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Compactness of first-order fuzzy logics

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Abstract

One of the nice properties of the first-order logic is the compactness of satisfiability. It states that a finitely satisfiable theory is satisfiable. However, different degrees of satisfiability in fuzzy logics will pose various kinds of compactness in these logics. In this article, after an overview on the results around the compactness of satisfiability and compactness of K-satisfiability in Hájek Basic logic, some new results are given around this issue. It will be shown that there are topologies on $[0,1]$ and $[0,1]^2$ for which the interpretation of all logical connectives of the Basic logic is continuous. Furthermore, a topology on first-order structures will be introduced for any similarity relation as well. Then by the same ideas as in continuous logic, the results around the compactness of satisfiability will be extended for Basic logic.

Keywords: Basic logic, first-order Basic logic, compactness, K-compactness, ultraproduct method, similarity topology.

1 Introduction

The compactness of satisfiability in classical first-order logic states that a finitely satisfiable theory is satisfiable. In the case of fuzzy logic, switching from a bivalent setting to a many-valent one poses different kinds of fuzzy logics as well as various kinds of the compactness of satisfiability in these logics. It is possible to state that truth value set, basic set of logical connectives, interpretations of logical connectives, and different kinds of satisfiability are the most significant factors that impact a logic.

The class of all fuzzy logics is very large to study. However, in the context of fuzzy logics, the prominent class of Hájek Basic logic, **BL**, in both propositional and first-order case is known as a basis for many well-known fuzzy logics [12]. **BL** can be cast as an extension of the Łukasiewicz, Gödel, and Product logics and furthermore it is the logic of all continuous t-norms. Recall that a continuous t-norm $*$ is a continuous function from $[0,1]^2$ into $[0,1]$ ($[0,1]$ and $[0,1]^2$ with the Euclidean topology) which is commutative, associative, non-decreasing in both arguments, and $1 * x = x$ for all $x \in [0,1]$. The significant continuous t-norms are Łukasiewicz, Gödel, and Product t-norm. Continuous t-norm-based fuzzy logics are axiomatized in the language $\{\&, \rightarrow, \perp\}$ by $[0,1]$-valued calculi using a continuous t-norm as the truth-function for & and its residua as the truth-function for $\rightarrow$ and 0 for the interpretation of $\perp$. **BL** also can be cast in the theme of algebraic logics and the corresponding algebraic part of **BL** is BL-algebras. It is known that when **BL** is endowed with the standard BL-algebra $[0,1]$, the truth-function of & is always a continuous t-norm [2]. The logic of left-continuous t-norms [10, 11], **MTL**, is a generalization of **BL** which is axiomatized in the language $\{\&, \wedge, \rightarrow, \perp\}$ and also is an appropriate class of fuzzy logics to study.

On the other hand, there are various kinds of satisfiability for fuzzy logics. In the case of the propositional fuzzy logics, a theory is called 1-satisfiable, whenever all of its propositions will be evaluated by 1 in the standard semantics. The K-satisfiability of a theory Σ for $K \subseteq [0,1]$ means that there exists an evaluation which evaluates all the propositions of Σ by values in K. In the case of **BL**, a theory Σ is called BL-satisfiable if there exists a BL-algebra **L** such that all the propositions of Σ become **L**-tautologies. In a similar way, 1-satisfiability, K-satisfiability, and BL-satisfiability have been defined for first-order fuzzy logics. Now, variants of the compactness of satisfiability will be defined with respect to each kind of satisfiability. So, "compactness of 1-satisfiability" or "usual compactness" for a logic means that "any finitely

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1-satisfiable theory is 1-satisfiable". The terms of "K-compactness" and "BL-compactness" are defined correspondingly. There are other forms for compactness property in a fuzzy logic. For example, the entailment compactness states that "for a theory Σ and a proposition φ, if $\Sigma \models \varphi$ then there exists a finite subset Σ_0 of Σ such that $\Sigma_0 \models \varphi$". Furthermore, several alternatives of the entailment notion, $\models$, pose various types of compactness of entailment in fuzzy logics.

In this paper, we study compactness of satisfiability for **BL**. So, by compactness we mean the compactness of satisfiability. **BL** in both propositional and first-order case, admits the BL-compactness. Indeed, the BL-compactness of **BL** and **BL**$\forall$ depend on the existence of an arbitrary BL-algebra **L** and finding an **L**-model for theories, which follows from [12, Theorem 2.4.3, Lemma 5.2.8, Theorem 5.2.9]. However, for the standard semantics of **BL** and **BL**$\forall$ in which the usual compactness and K-compactness are meaningful, there is not a universal answer.

In the case of **BL**, a systematic study has been done for the usual compactness as well as K-compactness in [7]. However, in the case of **BL**$\forall$ there is no such a comprehensive study. In many cases, in fact, even the usual compactness fails in these logics. Examples 5.7 and 5.8 shows that the usual compactness fails in the Gödel and Product logic whose set of truth values is the continuous scale $[0, 1]$. In spite of these examples, however, changing the truth value set or generalizing the concept of satisfiability to K-satisfiability leads to some versions of compactness.

Here, the ideas in [6, 7, 14, 20] are extended to derive new results on usual compactness and K-compactness of **BL** and **BL**$\forall$. To some extent, this article tries to answer the problem stated in [8, Point2, Sectin 5] about a systematic study for the usual compactness and K-compactness of **BL**$\forall$ such as the one attended for **BL** in [7]. Besides improving the results around compactness, some topologies on both of truth value sets and structures are introduced which may be interesting to study.

It seems that the only fuzzy logic that satisfies the usual compactness as well as the K-compactness for any compact subset K of the unit interval $[0, 1]$, in both propositional and first-order cases, is Łukasiewicz logic [4, 7, 20]. In the case of propositional Łukasiewicz logic an easy application of the Tychonoff theorem leads to the result [4, 7]. In the first-order case, there are several methods, of which the most significant one is the "Ultraproduct method" [8, 20]. In fact, the main reason behind this is the continuity of the truth function of logical connectives of the Łukasiewicz logic with respect to the Euclidean topology on $[0, 1]$.

For a continuous t-norm $*$, assume that the interpretation of $\leftrightarrow$ is denoted by e_*. An easy argument show that the truth function of $\neg(p \leftrightarrow q)$ in Łukasiewicz logic is $1 - e_*(a, b) = |a - b|$ which is the Euclidean metric, while in the Gödel or Product logic this gives only the discrete metric. The interpretation of $*$ and its residua in Łukasiewicz logic are continuous functions from $[0, 1]^2$ into $[0, 1]$ with the Euclidean topology on $[0, 1]^2$ and $[0, 1]$. However, for arbitrary continuous t-norm $*$, the interpretation of the residua of $*$ is not necessarily a continuous functions. If one consider a topology T_* on $[0, 1]$ whose base is the collection of balls $B_r(a) = \{b : e_*(a, b) > r\}$, then in the case of Łukasiewicz logic this topology again is the Euclidean topology on $[0, 1]$, but in the case of Gödel or Product logic this topology does not coincide with the discrete topology. It can be shown that for any continuous t-norm $*$, a topology $\mathbf{T}_*$ also could be defined on $[0, 1]^2$ such that not only $*$ but also its residua becomes continuous functions from $([0, 1]^2, \mathbf{T}_*)$ into $([0, 1], T_*)$. Using this fact, a version of K-compactness derived for **BL**. In addition, for any similarity relation ρ, a topology on structures is introduced which is called the similarity topology and then a K-compactness for **BL**$\forall$ is obtained.

In the case of **BL**, there is nothing new unless the introduced topologies on $[0, 1]$ and $[0, 1]^2$ and a K-compactness result is proved by the same method as in Łukasiewicz logic which appeared first in [4] and later in [7]. Then a K-compactness result for **BL**$\forall$ is proved by an ultraproduct construction as in [20]. The main idea goes back to [6] but here $([0, 1], T_*)$ is not necessarily a compact space and so we prove the K-compactness in the case that K is a compact subspace of a compact subalgebra of $([0, 1], T_*)$. Indeed, the topologies that we introduced on $[0, 1]$ and $[0, 1]^2$ along with presenting the similarity topology on structures and the order structure of $[0, 1]$, provides the necessary conditions to use the ultraproduct construction stated in [6].

As the first step after introduction, in Section 2 we are going to present **BL**. Then in Section 3 we will review some facts about compactness in three fundamental continuous t-norm-based fuzzy logics (Łukasiewicz, Gödel, and Product logic). Section 4, presents the $*$-open ball topologies on $[0, 1]$ and $[0, 1]^2$ for which the interpretation of all logical connectives become continuous functions. Finally, in Section 5 we will prove some variant of compactness for the Basic logic and especially for these three Basic logics.

2 Basic logic

Basic logic can be presented by a semantics on BL-algebras. Basic logical connectives of Basic logic are in the set $\{\&, \rightarrow, \perp\}$.

Definition 2.1. [12] *Let $P = \{p_i\}_{i\in I}$ be a set of atomic propositions. Assume that Prop is generated from P by the formal binary operations $\{\&, \to\}$ and the unary operation $\bot$. Prop is called a propositional Basic logic and denoted by* **BL**. *Any set of Prop is is called a theory over* **BL** *or shortly a theory.*

The algebraic semantics of **BL** is based on BL-algebras. Recall that a BL-algebra is an algebra $\mathbf{L} = (L, \cap, \cup, \star, \rightarrowtail, 0_\mathbf{L}, 1_\mathbf{L})$ of type $(2, 2, 2, 2, 0, 0)$ such that $(L, \cap, \cup, 0_\mathbf{L}, 1_\mathbf{L})$ is a bounded lattice with greatest element $1_\mathbf{L}$ and smallest element $0_\mathbf{L}$, $(L, \star, 1_\mathbf{L})$ is an Abelian monoid, $\rightarrowtail$ is the residua of $\star$ that is "$c \leq a \rightarrowtail b$ iff $c \star a \leq b$ for all $a, b, c \in L$", L is pre-linear that is "$(a \rightarrowtail b) \cup (b \rightarrowtail a) = 1$ for all $a, b \in L$", and finally $a \cap b = a \star (a \rightarrowtail b)$ for all $a, b \in L$ [12, Chapter 2].

Definition 2.2. [12] *Let $\mathbf{L} = (L, \cap, \cup, \star, \rightarrowtail, 0_\mathbf{L}, 1_\mathbf{L})$ be a BL-algebra. Any function $v_0 : P \to \mathbf{L}$ could be extended to a unique function $v : Prop \to \mathbf{L}$ called an* **L***-evaluation by the following rules:*

$$v(\bot) = 0_\mathbf{L},\ v(\varphi\&\psi) = v(\varphi) \star v(\psi),\ \text{and}\ v(\varphi \to \psi) = v(\varphi) \rightarrowtail v(\psi).$$

If $[0, 1]_\star = ([0, 1], \cap, \cup, \star, \rightarrowtail, 0, 1)$ is a BL-algebra, then $\star$ and $\rightarrowtail$ becomes respectively a continuous t-norm and the residua of $\star$ that is $x \rightarrowtail y = \max\{z : z \star x \leq y\}$ [2]. For a continuous t-norm $*$ the residua of $*$ is denoted by $\Rightarrow_*$. So, in the standard semantics, the strong conjunction & is interpreted by a continuous t-norm $*$, the implication is interpreted by $\Rightarrow_*$ and the zero function plays the role of $\bot$. Well-known continuous t-norm-based fuzzy logics are Łukasiewicz, Gödel, and Product logic. In the rest of the paper assume that $[0, 1]_* = ([0, 1], \cap, \cup, *, \Rightarrow_*, 0, 1)$ is the standard BL-algebra based on an arbitrary continuous t-norm $*$.

Further notable logical connectives, defined by the set of basic logical connectives are

$$\varphi \wedge \psi := \varphi\&(\varphi \to \psi) \quad , \quad \varphi \vee \psi := \big((\varphi \to \psi) \to \psi\big) \wedge \big((\psi \to \varphi) \to \varphi\big),$$
$$\neg\varphi := \varphi \to \bot \quad , \quad \varphi \leftrightarrow \psi := (\varphi \to \psi)\&(\psi \to \varphi) \quad , \quad \top := \neg\bot.$$

Continuity of $*$ implies that $v(\varphi \vee \psi) = \max\{v(\varphi), v(\psi)\}$ and $v(\varphi \wedge \psi) = \min\{v(\varphi), v(\psi)\}$ for every **L**-evaluation v.

Definition 2.3. [12] *Let* **L** *be a BL-algebra, v be an* **L***-evaluation and $\Sigma \cup \{\varphi\} \subseteq Prop$. If $v(\varphi) = 1_\mathbf{L}$ we say that v is an* **L***-model of φ, in symbols $v \models_\mathbf{L} \varphi$. $v \models_\mathbf{L} \Sigma$, whenever $v \models_\mathbf{L} \psi$ for all $\psi \in \Sigma$. When a proposition or theory has an* **L***-model, it is called* **L***-satisfiable. $\Sigma \models_\mathbf{L} \varphi$ means that any* **L***-model of Σ is an* **L***-model of φ. $[0, 1]_*$-satisfiable sentences and theories are called 1-satisfiable.*

In the standard semantics of fuzzy logics, the set of truth values is assumed to be $[0, 1]$. So, one could consider different kinds of satisfiability. Therefore, the concept of 1-satisfiability is, to some extent, a crisp notion. One of the generalizations of this concept to a fuzzy concept, is K-satisfiability.

Definition 2.4. *Let $*$ be a continuous t-norm,* **L** *be a subalgebra of $[0, 1]_*$, and $K \subseteq L \subseteq [0, 1]$. A proposition φ is said to be $K^\mathbf{L}$-satisfiable if there exists an* **L***-evaluation v such that $v(\varphi) \in K$. In this way, v is called a $K^\mathbf{L}$-model of φ. In the case that $\mathbf{L} = [0, 1]_*$, φ is called a K-satisfiable proposition and v is called a K-model of φ [4]. A theory whose propositions are satisfied by a $K^\mathbf{L}$-model (K-model) v, is called a $K^\mathbf{L}$-satisfiable (K-satisfiable) theory.*

Given a first-order language $\mathcal{L}$ consisting of function symbols $\{f_i\}_{i\in I}$ and predicate symbols $\{P_j\}_{j\in J}$, the notion of a structure in **BL**∀ is defined as follows.

Definition 2.5. *Let* **L** *be a BL-algebra and $\mathcal{L} = \{\{P_j\}_{j\in J}, \{f_i\}_{i\in I},\}$ be a first-order language for arbitrary index sets J and I. An* **L***-structure $\mathcal{M}$ for $\mathcal{L}$ is a nonempty set M together with a set of functions $\{(f_\mathcal{M})_i : M^{n_i} \to M\}_{i\in I}$ and a set of partial functions $\{(P_\mathcal{M})_j : M^{n_j} \to L\}_{j\in J}$ as the interpretations of language symbols, assuming that whenever $n_i = 0$, $(f_\mathcal{M})_i$ is an element of M and whenever $n_j = 0$, $(P_\mathcal{M})_j$ is a truth constant in L. Note that nullary function symbols of the language $\mathcal{L}$ are commonly called constant symbols and denoted by c_i instead of f_i.*

As usual logical symbols of **BL**∀ are assumed to be object variables $x, y, z, ...$, basic logical connectives $\{\&, \to, \bot\}$, and quantifiers $\{\forall, \exists\}$. Given a first-order language $\mathcal{L}$, $\mathcal{L}$-terms or in briefly terms are object variables, constant symbols of $\mathcal{L}$, and $f(t_1, ..., t_n)$ for each n-ary function symbol $f \in \mathcal{L}$ and $\mathcal{L}$-terms $t_1, ..., t_n$. Furthermore, given a BL-algebra **L** and an **L**-structure $\mathcal{M}$, any mapping v from the set of object variables into the underlying set (or briefly universe) of $\mathcal{M}$ is called an $\mathcal{M}$-evaluation. For an $\mathcal{M}$-evaluation v and an object variable x and $a \in M$ by $v[x \mapsto a]$ we mean the $\mathcal{M}$-evaluation v is such that $v[x \mapsto a](x) = a$ and $v[x \mapsto a](y) = v(y)$ for each $y \neq x$.

Definition 2.6. *Let $\mathcal{L}$ be a first-order language,* **L** *be a BL-algebra, $\mathcal{M}$ be an* **L***-structure for $\mathcal{L}$, and v be an $\mathcal{M}$-evaluation. For an n-tuple $\bar{x} = (x_1, ..., x_n)$, the interpretation of term $t(\bar{x})$ is defined inductively by 1) if $t(\bar{x}) = x_i$ then $\|t(\bar{x})\|_{\mathcal{M},v} = v(x_i)$, 2) if $t(\bar{x}) = c$ for some constant symbol $c \in \mathcal{L}$ then $\|t(\bar{x})\|_{\mathcal{M},v} = c_\mathcal{M}$, and 3) if $t(\bar{x}) = f(t_1(\bar{x}), ..., t_n(\bar{x}))$ for some n-ary function symbol $f \in \mathcal{L}$ then $\|t(\bar{x})\|_{\mathcal{M},v} = f_\mathcal{M}(\|t_1(\bar{x})\|_{\mathcal{M},v}, ..., \|t_n(\bar{x})\|_{\mathcal{M},v})$. The interpretation of a formula $\varphi(\bar{x})$ is defined inductively as follows:*

- $\|\bot\|^{\mathbf{L}}_{\mathcal{M},v} = 0_{\mathbf{L}}$.
- *For every n-ary predicate symbol P,* $\|P(t_1,...,t_n)\|^{\mathbf{L}}_{\mathcal{M},v} = P_{\mathcal{M}}(\|t_1\|_{\mathcal{M},v},...,\|t_n\|_{\mathcal{M},v})$.
- $\|(\varphi\&\psi)\|^{\mathbf{L}}_{\mathcal{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathcal{M},v} \star \|\psi\|^{\mathbf{L}}_{\mathcal{M},v}$.
- $\|\varphi \to \psi\|^{\mathbf{L}}_{\mathcal{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathcal{M},v} \rightarrowtail \|\psi\|^{\mathbf{L}}_{\mathcal{M},v}$.
- *For* $\varphi(\bar{x}) = \forall x\ \psi(x)$, $\|\varphi\|^{\mathbf{L}}_{\mathcal{M},v} = \inf_{a\in M} \|\psi\|^{\mathbf{L}}_{\mathcal{M},v[x\mapsto a]}$.
- *For* $\varphi(\bar{x}) = \exists x\ \psi(x)$, $\|\varphi\|^{\mathbf{L}}_{\mathcal{M},v} = \sup_{a\in M} \|\psi\|^{\mathbf{L}}_{\mathcal{M},v[x\mapsto a]}$.

The truth value of a formula φ *in* $\mathcal{M}$ *is a partial function* $\|\varphi\|^{\mathbf{L}}_{\mathcal{M}} : M^n \to L$ *defined by* $\|\varphi\|^{\mathbf{L}}_{\mathcal{M}} = \inf\{\|\varphi\|^{\mathbf{L}}_{\mathcal{M},v} :$ *v is an* $\mathcal{M}$*-evaluation*$\}$.

When the truth value of all formulas are total functions, we call $\mathcal{M}$ a safe $\mathbf{L}$-structure. If $*$ is a continuous t-norm and all of the predicate symbols of $\mathcal{L}$ interpreted by total function in $\mathcal{M}$, then $\mathcal{M}$ becomes a safe $[0,1]_*$-structure. We assume that all $[0,1]_*$-structures are safe. For $[0,1]_*$-structure $\mathcal{M}$ and $\mathcal{M}$-evaluation v, we use $\|\varphi\|_{\mathcal{M}}$ and $\|\varphi\|_{\mathcal{M},v}$ instead of $\|\varphi\|^{[0,1]_*}_{\mathcal{M}}$ and $\|\varphi\|^{[0,1]_*}_{\mathcal{M},v}$.

Bellow we introduce the concept of a model in the first-order Basic logic. This definition for safe structures is introduced in [12, Definition 5.1.11]

Definition 2.7. *Let* $\mathcal{L}$ *be a language,* $\mathbf{L}$ *be a BL-algebra,* φ *be an* $\mathcal{L}$*-sentence, and* Σ *be an* $\mathcal{L}$*-theory, i.e., a set of* $\mathcal{L}$*-sentences. We say that* $\mathcal{M}$ *is an* $\mathbf{L}$*-model of* φ *or* φ *is* $\mathbf{L}$*-satisfiable, whenever there exists an* $\mathbf{L}$*-structure* $\mathcal{M}$ *for* $\mathcal{L}$ *such that* $\|\varphi\|^{\mathbf{L}}_{\mathcal{M}} \downarrow = 1_{\mathbf{L}}$ *and denoted by* $\mathcal{M} \models_{\mathbf{L}} \varphi$*. Here, "*$\downarrow$*" indicates that* $\|\varphi\|^{\mathbf{L}}_{\mathcal{M}}$ *is defined.* $\mathcal{M} \models_{\mathbf{L}} \Sigma$ *means that* $\mathcal{M} \models_{\mathbf{L}} \psi$ *for any* $\psi \in \Sigma$ *and in this case* Σ *is call* $\mathbf{L}$*-satisfiable.* $\Sigma \models_{\mathbf{L}} \varphi$ *means that any* $\mathbf{L}$*-model of* Σ *is an* $\mathbf{L}$*-model of* φ*. In the case of standard models the prefix or index* $\mathbf{L}$ *will be omitted.*

Definition 2.8. *Let* $*$ *be a continuous t-norm,* $\mathbf{L}$ *be a subalgebra of* $[0,1]_*$*, and* $K \subseteq L \subseteq [0,1]$*. An* $\mathcal{L}$*-sentence* φ *is called* $K^{\mathbf{L}}$*-satisfiable if there exists an* $\mathbf{L}$*-structure* $\mathcal{M}$ *for* $\mathcal{L}$ *such that* $\|\varphi\|^{\mathbf{L}}_{\mathcal{M}} \downarrow\in K$*, and* $\mathcal{M}$ *is called a* $K^{\mathbf{L}}$*-model of* φ*. If* $\mathbf{L} = [0,1]_*$ *then* φ *is called a* K*-satisfiable proposition and* $\mathcal{M}$ *is called a* K*-model of* φ *[20]. The concept of a* $K^{\mathbf{L}}$*-satisfiable (*K*-satisfiable) theory is defined in a similar way.*

3 Compactness and K-compactness in Basic logic (well-known results)

As usual, a theory Σ is finitely satisfiable means that every finite subset of Σ is satisfiable. A logic is said to satisfy the compactness property if every finitely satisfiable theory is satisfiable. Finitely K-satisfiable theory and K-compactness are defined in a similar way. In the following section, some well-known facts about compactness in Basic logic are reviewed.

3.1 Łukasiewicz logic

Let **Ł** and **Ł**$\forall$ be abbreviations for propositional and first-order Łukasiewicz logic with standard semantics on $[0,1]$, respectively.

Fact 3.1. *Let* K *be a compact subset of* $[0,1]$ *with the Euclidean topology. Every finitely* K*-satisfiable theory over* **Ł** *is* K*-satisfiable.*

Fact 3.2. *Let* K *be a noncompact subset of* $[0,1]$ *with the Euclidean topology. There is a finitely* K*-satisfiable theory over* **Ł** *such that it is not* K*-satisfiable.*

Fact 3.3. *Let* K *be a compact subset of* $[0,1]$ *with the Euclidean topology. Every finitely* K*-satisfiable theory over* **Ł**$\forall$ *is* K*-satisfiable.*

The main reason behind Fact 3.1 and Fact 3.3 is the continuity of the interpretation of logical connectives in **Ł** and **Ł**$\forall$. For $K = \{1\}$, Fact 3.1 is the standard compactness and it is an easy consequence of the completeness theorem which has been proved independently in [19, Theorem 13.22] and [5, final Theorem]. For arbitrary compact subset K of $[0,1]$, the sufficiency condition for the K-compactness of **Ł**, Fact 3.1, has been established in [4, Theorem 3.3][7, Theorem 4.1] and the necessity condition, Fact 3.2, has appeared in [7, Theorem 4.4]. Fact 3.3 for $K = \{1\}$, is the standard compactness theorem for **Ł**$\forall$ that was initially proved in [1, Theorem 9]. The K-compactness of **Ł**$\forall$ for arbitrary compact subset K of $[0,1]$, which appeared in Fact 3.3 is proved in [20, Theorem 3.9].

3.2 Gödel logic and product logic

The non-continuity of the interpretation of the implication connective in Gödel logic as well as Product logic, break down getting a general result about the compactness in these logics. However, some partial results are obtained in the literature.

Let **G**, **G**$\forall$, $\mathbf{\Pi}$, and $\mathbf{\Pi}\forall$ be abbreviations for propositional and first-order Gödel and Product logic with standard semantics on $[0,1]$, respectively. In the cases that a subalgebra $\mathbf{L}$ of $[0,1]_G$ or $[0,1]_\pi$ is considered as the truth value set, we use the abbreviations $\mathbf{G_L}$, $\mathbf{G_L}\forall$, $\mathbf{\Pi_L}$, and $\mathbf{\Pi_L}\forall$ for the corresponding logics.

Fact 3.4. *Let K be an arbitrary subset of $[0,1]$ and the set of atomic propositions be finite. Then every finitely K-satisfiable theory over* **G** *is K-satisfiable.*

Fact 3.5. *Assume that the set of atomic propositions is at most countable. Then every finitely 1-satisfiable theory over* **G** *is 1-satisfiable.*

Fact 3.6. *Assume that $\mathcal{L}$ is an at most countable first-order language. Every finitely 1-satisfiable $\mathcal{L}$-theory over* **G**$\forall$ *is 1-satisfiable.*

Fact 3.7. *Let K be a finite subset of $[0,1]$.* **G** *with at most countable set of atomic propositions and* **G**$\forall$ *with at most countable underlying language are K-compact.*

Fact 3.8. *Let $\mathcal{L}$ be an at most countable first-order language and K be a closed subset of $[0,1]$.* **G**$\forall$ *is not K-compact if and only if K is infinite and $1 \notin K$.*

Fact 3.9. *Assume that $K \subseteq (0,1]$ contains 1. Then* **G** *and* $\mathbf{\Pi}$ *are K-compact.*

Fact 3.4 is an easy consequence of the semantics of Gödel logic. Indeed, since the set of atomic propositions is finite, we can only form finitely many formulas with different semantics [7, Proposition 5.3]. The common idea in the proof of Fact 3.5 and Fact 3.6 is that the Gödel algebra of equivalent formulas could be embedded into the standard Gödel algebra $[0,1]_G$. It seems that this idea is originated by Dummet [9] who employed it to prove the completeness theorem for **G** which implies Fact 3.5 (see also [12]). This idea is also used by Horn [13] to prove the completeness theorem for **G**$\forall$ which argues Fact 3.6 (again, see also [12]). An easy consequence of Facts 3.5 and 3.6 is Fact 3.7 [7]. A more interesting consequence of the Fact 3.6 is derived by [18] which is given in Fact 3.8. Fact 3.9 is proved using the interpretation of double negation and the usual compactness theorem in classical logic [7].

In the rest of the paper we develop the results about compactness and K-compactness for continuous t-norm-based fuzzy logics.

4 $*$-Open ball topology

In this section, for any subalgebra $\mathbf{L}$ of the standard BL-algebra $[0,1]_*$, two topologies on L and L^2 are introduced for which all of the operators of $\mathbf{L}$ become continuous functions. Let's remind some facts about subalgebras of standard BL-algebras.

Proposition 4.1. *Let $*$ be a continuous t-norm and $[0,1]_* = ([0,1], \cap, \cup, *, \Rightarrow_*, 0, 1)$ be a BL-algebra. Then the following properties hold in any subalgebra $\mathbf{L}$ of $[0,1]_*$.*

(B1) $a \leq b$ *iff* $a \Rightarrow_* b = 1_{\mathbf{L}}$,

(B2) $(a \Rightarrow_* b) \leq \big((b \Rightarrow_* c) \Rightarrow_* (a \Rightarrow_* c)\big)$,

(B3) $(a \Rightarrow_* b) * (b \Rightarrow_* c) \leq (a \Rightarrow_* c)$,

(B4) $a \Rightarrow_* a = 1_{\mathbf{L}}$,

(B5) $(a \Rightarrow_* b) * (c \Rightarrow_* d) \leq (a * c) \Rightarrow_* (b * d)$,

Proof. See [12, Chapter 2]. □

Interpretation of the equivalence relation and its properties plays a key role in the rest of the article.

Definition 4.2. *Let $*$ be a continuous t-norm, $\Rightarrow_*$ be its residua, and $\mathbf{L}$ be a subalgebra of $[0,1]_*$. Interpretation of the equivalence relation is denoted by e_*, i.e., $e_* : L^2 \to L$ is defined by $e_*(a,b) = (a \Rightarrow_* b) * (b \Rightarrow_* a)$. The binary operator $\mathbf{e}_*$ on L^2 is defined by $\mathbf{e}_*(\bar{a}, \bar{b}) = e_*(a_1, b_1) * e_*(a_2, b_2)$ in which $\bar{a} = (a_1, a_2)$ and $\bar{b} = (b_1, b_2)$.*

Lemma 4.3. *Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$. Then the followings holds.*

1. *Both of e_* and $\mathbf{e}_*$ are symmetric operators.*
2. *$e_*(a,a) = \mathbf{e}_*(\bar{a},\bar{a}) = 1_{\mathbf{L}}$ for any $a \in L$ and $\bar{a} \in L^2$.*
3. *$e_*(a,b) = 1_{\mathbf{L}}$ if and only if $a = b$ for any $a, b \in L$.*
4. *for each $a \neq b \in L$ there exists $c \in L \setminus \{1_{\mathbf{L}}\}$ such that $e_*(a,b) < c * c$.*
5. *$e(a,b) \geq e(a,c) * e(c,b)$ for any $a, b, c \in L$.*
6. *$\mathbf{e}_*(\bar{a},\bar{b}) \geq \mathbf{e}_*(\bar{a},\bar{c}) * \mathbf{e}_*(\bar{c},\bar{b})$ for any $\bar{a},\bar{b},\bar{c} \in L^2$.*

Proof. (1) is obvious. (2) follows from (B4). (3) is an easy consequence of (B1) and the fact that $a * 1_{\mathbf{L}} = a$ for any $a \in L$. For (4), if by contradiction for each $c \in L \setminus \{1_{\mathbf{L}}\}$, $e_*(a,b) \geq c * c$ then continuity of $*$ implies that $e_*(a,b) = 1$ and therefore by (3) a=b, a contradiction. For (5), by (B3) we have

$$\begin{aligned} e_*(a,b) &= (a \Rightarrow_* b) * (b \Rightarrow_* a) \\ &\geq (a \Rightarrow_* c) * (c \Rightarrow_* b) * (b \Rightarrow_* c) * (c \Rightarrow_* a) \\ &= (a \Rightarrow_* c) * (c \Rightarrow_* a) * (c \Rightarrow_* b) * (b \Rightarrow_* c) \\ &= e_*(a,c) * e_*(c,b). \end{aligned}$$

Finally, (6) follows from Definition 4.2 and (5). □

Now, for any continuous t-norm $*$ and subalgebra $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow_*, 0_{\mathbf{L}}, 1_{\mathbf{L}})$ of $[0,1]_*$, we introduce topologies on L and L^2 which are similar to the usual open ball topology arising from a metric.

Definition 4.4. *Let $*$ be a continuous t-norm and $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow_*, 0_{\mathbf{L}}, 1_{\mathbf{L}})$ be a subalgebra of $[0,1]_*$. For any $a \in L$ ($\bar{a} \in L^2$) and $r \in L \setminus \{1_{\mathbf{L}}\}$ the set $B_r(a) = \{b \in L : e_*(a,b) > r\}$ ($\mathbf{B}_r(\bar{a}) = \{\bar{b} \in L^2 : \mathbf{e}_*(\bar{a},\bar{b}) > r\}$) is called the $*$-ball around a ($*$-ball around $\bar{a}$) of radius r. A subset G of L (L^2) is called a $*$-open set if for every $a \in G$ ($\bar{a} \in G$) there exists $r \in L \setminus \{1_{\mathbf{L}}\}$ such that $B_r(a) \subseteq G$ ($\mathbf{B}_r(\bar{a}) \subseteq G$). The $*$-open ball topology on L (L^2) is $T_* = \{G : G$ is a $*$-open subset of $L\}$ ($\mathbf{T}_* = \{G : G$ is a $*$-open subset of $L^2\}$).*

Theorem 4.5. *Let $*$ be a continuous t-norm and $\mathbf{L}$ be a subalgebra of $[0,1]_*$. The sets T_* and $\mathbf{T}_*$ form topologies on L and L^2. Furthermore, (L, T_*) is a Hausdorff space.*

Proof. We do the proof for T_*, the proof for $\mathbf{T}_*$ is similar. Clearly, the definition of the $*$-open sets implies that $\emptyset$ and L belongs to T_*. If $\{A_i\}_{i \in I}$ is a family of $*$-open sets, again the definition of $*$-open sets implies that $\cup_{i \in I} A_i$ is a $*$-open set. Now, if $A, B \in T_*$ and $a \in A \cap B$, then since A and B are $*$-open sets, there exist $0_{\mathbf{L}} \leq r_A, r_B < 1_{\mathbf{L}}$ such that $B_{r_A}(a) \subseteq A$ and $B_{r_B}(a) \subseteq B$. If $r = \max\{r_A, r_B\}$ and $x \in B_r(a)$ then $e_*(x,a) > r \geq r_A$ (also $e_*(x,a) > r \geq r_B$) and therefore $x \in B_{r_A}(a)$ (also $x \in B_{r_B}(a)$). Hence $B_r(a) \subseteq B_{r_A}(a) \cap B_{r_B}(a) \subseteq A \cap B$, that is $A \cap B$ is a $*$-open set.

Now, let a and b be two distinct points of L. By Lemma 4.3 (4) there exists $c \in L \setminus \{1_{\mathbf{L}}\}$ such that $e_*(a,b) < c * c$. We show that $B_c(a) \cap B_c(b) = \emptyset$. To this end suppose that $x \in B_c(a) \cap B_c(b)$. Then by Lemma 4.3 (5), $e_*(a,b) \geq e_*(a,c) * e_*(c,b) > c * c$, a contradiction with the chosen of c. □

The following examples describe T_* more precisely in the case of well-known continuous t-norms. We denote T_* for Łukasiewicz, Gödel, and Product t-norm by T_L, T_G, and T_π, respectively.

Example 4.6. *Let $*$ be the Łukasiewicz t-norm $a * b = \max\{0, a+b-1\}$. The residua of $*$ is $a \Rightarrow_* b = \min\{1, 1-a+b\}$. If $a \leq b$ then $a \Rightarrow_* b = 1$, $b \Rightarrow_* a = 1 - b + a$, and consequently $e_*(a,b) = 1 * (1-b+a) = 1 - b + a$. Similarly, if $b \leq a$ then $e_*(a,b) = 1 - a + b$. Hence for the Łukasiewicz t-norm we have $e_*(a,b) = 1 - |b-a|$. So,*

$$B_r(a) = \{b : e_*(a,b) > r\} = \{b : 1 - |a-b| > r\} = \{b : |a-b| < 1-r\},$$

which shows T_L is the Euclidean topology on $[0,1]$.

Example 4.7. *For the Gödel t-norm which is defined by $a * b = \min\{a,b\}$, the value of $a \Rightarrow_* b$ is b for $a > b$, and is 1 for $a \leq b$. So, $e_*(a,b) = \begin{cases} \min\{a,b\} & a \neq b \\ 1 & a = b \end{cases}$ which together with the fact that $e_*(a,a) = 1$, implies that*

$$B_r(a) = \{b : e_*(a,b) > r\} = \begin{cases} (r,1] & a > r \\ \{a\} & a \leq r \end{cases}.$$

For example $B_{0.2}(0.5) = (0.2, 1]$, $B_{0.2}(0.1) = \{0.1\}$. *It can be seen that the, topology on* $[0, 1)$ *is discrete. However, since in* $([0, 1], T_G)$ *the singleton set* $\{1\}$ *is not open,* T_G *is a little coarser than the discrete topology on* $[0, 1]$.

Example 4.8. *Let* $*$ *be the Product t-norm defined by* $a * b = a.b$. *The residua of* $*$ *is* $a \Rightarrow_* b = \begin{cases} b/a & a > b \\ 1 & a \leq b \end{cases}$. *So, we have* $e_*(a, b) = \begin{cases} \min\{a, b\}/\max\{a, b\} & a, b \neq 0 \\ 1 & a, b = 0 \end{cases}$. *Accordingly,* $B_r(0) = \{0\}$ *for any* $r \in [0, 1)$, *and*

$$B_r(a) = \{b : e_*(a, b) > r\} = \{b : \min\{a, b\} > r. \max\{a, b\}\} = \begin{cases} (r.a, 1] & a > r \\ (r.a, a/r) & a \leq r \end{cases}$$

for any $a > 0$. *An easy argument shows that* T_π *on* $(0, 1]$ *is equivalent to the Euclidean topology on* $(0, 1]$. *Note for any open ball* $(a - r, a + r)$ *of* $[0, 1]$ *with the Euclidean topology, if we choose* $s > \max\{\frac{a-r}{a}, \frac{a}{a+r}, a\}$, *then the open ball* $B_s(a)$ *of* $([0, 1], T_\pi)$ *is a subset of* $(a - r, a + r)$. *Furthermore,* $\{0\}$ *is an open set of* $([0, 1], T_\pi)$ *but it is not an open set of* $[0, 1]$ *with the Euclidean topology. Therefore* T_π *is strictly finer than the Euclidean topology on* $[0, 1]$.

The following theorem shows that for each continuous t-norm $*$ and subalgebra $\mathbf{L}$ of $[0, 1]_*$, the interpretation of all logical connectives in Basic logic with the truth value set $\mathbf{L}$ are continuous functions with respect to the $*$-open ball topologies on L and L^2.

Theorem 4.9. *Let* $*$ *be a continuous t-norm and* $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow_*, 0_\mathbf{L}, 1_\mathbf{L})$ *be a subalgebra of* $[0, 1]_*$. *Then the functions* $* : (L^2, \mathbf{T}_*) \to (L, T_*)$ *and* $\Rightarrow_*: (L^2, \mathbf{T}_*) \to (L, T_*)$ *are continuous functions.*

Proof. Firstly, we show that $*^{-1}(A)$ is a $*$-open subset of L^2 for any $*$-open set $A \in T_*$. Consider a point $\bar{a} \in *^{-1}(A)$, that is $a_1 * a_2 \in A$. Since $A \in T_*$ there exists $r \in L \setminus \{1_\mathbf{L}\}$ such that $B_r(a_1 * a_2) \subseteq A$. We show that $\mathbf{B}_r(\bar{a}) \subseteq *^{-1}(A)$. To this end, assume that $\bar{b} \in \mathbf{B}_r(\bar{a})$. So, $e_*(a_1, b_1) * e_*(a_2, b_2) = \mathbf{e}_*(\bar{a}, \bar{b}) > r$. Additionally, (B5) implies that

$$(a_1 * a_2) \Rightarrow_* (b_1 * b_2) \geq (a_1 \Rightarrow_* b_1) * (a_2 \Rightarrow_* b_2),$$

and similarly

$$(b_1 * b_2) \Rightarrow_* (a_1 * a_2) \geq (b_1 \Rightarrow_* a_1) * (b_2 \Rightarrow_* a_2).$$

Now, monotonicity of $*$ gives

$$\begin{aligned} e_*(a_1 * a_2, b_1 * b_2) &= \big((a_1 * a_2) \Rightarrow_* (b_1 * b_2)\big) * \big((b_1 * b_2) \Rightarrow_* (a_1 * a_2)\big) \\ &\geq \big((a_1 \Rightarrow_* b_1) * (a_2 \Rightarrow_* b_2)\big) * \big((b_1 \Rightarrow_* a_1) * (b_2 \Rightarrow_* a_2)\big) \\ &= \big((a_1 \Rightarrow_* b_1) * (b_1 \Rightarrow_* a_1)\big) * \big((a_2 \Rightarrow_* b_2) * (b_2 \Rightarrow_* a_2)\big) \\ &= e_*(a_1, b_1) * e(a_2, b_2) \\ &> r. \end{aligned}$$

Thus $b_1 * b_2 \in B_r(a_1 * a_2) \subseteq A$ that is $\bar{b} \in *^{-1}(A)$ which completes the first part of the proof.

Secondly, we prove that $\Rightarrow_*^{-1}(A) \in \mathbf{T}_*$ for each $A \in T_*$. Consider a point $\bar{a} \in \Rightarrow_*^{-1}(A)$. Since, $a_1 \Rightarrow_* a_2 \in A$ and $A \in T_*$, there exists $r \in L \setminus \{1_\mathbf{L}\}$ such that $B_r(a_1 \Rightarrow_* a_2) \subseteq A$. In order to finalize the proof we demonstrate that $\mathbf{B}_r(\bar{a}) \subseteq \Rightarrow_*^{-1}(A)$. If $\bar{b} \in \mathbf{B}_r(\bar{a})$,then $\mathbf{e}_*(\bar{a}, \bar{b}) > r$. By (B3) we have $(a_1 \Rightarrow_* b_1) * (b_1 \Rightarrow_* b_2) \leq (a_1 \Rightarrow_* b_2)$, and by (B2) we have $a_1 \Rightarrow_* b_2 \leq (b_2 \Rightarrow_* a_2) \Rightarrow_* (a_1 \Rightarrow_* a_2)$, together implies that

$$(a_1 \Rightarrow_* b_1) * (b_1 \Rightarrow_* b_2) \leq (b_2 \Rightarrow_* a_2) \Rightarrow_* (a_1 \Rightarrow_* a_2).$$

Now applying residuation property, commutativity of $*$, and again residuation property we get

$$(b_1 \Rightarrow_* b_2) \Rightarrow_* (a_1 \Rightarrow_* a_2) \geq (a_1 \Rightarrow_* b_1) * (b_2 \Rightarrow_* a_2). \tag{1}$$

A similar argument show that

$$(a_1 \Rightarrow_* a_2) \Rightarrow_* (b_1 \Rightarrow_* b_2) \geq (b_1 \Rightarrow_* a_1) * (a_2 \Rightarrow_* b_2). \tag{2}$$

Therefore 1, 2, and monotonicity of $*$ poses that

$$\begin{aligned} &e(a_1 \Rightarrow_* a_2, b_1 \Rightarrow_* b_2) \\ &\quad = \big((a_1 \Rightarrow_* a_2) \Rightarrow_* (b_1 \Rightarrow_* b_2)\big) * \big((b_1 \Rightarrow_* b_2) \Rightarrow_* (a_1 \Rightarrow_* a_2)\big) \\ &\quad \geq \big((b_1 \Rightarrow_* a_1) * (a_2 \Rightarrow_* b_2)\big) * \big((a_1 \Rightarrow_* b_1) * (b_2 \Rightarrow_* a_2)\big) \\ &\quad = \big((b_1 \Rightarrow_* a_1) * (a_1 \Rightarrow_* b_1)\big) * \big((a_2 \Rightarrow_* b_2) * (b_2 \Rightarrow_* a_2)\big) \\ &\quad = e(a_1, b_1) * e(a_2, b_2) = \mathbf{e}_*(\bar{a}, \bar{b}) > r. \end{aligned}$$

Hence $(b_1 \Rightarrow_* b_2) \in B_r(a_1 \Rightarrow_* a_2) \subseteq A$ which means that $\bar{b} \in \Rightarrow_*^{-1} (A)$. □

5 Compactness and K-compactness in basic logic (new results)

For a continuous t-norm $*$ and subalgebra $\mathbf{L}$ of $[0,1]_*$, continuity of the interpretation of the logical connectives with respect to the topologies T_* and $\mathbf{T}_*$ provides some versions of the K-compactness for the Basic logic.

5.1 Propositional basic logic

In the propositional Basic logic, K-compactness could be proved as in the propositional Łukasiewicz logic [4, 7].

Theorem 5.1. *Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, (L, T_*) be a compact topological space, and K be compact subset of (L, T_*). Then in the Basic logic every finitely $K^{\mathbf{L}}$-satisfiable theory over $\mathbf{BL}$ is $K^{\mathbf{L}}$-satisfiable.*

Proof. Let P and $Prop$ be as in the Definition 2.1. Since every assignment $v_0 : P \to L$ determines a unique evaluation $v : Prop \to L$, so L^I determines the set of all evaluations. Now as by Theorem 4.9, logical connectives are interpreted by continuous functions, each $\varphi \in Prop$ can be identified by a continuous function $\hat{\varphi} : L^I \to L$ defined by $\hat{\varphi}(v) = v(\varphi)$.

Assume that Σ is a finitely $K^{\mathbf{L}}$-satisfiable theory. Thus, for each finite subset Σ_0 of Σ, $\bigcap_{\varphi\in\Sigma_0} \hat{\varphi}^{-1}(K) \neq \emptyset$. But, for each $\varphi \in \Sigma$, $\hat{\varphi}^{-1}(K)$ is a closed subset of L^I and furthermore, (L, T_*) is a compact topological space which besides the Tychonoff's theorem provides L^I to be compact. Therefore $\hat{\varphi}^{-1}(K)$ is a compact subset of L^I and so is $\bigcap_{\varphi\in\Sigma_0} \hat{\varphi}^{-1}(K)$. Now, finite intersection property of compact sets implies that, $\bigcap_{\varphi\in\Sigma} \hat{\varphi}^{-1}(K) \neq \emptyset$, that is Σ is $K^{\mathbf{L}}$-satisfiable. □

Remark 5.2. *In the case of Łukasiewicz logic, the compactness of K in $[0,1]$ with the Euclidean topology is equivalent to it's closeness. Therefore, in the case of Łukasiewicz logic, Theorem 5.1 holds for closed subset K of (L, T_L). A question raised here: "Is Theorem 5.1 holds for any arbitrary continuous t-norm $*$ and closed subset K of (L, T_*)?"*

Fact 3.2 shows that in the Łukasiewicz logic, K-compactness fails for any noncompact subset K of $([0,1], T_L)$. However, this does not hold in Basic logic. Indeed, the expressive power of the language of logic and the forms of the compact subsets of $[0,1]_*$, imposes some limitations in the results.

In the case of Gödel logic and Product logic, this limitation is stated in Fact 3.9. For example if we set $K = (0,1]$, then $\mathbf{G}$ is K-compact but K is not a compact subset of $([0,1], T_G)$ or $([0,1], T_\pi)$. Let us describe compact subsets of $([0,1], T_G)$ and $([0,1], T_\pi)$ more precisely.

Lemma 5.3. *The compact subsets of $([0,1], T_G)$ are finite sets together with infinite sets containing 1 whose only limit point with respect to the Euclidean topology is 1. In particular, $([0,1], T_G)$ is not a compact topological space.*

Proof. Let $A \subseteq [0,1]$ be an infinite set that contains a limit point $\alpha < 1$ (with respect to the Euclidean topology on $[0,1]$). If r is chosen such that $\alpha < r < 1$, then as we describe the $*$-open balls of T_G in Example 4.7, the open cover $\mathcal{U} = \{B_r(x)\}_{x\in A}$ of A has not any finite subcover for A, that is A is not a compact subset of $([0,1], T_G)$.

On the other hand, if $A \subseteq [0,1]$ is an infinite set whose only limit point is 1 (with respect to the Euclidean topology on $[0,1]$) and $1 \in A$, then A is a compact subset of $([0,1], T_G)$. Indeed, since $B_r(1)$ for any $r < 1$ contains infinitely many elements of A, every open set containing 1 also contains infinitely many elements of A which shows that every open cover of A has a finite subcover. □

Despite of the expressive power of the language of Gödel logic which implies Fact 3.9, we can state a weak version of necessary condition for the K-compactness of $\mathbf{G}$.

Lemma 5.4. *Let K be any non-compact subset of $([0,1], T_G)$ which does not contain 1. Then $\mathbf{G}$ does not admit the K-compactness.*

Proof. Let $T = \{p_i \to p_j\}_{i \leq j, i,j \in \omega_1+1}$. As K is infinite, every finite subset T_f of T is K-satisfiable. Indeed if we set

$$m = \min\{i : (p_i \to p_j) \in T_f \text{ for some } j\},$$
$$M = \max\{j : (p_i \to p_j) \in T_f \text{ for some } i\},$$

then there is a K-evaluation v such that $v(p_m) > ... > v(p_M)$, and so v is a K-model of T_f. However, satisfiability of T contradicts with the order type of $([0,1], <)$. □

If $*$ is the Gödel t-norm, then the algebraic operators of $[0,1]_*$ implies that every closed subset of $[0,1]$ (with respect to the Euclidean topology) becomes a subalgebra of $[0,1]_*$. So, the following corollary summarizes the results of Theorem 5.1, Fact 3.9, Lemma 5.3, and Lemma 5.4 for the propositional Gödel logic.

Corollary 5.5. *Let $*$ be the Gödel t-norm and L be a subset of $[0,1]$ such that $L' = \emptyset$ or $L' \cap L = \{1\}$ where L' is the set of limit points of L with respect to the Euclidean topology on $[0,1]$. Furthermore, let $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow_*, 0_{\mathbf{L}}, 1_{\mathbf{L}})$. Then $\mathbf{G_L}$ admits the $K^{\mathbf{L}}$-compactness if and only if K is a compact subset of (L, T_G).*

In order to investigate the K-compactness of Product logic, we need a description of compact subsets of $([0,1], T_\pi)$.

Lemma 5.6. *The set $K \subseteq [0,1]$ is a compact subsets of $([0,1], T_\pi)$ if and only if there exists a compact subset K' of $(0,1]$ with respect to the Euclidean topology such that either $K = K'$ or $K = K' \cup \{0\}$. In particular, $([0,1], T_\pi)$ is not a compact topological space.*

Proof. As T_π is described in Example 4.8, T_π on $(0,1]$ is equivalent to the Euclidean topology on $(0,1]$. Therefore, a set K is a compact subset of $(0,1]$ with respect to the Euclidean topology if and only if it is a compact subset of $((0,1], T_\pi)$. Furthermore, since the only $*$-open balls which contain 0 are $B_r(0)$ for $0 \leq r < 1$, thus only if 0 is added to a compact subset K of $((0,1], T_\pi)$, $K \cup \{0\}$ remains compact in $([0,1], T_\pi)$. specifically, since $(0,1]$ is not compact with respect to the Euclidean topology on $(0,1]$, $[0,1]$ is not compact with respect to T_π. □

If $*$ is the Product t-norm and $\mathbf{L}$ is a subalgebra of $[0,1]_*$, then for any $a \in L$ such that $0 < a < 1$, $a * a = a^2 < a$ and therefore $a^2 \in L$. This implies that every subalgebra $\mathbf{L}$ of $[0,1]_*$ contains a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $\lim a_n = 0$ and therefore by an argument such as in the proof of Lemma 5.6, (L, T_*) it is not a compact space. Thus, Theorem 5.1 does not improve the results around the K-compactness for the Product logic.

5.2 First-order basic logic

In this subsection, we present a result such as the Fact 3.3 for $\mathbf{BL}\forall$. The following two examples show that the usual compactness fails in $\mathbf{G}\forall$ and $\mathbf{\Pi}\forall$.

Example 5.7. [14, Modification of Example 4.1] *Uncountability of the underlying language in Fact 3.6 leads to the failure of compactness over $\mathbf{G}\forall$. Let $\mathcal{L}$ be a relational language containing uncountably many unary predicate symbols $\{\rho_i(x)\}_{i\in\omega_1+1}$ Set,*

$$T = \Big\{\neg\forall x\, \rho_{\omega_1}(x)\Big\} \cup \Big\{\forall x \Big(\big(\rho_j(x) \to \rho_i(x)\big) \to \rho_j(x)\Big)\Big\}_{i<j\in\omega_1+1}.$$

Recall that in Gödel logic

$$\|\neg\varphi(\bar a)\|_{\mathcal{M}} = \begin{cases} 1 & \|\varphi(\bar a)\|_{\mathcal{M}} = 0 \\ 0 & \|\varphi(\bar a)\|_{\mathcal{M}} > 0 \end{cases}, \quad \|((\varphi \to \psi) \to \varphi)(\bar a)\|_{\mathcal{M}} = \begin{cases} 1 & \|\psi(\bar a)\|_{\mathcal{M}} < \|\varphi(\bar a)\|_{\mathcal{M}} < 1 \\ \|\varphi(\bar a)\|_{\mathcal{M}} & \|\psi(\bar a)\|_{\mathcal{M}} \geq \|\varphi(\bar a)\|_{\mathcal{M}} \end{cases}.$$

Let $\mathcal{M} \models_{[0,1]_G} T$. Then $\mathcal{M} \models_{[0,1]_G} \neg\forall x\, \rho_{\omega_1}(x)$ and therefore there is an element $a \in M$ such that $\|\rho_{\omega_1}(a)\|_{\mathcal{M}} < 1$. Furthermore, $\mathcal{M} \models_{[0,1]_G} \forall x \Big(\big(\rho_j(x) \to \rho_i(x)\big) \to \rho_j(x)\Big)$ implies that $\|\rho_i(a)\|_{\mathcal{M}} < \|\rho_j(a)\|_{\mathcal{M}}$ for every $i < j \in \omega_1 + 1$. Hence

$$\|\rho_1(a)\|_{\mathcal{M}} < \|\rho_2(a)\|_{\mathcal{M}} < ... < \|\rho_{\omega_1}(a)\|_{\mathcal{M}} < 1.$$

Clearly T is finitely 1-satisfiable. However, 1-satisfiability of T provides an embedding of the algebra $(\omega_1 + 2, <)$ into $([0,1], <)$ which is impossible.

Example 5.8. *K-Compactness fails over $\mathbf{\Pi}$ even for finitely many atomic symbols [7][Theorem 6.2]. This example shows a similar situation for $\mathbf{\Pi}\forall$. Let $\mathcal{L} = \{R, \rho\}$ be a relational language in which R and ρ are unary predicate symbols. Assume that*

$$T = \Big\{\neg\forall x\,\big(R(x) \vee \rho(x)\big), \neg\neg\forall x\, R(x)\Big\} \bigcup \Big\{\forall x\,\big(R(x) \to \rho^n(x)\big)\Big\}_{n\in\mathbb{N}}.$$

If $\mathcal{M} \models_{[0,1]_\pi} T$, then $\mathcal{M} \models_{[0,1]_\pi} \neg\forall x\,\big(R(x)\vee\rho(x)\big)$ and so there is an element $b \in M$ such that $\max\{\|R(b)\|_{\mathcal{M}}, \|\rho(b)\|_{\mathcal{M}}\} < 1$. Furthermore, $\mathcal{M} \models_{[0,1]_\pi} \neg\neg\forall x\, R(x)$ and so $\|R(a)\|_{\mathcal{M}} > 0$ for all $a \in M$. In particular $0 < \|R(b)\|_{\mathcal{M}} < 1$. On the other hand, $\mathcal{M} \models_{[0,1]_\pi} \forall x\,\big(R(x) \to \rho^n(x)\big)$ for each $n \geq 1$, which implies that for each $n \geq 1$, $\inf\limits_{a\in M}\big(\|R(a)\|_{\mathcal{M}} \Rightarrow_\pi \|\rho^n(a)\|_{\mathcal{M}}\big) = 1$. Hence $\big(\|R(b)\|_{\mathcal{M}} \Rightarrow_\pi \|\rho^n(b)\|_{\mathcal{M}}\big) = 1$. Therefore, $0 < \|R(b)\|_{\mathcal{M}} \leq \|\rho^n(b)\|_{\mathcal{M}} < \|\rho(b)\|_{\mathcal{M}} < 1$ for all $n \geq 1$, which is impossible. Thus T is not 1-satisfiable. However, obviously T is finitely 1-satisfiable.

There are several approaches to prove the compactness of the first-order logics. In Łukasiewicz logic [1] using continuity of the interpretation of logical connectives, showed that consistency and satisfiability are equivalent concepts which leads to the compactness of first-order Łukasiewicz logic. In Gödel logic [9] proving the completeness theorem conclude the compactness theorem (Fact 3.5 and Fact 3.6). [15, 16, 17] adding some nullary connectives to the Łukasiewicz logic and showing that truth degree of any sentence is equal to its provability degree, deduce the compactness theorem for Łukasiewicz logic enriched with the set $\{\bar{r} : r \in (0,1)\}$ of nullary connectives. In [20] the ultraproduct method is used to derive the K-compactness for Łukasiewicz logic (Fact 3.3). Here we use the ultraproduct method to prove the K-compactness for **BL**$\forall$.

5.2.1 Similarity relation and extensionality axioms

To use the ultraproduct method, we need a similarity relation in the underlying language. So, from now on, assume that the underlying language of any theory contains a binary predicate symbol ρ as the similarity relation. The axioms of similarity are as follows [12, Chapter 5].

(S1) $\forall x \rho(x,x)$.

(S2) $\forall x\forall y\big(\rho(x,y) \to \rho(y,x)\big)$.

(S3) $\forall x\forall y\forall z\big[\big(\rho(x,y)\&\rho(y,z)\big) \to \rho(x,z)\big]$.

For a similarity relation ρ, the binary n-tuple similarity relation is defined by

$$\rho_n(\bar{x},\bar{y}) = \rho(x_1,y_1)\&\rho(x_2,y_2)\&...\&\rho(x_n,y_n).$$

Existence of a similarity relation in a language, induced a topology on any structure in that language.

Definition 5.9. *Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, and $\mathcal{M}$ be an $\mathbf{L}$-structure satisfying the axioms of similarity. For $r \in [0_{\mathbf{L}}, 1_{\mathbf{L}})$ and $a \in M$ the ρ-open ball around a is $B_r^\rho(a) = \{m \in M : \|\rho(a,m)\|_{\mathcal{M}}^{\mathbf{L}} > r\}$. For $r \in [0_{\mathbf{L}}, 1_{\mathbf{L}})$ and $\bar{a} \in M^n$ the ρ-open ball around $\bar{a}$ is $B_r^\rho(\bar{a}) = \{\bar{m} \in M^n : \|\rho_n(\bar{a},\bar{m})\|_{\mathcal{M}}^{\mathbf{L}} > r\}$. A subset $G \subseteq M$ ($G \subseteq M^n$) is called ρ-open if for each $a \in G$ ($\bar{a} \in G$) there exists $r \in [0_{\mathbf{L}}, 1_{\mathbf{L}})$ such that $B_r^\rho(a) \subseteq G$ ($B_r^\rho(\bar{a}) \subseteq G$). The similarity topologies on M and M^n are induced by the set of all ρ-open subsets of M and M^n which are denoted by T_ρ and T_ρ^n, respectively.*

Theorem 5.10. *Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, and $\mathcal{M}$ be an $\mathbf{L}$-structure satisfying the axioms of similarity. T_ρ and T_ρ^n form topologies on M and M^n.*

Proof. Clearly, $\emptyset, M \in T_\rho$. Let $\{G_i\}_{i\in I}$ be a family of elements of T_ρ and take an arbitrary $a \in \cup_{i\in I}G_i$. Since there is an $i \in I$ such that $a \in G_i$, there exists $r \in [0_{\mathbf{L}}, 1_{\mathbf{L}})$ such that $B_r^\rho \subseteq G_i \subseteq \cup_{i\in I}G_i$ which shows that $\cup_{i\in I}G_i \in T_\rho$. If $G, H \in T_\rho$ and $a \in G \cap H$, then there exist $r_G, r_H \in [0_{\mathbf{L}}, 1_{\mathbf{L}})$ such that $B_{r_G}^\rho(a) \subseteq G$ and $B_{r_H}^\rho(a) \subseteq H$. So for $r = \max\{r_G, r_H\}$ and $x \in B_r^\rho(a)$ we have $\|\rho(x,a)\|_{\mathcal{M}}^{\mathbf{L}} > r \geq r_G$ and $\|\rho(x,a)\|_{\mathcal{M}}^{\mathbf{L}} > r \geq r_H$. Therefore $x \in B_{r_G}^\rho(a)$ and $x \in B_{r_H}^\rho(a)$ that is $B_r^\rho(a) \subseteq B_{r_G}^\rho(a) \cap B_{r_H}^\rho(a) \subseteq G \cap H$. Hence $G \cap H \in T_\rho$. A similar argument shows that T_ρ^n is a topology on M^n. □

For any n-ary predicate symbol P and n-ary function symbol f, the extensionality axioms with respect to the similarity relation ρ are as follows [12, Chapter 5].

(E1) $\forall\bar{x}, \forall\bar{y}\big[\rho_n(\bar{x},\bar{y}) \to \big(P(\bar{x}) \leftrightarrow P(\bar{y})\big)\big]$.

(E2) $\forall\bar{x}, \forall\bar{y}\big[\rho_n(\bar{x},\bar{y}) \to \rho\big(f(\bar{x}), f(\bar{y})\big)\big]$.

Theorem 5.11. *Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, $\mathcal{L}$ be a first-order language, and $\mathcal{M}$ be an $\mathbf{L}$-structure for $\mathcal{L}$ satisfying the axioms of similarity. If for some n-ary predicate symbol $P \in \mathcal{L}$ and some n-ary function symbol $f \in \mathcal{L}$, $\mathcal{M}$ satisfies (E1) and (E2), then $P_{\mathcal{M}} : (M^n, T_\rho^n) \to (\mathbf{L}, T_*)$ and $f_{\mathcal{M}} : (M^n, T_\rho^n) \to (M, T_\rho)$ are continuous functions.*

Proof. For continuity of $P_{\mathcal{M}}$ we show that for each $G \in T^*$, $P_{\mathcal{M}}^{-1}(G) \in T_\rho^n$. To this end, for an arbitrary element $\bar{a} \in P_{\mathcal{M}}^{-1}(G)$ we must find $r \in [0_{\mathbf{L}}, 1_{\mathbf{L}})$ such that $B_r^\rho(\bar{a}) \subseteq P_{\mathcal{M}}^{-1}(G)$. But, $\bar{a} \in P_{\mathcal{M}}^{-1}(G)$ means that $P_{\mathcal{M}}(\bar{a}) \in G$ and therefore there exists $r \in [0_{\mathbf{L}}, 1_{\mathbf{L}})$ such that $B_r(P_{\mathcal{M}}(\bar{a})) \subseteq G$. We show that this r is that we search for it. If $\bar{b} \in B_r^\rho(\bar{a})$ then $\|\rho_n(\bar{b}, \bar{a})\|_{\mathcal{M}}^{\mathbf{L}} > r$. But since $\mathcal{M} \models$ E1,

$$\inf_{\bar{a},\bar{b} \in M^n} \left(\|\rho_n(\bar{a}, \bar{b})\|_{\mathcal{M}}^{\mathbf{L}} \Rightarrow_* e_* \left(P_{\mathcal{M}}(\bar{a}), P_{\mathcal{M}}(\bar{b})\right)\right) = 1_{\mathbf{L}}.$$

Therefore, $e_* \left(P_{\mathcal{M}}(\bar{a}), P_{\mathcal{M}}(\bar{b})\right) \geq \|\rho_n(\bar{b}, \bar{a})\|_{\mathcal{M}}^{\mathbf{L}} > r$ which means that $P_{\mathcal{M}}(\bar{b}) \in B_r(P_{\mathcal{M}}(\bar{a}))$. So, $P_{\mathcal{M}}(\bar{b}) \in G$ that is $\bar{b} \in P_{\mathcal{M}}^{-1}(G)$. Hence, $B_r^\rho(\bar{a}) \subseteq P_{\mathcal{M}}^{-1}(G)$.

Similarly, for continuity of $f_{\mathcal{M}}$ we demonstrate that for each $G \in T_\rho$, $f_{\mathcal{M}}^{-1}(G) \in T_\rho^n$. So, for $\bar{a} \in f_{\mathcal{M}}^{-1}(G)$ we shall find $r \in [0_{\mathbf{L}}, 1_{\mathbf{L}})$ such that $B_r^\rho(\bar{a}) \subseteq f_{\mathcal{M}}^{-1}(G)$. $\bar{a} \in f_{\mathcal{M}}^{-1}(G)$ implies that $f_{\mathcal{M}}(\bar{a}) \in G$ and so there exists $r \in [0_{\mathbf{L}}, 1_{\mathbf{L}})$ such that $B_r^\rho(f_{\mathcal{M}}(\bar{a})) \subseteq G$. We show that $B_r^\rho(\bar{a}) \subseteq f_{\mathcal{M}}^{-1}(G)$. If $\bar{b} \in B_r^\rho(\bar{a})$ then $\|\rho_n(\bar{b}, \bar{a})\|_{\mathcal{M}}^{\mathbf{L}} > r$. Now, $\mathcal{M} \models$ E2 implies that

$$\inf_{\bar{a},\bar{b} \in M^n} \left(\|\rho_n(\bar{a}, \bar{b})\|_{\mathcal{M}}^{\mathbf{L}} \Rightarrow_* \|\rho\left(f_{\mathcal{M}}(\bar{a}), f_{\mathcal{M}}(\bar{b})\right)\|_{\mathcal{M}}^{\mathbf{L}}\right) = 1_{\mathbf{L}}.$$

Hence, $\|\rho\left(f_{\mathcal{M}}(\bar{a}), f_{\mathcal{M}}(\bar{b})\right)\|_{\mathcal{M}}^{\mathbf{L}} \geq \|\rho_n(\bar{b}, \bar{a})\|_{\mathcal{M}}^{\mathbf{L}} > r$. So, $f_{\mathcal{M}}(\bar{b}) \in B_r^\rho(f_{\mathcal{M}}(\bar{a})) \subseteq G$ that is $\bar{b} \in f_{\mathcal{M}}^{-1}(G)$ which completes the proof. □

Definition 5.12. *Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, and $\mathcal{L}$ be a first-order language containing a similarity relation ρ. An $\mathbf{L}$-structure $\mathcal{M}$ for $\mathcal{L}$ which satisfies the similarity axioms for ρ, and furthermore, satisfies the extensionality axioms with respect to the similarity relation ρ for every function symbol and predicate symbol, is called a continuous $\mathbf{L}$-structure for $\mathcal{L}$.*

5.2.2 Some facts about filters

To manage the compactness theorem with ultraproduct method we remind some facts about filters.

Definition 5.13. [3, Section 6 of Chapter 1, Definition 1 and Definition 4] *For a set X, a nonempty family $\mathfrak{F}$ of subset of X is said to be a filter on X whenever,*

- *the empty set is not an element of $\mathfrak{F}$,*
- *$A \cap B \in \mathfrak{F}$, for any $A, B \in \mathfrak{F}$,*
- *if $A \in \mathfrak{F}$ and $A \subseteq C$ then $C \in \mathfrak{F}$.*

A filter $\mathfrak{F}$ on X is called an ultrafilter whenever for any subset A of X, either A or $X \setminus A$ is an element of $\mathfrak{F}$. Any filter is contained in an ultrafilter (See [3, Section 6 of Chapter 1, Theorem 1]).

Fact 5.14. [3, Section 6 of Chapter 1, Proposition 1] *Any filter has the finite intersection property. On the other hand, when a nonempty family $\mathfrak{D}$ of nonempty subset of X admits the finite intersection property, then there exists a filter $\mathfrak{F}$ containing $\mathfrak{D}$. The intersection of all such filters is called the filter generated by $\mathfrak{D}$.*

Definition 5.15. [3, Section 7 of Chapter 1, a modification of Definition 1] *A filter $\mathfrak{F}$ on a topological space X, converges to an element $x \in X$, whenever for each open set U containing x, U is an element of $\mathfrak{F}$. This is denoted by $\mathfrak{F} \to x$ and x is called a limit point of $\mathfrak{F}$.*

Fact 5.16. [3, Section 9 of Chapter 1, equivalents of Definition 1] *X is a compact topological space if and only if every filter $\mathcal{F}$ on X can be extended to a convergent filter on X, and especially X is compact if and only if every ultrafilter on X converges.*

Fact 5.17. [3, Section 8 of Chapter 1, Proposition 1] *X is a Hausdorff topological space if and only if every filter $\mathcal{F}$ on X has at most one limit point.*

Fact 5.18. [3, Section 7 of Chapter 1, Definition 3 and Corollary 1] *Let $f : X \to Y$ be a continuous function at $x_0 \in X$, $\mathfrak{F}$ be a filter on X convergent to x_0, and $f(\mathfrak{F})$ be the filter on Y generated by the set $\{f(A) : A \in \mathfrak{F}\}$. Then $f(\mathfrak{F}) \to f(x_0)$.*

Definition 5.19. *Let X be a topological space, I be a nonempty set, $\mathfrak{F}$ be a filter on I, the range of $\mathbf{x} \in X^I$ be $\{x_i\}_{i\in I}$, and $\mathbf{x}^*(\mathfrak{F}) = \{A \subseteq X : \mathbf{x}^{-1}(A) \in \mathfrak{F}\}$. If $\mathbf{x}^*(\mathfrak{F})$ is convergent to $x \in X$, then we call x the $\mathfrak{F}$-limit of the family $\{x_i\}_{i\in I}$ and write $\lim_{\mathfrak{F}} x_i = x$.*

Remark 5.20. *By Definition 5.19, $\lim_{\mathfrak{F}} x_i = x$ if and only if for every neighbourhood U of x, $\{i : x_i \in U\}$ is an element of $\mathfrak{F}$.*

Remark 5.21. *A consequence of Fact 5.16 and Fact 5.17 implies that with the notions in Definition 5.19, X is a compact Hausdorff space if and only if $\lim_{\mathfrak{F}} x_i$ is convergent to a unique point $x \in X$.*

Another version of Fact 5.18 is the following.

Corollary 5.22. *Let $f : X \to Y$ be a continuous function at $x_0 \in X$, I be a nonempty set, and $\mathfrak{F}$ be a filter on I. If $\lim_{\mathfrak{F}} x_i = x_0$ then $\lim_{\mathfrak{F}} f(x_i) = f(x_0)$.*

Proof. Assume that $\{x_i\}_{i\in I}$ is the range of the function $\mathbf{x} \in X^I$. So, $\mathbf{x}^*(\mathfrak{F}) \to x_0$ and by Fact 5.18 $f\big(\mathbf{x}^*(\mathfrak{F})\big) \to f(x_0)$. Now, if we show that $f\big(\mathbf{x}^*(\mathfrak{F})\big) \subseteq (\mathbf{x} \circ f)^*(\mathfrak{F})$, then we have $(\mathbf{x} \circ f)^*(\mathfrak{F}) \to f(x_0)$ which fulfills the proof.

Let $B \in f\big(\mathbf{x}^*(\mathfrak{F})\big)$. So, there exists $A \in \mathbf{x}^*(\mathfrak{F})$ such that $f(A) \subseteq B$. Hence, $A \subseteq f^{-1}(B)$ and therefore $\mathbf{x}^{-1}(A) \subseteq \mathbf{x}^{-1}\big(f^{-1}(B)\big)$. But $A \in \mathbf{x}^*(\mathfrak{F})$ and so $\mathbf{x}^{-1}(A) \in \mathfrak{F}$ which implies that $\mathbf{x}^{-1}\big(f^{-1}(B)\big) \in \mathfrak{F}$. Hence, $B \in (\mathbf{x} \circ f)^*(\mathfrak{F})$. ☐

5.2.3 Ultraproduct of structures

For a family of continuous structures $\{\mathcal{M}_i\}_{i\in I}$ and a filter $\mathfrak{F}$ on I, we introduce the $\mathfrak{F}$-product of family $\{\mathcal{M}_i\}_{i\in I}$.

Definition 5.23. *Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, $\mathcal{L}$ be a first-order language, $\{\mathcal{M}_i\}_{i\in I}$ be a family of continuous $\mathbf{L}$-structures for $\mathcal{L}$, $\mathfrak{F}$ be a filter on I, and $M_I = \prod_{i\in I} M_i$ be the direct product of family $\{\mathcal{M}_i\}_{i\in I}$ whose elements denoted by $\mathbf{a} = \{\mathbf{a}(i)\}_{i\in I}$. Furthermore, Let $\sim_{\mathfrak{F}}$ be the equivalence relation on M_I defined by "$\mathbf{a} \sim_{\mathfrak{F}} \mathbf{b}$ if and only if $\big\{i \in I : \mathcal{M}_i \models_{\mathbf{L}} \rho\big(\mathbf{a}(i), \mathbf{b}(i)\big)\big\} \in \mathfrak{F}$" (i.e., $\mathbf{a} \sim_{\mathfrak{F}} \mathbf{b}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are similar on a big set of structures indexed by an element of $\mathfrak{F}$), $[\mathbf{a}]_{\mathfrak{F}}$ be the equivalence classes of $\mathbf{a} \in M_I$, and $M_{I,\mathfrak{F}}$ be the set of equivalence classes of M_I under $\sim_{\mathfrak{F}}$. The $\mathfrak{F}$-product of family $\{\mathcal{M}_i\}_{i\in I}$ which is denoted by $\mathcal{M}_{I,\mathfrak{F}}$ is as follows:*

- *the underlying universe of $\mathcal{M}_{I,\mathfrak{F}}$ is $M_{I,\mathfrak{F}}$,*
- *for any n-ary predicate symbol $P \in \mathcal{L}$, suppose that $P_{\mathcal{M}_{I,\mathfrak{F}}} : M^n_{I,\mathfrak{F}} \to L$ be the partial function defined by $P_{\mathcal{M}_{I,\mathfrak{F}}}\big([\mathbf{x}_1]_{\sim_{\mathfrak{F}}}, ..., [\mathbf{x}_n]_{\sim_{\mathfrak{F}}}\big) = \lim_{\mathfrak{F}} P_{\mathcal{M}_i}\big(\mathbf{x}_1(i), ..., \mathbf{x}_n(i)\big)$ where the limit is in the topology T_* of L,*
- *for any n-ary function symbol $f \in \mathcal{L}$, let $f_{\mathcal{M}_{I,\mathfrak{F}}} : M^n_{I,\mathfrak{F}} \to M_{I,\mathfrak{F}}$ is defined by $f_{\mathcal{M}_{I,\mathfrak{F}}}\big([\mathbf{x}_1]_{\sim_{\mathfrak{F}}}, ..., [\mathbf{x}_n]_{\sim_{\mathfrak{F}}}\big) = [\mathbf{f}_{\mathbf{x}}]_{\sim_{\mathfrak{F}}}$ in which $\mathbf{f}_{\mathbf{x}}(i) = f_{\mathcal{M}_i}\big(\mathbf{x}_1(i), ..., \mathbf{x}_n(i)\big)$.*

When $\mathfrak{F}$ is an ultrafilter, we call $\mathcal{M}_{I,\mathfrak{F}}$ the ultraproduct of $\{\mathcal{M}_i\}_{i\in I}$.

The following lemma shows that $\mathcal{M}_{I,\mathfrak{F}}$ is well-defined.

Lemma 5.24. *Let $*$ be a continuous t-norm and $\mathbf{L}$ be a subalgebra of $[0,1]_*$. For any set of continuous $\mathbf{L}$-structures $\{\mathcal{M}_i\}_{i\in I}$ and any filter $\mathfrak{F}$ on I, $\mathcal{M}_{I,\mathfrak{F}}$ is well-defined.*

Proof. We show that the interpretation of any n-ary predicate and function symbol is well-defined. Consider two n-tuples $([\mathbf{a}_1]_{\mathfrak{F}}, [\mathbf{a}_2]_{\mathfrak{F}}, \cdots, [\mathbf{a}_n]_{\mathfrak{F}})$ and $([\mathbf{b}_1]_{\mathfrak{F}}, ([\mathbf{b}_2]_{\mathfrak{F}}, \cdots, ([\mathbf{b}_n]_{\mathfrak{F}})$ in $M^n_{I,\mathfrak{F}}$ and assume that $[\mathbf{a}_j]_{\mathfrak{F}} = [\mathbf{b}_j]_{\mathfrak{F}}$ for all $1 \le j \le n$. So, $A_j = \big\{i \in I : \mathcal{M}_i \models_{\mathbf{L}} \rho\big(\mathbf{a}_j(i), \mathbf{b}_j(i)\big)\big\} \in \mathfrak{F}$ for any $1 \le j \le n$. Therefore, by the finite intersection property of filters, $A = \bigcap_{j=1}^n A_j$ is also belong to $\mathfrak{F}$. Hence, $\|\rho\left(\mathbf{a}_j(i), \mathbf{b}_j(i)\right)\|^{\mathbf{L}}_{\mathcal{M}_i} = 1_{\mathbf{L}}$ for any $i \in A$ and $1 \le j \le n$. Now, If we denote $\big(x_1(i), ..., x_n(i)\big)$ by $\overline{x(i)}$, then the following two paragraphs show that n-ary predicates and n-ary functions of the $\mathfrak{F}$-product structure are well-defined.

1. Well-definition of predicate $P_{\mathcal{M}_{I,\mathfrak{F}}}$: for each $i \in A$, $\mathcal{M}_i \models_{\mathbf{L}} E1$, therefore

$$\begin{aligned} e_*\left(P_{\mathcal{M}_i}\big(\overline{\mathbf{a}(i)}\big), P_{\mathcal{M}_i}\big(\overline{\mathbf{b}(i)}\big)\right) &\ge \|\rho_n\Big(\overline{\mathbf{a}(i)}, \overline{\mathbf{b}(i)}\Big)\|^{\mathbf{L}}_{\mathcal{M}_i} \\ &= \|\rho\big(\mathbf{a}_1(i), \mathbf{b}_1(i)\big)\|^{\mathbf{L}}_{\mathcal{M}_i} * ... * \|\rho\big(\mathbf{a}_n(i), \mathbf{b}_n(i)\big)\|^{\mathbf{L}}_{\mathcal{M}_i} \\ &= 1_{\mathbf{L}}. \end{aligned}$$

Hence, by Lemma 4.3 (3), we deduced that $P_{\mathcal{M}_i}\big(\overline{\mathbf{a}(i)}\big) = P_{\mathcal{M}_i}\big(\overline{\mathbf{b}(i)}\big)$ for any $i \in A$. Accordingly, $\big\{i \in I : P_{\mathcal{M}_i}\big(\overline{\mathbf{a}(i)}\big) = P_{\mathcal{M}_i}\big(\overline{\mathbf{b}(i)}\big)\big\} \in \mathfrak{F}$. Thus $\lim_{\mathfrak{F}} P_{\mathcal{M}_i}\big(\overline{\mathbf{a}(i)}\big)$ exists if and only if $\lim_{\mathfrak{F}} P_{\mathcal{M}_i}\big(\overline{\mathbf{b}(i)}\big)$ exists and since (L, T_*) is a Hausdorff space,

$$P_{\mathcal{M}_{I,\mathfrak{F}}}\downarrow\big([\mathbf{a}_1]_{\sim_\mathfrak{F}},....,[\mathbf{a}_n]_{\sim_\mathfrak{F}}\big)=P_{\mathcal{M}_{I,\mathfrak{F}}}\downarrow\big([\mathbf{b}_1]_{\sim_\mathfrak{F}},....,[\mathbf{b}_n]_{\sim_\mathfrak{F}}\big).$$

Thus, whenever $P_{\mathcal{M}_{I,\mathfrak{F}}}$ is defined, it is well-defined.
2. Well-definition of function $f_{\mathcal{M}_{I,\mathfrak{F}}}$: for each $i\in A$, $\mathcal{M}_i\models_{\mathbf{L}} E2$, so

$$\begin{aligned}\left\|\rho\left(f_{\mathcal{M}_i}(\overline{\mathbf{a}(i)}),f_{\mathcal{M}_i}(\overline{\mathbf{b}(i)})\right)\right\|^{\mathbf{L}}_{\mathcal{M}_i} &\geq \|\rho_n(\overline{\mathbf{a}(i)},\overline{\mathbf{b}(i)})\|^{\mathbf{L}}_{\mathcal{M}_i}\\ &= \|\rho(\mathbf{a}_1(i),\mathbf{b}_1(i))\|^{\mathbf{L}}_{\mathcal{M}_i}*...*\|\rho(\mathbf{a}_n(i),\mathbf{b}_n(i))\|^{\mathbf{L}}_{\mathcal{M}_i}\\ &= 1_{\mathbf{L}}.\end{aligned}$$

Therefore, $\big\{i\in I:\|\rho(f_{\mathcal{M}_i}(\overline{\mathbf{a}(i)}),f_{\mathcal{M}_i}(\overline{\mathbf{b}(i)}))\|^{\mathbf{L}}_{\mathcal{M}_i}=1_{\mathbf{L}}\big\}$ contains A which implies that $\big\{i\in I:\mathcal{M}_i\models_{\mathbf{L}}\rho(\mathbf{f_a}(i),\mathbf{f_b}(i))\big\}\in\mathfrak{F}$. Hence, $\mathbf{f_a}\sim_\mathfrak{F}\mathbf{f_b}$ that is

$$f_{\mathcal{M}_{I,\mathfrak{F}}}\big([\mathbf{a}_1]_{\sim_\mathfrak{F}},....,[\mathbf{a}_n]_{\sim_\mathfrak{F}}\big)=f_{\mathcal{M}_{I,\mathfrak{F}}}\big([\mathbf{b}_1]_{\sim_\mathfrak{F}},....,[\mathbf{b}_n]_{\sim_\mathfrak{F}}\big).$$

□

The following lemma, which is used in the Łoś Theorem (Theorem 5.26), shows the reason why we use the ultraproduct of structures, instead of the $\mathfrak{F}$-product.

Lemma 5.25. *Let $*$ be a continuous t-norm and $\mathbf{L}$ be a subalgebra of $[0,1]_*$. If $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are two families of elements of the topological space (L,T_*) and $\mathfrak{F}$ is an ultrafilter on I, then "$\lim_\mathfrak{F} x_i\leq\lim_\mathfrak{F} y_i$ iff $\{i:x_i\leq y_i\}\in\mathfrak{F}$".*

Proof. Let $E=\{i:x_i\leq y_i\}\in\mathfrak{F}$. Thus, for each $i\in E$, $x_i\Rightarrow_* y_i=1_{\mathbf{L}}$. Therefore, $\{i:e_*(x_i\Rightarrow_* y_i,1_{\mathbf{L}})=1_{\mathbf{L}}\}\in\mathfrak{F}$. Hence, $\{i:e_*(x_i\Rightarrow_* y_i,1_{\mathbf{L}})>r\}\in\mathfrak{F}$ for any $r<1_{\mathbf{L}}$, which concludes that $lim_\mathfrak{F}(x_i\Rightarrow_* y_i)=1_{\mathbf{L}}$. Now, continuity of $\Rightarrow_*$ implies that $lim_\mathfrak{F}x_i\Rightarrow_* lim_\mathfrak{F}y_i=1_{\mathbf{L}}$. Thus $lim_\mathfrak{F}x_i\leq lim_\mathfrak{F}y_i$.

Conversely, let $lim_\mathfrak{F}x_i\leq lim_\mathfrak{F}y_i$, and by contradiction assume that $E=\{i:x_i\leq y_i\}\notin\mathfrak{F}$. Firstly, note that $\{i:x_i=y_i\}\subseteq E\notin\mathfrak{F}$ and so $lim_\mathfrak{F}x_i\neq lim_\mathfrak{F}y_i$. On the other hand, as $\mathfrak{F}$ is an ultrafilter, $(I\setminus E)\in\mathfrak{F}$. So, $\{i:x_i>y_i\}\in\mathfrak{F}$. Thus, by the same reason as the first part of the proof, $lim_\mathfrak{F}x_i\geq lim_\mathfrak{F}y_i$ which together with our assumption implies that $lim_\mathfrak{F}x_i=lim_\mathfrak{F}y_i$, a contradiction. □

Theorem 5.26. *(Łoś Theorem) Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, (L,T_*) be a compact space, $\{\mathcal{M}_i\}_{i\in I}$ be a set of continuous $\mathbf{L}$-structures, and $\mathfrak{F}$ be an ultrafilter on I. Then, for any formula $\varphi(\bar{x})$ and $\big\{[\mathbf{a}_i]_{\sim_\mathfrak{F}}\big\}_{i=1}^n\subseteq M_{I,\mathfrak{F}}$,*

$$\big\|\varphi\big([\mathbf{a}_1]_{\sim_\mathfrak{F}},....,[\mathbf{a}_n]_{\sim_\mathfrak{F}}\big)\big\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}}=\lim_\mathfrak{F}\big\|\varphi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\big\|^{\mathbf{L}}_{\mathcal{M}_i}.$$

Proof. The proof is by induction on formulas.

- Clearly, for every atomic formula, by definition of the $\mathfrak{F}$-product of family $\{\mathcal{M}_i\}_{i\in I}$,

$$\big\|P\big([\mathbf{a}_1]_{\sim_\mathfrak{F}},....,[\mathbf{a}_n]_{\sim_\mathfrak{F}}\big)\big\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}}=\lim_\mathfrak{F}\big\|P\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\big\|^{\mathbf{L}}_{\mathcal{M}_i}.$$

- Let $\varphi(\bar{x})=\theta(\bar{x})\Box\psi(\bar{x})$, where $\Box\in\{\to,\&\}$ and the induction hypothesis holds for $\theta(\bar{x})$ and $\psi(\bar{x})$. By Theorem 4.9, the interpretation of $\Box$ is continuous and so by Corollary 5.22, the statement holds for $\varphi(\bar{x})$.

- Let $\varphi(\bar{x})=\forall y\,\psi(\bar{x},y)$, where for each $[\mathbf{c}]_{\sim_\mathfrak{F}}\in M_{I,\mathfrak{F}}$ and $\big\{[\mathbf{a}_i]_{\sim_\mathfrak{F}}\big\}_{i=1}^n\subseteq M_{I,\mathfrak{F}}$,

$$\big\|\psi\big([\mathbf{a}_1]_{\sim_\mathfrak{F}},....,[\mathbf{a}_n]_{\sim_\mathfrak{F}},[\mathbf{c}]_{\sim_\mathfrak{F}}\big)\big\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}}=\lim_\mathfrak{F}\big\|\psi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i),\mathbf{c}(i)\big)\big\|^{\mathbf{L}}_{\mathcal{M}_i}.$$

  For each $i\in I$ and $[\mathbf{c}]_{\sim_\mathfrak{F}}\in M_{I,\mathfrak{F}}$ and $\big\{[\mathbf{a}_i]_{\sim_\mathfrak{F}}\big\}_{i=1}^n\subseteq M_{I,\mathfrak{F}}$,

$$\big\|\psi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i),\mathbf{c}(i)\big)\big\|^{\mathbf{L}}_{\mathcal{M}_i}\geq\inf_{\mathbf{c}(i)\in M_i}\big\|\psi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\big\|^{\mathbf{L}}_{\mathcal{M}_i},$$

  which implies that

$$\big\|\psi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i),\mathbf{c}(i)\big)\big\|^{\mathbf{L}}_{\mathcal{M}_i}\geq\big\|\varphi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\big\|^{\mathbf{L}}_{\mathcal{M}_i}.$$

  Compactness of (L,T_*) implies that $\lim_\mathfrak{F}\big\|\varphi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\big\|^{\mathbf{L}}_{\mathcal{M}_i}$ exists and therefore by Lemma 5.25,

$$\lim_{\mathfrak{F}} \left\|\psi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i),\mathbf{c}(i)\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i} \geq \lim_{\mathfrak{F}} \left\|\varphi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i},$$

that is

$$\left\|\psi\big([\mathbf{a}_1]_{\sim_{\mathfrak{F}}},....,[\mathbf{a}_n]_{\sim_{\mathfrak{F}}},[\mathbf{c}]_{\sim_{\mathfrak{F}}}\big)\right\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}} \geq \lim_{\mathfrak{F}} \left\|\varphi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i}.$$

So,

$$\inf_{[\mathbf{c}]_{\sim_{\mathfrak{F}}}\in M_{I,\mathfrak{F}}} \left\|\psi\big([\mathbf{a}_1]_{\sim_{\mathfrak{F}}},....,[\mathbf{a}_n]_{\sim_{\mathfrak{F}}},[\mathbf{c}]_{\sim_{\mathfrak{F}}}\big)\right\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}} \geq \lim_{\mathfrak{F}} \left\|\varphi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i}.$$

Therefore

$$\left\|\varphi\big([\mathbf{a}_1]_{\sim_{\mathfrak{F}}},....,[\mathbf{a}_n]_{\sim_{\mathfrak{F}}}\big)\right\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}} \geq \lim_{\mathfrak{F}} \left\|\varphi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i}.$$

For the reverse inequality, for each $v \in L$, we show that

$$\text{if } \left\|\varphi\big([\mathbf{a}_1]_{\sim_{\mathfrak{F}}},....,[\mathbf{a}_n]_{\sim_{\mathfrak{F}}}\big)\right\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}} \geq v \text{ then } \lim_{\mathfrak{F}} \left\|\varphi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i} \geq v.$$

Suppose for the propose of contradiction that $\left\|\varphi\big([\mathbf{a}_1]_{\sim_{\mathfrak{F}}},....,[\mathbf{a}_n]_{\sim_{\mathfrak{F}}}\big)\right\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}} \geq v$ but $\lim_{\mathfrak{F}} \left\|\varphi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i} < v$. Thus Lemma 5.25 implies that

$$E = \big\{i : \left\|\varphi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i)\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i} < v\big\} \in \mathfrak{F}.$$

So, for each $i \in E$, $\inf_{\mathbf{c}(i)\in M_i} \left\|\psi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i),\mathbf{c}(i)\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i} < v$ which means that for each $i \in E$ there is $b_i \in M_i$ such that $\left\|\psi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i),b_i\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i} < v$. Now,

$$\big\{i : \left\|\psi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i),b_i\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i} < v\big\},$$

contains E and therefore is an element of $\mathfrak{F}$. Thus, by Lemma 5.25

$$\lim_{\mathfrak{F}} \left\|\psi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i),b_i\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i} < v.$$

So, if $\mathbf{b} = \{\mathbf{b}(i)\}_{i\in I}$ is defined by $\mathbf{b}(i) = b_i$, then

$$\begin{aligned}
\left\|\varphi\big([\mathbf{a}_1]_{\sim_{\mathfrak{F}}},....,[\mathbf{a}_n]_{\sim_{\mathfrak{F}}}\big)\right\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}} &= \inf_{[\mathbf{c}]_{\sim_{\mathfrak{F}}}\in M_{I,\mathfrak{F}}} \left\|\psi\big([\mathbf{a}_1]_{\sim_{\mathfrak{F}}},....,[\mathbf{a}_n]_{\sim_{\mathfrak{F}}},[\mathbf{c}]_{\sim_{\mathfrak{F}}}\big)\right\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}} \\
&\leq \left\|\psi\big([\mathbf{a}_1]_{\sim_{\mathfrak{F}}},....,[\mathbf{a}_n]_{\sim_{\mathfrak{F}}},[\mathbf{b}]_{\sim_{\mathfrak{F}}}\big)\right\|^{\mathbf{L}}_{\mathcal{M}_{I,\mathfrak{F}}} \\
&= \lim_{\mathfrak{F}} \left\|\psi\big(\mathbf{a}_1(i),...,\mathbf{a}_n(i),b_i\big)\right\|^{\mathbf{L}}_{\mathcal{M}_i} \\
&< v,
\end{aligned}$$

a contradiction.

- $\varphi(x_1,...,x_n) = \exists y\, \psi(x_1,...,x_n,y)$, the argument is similar to the previous item.

□

Theorem 5.27. *(Compactness theorem) Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, (L, T_*) be a compact space, and K be a compact subset of (L, T_*). Then, every finitely $K^{\mathbf{L}}$-satisfiable theory over $\mathbf{BL}\forall$ by continuous structures is $K^{\mathbf{L}}$-satisfiable.*

Proof. Let T be a finitely $K^{\mathbf{L}}$-satisfiable theory by continuous structures and $\mathcal{P}_{fin}(T)$ be the set of all finite subsets of T. For each $\varphi \in T$, let $\overline{\varphi} = \{\Sigma : \varphi \in \Sigma \text{ and } \Sigma \in \mathcal{P}_{fin}(T)\}$. The set $\mathfrak{D} = \{\overline{\varphi} : \varphi \in T\}$ has the finite intersection property and so there exists an ultrafilter $\mathfrak{F}$ on $\mathcal{P}_{fin}(T)$ containing $\mathfrak{D}$.

For any $\Sigma = \{\varphi_1,...,\varphi_n\} \in \mathcal{P}_{fin}(T)$, since T is finitely K-satisfiable by continuous structures, there exists a continuous $K^{\mathbf{L}}$-model $\mathcal{M}_\Sigma$ of Σ. In addition, for an arbitrary $\varphi \in T$, if $\Sigma \in \overline{\varphi}$ then $\mathcal{M}_\Sigma$ is a continuous $K^{\mathbf{L}}$-model of φ. Hence, for each $\varphi \in T$, $\{\Sigma : \mathcal{M}_\Sigma \text{ is a continuous } K^{\mathbf{L}} \text{ -model of } \varphi\} \supseteq \overline{\varphi} \in \mathfrak{F}$. Thus, since $\|\varphi\|^{\mathbf{L}}_{\mathcal{M}_\Sigma} \in K$ and K is a compact subset of (L, T^*), by Remark 5.20 and Remark 5.21, $\lim_{\mathfrak{F}} \|\varphi\|^{\mathbf{L}}_{\mathcal{M}_\Sigma}$ uniquely exists in K. Suppose that $\mathcal{M}$ be the $\mathfrak{F}$-ultraproduct of $\{\mathcal{M}_\Sigma\}_{\Sigma\in\mathcal{P}_{fin}(T)}$. By the Łoś theorem $\|\varphi\|^{\mathbf{L}}_{\mathcal{M}} = \lim_{\mathfrak{F}} \|\varphi\|^{\mathbf{L}}_{\mathcal{M}_\Sigma} \in K$ and therefore, $\mathcal{M}$ is a $K^{\mathbf{L}}$-model of T. □

6 Final remarks and further works

We study the compactness of satisfiability for the Hájek Basic logic. In our study, for any continuous t-norm $*$ we introduce two topologies on $[0,1]$ and $[0,1]^2$ and then we show that the interpretation of all of logical connectives are continuous functions with respect to these topologies. Using this fact, we prove a version of K-compactness theorem for **BL**. Continuing our study for **BL**$\forall$, we introduce the similarity topology on structures and then we establish a version of Łoś Theorem and finally K-compactness theorem proved for **BL**$\forall$. This article could be continued by a study of the Łoś Theorem for deriving a version of Keisler-Shelah isomorphism Theorem for **BL**$\forall$. A more interesting subject matter is the study of omitting types Theorem for **BL**$\forall$. Beside the possible application of the compactness, another interesting subject matter is studying the properties and application of the similarity topology on structures and also studying the induced topology of the interpretation of the equivalence relation on truth value sets.

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