research areas, such as graphs, metric spaces, matroids, lattices and so on. With the deepening of research, the properties of convex sets are abstracted and an axiomatic convex structure is established. A family of subsets, denoted by $\mathcal{C}$, is called a convexity [26] provided that $\mathcal{C}$ contains the empty set and the universal set, and it is closed under intersections and directed joins (equivalently, totally ordered joins). The relationship between convex structures and matroids is very close. Matroids can be regarded as convex structures with exchange law and convex sets are exactly the flat sets in matroids. Polyhedrons are typical convex sets. And the betweenness is to describe whether a singleton belongs to a polyhedron.

With the development of fuzzy mathematics, many structures have been combined with fuzzy set theory, such as fuzzy convergence [1, 3, 15], fuzzy topology [25, 38], fuzzy order [35, 37] and so on. Similarly, the combination of convex structures and fuzzy set theory has also been greatly developed. There are generally the following research directions: M-fuzzifying convexities [7, 10, 17, 21, 23, 28, 29, 32, 33, 36], L-convexities [4, 9, 12, 13, 14, 16, 20, 22, 34] and (L, M)-fuzzy convexities [6, 11, 24, 31, 39]. De Morgan algebra plays an important role in the study of relationships between fuzzy convexities and other structures. But De Morgan algebra is a very strong condition. In this paper, we will use LRG-Galois connections combined with the wedge-below relation to discuss the relationship between different structures. In addition, Liu and Shi [8] refer to Wang's method [27] and directly consider the convexities on a complete lattice and propose the notion of generalized convexities. In this method, the morphism between two convex structures is a left adjoint of an order-homomorphism, which is called generalized order-homomorphism [27]. On this basis, Wu [30] proposes fuzzy generalized convexity with respect to a mapping defined on a completely contributive lattice. All these show the feasibility of constructing convexities on lattices.

This paper is organized as follows. In Section 2, we recall some necessary concepts of lattices and fuzzy generalized convexities. In Section 3, we will present the notion of fuzzy betweenness spaces on continuous lattices. In Section 4, by using a special antitone Galois connection, we will consider the categorical relationship between fuzzy betweenness spaces and fuzzy generalized convex spaces.

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2 Preliminaries

2.1 Completely distributive lattice and continuous lattice

In this subsection, we will present some basic concepts of lattices. The pair $(L, \leq)$ is called a lattice [5], provided that for every two elements $a, b \in L$, there exists a supremum $a \vee b$ and an infimum $a \wedge b$. A lattice $(L, \leq)$ is complete if for any subset A of L, the supremum $\bigvee A$ and infimum $\bigwedge A$ exist. The largest element and the smallest element in L are denoted by 1 and 0, respectively. For a partially ordered set P, $S \subseteq P$ is directed provided that for any $a, b \in S$, there exists $c \in S$ such that $c \geq a$ and $c \geq b$.

The binary relation way below $\ll$ on L is defined as follows: for $a, b \in L$, $a \ll b$ if and only if for all directed subset $D \subseteq L$, $b \leq \bigvee D$ implies that there exists $d \in D$ such that $a \leq d$ [2]. L is a continuous lattice [2] if and only if for each $a \in L$, $a = \bigvee\{b \in L \mid b \ll a\}$. If $a \ll b$, then there exists $c \in L$ such that $a \ll c \ll b$. If $a \leq b \ll c \leq d$, then $a \ll d$.

The binary relation wedge below $\lhd$ on L is defined as follows: for $a, b \in L$, $a \lhd b$ if and only if for any $D \subseteq L$, $b \leq \bigvee D$ implies that there exists $d \in D$ such that $a \leq d$ [27]. $\beta(b) = \{a \in L \mid a \lhd b\}$ is called the greatest minimal family [27] of b. Moreover, the relation $\lhd^{op}$ on L is defined as follows: for $a, b \in L$, $b \lhd^{op} a$ if and only if for any $D \subseteq L$, $b \geq \bigwedge D$ implies that there exists $d \in D$ such that $a \geq d$. $\alpha(b) = \{a \in L \mid b \lhd^{op} a\}$ is called the greatest maximal family [27] of b. In [27], it is proved that L is completely distributive lattice [27] if and only if for each $b \in L$, $\alpha(b)$ and $\beta(b)$ exist and $b = \bigvee \beta(b) = \bigwedge \alpha(b)$. If $a \lhd b$, then there exists $c \in L$ such that $a \lhd c \lhd b$. If $a \leq b \lhd c \leq d$, then $a \lhd d$. So does $\lhd^{op}$.

Let L_1 and L_2 be two complete lattices. If a pair of mappings $f : L_1 \to L_2$ and $g : L_2 \to L_1$ ($f : L_1 \rightleftarrows L_2 : g$ for short) satisfies $f(a) \geq b$ if and only if $g(b) \geq a$, then we call (f, g) an antitone Galois connection.

Lemma 2.1. [27] *Let L be a completely distributive lattice and $\{a_i \mid i \in J\} \subseteq L$. Then*

(1) $\beta\left(\bigvee_{i\in J} a_i\right) = \bigcup_{i\in J} \beta(a_i)$, *i.e., β is a union-preserving mapping.*

(2) $\alpha\left(\bigwedge_{i\in J} a_i\right) = \bigcup_{i\in \omega} \alpha(a_i)$, *i.e., α is a $\wedge - \cup$ mapping.*

Definition 2.2. [20] *Let $A \in L^X$, $a \in L$. Define*

$$A_{[a]} = \{x \in X \mid A(x) \geq a\}, \qquad A^{[a]} = \{x \in X \mid a \notin \alpha(A(x))\}.$$

2.2 Fuzzy generalized convexity

Liu and Shi [8] proposed convex structures on a complete lattice, called generalized convex structure. Wu [30] introduced the fuzzy generalized convex structure as follows:

Definition 2.3. [30] *Let L and M be completely distributive lattices. A mapping $\mathcal{C} : L \to M$ is called a fuzzy generalized convexity on L and the pair $(L, \mathcal{C})$ is called a fuzzy generalized convex space if:*

(*FGC*1) $\mathcal{C}(0_L) = \mathcal{C}(1_L) = 1_M$;

(*FGC*2) *For any subset $\{a_i \mid i \in J\} \subseteq L$, $\mathcal{C}\left(\bigwedge_{i\in J} a_i\right) \geq \bigwedge_{i\in J} \mathcal{C}(a_i)$;*

(*FGC*3) *For any directed subset $\{a_i \mid i \in J\} \subseteq L$, $\mathcal{C}\left(\bigvee_{i\in J} a_i\right) \geq \bigwedge_{i\in J} \mathcal{C}(a_i)$.*

Remark 2.4. *When $M = \{0, 1\}$, a fuzzy general convex structure become a general convex structure which satisfies (GC1), (GC2) and (GC3) in* [8].

The next definition is proposed by Wang [27].

Definition 2.5. [27] *Let $(I, \leq)$ and $(L, \leq)$ be complete lattices. A mapping $h : I \to L$ is said to be a generalized order-homomorphism if there exists a join-preserving mapping $h^{\dashv} : L \to I$ such that $(h, h^{\dashv})$ is a Galois adjunction.*

Proposition 2.6. [27] *Let $h : I \to L$ be a generalized order-homomorphism. Then*

(1) *h is join-preserving.*

(2) *$h^{\dashv}h(a) \geq a$ for each $a \in I$.*

(3) *$hh^{\dashv}(b) \leq b$ for each $b \in L$.*

(4) *$h(a) = \bigwedge\{b \in L \mid h^{\dashv}(b) \geq a\}$ for each $a \in I$.*

(5) *$h^{\dashv}(b) = \bigvee\{a \in I \mid h(a) \leq b\}$ for each $b \in L$.*

Let **FGC** be the category of all fuzzy generalized convex spaces and all generalized order-homomorphism $h : I \to L$ satisfying $\mathcal{C}_I(h^{\dashv}(y)) \geq \mathcal{C}_L(y)$ (which is called a fuzzy generalized convexity-preserving mapping).

3 Fuzzy betweenness space on continuous lattice

In [8] and [30], requirements for the lattice L are different. And Shen et al. [19] point out the interesting connection between convex spaces and domain theory. Comprehensive consideration, L prefers to be a continuous lattice in this paper. For $a \in L$, a is called a compact element if $a \ll a$. The collection of compact elements of L is denoted by $F(L)$. And M is a completely distributive lattice. Next we propose the notion of fuzzy betweenness spaces on continuous lattices.

Definition 3.1. *Suppose L is a continuous lattice and M is a completely distributive lattice. A mapping $\mathcal{B} : L\times L \to M$ is called a fuzzy betweenness on L provided:*

($FB1$) *$\mathcal{B}(x, 0) = 0$ for all $x \in L - \{0\}$;*

($FB2$) *For all $x \in L$, $\mathcal{B}(x, x) = 1$;*

($FB3$) *For all $x, y, z \in L$, $\mathcal{B}(x, z) \geq \mathcal{B}(x, y) \wedge \mathcal{B}(y, z)$;*

($FB4$) *For all $z \in L$ and $\{x_i \mid i \in I\} \subseteq L$, $\mathcal{B}\left(\bigvee_{i\in J} x_i, z\right) = \bigwedge_{i\in J} \mathcal{B}(x_i, z)$;*

($FB5$) *For all $x, y, z \in L$, $\mathcal{B}(x, y) = \bigwedge_{u\ll x} \bigvee_{v\ll y} \mathcal{B}(u, v)$.*

For a fuzzy betweenness on L, the pair $(L, \mathcal{B})$ is called a fuzzy betweenness space.

Remark 3.2. *In (FB5), assume $x, y, z \in L$. For each u with $u \ll x$, the set $\{\mathcal{B}(u, v) \mid v \ll y\}$ is directed. This is because the subset $\{v \in L \mid v \ll y\}$ is directed, and*

$$\mathcal{B}(x, y) = \bigwedge_{u\ll x} \bigvee_{v\ll y} \mathcal{B}(u, v),$$

implies

($FB6$) *If $u, s, t \in L$ with $s \leq t$, then $\mathcal{B}(u, s) \leq \mathcal{B}(u, t)$.*

By (FB2) and (FB6), we know that for $(x, y) \in L \times L$ with $x \leq y$, $1 = \mathcal{B}(x, x) \leq \mathcal{B}(x, y)$. Let X be a set, the power set 2^X can be regarded as a continuous lattice. And for each $A \in 2^X$, A is a compact element if and only if A is a finite set. For each subset B, it is a join of some compact elements in 2^A. This does not hold in general continuous lattices. Thus (FB5) is so complicated.

Proposition 3.3. *Suppose L is a continuous lattice and M is a completely distributive lattice. Let $\mathcal{B} : L \times L \to M$ be a mapping. Then (FB5) implies*

(FB7) *For each $x, y \in L \times L$, $\mathcal{B}(x, y) = \bigwedge_{u\ll x} \mathcal{B}(u, y)$.*

Proof. By (FB6),

$$\bigwedge_{u\ll x}\mathcal{B}(u,y)\geq\bigwedge_{u\ll x}\bigvee_{v\ll y}\mathcal{B}(u,v)=\mathcal{B}(x,y).$$

Conversely, take each $a\in M$ such that

$$a\lhd\bigwedge_{u\ll x}\mathcal{B}(u,y)=\bigwedge_{u\ll x}\bigwedge_{w\ll u}\bigvee_{v\ll y}\mathcal{B}(w,v).$$

Then for each $u\ll x$, each $w\ll u$, there exists $v_{u,w}\ll y$, such that $a\lhd\mathcal{B}(w,v_{u,w})$. Since for each $w_1\ll x$, there exists w_2 with $w_1\ll w_2\ll x$. Furthermore, there exists $v_{w_1,w_2}\ll y$ such that $a\lhd\mathcal{B}(w_1,v_{w_1,w_2})$. Thus $a\leq\bigwedge_{w\ll x}\bigvee_{v\ll y}\mathcal{B}(w,v)=\mathcal{B}(x,y)$. By the arbitrariness of a,

$$\bigwedge_{u\ll x}\mathcal{B}(u,y)\leq\mathcal{B}(x,y).$$

Thus we obtain $\bigwedge_{u\ll x}\mathcal{B}(u,y)=\mathcal{B}(x,y)$, as desired. □

Remark 3.4. *By Proposition 3.3, we have:*

($FB8$) *If $x,y\in L$ with $x\leq y$, then $\mathcal{B}(x,z)\geq\mathcal{B}(y,z)$.*

The next corollary comes from Proposition 3.3 and (FB3) directly.

Corollary 3.5. *Suppose L is a continuous lattice and M is a completely distributive lattice. Let $\mathcal{B}:L\times L\to M$ be a mapping. Then (FB3) and (FB5) implies*

(FB9) *For each $(x,z)\in L\times L$, $\mathcal{B}(x,z)\geq\mathcal{B}(x,y)\wedge\bigwedge_{v\ll y}\mathcal{B}(v,z)$.*

Remark 3.6. *Given a mapping $\mathcal{B}:L\times L\to M$, then by Proposition 3.3, we know that (FB9) and (FB5) imply (FB3). Hence in Definition 3.1, we can use (FB9) instead of (FB3).*

Proposition 3.7. *Suppose L is a continuous lattice and M is a completely distributive lattice. Let $\mathcal{B}:L\times L\to M$ be a fuzzy betweenness on L, then*

(FB10) $\mathcal{B}(x,z)=\bigwedge_{x\nleq y\geq z}\left(\bigvee_{p\nleq y\geq q}\mathcal{B}(p,q)\right).$

Proof. It is clear that the left-hand side is smaller than the right-hand side. It suffices to show that the left-hand side is greater. Take each $a\in L$ such that

$$a\lhd\bigwedge_{x\nleq y\geq z}\left(\bigvee_{p\nleq y\geq q}\mathcal{B}(p,q)\right).$$

Then for each $y\in L$ with $x\nleq y\geq z$, there exists p and q in L with $p\nleq y\geq q$ such that $a\lhd\mathcal{B}(p,q)$. Suppose $a\nleq\mathcal{B}(x,z)$. Let $Y_{az}=\bigvee\{v\in L\mid a\leq\mathcal{B}(v,z)\}$. By (FB2) $\mathcal{B}(z,z)=1$, thus $z\leq Y_{az}$. Then we have $x\nleq Y_{az}$, otherwise $x\leq Y_{az}$. By (FB7) and (FB4),

$$\mathcal{B}(x,z)\geq\mathcal{B}(Y_{az},z)=\bigwedge_{a\leq\mathcal{B}(v,z)}\mathcal{B}(v,z)\geq a,$$

this is a contradiction. Hence $x\nleq Y_{az}\geq z$. Furthermore, there exist $p,q\in L$ with $p\nleq Y_{az}\geq q$ such that $a\lhd\mathcal{B}(p,q)$. By (FB7) and (FB4),

$$\mathcal{B}(q,z)\geq\mathcal{B}(Y_{az},z)=\bigwedge_{a\leq\mathcal{B}(v,z)}\mathcal{B}(v,z)\geq a$$

It follows from (FB3) that $\mathcal{B}(p,z)\geq\mathcal{B}(p,q)\wedge\mathcal{B}(q,z)\geq a$, which implies $p\leq Y_{az}$. This is a contradiction. Hence the assumption that $a\nleq\mathcal{B}(x,z)$ fails to hold. Thus we see that

$$a\lhd\bigwedge_{x\nleq y\geq z}\left(\bigvee_{p\nleq y\geq q}\mathcal{B}(p,q)\right),$$

implies $a \le \mathcal{B}(x,z)$. By the arbitrariness of a, we obtain $\mathcal{B}(x,z) \ge \bigwedge_{x \not\le y \ge z} \left(\bigvee_{p \not\le y \ge q} \mathcal{B}(p,q) \right)$. It follows that

$$\mathcal{B}(x,z) = \bigwedge_{x \not\le y \ge z} \left(\bigvee_{p \not\le y \ge q} \mathcal{B}(p,q) \right).$$

□

Proposition 3.8. *Suppose L is a continuous lattice and M is a completely distributive lattice. Let $\mathcal{B} : L \times L \to M$ be a mapping. Then (FB5) and (FB10) imply (FB4).*

Proof. By (FB8) in Remark 3.4, it suffices to show

$$a \in \alpha \left(\mathcal{B} \left(\bigvee_{i \in J} x_i, z \right) \right) \Rightarrow a \in \alpha \left(\bigwedge_{i \in J} \mathcal{B}(x_i, z) \right).$$

By (FB10),

$$a \in \alpha \left(\mathcal{B} \left(\bigvee_{i \in J} x_i, z \right) \right) \Rightarrow \bigwedge_{\bigvee_{i \in J} x_i \not\le y \ge z} \left(\bigvee_{u \not\le y \ge v} \mathcal{B}(u,v) \right) \lhd^{op} a,$$

it follows that there exists $y \in L$ with $\bigvee_{i \in J} x_i \not\le y \ge z$, for each $u, v \in L$ with $u \not\le y \ge v$, such that $\mathcal{B}(u,v) \lhd^{op} a$. Since $\bigvee_{i \in J} x_i \not\le y$ implies that there exists $i_0 \in J$ such that $x_{i_0} \not\le y$, we can know there exists $i \in J$ and $y \in L$ with $x_i \not\le y \ge z$, for each $u, v \in L$ with $u \not\le y \ge v$, such that $\mathcal{B}(u,v) \lhd^{op} a$. This implies that

$$\bigwedge_{i \in J} \mathcal{B}(x_i, z) = \bigwedge_{i \in J} \bigwedge_{x_i \not\le y \ge z} \bigvee_{u \not\le y \ge v} \mathcal{B}(u,v) \lhd^{op} a$$

It follows from

$$\alpha \left(\mathcal{B} \left(\bigvee_{i \in J} x_i, z \right) \right) \subseteq \alpha \left(\bigwedge_{i \in J} \mathcal{B}(x_i, z) \right),$$

that

$$\mathcal{B} \left(\bigvee_{i \in J} x_i, z \right) \ge \bigwedge_{i \in J} \mathcal{B}(x_i, z),$$

as desired. □

Remark 3.9. *By Proposition 3.7 and Proposition 3.8, we can use (FB10) instead of (FB4) in Definition 3.1.*

Example 3.10. *In Definition 3.1, let X be a set, $(L, \le) = (2^X, \subseteq)$ and $M = \{0,1\}$, Restricting $\mathcal{B}$ on $L \times F(L) = 2^X \times 2^X_{fin}$ and $\mathcal{B}$ satisfies (FB1)-(FB5). $\mathcal{B}_h = \{(x,F) \in X \times 2^X_{fin} \mid \mathcal{B}(A,F) = 1$ and $|A| = 1\}$ is a betweenness relation on $X \times 2^X_{fin}$. Conversely, if $\mathcal{B}_h$ is a betweenness on X, then $\mathcal{B} = \{(A,F) \in 2^X \times 2^X_{fin} \mid \forall x \in A, (x,F) \in \mathcal{B}_h.\}$ is a fuzzy betweenness restricted on $2^X \times 2^X_{fin}$.*

Example 3.11. *Let L be a lattice as shown in Fig. 1 and M be a lattice as shown in Fig. 2. Define $\mathcal{B} : L \times L \to M$ as follows:*

$$\mathcal{B}(x,y) = \begin{cases} 0 & x \neq 0 \text{ and } y = 0; \\ 1 & x \le y; \\ s & (x,y) = (a,b); \\ t & (x,y) = (b,a); \\ s & (x,y) = (1,b); \\ t & (x,y) = (1,a). \end{cases}$$

Then $\mathcal{B}$ is a fuzzy betweenness on L.

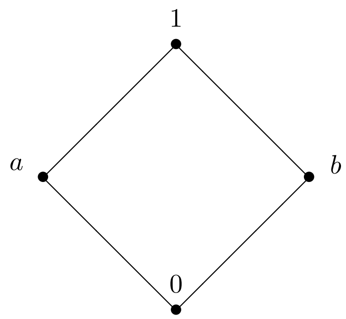

Fig.1

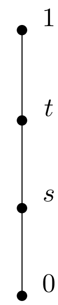

Fig.2

Assume X is a set and L is a continuous lattice, then L^X is a continuous lattice. Fuzzy betweenness on continuous lattice can be regarded as a generalization of $(L, M)-$betweenness. By the generalized order-homomorphism, we can consider the category of fuzzy betweenness spaces on continuous lattices. Before that, we list two related propositions.

Proposition 3.12. *Let $h : I \to L$ be a generalized order-homomorphism, where I and L are continuous lattices. If $x \ll z$, then $h(x) \ll h(z)$.*

Proof. This is a corollary of Proposition 2.6. For each directed subset $\{w_k \mid k \in K\} \subseteq L$, suppose $h(z) \leq \bigvee_{k\in K} w_k$. Then since

$$x \ll z \leq h^{\dashv}h(z) \leq h^{\dashv}\left(\bigvee_{k\in K} w_k\right) \leq \bigvee_{k\in K} h^{\dashv}(w_k),$$

there exists $k_0 \in K$ such that $x \leq h^{\dashv}(w_{k_0})$. Then $h(x) \leq hh^{\dashv}(w_{k_0}) \leq w_{k_0}$, which implies that $h(x) \ll h(z)$. □

Proposition 3.13. *Assume I and L are continuous lattices. Let $h : I \to L$ be a generalized order-homomorphism, $(x, z) \in I \times I$, and $y \in L$. Then $h(x) \nleq y \geq h(z)$ if and only if $x \nleq h^{\dashv}(y) \geq z$.*

Proof. This is also a corollary of Proposition 2.6. On one hand, $y \geq h(z)$ implies $h^{\dashv}(y) \geq h^{\dashv}h(z) \geq z$. $h^{\dashv}(y) \geq z$ implies $y \geq hh^{\dashv}(y) \geq h(z)$. On the other hand, by the converse-negative proposition, $y \ngeq h(z)$ if and only if $h^{\dashv}(y) \ngeq z$. □

Now we can define the category of fuzzy betweenness spaces on continuous lattices.

Definition 3.14. *Given two fuzzy betweenness spaces on continuous lattices $(I, \mathcal{B}_I)$ and $(L, \mathcal{B}_L)$, a generalized order-homomorphism $h : I \to L$ is a fuzzy betweenness-preserving mapping provided:*

$$\text{for all } (x, z) \in I \times I,\ \mathcal{B}_I(x, z) \leq \mathcal{B}_L(h(x), h(z)).$$

FBS denotes the category of all fuzzy betweenness spaces on continuous lattices and all fuzzy betweenness-preserving generalized order-homomorphisms.

4 Relationship between fuzzy generalized convex spaces and fuzzy betweenness spaces on continuous lattices

Before studying the relationship between fuzzy generalized convex and fuzzy betweenness spaces on continuous lattices without De Morgan algebra, we consider a special antitone Galois connection.

4.1 LRG-Galois connection

Definition 4.1. *Let $(P, \leq), (Q, \leq)$ be two completely distributive lattices. Suppose $f : P \to Q$ and $g : Q \to P$ are two mappings. The pair (f, g) is called an LRG adjunction if it satisfies the following condition:*

(LRG) *For each $a \in P$ and $b \in Q$, $f(a) \lhd^{op} b \Leftrightarrow g(b) \lhd a$.*

Lemma 4.2. *Let $(P, \leq), (Q, \leq)$ be two completely distributive lattices. Suppose a pair of mappings $f : P \rightleftarrows Q : g$ satisfies (LRG). Then f is an antitone mapping, and for any subset $A \subseteq P$, $f(\bigvee A) = \bigwedge f(A)$ (so-called $\bigvee - \bigwedge$ mapping).*

Proof. (1) Assume $a, b \in P$ with $a \leq b$. Take each $x \in Q$ such that $f(a) \lhd^{op} x$. By (LRG), we have $g(x) \lhd a \leq b$. Then it follows from (LRG) that $f(b) \lhd^{op} x$. This implies

$$f(a) = \bigwedge \{x \in Q \mid f(a) \lhd^{op} x\} \geq \bigwedge \{x \in Q \mid f(b) \lhd^{op} x\} = f(b).$$

(2) By (1), it follows that $f\left(\bigvee\limits_{a \in A} a\right) \leq \bigwedge\limits_{a \in A} f(a)$. In order to show the conversely, take each $x \in Q$ such that

$$f\left(\bigvee_{a \in A} a\right) \lhd^{op} x.$$

By (LRG), we have $g(x) \lhd \bigvee\limits_{a \in A} a$. Then there exists $a_0 \in A$ such that $g(x) \lhd a_0$, which is equivalent to $f(a_0) \lhd^{op} x$. This implies $\bigwedge\limits_{a \in A} f(a) \lhd^{op} x$. By the arbitrariness of x, we obtain

$$\bigwedge_{a \in A} f(a) \leq f\left(\bigvee_{a \in A} a\right),$$

as desired. This shows f is a $\bigvee - \bigwedge$ mapping. □

Corollary 4.3. *Let $(P, \leq), (Q, \leq)$ be two completely distributive lattices. Suppose a pair of mappings $f : P \rightleftarrows Q : g$ satisfies (LRG). Then $f(0) = 1$.*

Definition 4.4. *Suppose $(P, \leq), (Q, \leq)$ are two completely distributive lattices, and a pair of mappings $f : P \rightleftarrows Q : g$ forms an antitone Galois connection with $f(1) = 0$. We call (f, g) an LRG-Galois connection if it satisfies (LRG).*

Remark 4.5. *Let $(P, \leq), (Q, \leq)$ be two completely distributive lattices. If mappings $f : P \rightleftarrows Q : g$ form an antitone Galois connection. Then $fg \geq id_Q$, $gf \geq id_P$.*

Example 4.6. (1) *Let $P = Q = [0, 1]$. Assume $f(x) = g(x) = 1 - x$ (resp., $f(x) = 1 - x^3$ and $g(x) = \sqrt[3]{1 - x}$), then (f, g) is an LRG-Galois connection. Since $x \lhd y$ ($x \lhd^{op} y$) if and only if $x < y$.*

(2) *Any antitone Galois connection (f, g) on $[0, 1]$ is actually an LRG-Galois connection.*

(3) *Let $P = \{1, 2, 3, 4\}$ with $4 \leq 3 \leq 2 \leq 1$, and $Q = \{\top, s, \bot\}$ with $\bot \leq s \leq \top$. Define mapping $f : P \to Q$ as follows:*

$$f(x) = \begin{cases} \top & x = 4; \\ s & x = 3; \\ \bot & x = 1, 2. \end{cases}$$

Define mapping $g : Q \to P$ as follows: $g(\top) = 4$, $g(\bot) = 1$ and $g(s) = 3$. Then (f, g) is an LRG-Galois connection.

4.2 Fuzzy generalized convexity and fuzzy betweenness on continuous lattice

In the rest of this paper, if not emphasized, we assume that $(P, \leq)$ and $(Q, \leq)$ are completely distributive lattices, $\mathcal{C}$ is a mapping from L to P and $\mathcal{B}$ is a mapping from $L \times L$ to Q. Next, let us consider the mutual induction of fuzzy generalized convex spaces and fuzzy betweenness spaces on lattices.

Theorem 4.7. *Suppose L is a continuous lattice, and P, Q are completely distributive lattices. Assume $f : P \rightleftarrows Q : g$ is a pair of mappings satisfying (LRG) and $f(1) = 0$. Let $\mathcal{C} : L \to P$ be a fuzzy generalized convexity on L. Define $\mathcal{B}_{\mathcal{C}} : L \times L \to Q$ by*

$$\mathcal{B}_{\mathcal{C}}(x, z) = \bigwedge_{x \nleq y \geq z} f(\mathcal{C}(y)),$$

Then $(L, \mathcal{B}_{\mathcal{C}})$ is a fuzzy betweenness space on L.

Proof. It suffices to show $\mathcal{B}_\mathcal{C}$ satisfies (FB1)(FB5).

(FB1) For each $x \neq 0$, $\mathcal{B}_\mathcal{C}(x,0) \leq f(\mathcal{C}(0)) = 0$.

(FB2) For each $x \in L$, $\mathcal{B}_\mathcal{C}(x,x) = \bigwedge \emptyset = 1$.

(FB3) For each $a \in M$,

$$a \in \alpha\left(\mathcal{B}_\mathcal{C}(x,z)\right) \Leftrightarrow a \in \alpha\left(f\left(\bigvee_{x \not\leq y \geq z} \mathcal{C}(y)\right)\right) \Leftrightarrow f\left(\bigvee_{x \not\leq y \geq z} \mathcal{C}(y)\right) \lhd^{op} a \Leftrightarrow g(a) \lhd \bigvee_{x \not\leq y \geq z} \mathcal{C}(y). \qquad (*)$$

(*) holds if and only if there exists y_0 with $x \not\leq y_0 \geq z$ such that $g(a) \lhd \mathcal{C}(y_0)$. It follows from $g(a) \lhd \mathcal{C}(y_0)$ that there exists $t \in P$ such that $g(a) \lhd t \lhd \mathcal{C}(y_0)$. It is clear that $t \lhd \mathcal{C}(y_0)$ implies $t \leq \mathcal{C}(y_0)$. Then we have $x \not\leq \bigwedge_{\substack{w \geq z \\ \mathcal{C}(w) \geq t}} w = W$. Otherwise, $x \leq y_0$, this is a contradiction. Let $U = \bigwedge_{\substack{u \geq y \\ \mathcal{C}(u) \geq t}} u$. By (FGC2), $\mathcal{C}(W) \geq t$ and $\mathcal{C}(U) \geq t$. Since $x \leq U$ and $y \leq W$ implies $x \leq W$, we can know $x \not\leq W$ implies $x \not\leq U$ or $y \not\leq W$. It follows from $x \not\leq U$ that there exists $w \in L$ with $x \not\leq w \geq y$ such that $\mathcal{C}(w) \geq t \rhd g(a)$. It follows from $y \not\leq W$ that there exists $u \in L$ with $y \not\leq u \geq z$ such that $\mathcal{C}(u) \geq t \rhd g(a)$. Hence we have

$$\begin{aligned}(*) \Rightarrow & g(a) \lhd \bigvee_{x \not\leq w \geq y} \mathcal{C}(w) \vee \bigvee_{y \not\leq u \geq z} \mathcal{C}(u) \Leftrightarrow f\left(\bigvee_{x \not\leq w \geq y} \mathcal{C}(w) \vee \bigvee_{y \not\leq u \geq z} \mathcal{C}(u)\right) \lhd^{op} a \\ \Leftrightarrow & a \in \alpha\left(f\left(\bigvee_{x \not\leq w \geq y} \mathcal{C}(w) \vee \bigvee_{y \not\leq u \geq z} \mathcal{C}(u)\right)\right) \Leftrightarrow a \in \alpha\left(\mathcal{B}_\mathcal{C}(x,y) \wedge \mathcal{B}_\mathcal{C}(y,z)\right) \ (f \text{ is a } \bigvee - \bigwedge \text{ mapping}).\end{aligned}$$

It follows from $\alpha\left(\mathcal{B}_\mathcal{C}(x,z)\right) \subseteq \alpha\left(\mathcal{B}_\mathcal{C}(x,y) \wedge \mathcal{B}_\mathcal{C}(y,z)\right)$ that $\mathcal{B}_\mathcal{C}(x,z) \geq \mathcal{B}_\mathcal{C}(x,y) \wedge \mathcal{B}_\mathcal{C}(y,z)$.

(FB4) Take each $u, v, w \in L$ such that $u \geq v$ and $\mathcal{B}_\mathcal{C}(u,w) \leq \mathcal{B}_\mathcal{C}(v,w)$, then

$$\mathcal{B}_\mathcal{C}\left(\bigvee_{i \in J} x_i, z\right) \leq \bigwedge_{i \in J} \mathcal{B}_\mathcal{C}(x_i, z).$$

Conversely, suppose

$$\bigwedge_{\bigvee_{i \in J} x_i \not\leq w \geq z} f(\mathcal{C}(w)) = \mathcal{B}_\mathcal{C}\left(\bigvee_{i \in J} x_i, z\right) \lhd^{op} a,$$

then there exists $w \in L$ such that $\bigvee_{i \in J} x_i \not\leq w \geq z$ and $f(\mathcal{C}(w)) \lhd^{op} a$. It follows that there exists $i_0 \in J$ such that $x_i \not\leq w \geq z$ and $f(\mathcal{C}(w)) \lhd^{op} a$. This implies

$$\bigwedge_{i \in J} \mathcal{B}_\mathcal{C}(x_i, z) = \bigwedge_{i \in J} \bigwedge_{x_i \not\leq w \geq z} f(\mathcal{C}(w)) \lhd^{op} a.$$

Therefore, it follows from

$$\alpha\left(\mathcal{B}_\mathcal{C}\left(\bigvee_{i \in J} x_i, z\right)\right) \subseteq \alpha\left(\bigwedge_{i \in J} \mathcal{B}_\mathcal{C}(x_i, z)\right),$$

that $\mathcal{B}_\mathcal{C}\left(\bigvee_{i \in J} x_i, z\right) \geq \bigwedge_{i \in J} \mathcal{B}_\mathcal{C}(x_i, z)$. Hence we obtain $\mathcal{B}_\mathcal{C}\left(\bigvee_{i \in J} x_i, z\right) = \bigwedge_{i \in J} \mathcal{B}_\mathcal{C}(x_i, z)$.

(FB5) By (FB4), $\mathcal{B}_\mathcal{C}(x,z) = \bigwedge_{y \ll x} \mathcal{B}_\mathcal{C}(y,z)$. It is not difficult to see that for $u, v, w \in L$ with $v \geq w$, we obtain

$$\mathcal{B}_\mathcal{C}(u,v) \geq \mathcal{B}_\mathcal{C}(u,w).$$

Then $\mathcal{B}_\mathcal{C}(x,z) = \bigwedge_{u \ll x} \mathcal{B}_\mathcal{C}(u,z) \geq \bigwedge_{u \ll x} \bigvee_{v \ll z} \mathcal{B}_\mathcal{C}(u,v)$.

Conversely, suppose

$$\bigwedge_{u \ll x} \bigvee_{v \ll z} \bigwedge_{u \not\leq w \geq v} f(\mathcal{C}(w)) = \bigwedge_{u \ll x} \bigvee_{v \ll z} \mathcal{B}(u,v) \lhd^{op} a.$$

This implies that there exists u_0 with $u_0 \ll x$, for all v with $v \ll z$, there exists $w_{0v} \in L$ with $u_0 \nleq w_{0v} \geq v$, such that $g(a) \lhd \mathcal{C}(w_{0v})$. It follows that there exists t with $g(a) \lhd t \lhd \mathcal{C}(w_{0v})$. Let $W_{0v} = \bigwedge_{w \geq v,\ \mathcal{C}(w) \geq t} w$. Thus $W_{0v} \geq v$. We obtain $u_0 \nleq W_{0v}$, otherwise $u_0 \leq w_{0v}$. This is a contradiction. By (FGC2), $\mathcal{C}(W_{0v}) \geq t$. Then it follows from $W_{0v_1 \vee v_2} \geq W_{0v_1} \vee W_{0v_2}$ (where $v_1, v_2 \ll z$) that $\{W_{0v} \mid v \ll z\}$ is directed. Let $W_0 = \bigvee_{v \ll z} W_{0v}$. Then $x \nleq W_0$, otherwise it follows from $u_0 \ll x \leq W_0$ that there exists $v_3 \ll z$ such that $u_0 \leq W_{0v_3}$. This is a contradiction. By (FGC3), $\mathcal{C}(W_0) \geq t$. Now we obtain that there exists $W \in L$ such that $x \nleq W \geq z$ and $g(a) \lhd t \leq \mathcal{C}(W)$, which implies that

$$g(a) \lhd \bigvee_{x \nleq w \geq z} \mathcal{C}(w),$$

i.e.,

$$\mathcal{B}_{\mathcal{C}}(x, z) = \bigwedge_{x \nleq w \geq z} f(\mathcal{C}(w)) = f\Big(\bigvee_{x \nleq w \geq z} \mathcal{C}(w)\Big) \lhd^{op} a.$$

Therefore, it follows from

$$\alpha\left(\bigwedge_{u \ll x} \bigvee_{v \ll z} \mathcal{B}(u, v)\right) \subseteq \alpha\left(\mathcal{B}_{\mathcal{C}}(x, z)\right),$$

that $\bigwedge_{u \ll x} \bigvee_{v \ll z} \mathcal{B}(u, v) \geq \mathcal{B}_{\mathcal{C}}(x, z)$, as desired. Thus (FB5) holds. □

Theorem 4.8. *Suppose L is a continuous lattice, and P, Q are completely distributive lattices. Let $\mathcal{B} : L \times L \to Q$ be a mapping and $g : Q \to P$ be a $\bigvee - \bigwedge$ mapping with $g(0) = 1$. Assume $(L, \mathcal{B})$ is a fuzzy betweenness space on L. Define $\mathcal{C}_{\mathcal{B}} : L \to P$ by*

$$\mathcal{C}_{\mathcal{B}}(y) = \bigwedge_{x \nleq y \geq z} g(\mathcal{B}(x, z)).$$

Then $(L, \mathcal{C}_{\mathcal{B}})$ is a fuzzy generalized convex space.

Proof. It suffices to show $\mathcal{C}_{\mathcal{B}}$ satisfies (FGC1)(FGC3).

(FGC1) $\mathcal{C}_{\mathcal{B}}(1) = \bigwedge \emptyset = 1$, $\mathcal{C}_{\mathcal{B}}(0) = \bigwedge g(0) = 1$.

(FGC2) Given an $a \in P$ and a family of elements $\{w_i \mid i \in J\} \subseteq L$,

$$\mathcal{C}_{\mathcal{B}}\left(\bigwedge_{i \in J} w_i\right) \lhd^{op} a \Leftrightarrow \bigwedge_{x \nleq (\bigwedge_{i \in J} w_i) \geq z} g(\mathcal{B}(x, z)) \lhd^{op} a,$$

which is equivalent to that there exist $(x, z) \in L \times L$ such that $x \nleq \left(\bigwedge_{i \in J} w_i\right) \geq z$ and $g(\mathcal{B}(x, z)) \lhd^{op} a$. Then there exist w_{i_0}, x and z such that $x \nleq w_{i_0} \geq z$ and $g(\mathcal{B}(x, z)) \lhd^{op} a$. Hence

$$\bigwedge_{i \in J} \bigwedge_{x \nleq w_i \geq z} g(\mathcal{B}(x, z)) \lhd^{op} a \Leftrightarrow \bigwedge_{i \in J} \mathcal{C}_{\mathcal{B}}(w_i) \lhd^{op} a.$$

Therefore, it follows from

$$\alpha\left(\mathcal{C}_{\mathcal{B}}\left(\bigwedge_{i \in J} w_i\right)\right) \subseteq \alpha\left(\bigwedge_{i \in J} \mathcal{C}_{\mathcal{B}}(w_i)\right),$$

that

$$\mathcal{C}_{\mathcal{B}}\left(\bigwedge_{i \in J} w_i\right) \geq \bigwedge_{i \in J} \mathcal{C}_{\mathcal{B}}(w_i).$$

(FGC3) Given an $a \in P$ and a directed subset $\{w_i \mid i \in J\} \subseteq L$,

$$\bigwedge_{x \nleq \bigvee_{i \in J} w_i \geq y} g(\mathcal{B}(x, y)) = \mathcal{C}_{\mathcal{B}}\left(\bigvee_{i \in J} w_i\right) \lhd^{op} a,$$

holds if and only if there exist $x, y \in L$ such that $x \nleq \bigvee_{i\in J} w_i \geq y$ and $g(\mathcal{B}(x,y)) \lhd^{op} a$. Since

$$\begin{aligned}
& g(\mathcal{B}(x,y)) \lhd^{op} a \\
\Leftrightarrow & g\left(\bigwedge_{u\ll x}\bigvee_{v\ll y}\mathcal{B}(u,v)\right) \lhd^{op} a \\
\Rightarrow & \forall u \ll x,\ \bigwedge_{v\ll y} g\left(\mathcal{B}(u,v)\right) = g\left(\bigvee_{v\ll y}\mathcal{B}(u,v)\right) \lhd^{op} a \\
\Leftrightarrow & \forall u \ll x,\ \exists v \ll y,\ g(\mathcal{B}(u,v)) \lhd^{op} a.
\end{aligned}$$

It follows from $x \nleq \bigvee\limits_{i\in J} w_i$ that there exists u_1 with $u_1 \ll x$ such that $u_1 \nleq w_i$ for each $i \in J$. Furthermore, there exists $v_1 \in L$ with $v_1 \ll y$ such that $g(\mathcal{B}(u_1,v_1)) \lhd^{op} a$. It follows from $v_1 \ll y \leq \bigvee\limits_{i\in J} w_i$ that there exists $w_{i_1} \in \{w_i \mid i \in J\}$ such that $v_1 \leq w_{i_1}$. This implies that there exists $i_1 \in J$, $v_1 \ll y$ and $u_1 \ll x$ with $u_1 \nleq w_{i_1} \geq v_1$ such that $g(\mathcal{B}(u_1,v_1)) \lhd^{op} a$. Hence we have

$$\bigwedge_{i\in J}\mathcal{C}_{\mathcal{B}}(w_i) \leq \mathcal{C}_{\mathcal{B}}(w_{i_1}) = \bigwedge_{\mu\nleq w_{i_1}\geq\nu} g(\mathcal{B}(\mu,\nu)) \lhd^{op} a.$$

Therefore

$$\alpha\left(\mathcal{C}_{\mathcal{B}}\left(\bigvee_{i\in J} w_i\right)\right) \subseteq \alpha\left(\bigwedge_{i\in J}\mathcal{C}_{\mathcal{B}}(w_i)\right).$$

Thus (FGC3) holds. □

g is the right adjoint of an LRG-Galois connection, then g is a $\bigvee - \bigwedge$ mapping and $g(1) = 0$, $g(0) = 1$. These are useful properties that we will need in the next part.

Theorem 4.9. *Suppose L is a continuous lattice, and P, Q are completely distributive lattices. Let the pair of mappings $f : P \rightleftarrows Q : g$ form an LRG-Galois connection. Then $\mathcal{B}_{\mathcal{C}_{\mathcal{B}}}(x,z) \geq \mathcal{B}(x,z)$ for all $(x,z) \in L \times L$.*

Proof. Given $x, z \in L$,

$$\mathcal{B}_{\mathcal{C}_{\mathcal{B}}}(x,z) = \bigwedge_{x\nleq y\geq z} f(\mathcal{C}_{\mathcal{B}}(y)) = \bigwedge_{x\nleq y\geq z} f\left(\bigwedge_{x_0\nleq y\geq z_0} g(\mathcal{B}(x_0,z_0))\right).$$

Suppose $a \in \alpha\left(\mathcal{B}_{\mathcal{C}_{\mathcal{B}}}(x,z)\right)$, then there exists y_0 such that $x \nleq y_0 \geq z$ and

$$f\left(\bigwedge_{x_0\nleq y_0\geq z_0} g(\mathcal{B}(x_0,z_0))\right) \lhd^{op} a.$$

By (LRG), $g(a) \lhd \bigwedge\limits_{x_0\nleq y_0\geq z_0} g(\mathcal{B}(x_0,z_0))$. This implies that for all x_0 and z_0 with $x_0 \nleq y_0 \geq z_0$, we have $g(a) \lhd g(\mathcal{B}(x_0,z_0))$. Since (f,g) is antitone Galois connection, we obtain $fg \geq id$. By (LRG),

$$\mathcal{B}(x_0,z_0) \leq fg(\mathcal{B}(x_0,z_0)) \lhd^{op} a.$$

Hence for each x_0 and z_0 with $x_0 \nleq y_0 \geq z_0$, we have $\mathcal{B}(x_0,z_0) \lhd^{op} a$. Let $x_0 = x$ and $z_0 = z$, then $\mathcal{B}(x,z) \lhd^{op} a$. It follows from

$$\alpha(\mathcal{B}_{\mathcal{C}\mathcal{B}}(x,z)) \subseteq \alpha(\mathcal{B}(x,z)),$$

that $\mathcal{B}_{\mathcal{C}\mathcal{B}}(x,z) \geq \mathcal{B}(x,z)$. □

Corollary 4.10. *Suppose L is a continuous lattice, and P, Q are completely distributive lattices. Assume $\mathcal{B} : L \times L \to Q$ is a fuzzy betweenness on L and $\mathcal{C} : L \to P$ is a fuzzy generalized convexity. Let mappings $f : P \rightleftarrows Q : g$ satisfying $f \circ g = id_Q$ form an LRG-Galois connection, then $\mathcal{B}_{\mathcal{C}\mathcal{B}} = \mathcal{B}$.*

This is a corollary of (FB10), Theorem 4.7, Theorem 4.8 and Theorem 4.9.

Theorem 4.11. *Suppose L is a continuous lattice, and P, Q are completely distributive lattices. Let the pair of mappings $f : P \rightleftarrows Q : g$ form an LRG-Galois connection, and mapping $\mathcal{C} : L \to P$ be a fuzzy general convexity. Then $\mathcal{C}_{\mathcal{BC}}(y) \geq \mathcal{C}(y)$ for all $y \in L$.*

Proof. Given a $y \in L$,

$$\mathcal{C}_{\mathcal{B}_{\mathcal{C}}}(y) = \bigwedge_{x \not\leq y \geq z} g \left(\bigwedge_{x \not\leq y' \geq z} f(\mathcal{C}(y')) \right).$$

Take each (x, z) such that $x \not\leq y \geq z$, we have $\bigwedge_{x \not\leq y' \geq z} f(\mathcal{C})(y') \leq f(\mathcal{C}(y))$, Since g is antitone,

$$g \left(\bigwedge_{x \not\leq y' \geq z} f(\mathcal{C}(y')) \right) \geq gf(\mathcal{C}(y)) \geq \mathcal{C}(y).$$

By the arbitrariness of x and z, $\mathcal{C}_{\mathcal{BC}}(y) \geq \mathcal{C}(y)$. □

Corollary 4.12. *Suppose L is a continuous lattice, and P, Q are completely distributive lattices. Assume $\mathcal{B} : L \times L \to Q$ is a fuzzy betweenness on L and $\mathcal{C} : L \to P$ is a fuzzy generalized convexity. Let mappings $f : P \rightleftarrows Q : g$ satisfying $g \circ f = id_p$ form an LRG-Galois connection, then $\mathcal{C}_{\mathcal{BC}} = \mathcal{C}$.*

Proof. By Theorem 4.11, we need only show $\mathcal{C}_{\mathcal{BC}} \leq \mathcal{C}$. Suppose $g \circ f = id_p$. Then

$$\mathcal{C}_{\mathcal{B}_{\mathcal{C}}}(y) = \bigwedge_{x \not\leq y \geq z} g(\mathcal{B}_{\mathcal{C}}(x, z)) \leq \bigwedge_{x \not\leq y} g(\mathcal{B}_{\mathcal{C}}(x, y)) = \bigwedge_{x \not\leq y} \left(\bigvee_{x \not\leq y' \geq y} (\mathcal{C}(y')) \right).$$

For each $x, y \in L$, let $Y_{xy} = \{u \in L \mid x \not\leq u \geq y\}$. By the completely distributive law,

$$\begin{aligned}
\bigwedge_{x \not\leq y} \left(\bigvee_{x \not\leq y' \geq y} (\mathcal{C}(y')) \right) &= \bigvee_{p \in \prod_{x \not\leq y} Y_{xy}} \left(\bigwedge_{x \not\leq y} (\mathcal{C}(p(x))) \right) \\
&\leq \bigvee_{p \in \prod_{x \not\leq y} Y_{xy}} \left(\mathcal{C} \left(\bigwedge_{x \not\leq y} p(x) \right) \right) \\
&= \bigvee_{p \in \prod_{x \not\leq y} Y_{xy}} (\mathcal{C}\,(y)) = \mathcal{C}(y).
\end{aligned}$$

□

Theorem 4.13. *Suppose the pair of mappings $f : P \rightleftarrows Q : g$ forms an LRG-Galois connection.*

(1) *Suppose $(L, \mathcal{C}_1)$ and $(L, \mathcal{C}_2)$ are fuzzy generalized convex spaces with $\mathcal{C}_1 \geq \mathcal{C}_2$. Then $\mathcal{B}_{\mathcal{C}_1} \leq \mathcal{B}_{\mathcal{C}_2}$.*

(2) *Suppose $(L, \mathcal{B}_1)$ and $(L, \mathcal{B}_2)$ are fuzzy betweenness spaces on L with $\mathcal{B}_1 \geq \mathcal{B}_2$. Then $\mathcal{C}_{\mathcal{B}_1} \leq \mathcal{C}_{\mathcal{B}_2}$.*

This is because f, g are antitone mappings.

Theorem 4.14. *Suppose L is a continuous lattice, P, Q are completely distributive lattices, and the pair of mappings $f : P \rightleftarrows Q : g$ forms an LRG-Galois connection. Then there exists a pair of functors $K :$ **FGC** $\rightleftarrows$ **FBS** $: H$, in which $K(\mathcal{C}) = \mathcal{B}_{\mathcal{C}}$ and $H(\mathcal{B}) = \mathcal{C}_{\mathcal{B}}$. And there exists a natural isomorphism between $hom_{FGC}(K(\mathcal{C}), \mathcal{B})$ and $hom_{FB}(\mathcal{C}, H(\mathcal{B}))$.*

Proof. (1) Given an **FGC**-morphism $h : (I, \mathcal{C}_I) \to (L, \mathcal{C}_L)$, since Proposition 3.13 and f is antitone,

$$\begin{aligned}
\mathcal{B}_{I\mathcal{C}_I}(x, z) &= \bigwedge_{x \not\leq s \geq z} f(\mathcal{C}_I(s)) \leq \bigwedge_{x \not\leq h^{\dashv}(y) \geq z} f(\mathcal{C}_I(h^{\dashv}(y))) \\
&= \bigwedge_{h(x) \not\leq y \geq h(z)} f(\mathcal{C}_I(h^{\dashv}(y))) \leq \bigwedge_{h(x) \not\leq y \geq h(z)} f(\mathcal{C}_L(y)) \\
&= \mathcal{B}_{L\mathcal{C}_L}(h(x), h(z)).
\end{aligned}$$

Hence $K(h)=h:(I,\mathcal{B}_{I\mathcal{C}_I})\to(L,\mathcal{B}_{L\mathcal{C}_L})$ is an **FBS**-morphism.

Given an **FBS**-morphism $h:(I,\mathcal{B}_I)\to(L,\mathcal{B}_L)$. Since Proposition 3.13 and g is antitone,

$$\begin{aligned}\mathcal{C}_{L\mathcal{B}_L}(y)=&\bigwedge_{u\nleq y\geq v} g(\mathcal{B}_L(u,v))\leq\bigwedge_{h(x)\nleq y\geq h(z)} g(\mathcal{B}_L(h(x),h(z)))\\ &\leq\bigwedge_{h(x)\nleq y\geq h(z)} g(\mathcal{B}_I(x,z))=\bigwedge_{x\nleq h^{\dashv}(y)\geq z} g(\mathcal{B}_I(x,z))\\ &=\mathcal{C}_{I\mathcal{B}_I}(h^{\dashv}(y)).\end{aligned}$$

Hence $H(h)=h:(I,\mathcal{C}_{I\mathcal{B}_I})\to(L,\mathcal{C}_{L\mathcal{B}_L})$ is an **FGC**-morphism.

(2) The naturality of the transformation from $hom_{FGC}(K(-),-)$ to $hom_{FB}(-,H(-))$ is established by the compositions of mappings. Then we need only to show that for each fuzzy generalized convex space $(L,\mathcal{C}_L)$, each fuzzy betweenness space $(I,\mathcal{B}_I)$ on I, and each generalized order-homomorphism $h:I\to L$,

$$\mathcal{C}_{I\mathcal{B}_I}(h^{\dashv}(y))\geq\mathcal{C}_L(y) \text{ if and only if } \mathcal{B}_I(x,z)\leq\mathcal{B}_{L\mathcal{C}_L}(h(x),h(z)).$$

On one hand, by Proposition 3.13, we have:

$$\begin{aligned}&\mathcal{C}_{I\mathcal{B}_I}(h^{\dashv}(y))\geq\mathcal{C}_L(y)\\ \Rightarrow& f\left(\mathcal{C}_{I\mathcal{B}_I}(h^{\dashv}(y))\right)\leq f\left(\mathcal{C}_L(y)\right)\\ \Rightarrow&\bigvee_{x\nleq h^{\dashv}(y)\geq z}\mathcal{B}_I(x,z)\leq\bigvee_{x\nleq h^{\dashv}(y)\geq z} f\left(g(\mathcal{B}_I(x,z))\right)\\ &\leq f\left(\bigwedge_{x\nleq h^{\dashv}(y)\geq z} g(\mathcal{B}_I(x,z))\right)\leq f\left(\mathcal{C}_L(y)\right)\\ \Rightarrow&\mathcal{B}_I(x,z)\leq\bigwedge_{x\nleq h^{\dashv}(y)\geq z} f\left(\mathcal{C}_L(y)\right)\\ &=\bigwedge_{h(x)\nleq y\geq h(z)} f\left(\mathcal{C}_L(y)\right)=\mathcal{B}_{L\mathcal{C}_L}(h(x),h(z)).\end{aligned}$$

On the other hand, by Proposition 3.13, we have:

$$\begin{aligned}&\mathcal{B}_I(x,z)\leq\mathcal{B}_{L\mathcal{C}_L}(h(x),h(z))=\bigwedge_{h(x)\nleq y\geq h(z)} f\left(\mathcal{C}_L(y)\right)\\ \Rightarrow& g\left(\mathcal{B}(x,z)\right)\geq g\left(\bigwedge_{h(x)\nleq y\geq h(z)} f\left(\mathcal{C}_L(y)\right)\right)\\ &=gf\left(\bigvee_{h(x)\nleq y\geq h(z)}\left(\mathcal{C}_L(y)\right)\right)\geq\bigvee_{h(x)\nleq y\geq h(z)}\mathcal{C}_L(y)\\ \Rightarrow&\mathcal{C}_{I\mathcal{B}_I}(h^{\dashv}(y))=\bigwedge_{x\nleq h^{\dashv}(y)\geq z} g(\mathcal{B}_I(x,z))\\ &=\bigwedge_{h(x)\nleq y\geq h(z)} g(\mathcal{B}_I(x,z))\geq\mathcal{C}_L(y).\end{aligned}$$

□

Corollary 4.15. (1) *If there exists an LRG-Galois connection (f,g) between P and Q, then there exists an adjoint pair between* **FGC** *and* **FBS***;*

(2) *If LRG-Galois connection (f,g) satisfies $f\circ g=id_Q$ and $g\circ f=id_P$, then* **FGC** *and* **FBS** *are isomorphic.*

5 Conclusion

Replacing general fuzzy sets by elements of a continuous lattice, replacing $f^{\rightarrow}$ and $f^{\leftarrow}$ by generalized order-homomorphisms and their right adjoints, and considering the generalized convexity [8] on a complete lattice L are interesting ideas. These thoughts stem from G.J. Wang [27]. To some extent, fuzzy generalized convexities can be regarded as the dual space of a generalized convex space. The dual space of fuzzy generalized convexities is a structure on another continuous lattice again. In this view, we can study many areas of convexity theory combined with lattice theory.

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