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Original paper

(L, M)-fuzzy topological derived internal relations and (L, M)-fuzzy topological derived enclosed relations

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Abstract

In this paper, notions of (L, M)-fuzzy topological derived internal relation space and (L, M)-fuzzy topological derived interior space are introduced. It is proved that they are categorically isomorphic to (L, M)-fuzzy topological internal relation space and (L, M)-fuzzy topological space. Also, notions of (L, M)-fuzzy topological derived enclosed relation space and (L, M)-fuzzy topological derived closure space are introduced. It is proved that they are categorically isomorphic to (L, M)-fuzzy topological enclosed relation space and (L, M)-fuzzy topological space.

Keywords: (L, M)-fuzzy topological derived internal relation space, (L, M)-fuzzy topological derived interior space, (L, M)-fuzzy derived topological enclosed relation space, (L, M)-fuzzy topological derived closure space.

1 Introduction

Since Zadeh introduced the concept of fuzzy sets [42], many classic mathematical theories have been combined with fuzzy set theory such as fuzzy topology [3, 36, 43], fuzzy convergence [10, 11, 12, 14, 15, 17, 37], fuzzy matroid [6, 23, 24] and fuzzy convexity [15, 16, 18, 19, 29, 30, 32, 38]. As for the counterpart of fuzzy topology, Chang introduced fuzzy topology with the interval lattice I which is comprised by a crisp subset of I-powerset I^X [3]. Also, from a completely different point of view, Höhle introduced the notion of a fuzzy topology as an I-subset of a powerset 2^X [7]. Ying studied Höhle's topology by logically analyzing Chang's topology and called it fuzzifying topology [41]. Šostak and Kubiak respectively extended Höhle's topology to I-subset of I^X and M-subset of L^X [9, 31]. The extension of Höhle's topology in Kubiak's sense is usually called (L, M)-fuzzy topology whose characterizations have been discussed [5, 25, 26, 28].

Relations are frequently used to characterize fuzzy mathematical structures. In L-setting, Shi and Shi introduced L-topological internal relation and L-topological enclosed relation by which they characterized L-topologies [27]. Also, Liao and Wu introduced L-convex enclosed relation and characterized L-convex structures [13]. In (L, M)-fuzzy setting, Shi and Shi introduced (L, M)-fuzzy topological internal relation and (L, M)-fuzzy topological enclosed relation which are used to characterize (L, M)-fuzzy topologies [28]. Wu and Liao introduced (L, M)-fuzzy convex enclosed relation and characterized (L, M)-fuzzy convex structures [33]. Meanwhile, they also introduced (L, M)-fuzzy topological-convex enclosed relation by which they characterized (L, M)-fuzzy topological-convex spaces. Recently, Xiu and Li introduced (L, M)-fuzzy concave internal space and characterized (L, M)-fuzzy concave structures [35].

The notion of derived sets was originally defined by Georg Cantor in 1872 [8]. In a topological space, the derived set of a given set is composed by all of its adherent points. In addition, the closure of the given set is exact the union of itself and its derived set. In view of these properties, Shi presented an axiomatic concept of derived operators and studied its induced fuzzy derived operators [22]. Actually, Shi's derived operators were defined in the framework of topological spaces. By analogy, scholars further introduced some derived operators in M-fuzzifying convex structures and M-fuzzifying matroids [4, 21, 34, 44]. These derived operators have a common feature which take an axiom to show

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the relation between a set and its adherent points. In addition, they can be used to characterize M-fuzzifying convex structures or M-fuzzifying matroids. Also, from the perspective of the composition of derived sets, scholars extended adherent points and discussed Moore-Smith convergence theories in fuzzy topological spaces [1, 2, 20, 39, 40].

As mentioned above, derived sets are composed by adherent points which are associated with closure operators of topological spaces or hull operators of convex spaces. In addition, derived operators were only extended in mathematical structures of M-fuzzifying settings such as M-fuzzifying convex structures and M-fuzzifying matroids. So, some natural questions arise: can derived operators be extended in (L, M)-fuzzy topological spaces? Can they be obtained by interior operators of (L, M)-fuzzy topological spaces? Further, can they be extend in (L, M)-fuzzy internal relation spaces or (L, M)-fuzzy enclosed relation spaces? Can they be applied to characterize (L, M)-fuzzy topologies? Motivated by these problems, we present this paper. Specifically, we aim to seek reasonable notions to fulfill the following diagrams.

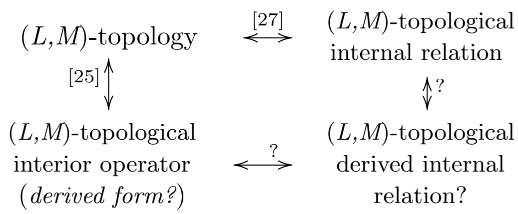


Figure 1: Problem 1

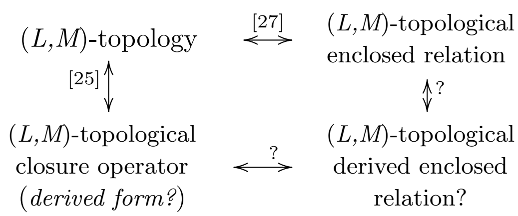


Figure 2: Problem 2

The arrangement of this paper is as follows. In Section 2, we recall some basic concepts, denotations and results. In Section 3, we introduce (L, M)-fuzzy topological derived internal relation space and (L, M)-fuzzy topological derived interior space by which we characterize (L, M)-fuzzy topological internal relation space and (L, M)-fuzzy topological space. In Section 4, we introduce (L, M)-fuzzy topological derived enclosed relation space and (L, M)-fuzzy topological derived closure space by which we characterize (L, M)-fuzzy topological enclosed relation space and (L, M)-fuzzy topological space.

2 Preliminaries

In this paper, X and Y are nonempty sets. The power set of X is denoted by 2^X. L and M are completely distributive lattice and M has an inverse involution $'$. The smallest (resp. largest) element in L and M is denoted by $\bot$ (resp. $\top$). An element $a \in L$ is called a co-prime, if for all $b, c \in L$, $a \le b \vee c$ implies $a \le b$ or $a \le c$. The set of all co-primes in $L\backslash\{\bot\}$ is denoted by $J(L)$. For any $a \in L$, there is an $L_1 \in J(L)$ such that $a = \bigvee_{b \in L_1} b$. Also, we adopt the convention that $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$. A binary relation $\prec$ on L is defined by $a \prec b$ iff for each $L_1 \subseteq L$, $b \le \bigvee L_1$ implies some $d \in L_1$ such that $a \le d$. The mapping $\beta : L \to 2^L$, defined by $\beta(a) = \{b : b \prec a\}$, satisfies $\beta(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \beta(a_i)$ for $\{a_i\}_{i \in I} \subseteq L$. In addition, $\beta(a)$ and $\beta^*(a) = \beta(a) \cap J(L)$ satisfy $a = \bigvee \beta(a) = \bigvee \beta^*(a)$ and $\beta^*(a) = \bigcup_{b \in \beta^*(a)} \beta^*(b)$ [25].

An L-fuzzy set on X is a mapping $A : X \to L$. The set of all L-fuzzy sets on X is denoted by L^X. The smallest (resp, largest) element in L^X is denoted by $\underline{\bot}$ (resp. $\underline{\top}$). An L-fuzzy point x_λ ($\lambda \in L\backslash\{\bot\}$) is an L-fuzzy set defined by $x_\lambda(x) = \lambda$ and $x_\lambda(y) = \bot$ for any $y \in X\backslash\{x\}$. The set of all L-fuzzy points on L^X is denoted by $Pt(L^X)$. Also, we denote $J(L^X) = \{x_\lambda \in Pt(L^X) : \lambda \in J(L)\}$ and $\beta^*(A) = \{x_\lambda \in Pt(L^X) : \lambda \in \beta^*(A(x))\}$ for $A \in L^X$. For an L-fuzzy set $A \in L^X$, the M-fuzzy set $\chi_A \in M^{L^X}$ is defined by: for any $B \in L^X$, $\chi_A(B) = \top$ whenever $B \le A$ and $\chi_A(B) = \bot$ otherwise. For a mapping $f : X \to Y$, the L-fuzzy mapping $f_L^{\rightarrow} : L^X \to L^Y$ is defined by $f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) : f(x) = y\}$ for $A \in L^X$ and $y \in Y$, and the mapping $f_L^{\leftarrow} : L^Y \to L^X$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$ for $B \in L^Y$ and $x \in X$ [9, 25].

Definition 2.1. [9, 25] *A mapping $\mathcal{T} : L^X \to M$ is called an (L, M)-fuzzy topology on L^X and the pair $(X, \mathcal{T})$ is called an (L, M)-fuzzy topological space if*
$(LMT1)$ $\mathcal{T}(\underline{\perp}) = \mathcal{T}(\underline{\perp}) = \top$;
$(LMT2)$ $\mathcal{T}(\bigvee_{i\in I} A_i) \geq \bigwedge_{i\in I} \mathcal{T}(A_i)$ *for any subset* $\{A_i\}_{i\in I} \subseteq L^X$;
$(LMT3)$ $\mathcal{T}(A \wedge B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ *for all* $A, B \in L^X$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be (L, M)-fuzzy topological spaces. A mapping $f : X \to Y$ is called an (L, M)-fuzzy continuous mapping if $\mathcal{T}_Y(B) \leq \mathcal{T}_X(f_L^{\leftarrow}(B))$ for any $B \in L^Y$. The category of (L, M)-fuzzy topological spaces and (L, M)-fuzzy continuous mappings is denoted by **LMTOP** [25].

Definition 2.2. [28] *A binary relation $\mathcal{I} : L^X \times L^X \to M$ is called an (L, M)-fuzzy topological internal relation on L^X and the pair $(X, \mathcal{I})$ is called an (L, M)-fuzzy topological internal space if*
$(LMTIR1)$ $\mathcal{I}(\underline{\top}, \underline{\top}) = \top$;
$(LMTIR2)$ $\mathcal{I}(A, B) \neq \perp$ *implies* $A \leq B$;
$(LMTIR3)$ $\mathcal{I}(\bigvee_{i\in I} A_i, B) = \bigwedge_{i\in I} \mathcal{I}(A_i, B)$;
$(LMTIR4)$ $\mathcal{I}(A, B) \leq \bigvee_{C\in L^X} \mathcal{I}(A, C) \wedge \mathcal{I}(C, B)$;
$(LMTIR5)$ $\mathcal{I}(A, B \wedge C) = \mathcal{I}(A, B) \wedge \mathcal{I}(A, C)$.

Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be (L, M)-fuzzy topological internal relation spaces. A mapping $f : X \to Y$ is called an (L, M)-fuzzy topological internal relation preserving mapping if $\mathcal{I}_Y(A, B) \leq \mathcal{I}_X(f_L^{\leftarrow}(A), f_L^{\leftarrow}(B))$ for all $A, B \in L^Y$. The category of (L, M)-fuzzy topological internal relation spaces and (L, M)-fuzzy internal relation preserving mappings is denoted by **LMTIR**. It is isomorphic to **LMTOP** [28].

Definition 2.3. [28] *A binary relation $\mathcal{E} : L^X \times L^X \to M$ is called an (L, M)-fuzzy topological enclosed relation on L^X and the pair $(X, \mathcal{E})$ is called an (L, M)-fuzzy topological enclosed relation space if*
$(LMTER1)$ $\mathcal{E}(\underline{\perp}, \underline{\perp}) = \top$;
$(LMTER2)$ $\mathcal{E}(A, B) \neq \perp$ *implies* $A \leq B$;
$(LMTER3)$ $\mathcal{E}(A, \bigwedge_{i\in I} B_i) = \bigwedge_{i\in I} \mathcal{E}(A, B_i)$;
$(LMTER4)$ $\mathcal{E}(A, B) \leq \bigvee_{C\in L^X} \mathcal{E}(A, C) \wedge \mathcal{E}(C, B)$;
$(LMTER5)$ $\mathcal{E}(A \vee B, C) = \mathcal{E}(A, C) \wedge \mathcal{E}(B, C)$.

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be (L, M)-fuzzy topological enclosed relation spaces. A mapping $f : X \to Y$ is called an (L, M)-fuzzy topological enclosed relation preserving mapping if $\mathcal{E}_Y(A, B) \leq \mathcal{E}_X(f_L^{\leftarrow}(A), f_L^{\leftarrow}(B))$ for $A, B \in L^Y$. The category of (L, M)-fuzzy topological enclosed relation spaces and (L, M)-fuzzy topological enclosed relation preserving mappings is denoted by **LMTER**. It is isomorphic to **LMTOP** [28].

3 (L, M)-fuzzy topological derived internal relation spaces

In this section, we introduce (L, M)-fuzzy topological derived internal relation space and (L, M)-fuzzy topological derived interior space by which we characterize (L, M)-fuzzy topological internal relation space and (L, M)-fuzzy topological space.

Definition 3.1. *A binary relation $\mathcal{I}^d : L^X \times L^X \to M$ is called an (L, M)-fuzzy topological derived internal relation and the pair $(X, \mathcal{I}^d)$ is called an (L, M)-fuzzy topological derived internal relation space, if*
$(LMTDIR1)$ $\mathcal{I}^d(\underline{\top}, \underline{\top}) = \top$;
$(LMTDIR2)$ $\mathcal{I}^d(A, B) = \bigwedge_{x_\lambda\in\beta^*(A)} \mathcal{I}^d(x_\lambda, B \vee x_\lambda)$;
$(LMTDIR3)$ $\mathcal{I}^d(\bigvee_{i\in I} A_i, B) = \bigwedge_{i\in I} \mathcal{I}^d(A_i, B)$;
$(LMTDIR4)$ $\mathcal{I}^d(A, B) \leq \bigvee_{A\wedge B\leq C} \mathcal{I}^d(A \wedge B, C) \wedge \mathcal{I}^d(C, B)$;
$(LMTDIR5)$ $\mathcal{I}^d(A, B \wedge C) = \mathcal{I}^d(A, B) \wedge \mathcal{I}^d(A, C)$.

It directly follows from (LMTDIR3) and (LMTDIR5) that $\mathcal{I}^d(C, D) \leq \mathcal{I}^d(A, B)$ for all $A, B, C, D \in L^X$ with $A \leq C$ and $D \leq B$.

Let $(X, \mathcal{I}^d_X)$ and $(Y, \mathcal{I}^d_Y)$ be (L, M)-fuzzy topological derived internal relation spaces. A mapping $f : X \to Y$ is called (L, M)-fuzzy topological derived internal relation preserving, if for all $A, B \in L^Y$,

$$\mathcal{I}^d_Y(A, B) \leq \mathcal{I}^d_X(f_L^{\leftarrow}(A \wedge B), f_L^{\leftarrow}(B)).$$

The category of (L, M)-fuzzy topological derived internal relation spaces and (L, M)-fuzzy topological derived internal relation preserving mappings is denoted by **LMTDIR**.

Next, we study relations between **LMTDIR** and **LMTIR**.

Theorem 3.2. *Let $(X,\mathcal{I}^d)$ be an (L,M)-fuzzy topological derived internal relation space. Define a binary mapping $\mathcal{I}_{\mathcal{I}^d}:L^X\times L^X\to M$ by*

$$\forall A,B\in L^X,\quad \mathcal{I}_{\mathcal{I}^d}(A,B)=\bigvee_{A=B\wedge C}\mathcal{I}^d(C,B).$$

Then $(X,\mathcal{I}_{\mathcal{I}^d})$ is an (L,M)-fuzzy topological internal relation space.

Proof. Let $A,B\in L^X$. Clearly, we have $\mathcal{I}_{\mathcal{I}^d}(A,B)=\mathcal{I}^d(A,B)$ whenever $A\leq B$ and $\mathcal{I}_{\mathcal{I}^d}(A,B)=\bot$ otherwise. Next, we check that $\mathcal{I}_{\mathcal{I}^d}$ satisfies (LMTIR1)–(LMTIR5).

Indeed, (LMTIR1)–(LMTIR3) are easy. Next, we prove that (LMTIR4) and (LMTIR5) hold for $\mathcal{I}_{\mathcal{I}^d}$.

(LMTIR4). By (LMTDIR4) and (LMTDIR5), we have

$$\begin{aligned}\mathcal{I}_{\mathcal{I}^d}(A,B) &= \bigvee_{A=B\wedge D}\mathcal{I}^d(D,B)\\ &\leq \bigvee_{A=B\wedge D\leq C}\mathcal{I}^d(A,C)\wedge\mathcal{I}^d(C,B)\wedge\mathcal{I}^d(A,B)\\ &\leq \bigvee_{A\leq B\wedge C}\mathcal{I}^d(A,C\wedge B)\wedge\mathcal{I}^d(B\wedge C,B)\\ &\leq \mathcal{I}_{\mathcal{I}^d}(A,B\wedge C)\wedge\mathcal{I}_{\mathcal{I}^d}(B\wedge C,B)\\ &\leq \bigvee_{G\in L^X}\mathcal{I}_{\mathcal{I}^d}(A,G)\wedge\mathcal{I}_{\mathcal{I}^d}(G,B).\end{aligned}$$

(LMTIR5). By (LMTDIR5), we have

$$\begin{aligned}\mathcal{I}_{\mathcal{I}^d}(A,B\wedge C) &= \bigvee_{A=(B\wedge C)\wedge D}\mathcal{I}^d(D,B\wedge C)\\ &\leq \mathcal{I}_{\mathcal{I}^d}(A,B)\wedge\mathcal{I}_{\mathcal{I}^d}(A,C)\\ &= \bigvee_{A=B\wedge D=C\wedge E}\mathcal{I}^d(D,B)\wedge\mathcal{I}^d(E,C)\\ &\leq \bigvee_{A=B\wedge D=C\wedge E}\mathcal{I}^d(D\wedge E,B\wedge C)\\ &\leq \mathcal{I}_{\mathcal{I}^d}(A,B\wedge C).\end{aligned}$$

Thus $\mathcal{I}_{\mathcal{I}^d}(A,B\wedge C)=\mathcal{I}_{\mathcal{I}^d}(A,B)\wedge\mathcal{I}_{\mathcal{I}^d}(A,C)$.

Therefore $\mathcal{I}_{\mathcal{I}^d}$ is an (L,M)-fuzzy topological internal relation. □

Theorem 3.3. *Let $(X,\mathcal{I}^d_X)$ and $(Y,\mathcal{I}^d_Y)$ be (L,M)-fuzzy topological derived internal relation spaces. If $f:X\to Y$ is an (L,M)-fuzzy topological derived internal relation preserving mapping, then $f:(X,\mathcal{I}_{\mathcal{I}^d_X})\to(Y,\mathcal{I}_{\mathcal{I}^d_Y})$ is an (L,M)-fuzzy topological internal relation preserving mapping.*

Proof. We have $\mathcal{I}_{\mathcal{I}^d_Y}(A,B)\leq\bigvee_{A=B\wedge C}\mathcal{I}^d_X(f_L^{\leftarrow}(B)\wedge f_L^{\leftarrow}(C),f_L^{\leftarrow}(B))\leq\mathcal{I}_{\mathcal{I}^d_X}(f_L^{\leftarrow}(A),f_L^{\leftarrow}(B))$. Thus f is an (L,M)-fuzzy topological internal relation preserving mapping. □

Theorem 3.4. *Let $(X,\mathcal{I})$ be an (L,M)-fuzzy topological internal relation space. Define a binary relation $\mathcal{I}^d_{\mathcal{I}}:L^X\times L^X\to M$ by*

$$\forall A\in L^X,\quad \mathcal{I}^d_{\mathcal{I}}(A,B)=\bigwedge_{x_\lambda\in\beta^*(A)}\mathcal{I}(x_\lambda,B\vee x_\lambda).$$

Then $\mathcal{I}^d_{\mathcal{I}}$ is an (L,M)-fuzzy topological derived internal relation.

Proof. Clearly, $\mathcal{I}^d_{\mathcal{I}}(C,D)\leq\mathcal{I}^d_{\mathcal{I}}(A,B)$ for all $A,B,C,D\in L^X$ with $A\leq C$ and $D\leq B$. In addition, for all $A,B\in L^X$ with $A\leq B$, we have $\mathcal{I}^d_{\mathcal{I}}(A,B)=\mathcal{I}(A,B)$ by (LMTIR3). Next, we check that $\mathcal{I}_{\mathcal{I}^d}$ satisfies (LMTDIR1)–(LMTDIR5).

(LMTDIR1). We have $\mathcal{I}^d_{\mathcal{I}}(\underline{\top},\underline{\top})=\mathcal{I}(\underline{\top},\underline{\top})=\top$ by (LMTIR1).

(LMTDIR2). By (LMTIR3), $\mathcal{I}^d_{\mathcal{I}}(A,B)=\bigwedge_{x_\lambda\in\beta^*(A)}\bigwedge_{\eta\in\beta^*(\lambda)}\mathcal{I}(x_\eta,B\vee x_\lambda)=\bigwedge_{x_\lambda\in\beta^*(A)}\mathcal{I}^d_{\mathcal{I}}(x_\lambda,B\vee x_\lambda)$.

(LMTDIR3). We have $\mathcal{I}^d_{\mathcal{I}}(\bigvee_{i\in I}A_i,B)=\bigwedge_{i\in I}\bigwedge_{x_\lambda\in\beta^*(A_i)}\mathcal{I}(x_\lambda,B\vee x_\lambda)=\bigwedge_{i\in I}\mathcal{I}^d_{\mathcal{I}}(A_i,B)$.

(LMTDIR4). Clearly, $\mathcal{I}^d_{\mathcal{I}}(A,B)\le \mathcal{I}(A\wedge B,B)$. In addition, by (LMTIR4) and (LMTIR2), we have

$$\begin{aligned}\mathcal{I}(A\wedge B,B) &\le \bigvee_{C\in L^X}\mathcal{I}(A\wedge B,C)\wedge\mathcal{I}(C,B)\\ &= \bigvee_{A\wedge B\le C\le B}\mathcal{I}(A\wedge B,C)\wedge\mathcal{I}(C,B)\\ &= \bigvee_{A\wedge B\le C\le B}\mathcal{I}^d_{\mathcal{I}}(A\wedge B,C)\wedge\mathcal{I}^d_{\mathcal{I}}(C,B)\\ &\le \bigvee_{A\wedge B\le C}\mathcal{I}^d_{\mathcal{I}}(A\wedge B,C)\wedge\mathcal{I}^d_{\mathcal{I}}(C,B).\end{aligned}$$

(LMTDIR5). By (LMTIR5), $\mathcal{I}^d_{\mathcal{I}}(A,B\wedge C)=\bigwedge_{x_\lambda\in\beta^*(A)}\mathcal{I}(x_\lambda,(B\vee x_\lambda)\wedge(C\vee x_\lambda))=\mathcal{I}^d_{\mathcal{I}}(A,B)\wedge\mathcal{I}^d_{\mathcal{I}}(A,C)$.
Therefore $\mathcal{I}^d_{\mathcal{I}}$ is an (L,M)-fuzzy topological derived internal relation. □

Theorem 3.5. *Let $(X,\mathcal{I}_X)$ and $(Y,\mathcal{I}_Y)$ be (L,M)-fuzzy topological internal relation spaces. If $f:X\to Y$ is an (L,M)-fuzzy topological internal relation preserving mapping, then $f:(X,\mathcal{I}^d_{\mathcal{I}_X})\to(Y,\mathcal{I}^d_{\mathcal{I}_Y})$ is an (L,M)-fuzzy topological derived internal relation preserving mapping.*

Proof. For $A,B\in L^Y$, $\mathcal{I}^d_{\mathcal{I}_Y}(A,B)\le\mathcal{I}^d_{\mathcal{I}_Y}(A\wedge B,B)\le\bigwedge_{f^{\to}_L(x_\lambda)\in\beta^*(A\wedge B)}\mathcal{I}_Y(f^{\to}_L(x_\lambda),B)\le\mathcal{I}^d_{\mathcal{I}_X}(f^{\leftarrow}_L(A\wedge B),f^{\leftarrow}_L(B))$. Therefore f is an (L,M)-fuzzy topological derived internal relation preserving mapping. □

Theorem 3.6. *We have $\mathcal{I}_{\mathcal{I}^d_{\mathcal{I}}}=\mathcal{I}$ for any (L,M)-fuzzy topological interval relation space $(X,\mathcal{I})$ and $\mathcal{I}^d_{\mathcal{I}_{\mathcal{I}^d}}=\mathcal{I}^d$ for any (L,M)-fuzzy topological derived internal relation space $(X,\mathcal{I}^d)$.*

Proof. Let $(X,\mathcal{I})$ be an (L,M)-fuzzy topological internal relation space. Let $A,B\in L^X$. If $A\not\le B$, then it is clear that $\mathcal{I}_{\mathcal{I}^d_{\mathcal{I}}}(A,B)=\mathcal{I}(A,B)=\perp$. If $A\le B$, then $\mathcal{I}_{\mathcal{I}^d_{\mathcal{I}}}(A,B)=\mathcal{I}^d_{\mathcal{I}}(A,B)=\mathcal{I}(A,B)$. Thus $\mathcal{I}_{\mathcal{I}^d_{\mathcal{I}}}=\mathcal{I}$.

Let $(X,\mathcal{I}^d)$ be an (L,M)-fuzzy topological derived internal relation space. For all $A,B\in L^X$, we have

$$\mathcal{I}^d_{\mathcal{I}_{\mathcal{I}^d}}(A,B)=\bigwedge_{x_\lambda\in\beta^*(A)}\mathcal{I}_{\mathcal{I}^d}(x_\lambda,B\vee x_\lambda)=\bigwedge_{x_\lambda\in\beta^*(A)}\mathcal{I}^d(x_\lambda,B\vee x_\lambda)=\mathcal{I}^d(A,B).$$

Therefore $\mathcal{I}^d_{\mathcal{I}_{\mathcal{I}^d}}=\mathcal{I}^d$. □

Based on Theorems 3.2 and 3.3, we define a functor $\mathbb{U}$:**LMTDIR**→**LMTIR** by:

$$\mathbb{U}((X,\mathcal{I}^d))=(X,\mathcal{I}_{\mathcal{I}^d}),\quad \mathbb{U}(f)=f.$$

Based on Theorems 3.2–3.6, $\mathbb{U}$ is an isomorphic functor. So we have the following conclusion.

Theorem 3.7. *The category* **LMTDIR** *is isomorphic to the category* **LMTIR**.

Based on Theorems 3.2–3.6, relations between **LMTDIR** and **LMTOP** are as follows.

Corollary 3.8. (1) *Let $(X,\mathcal{T})$ be an (L,M)-fuzzy topological space. Define $\mathcal{I}^d_{\mathcal{T}}:L^X\times L^X\to M$ by*

$$\forall A,B\in L^X,\quad \mathcal{I}^d_{\mathcal{T}}(A,B)=\bigwedge_{x_\lambda\in\beta^*(A)}\ \bigvee_{x_\lambda\le C\le B\vee x_\lambda}\mathcal{T}(C).$$

Then $\mathcal{I}^d_{\mathcal{T}}$ is an (L,M)-fuzzy topological derived internal relation.

(2) *Let $(X,\mathcal{I}^d)$ be an (L,M)-fuzzy derived internal relation space. Define a mapping $\mathcal{T}_{\mathcal{I}^d}:L^X\to M$ by*

$$\forall A\in L^X,\quad \mathcal{T}_{\mathcal{I}^d}(A)=\mathcal{I}^d(A,A).$$

Then $\mathcal{T}_{\mathcal{I}^d}$ is an (L,M)-fuzzy topology.

(3) *We have $\mathcal{T}_{\mathcal{I}^d_{\mathcal{T}}}=\mathcal{T}$ for any (L,M)-fuzzy topological space $(X,\mathcal{T})$ and $\mathcal{I}^d_{\mathcal{T}_{\mathcal{I}^d}}=\mathcal{I}^d$ for any (L,M)-fuzzy topological derived internal relation space $(X,\mathcal{I}^d)$.*

Corollary 3.9. *The category* **LMTDIR** *is isomorphic to the category* **LMTOP**.

To simply characterize (L,M)-fuzzy topological derived internal relation space, we introduce (L,M)-fuzzy topological derived interior space as follows.

Definition 3.10. *An (L,M)-fuzzy operator $Int^d : L^X \to M^{J(L^X)}$ is called an (L,M)-fuzzy topological derived interior operator on L^X and the pair (X, Int^d) is called an (L,M)-fuzzy topological derived interior space if for all $A, B \in L^X$ and any $x_\lambda \in J(L^X)$,*

(LMTDInt1) $Int^d(\underline{\top})(x_\lambda) = \top$;

(LMTDInt2) $Int^d(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} Int^d(A \vee x_\mu)(x_\mu)$;

(LMTDInt3) $Int^d(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{B\leq A} \bigwedge_{y_\eta\in\beta^*(B\vee x_\mu)} Int^d(B \vee x_\mu)(y_\eta)$;

(LMTDInt4) $Int^d(A \wedge B) = Int^d(A) \wedge Int^d(B)$.

Let (X, Int^d_X) and (Y, Int^d_Y) be (L,M)-fuzzy topological derived interior spaces. A mapping $f : X \to Y$ is called an (L,M)-fuzzy topological derived interior preserving mapping, if for all $B \in L^Y$ and $x_\lambda \in J(L^X)$,

$$Int^d_Y(B)(f^{\rightarrow}_L(x_\lambda)) \leq \bigwedge_{x_\mu\in\beta^*(x_\lambda\wedge f^{\leftarrow}_L(B))} Int^d_X(f^{\leftarrow}_L(B))(x_\mu).$$

The category of (L,M)-fuzzy topological derived interior spaces and (L,M)-fuzzy topological derived interior preserving mappings is denoted by **LMTDINT**. The proof of the following lemma is direct.

Lemma 3.11. *Let $\varphi : L^X \to M^{J(L^X)}$ be an operator.*

(1) $\varphi(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \varphi(A)(x_\mu)$ provided that φ satisfies (LMTDInt2).

(2) φ is monotonic provided that φ satisfies (LMTDInt4).

Proposition 3.12. *Let $\varphi : L^X \to M^{J(L^X)}$ be a monotonic operator satisfying (LMTDInt2). The following conditions are equivalent.*

(1) φ satisfies (LMTDInt3).

(2) $x_\lambda \leq A$ implies $\varphi(A)(x_\lambda) = \bigvee_{x_\lambda\leq B\leq A} \bigwedge_{y_\eta\in\beta^*(B)} \varphi(B)(y_\eta)$.

(3) $x_\lambda \leq A$ implies $\varphi(A)(x_\lambda) = \bigvee_{x_\lambda\leq B\leq A} [\varphi(B)(x_\lambda) \wedge \bigwedge_{y_\eta\in\beta^*(B)} \varphi(A)(y_\eta)]$.

(4) $\varphi(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{x_\mu\leq B\leq A\vee x_\mu} [\varphi(B)(x_\mu) \wedge \bigwedge_{y_\eta\in\beta^*(B)} \varphi(A \vee x_\mu)(y_\eta)]$.

Proof. (1) ⇒ (2). By (LMTDInt3), it is easy to check that (2) holds.

(2) ⇔ (3). Similar to Lemma 3.5 in [28].

(2) ⇒ (1). By (LMTDInt2) and (2), we have

$$\varphi(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{x_\mu\leq D\leq A\vee x_\mu} \bigwedge_{y_\eta\in\beta^*(D)} \varphi(D)(y_\eta) \geq \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{B\leq A} \bigwedge_{y_\eta\in\beta^*(B\vee x_\mu)} \varphi(B \vee x_\mu)(y_\eta).$$

Conversely, for any $\mu \in \beta^*(\lambda)$ and any $D \in L^X$ with $x_\mu \leq D \leq A \vee x_\mu$, we denote $B = D \wedge A$. Then $B \leq A$ and $D = B \vee x_\mu$. Thus, by (LMTDInt2) and (2), we have

$$\varphi(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{x_\mu\leq D\leq A\vee x_\mu} \bigwedge_{y_\eta\in\beta^*(D)} \varphi(D)(y_\eta) \leq \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{B\leq A} \bigwedge_{y_\eta\in\beta^*(B\vee x_\mu)} \varphi(B \vee x_\mu)(y_\eta).$$

Hence $\varphi(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{B\leq A} \bigwedge_{y_\eta\in\beta^*(B\vee x_\mu)} \varphi(B \vee x_\mu)(y_\eta)$. Therefore φ satisfies (LMTDInt3).

(3) ⇒ (4). By (LMTDInt2) and (3), we have

$$\varphi(A)(x_\lambda) \leq \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{x_\mu\leq B\leq A\vee x_\mu} [\varphi(B)(x_\mu) \wedge \bigwedge_{y_\eta\in\beta^*(B)} \varphi(A \vee x_\mu)(y_\eta)].$$

The inverse inclusion of the above inequality follows from the monotonicity of φ. So (4) holds.

(4) ⇒ (3). Let $x_\lambda \leq A$. By (4), $\varphi(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{x_\mu\leq B\leq A} [\varphi(B)(x_\mu) \wedge \bigwedge_{y_\eta\in\beta^*(B)} \varphi(A)(y_\eta)]$. Let $a \in \beta(\top)$ with $a \prec \varphi(A)(x_\lambda)$. For $\mu \in \beta^*(\lambda)$, there is a set $B_\mu \in L^X$ with $x_\mu \leq B_\mu \leq A$ such that $a \leq \varphi(B_\mu)(x_\mu)$ and $a \leq \bigwedge_{y_\eta\in\beta^*(B_\mu)} \varphi(A)(y_\eta)$. Let $D = \bigvee_{\mu\in\beta^*(\lambda)} B_\mu$. Then $x_\lambda \leq D \leq A$, $a \leq \bigwedge_{\mu\in\beta^*(\lambda)} \varphi(B_\mu)(x_\mu) \leq \bigwedge_{\mu\in\beta^*(\lambda)} \varphi(D)(x_\mu) = \varphi(D)(x_\lambda)$ and

$$a \leq \bigwedge_{\mu\in\beta^*(\lambda)} \bigwedge_{y_\eta\in\beta^*(B_\mu)} \varphi(A)(y_\eta) = \bigwedge_{y_\eta\in\beta^*(D)} \varphi(A)(y_\eta) \leq \bigvee_{x_\lambda\leq B\leq A} \bigwedge_{y_\eta\in\beta^*(B)} \varphi(A)(y_\eta).$$

So $a \leq \varphi(D)(x_\lambda) \wedge \bigwedge_{y_\eta\in\beta^*(D)} \varphi(A)(y_\eta)$. By the arbitrariness of $a \in \beta(\top)$, $\varphi(A)(x_\lambda) \leq \bigvee_{x_\lambda\leq B\leq A} [\varphi(B)(x_\lambda) \wedge \bigwedge_{y_\eta\in\beta^*(B)} \varphi(A)(y_\eta)]$. The inverse inclusion of the above inequality is clear. Therefore (3) holds. □

Next, we study relations between **LMTDIR** and **LMTDINT**.

Theorem 3.13. *Let (X, Int^d) be an (L,M)-fuzzy topological derived interior space. Define a binary relation $\mathcal{I}^d_{Int^d} : L^X \times L^X \to M$ by*

$$\forall A, B \in L^X, \quad \mathcal{I}^d_{Int^d}(A,B) = \bigwedge_{x_\lambda \in \beta^*(A)} Int^d(B)(x_\lambda).$$

Then $\mathcal{I}^d_{Int^d}$ is an (L,M)-fuzzy topological derived internal relation.

Proof. We check that $\mathcal{I}^d_{Int^d}$ satisfies (LMTDIR1)–(LMTDIR5).

(LMTDIR1). By (LMTDInt1), we have $\mathcal{I}^d_{Int^d}(\underline{\top}, \underline{\top}) = \bigwedge_{x_\lambda \in \beta^*(\underline{\top})} Int^d(\underline{\top})(x_\lambda) = \top$.

(LMTDIR2). By (LMTDInt2), $\bigwedge_{x_\lambda \in \beta^*(A)} \mathcal{I}^d_{Int^d}(x_\lambda, B \vee x_\lambda) = \bigwedge_{x_\mu \in \beta^*(A)} Int^d(B)(x_\mu) = \mathcal{I}^d_{Int^d}(A,B)$.

(LMTDIR3). We have $\mathcal{I}^d_{Int^d}(\bigvee_{i\in I} A_i, B) = \bigwedge_{i\in I} \bigwedge_{x_\lambda \in \beta^*(A_i)} Int^d(B)(x_\lambda) = \bigwedge_{i \in I} \mathcal{I}^d_{Int^d}(A_i, B)$.

(LMTDIR4). It is sufficient to prove that $\mathcal{I}_{Int^d}(A \wedge B, B) \le \bigvee_{A\wedge B \le C} \mathcal{I}^d_{Int^d}(A\wedge B, C) \wedge \mathcal{I}^d_{Int^d}(C, B)$.

Let $a \in \beta(\top)$ with $a \prec \bigwedge_{x_\lambda \in \beta^*(A\wedge B)} Int^d(B)(x_\lambda)$. We have $a \prec Int^d(B)(x_\lambda)$ for any $x_\lambda \in \beta^*(A \wedge B)$. By (2) of Proposition 3.12, there is a set $D_\lambda \in L^X$ with $x_\lambda = x_\lambda \wedge B \le D_\lambda \le B$ such that $a \le Int^d(D_\lambda)(x_\lambda)$ and $a \le Int^d(B)(y_\mu)$ for any $y_\mu \in \beta^*(D_\lambda)$. Let $D = \bigvee_{x_\lambda \in \beta^*(A \wedge B)} D_\lambda$. We have $A \wedge B \le D$ and $\bigwedge_{x_\lambda\in\beta^*(A\wedge B)} Int^d(D)(x_\lambda) \ge \bigwedge_{x_\lambda \in \beta^*(A\wedge B)} Int^d(D_\lambda)(x_\lambda) \ge a$ and

$$\bigwedge_{y_\mu \in \beta^*(D)} Int^d(B)(y_\mu) = \bigwedge_{x_\lambda \in \beta^*(A\wedge B)} \bigwedge_{y_\mu \in \beta^*(D_\lambda)} Int^d(B)(y_\mu) \ge a.$$

Hence we conclude that

$$a \le \bigvee_{A\wedge B\le C} [\bigwedge_{x_\lambda \in \beta^*(A\wedge B)} Int^d(C)(x_\lambda) \wedge \bigwedge_{y_\mu \in \beta^*(C)} Int^d(B)(y_\mu)] = \bigvee_{A\wedge B \le C} \mathcal{I}^d_{Int^d}(A\wedge B, C) \wedge \mathcal{I}^d_{Int^d}(C,B).$$

Therefore, by the arbitrariness of a, $\mathcal{I}^d_{Int^d}(A,B) \le \mathcal{I}^d_{Int^d}(A\wedge B, B) \le \bigvee_{A\wedge B\le C} \mathcal{I}^d_{Int^d}(A\wedge B, C) \wedge \mathcal{I}^d_{Int^d}(C,B)$.

(LMTDIR5). By (LMTDInt4), we have $\mathcal{I}^d_{Int^d}(A, B\wedge C) = \mathcal{I}^d_{Int^d}(A,B) \wedge \mathcal{I}^d_{Int^d}(A,C)$.

Therefore $\mathcal{I}^d_{Int^d}$ is an (L,M)-fuzzy topological derived internal relation. □

Theorem 3.14. *Let (X, Int^d_X) and (Y, Int^d_Y) be (L,M)-fuzzy topological derived interior spaces. If $f : X \to Y$ is an (L,M)-fuzzy topological derived interior preserving mapping, then $f : (X, \mathcal{I}^d_{Int^d_X}) \to (Y, \mathcal{I}^d_{Int^d_Y})$ is an (L,M)-fuzzy topological derived internal relation preserving mapping.*

Proof. For all $A, B \in L^Y$, $\mathcal{I}^d_{Int^d_Y}(A,B) \le \bigwedge_{x_\lambda \in \beta^*(f^\leftarrow_L(A\wedge B))} Int^d_X(f^\leftarrow_L(B))(x_\lambda) = \mathcal{I}^d_{Int^d_X}(f^\leftarrow_L(A\wedge B), f^\leftarrow_L(B))$. Therefore f is an (L,M)-fuzzy topological derived internal relation preserving mapping. □

Theorem 3.15. *Let $(X, \mathcal{I}^d)$ be an (L,M)-fuzzy topological derived internal relation space. Define an operator $Int^d_{\mathcal{I}^d} : L^X \to M^{J(L^X)}$ by*

$$\forall A \in L^X, \forall x_\lambda \in J(L^X), \quad Int^d_{\mathcal{I}^d}(A)(x_\lambda) = \mathcal{I}^d(x_\lambda, A).$$

Then $Int^d_{\mathcal{I}^d}$ is an (L,M)-fuzzy topological derived interior operator.

Proof. (LMTDInt1). By (LMTDIR1), $Int^d_{\mathcal{I}^d}(\underline{\top})(x_\lambda) = \mathcal{I}^d(x_\lambda, \underline{\top}) \ge \mathcal{I}^d(\underline{\top}, \underline{\top}) = \top$.

(LMTDInt2). By (LMTDIR2), $Int^d_{\mathcal{I}^d}(A)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} \mathcal{I}^d(x_\mu, A \vee x_\mu) = \bigwedge_{\mu\in\beta^*(\lambda)} Int^d_{\mathcal{I}^d}(A \vee x_\mu)(x_\mu)$.

(LMTDInt3). Clearly, $Int^d_{\mathcal{I}^d}$ is monotonic. Let $x_\lambda \le A$. By (LMTDIR3) and (LMTDIR5), we have

$$\begin{aligned} Int^d_{\mathcal{I}^d}(A)(x_\lambda) &\le \bigvee_{x_\lambda \le C \le A} \mathcal{I}^d(x_\lambda, C) \wedge \mathcal{I}^d(C, A) \\ &= \bigvee_{x_\lambda \le C \le A} Int^d_{\mathcal{I}^d}(C)(x_\lambda) \wedge \bigwedge_{y_\mu \in \beta^*(C)} Int^d_{\mathcal{I}^d}(A)(y_\mu) \\ &\le Int^d_{\mathcal{I}^d}(A)(x_\lambda). \end{aligned}$$

Therefore (LMTDInt3) directly follows from (3) of Proposition 3.12.

(LMTDInt4). By (LMTDIR5), we have $Int^d_{\mathcal{I}^d}(A \wedge B)(x_\lambda) = \mathcal{I}^d(x_\lambda, A) \wedge \mathcal{I}^d(x_\lambda, B) = Int^d_{\mathcal{I}^d}(A)(x_\lambda) \wedge Int^d_{\mathcal{I}^d}(B)(x_\lambda)$.

Therefore $Int^d_{\mathcal{I}^d}$ is an (L,M)-fuzzy topological derived interior operator. □

Theorem 3.16. *Let $(X,\mathcal{I}_X^d)$ and $(Y,\mathcal{I}_Y^d)$ be (L,M)-fuzzy topological derived internal relation spaces. If $f:X\to Y$ is an (L,M)-fuzzy topological derived internal relation preserving mapping, then $f:(X,Int^d_{\mathcal{I}_X^d})\to(Y,Int^d_{\mathcal{I}_Y^d})$ is an (L,M)-fuzzy topological derived interior preserving mapping.*

Proof. For $B\in L^Y$ and $x_\lambda\in J(L^X)$, we have $\bigwedge_{x_\mu\in\beta^*(x_\lambda\wedge f_L^{\leftarrow}(B))} Int^d_{\mathcal{I}_X^d}(f_L^{\leftarrow}(B))(x_\mu)\geq \mathcal{I}_X^d(f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda)\wedge B),f_L^{\leftarrow}(B))\geq Int^d_{\mathcal{I}_Y^d}(B)(f_L^{\rightarrow}(x_\lambda))$. Therefore f is an (L,M)-fuzzy topological derived interior preserving mapping. □

Theorem 3.17. *We have $\mathcal{I}^d_{Int^d_{\mathcal{I}^d}}=\mathcal{I}^d$ for any (L,M)-fuzzy topological derived internal relation space $(X,\mathcal{I}^d)$ and $Int^d_{\mathcal{I}^d_{Int^d}}=Int^d$ for any (L,M)-fuzzy topological derived interior space (X,Int^d).*

Proof. Let $(X,\mathcal{I}^d)$ be an (L,M)-fuzzy topological derived internal relation space. For all $A,B\in L^X$, $\mathcal{I}^d_{Int^d_{\mathcal{I}^d}}(A,B)=\bigwedge_{x_\lambda\in\beta^*(A)}\mathcal{I}^d(x_\lambda,B)=\mathcal{I}^d(A,B)$. Therefore $\mathcal{I}^d_{Int^d_{\mathcal{I}^d}}=\mathcal{I}^d$.

Let (X,Int^d) be an (L,M)-fuzzy topological derived interior space. For all $A\in L^X$ and $x_\lambda\in J(L^X)$, we have $Int^d_{\mathcal{I}^d_{Int^d}}(A)(x_\lambda)=\bigwedge_{x_\mu\in\beta^*(x_\lambda)}Int^d(A)(x_\mu)=Int^d(A)(x_\lambda)$. Thus $Int^d_{\mathcal{I}^d_{Int^d}}=Int^d$. □

From Theorems 3.15 and 3.16, we define a functor $\mathbb{W}$:**LMTDIR**→**LMTDINT** by:

$$\mathbb{W}((X,\mathcal{I}^d))=(X,Int^d_{\mathcal{I}^d}),\quad \mathbb{W}(f)=f.$$

Based on Theorems 3.13–3.17, $\mathbb{W}$ is an isomorphic functor. Thus we have the following result.

Theorem 3.18. *The category* **LMTDIR** *is isomorphic to the category* **LMTDINT**.

From Theorems 3.13–3.17 and Corollary 3.9, we have the following conclusions.

Corollary 3.19. (1) *Let (X,Int^d) be an (L,M)-fuzzy topological derived interior space. Define a mapping $\mathcal{T}_{Int^d}:L^X\to M$ by*

$$\forall A\in L^X,\quad \mathcal{T}_{Int^d}(A)=\bigwedge_{x_\lambda\in\beta^*(A)}Int^d(A)(x_\lambda).$$

Then $\mathcal{T}_{Int^d}$ is an (L,M)-fuzzy topology.

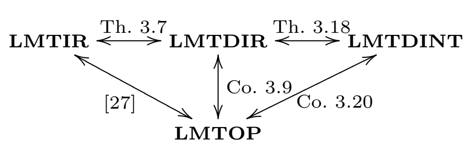
(2) *Let $(X,\mathcal{T})$ be an (L,M)-fuzzy topological space. Define an operator $Int^d_{\mathcal{T}}:L^X\to M^{J(L^X)}$ by*

$$\forall A\in L^X,\forall x_\lambda\in J(L^X),\quad Int^d_{\mathcal{T}}(A)(x_\lambda)=\bigvee_{x_\lambda\leq B\leq A\vee x_\lambda}\mathcal{T}(B).$$

Then $Int^d_{\mathcal{T}}$ is an (L,M)-fuzzy topological derived interior operator.

Corollary 3.20. *The category* **LMTDINT** *is isomorphic to the category* **LMTOP**.

Relations among categories mentioned in this section are shown by the following diagram.



Relations 1.

At the end of this section, we present a simple example to show that concepts of (L,M)-fuzzy derived internal relations and (L,M)-fuzzy derived interior operators are reasonable.

Example 3.21. *Let $X=\{x\}$ and $L=\{\bot,a,b,\top\}$ be a diamond lattice with two incomparable elements a,b. Let $M=\{\bot,\top\}=\{0,1\}$ be* **2***-lattice. Let $\mathcal{I}^d:L^X\times L^X\to M$ be a binary mapping defined by:*

$$\mathcal{I}^d(\underline{\bot},\underline{\bot})=\mathcal{I}^d(\underline{\bot},x_a)=\mathcal{I}^d(\underline{\bot},x_b)=\mathcal{I}^d(\underline{\bot},\underline{\top})=0;$$
$$\mathcal{I}^d(x_a,\underline{\bot})=\mathcal{I}^d(x_a,x_a)=\mathcal{I}^d(\underline{\top},\underline{\bot})=\mathcal{I}^d(\underline{\top},x_a)=0;$$
$$\mathcal{I}^d(x_b,\underline{\bot})=\mathcal{I}^d(x_b,x_a)=\mathcal{I}^d(x_b,x_b)=\mathcal{I}^d(x_b,\underline{\top})=1;$$
$$\mathcal{I}^d(x_a,x_b)=\mathcal{I}^d(x_a,\underline{\top})=\mathcal{I}^d(\underline{\top},x_b)=\mathcal{I}^d(\underline{\top},\underline{\top})=1.$$

It is routine to check that $\mathcal{I}^d$ is an (L,M)-fuzzy topological derived internal relation. In addition, the (L,M)-fuzzy topological internal relation $\mathcal{I}_{\mathcal{I}^d}$ and the (L,M)-fuzzy topological derived interior operator $cl^d_{\mathcal{I}^d}$ satisfy

$$\begin{aligned}
&\mathcal{I}_{\mathcal{I}^d}(\underline{\perp},\underline{\perp})=\mathcal{I}_{\mathcal{I}^d}(\underline{\perp},x_a)=\mathcal{I}_{\mathcal{I}^d}(\underline{\perp},x_b)=\mathcal{I}_{\mathcal{I}^d}(\underline{\perp},\underline{\top})=1;\\
&\mathcal{I}_{\mathcal{I}^d}(x_a,\underline{\top})=\mathcal{I}_{\mathcal{I}^d}(x_b,x_b)=\mathcal{I}_{\mathcal{I}^d}(x_b,\underline{\top})=\mathcal{I}_{\mathcal{I}^d}(\underline{\top},\underline{\top})=1;\\
&\mathcal{I}_{\mathcal{I}^d}(x_a,\underline{\perp})=\mathcal{I}_{\mathcal{I}^d}(x_a,x_a)=\mathcal{I}_{\mathcal{I}^d}(x_a,x_b)=\mathcal{I}_{\mathcal{I}^d}(x_b,\underline{\perp})=0;\\
&\mathcal{I}_{\mathcal{I}^d}(x_b,x_a)=\mathcal{I}_{\mathcal{I}^d}(\underline{\top},\underline{\perp})=\mathcal{I}_{\mathcal{I}^d}(\underline{\top},x_a)=\mathcal{I}_{\mathcal{I}^d}(\underline{\top},x_b)=0
\end{aligned}$$

and

$$\begin{aligned}
&Int^d_{\mathcal{I}^d}(\underline{\perp})(x_a)=Int^d_{\mathcal{I}^d}(x_a)(x_a)=0;\\
&Int^d_{\mathcal{I}^d}(\underline{\perp})(x_b)=Int^d_{\mathcal{I}^d}(x_a)(x_b)=Int^d_{\mathcal{I}^d}(x_b)(x_a)=Int^d_{\mathcal{I}^d}(x_b)(x_b)=Int^d_{\mathcal{I}^d}(\underline{\top})(x_a)=Int^d_{\mathcal{I}^d}(\underline{\top})(x_b)=1.
\end{aligned}$$

Also, the (L,M)-fuzzy topology $\mathcal{T}_{\mathcal{I}^d}$ satisfying $\mathcal{T}_{\mathcal{I}^d}(\underline{\perp})=\mathcal{T}_{\mathcal{I}^d}(x_b)=\mathcal{T}_{\mathcal{I}^d}(\underline{\top})=1$ and $\mathcal{T}_{\mathcal{I}^d}(x_a)=0$.

4 (L,M)-fuzzy topological derived enclosed relation spaces

In this section, we introduce (L,M)-fuzzy topological derived enclosed relation space and (L,M)-fuzzy topological derived closure space by which we characterize (L,M)-fuzzy topological enclosed relation space and (L,M)-fuzzy topological space. For $A\in L^X$ and $x_\lambda\in\beta^*(\underline{\top})$, we denote $A_{x_\lambda}=\bigvee\{y_\mu\in\beta^*(A):x_\lambda\not\leq y_\mu\}$ and $\beta^*_\lambda(L)=\{\mu\in\beta^*(\top):\lambda\in\beta^*(\mu)\}$. For convenience, we denote $x_\lambda\not\leq^* A$ provided that $x_\lambda\in\beta^*(\underline{\top})$ and $x_\lambda\not\leq A$. The following proposition is easy to check.

Proposition 4.1. *For all $x_\lambda,y_\eta\in\beta^*(\underline{\top})$, $\mu\in\beta^*_\lambda(L)$, $A,B\in L^X$ and $\{A_i\}_{i\in I}\subseteq L^X$, we have*
(1) $x_\lambda\not\leq^* A$ *implies* $A_{x_\lambda}=A$;
(2) $A\leq B$ *implies* $A_{x_\lambda}\leq B_{x_\lambda}$;
(3) $(A_{x_\lambda})_{x_\lambda}=A_{x_\lambda}$;
(4) $A_{x_\lambda}\leq A_{x_\mu}$ *and* $(A_{x_\mu})_{x_\lambda}=(A_{x_\lambda})_{x_\mu}=A_{x_\lambda}$;
(5) $y_\eta\not\leq^*\underline{\top}_{x_\lambda}$ *iff* $x=y$ *and* $\eta\in\beta^*_\lambda(L)$;
(6) $A=\bigwedge_{x_\lambda\not\leq^* A}\underline{\top}_{x_\lambda}$ *and* $\underline{\top}_{x_\lambda}=\bigwedge_{\eta\in\beta^*_\lambda(L)}\underline{\top}_{x_\eta}$;
(7) $(\bigvee_{i\in I}A_i)_{x_\lambda}=\bigvee_{i\in I}(A_i)_{x_\lambda}$.

Definition 4.2. *A binary relation $\mathcal{E}^d:L^X\times L^X\to M$ is called an (L,M)-fuzzy topological derived enclosed relation on L^X and the pair $(X,\mathcal{E}^d)$ is called an (L,M)-fuzzy topological derived enclosed relation space, if for all $A,B,C\in L^X$ and $x_\lambda\in\beta^*(\underline{\top})$,*
$(LMTDER1)$ $\mathcal{E}^d(\underline{\perp},\underline{\perp})=\top$;
$(LMTDER2)$ $\mathcal{E}^d(A,B)=\bigwedge_{x_\lambda\not\leq^* B}\bigwedge_{\mu\in\beta^*_\lambda(L)}\mathcal{E}^d(A_{x_\mu},\underline{\top}_{x_\lambda})\wedge\chi_{\underline{\top}_{x_\lambda}}(A_{x_\mu})$;
$(LMTDER3)$ $\mathcal{E}^d(A,\bigwedge_{i\in I}B_i)=\bigwedge_{i\in I}\mathcal{E}^d(A,B_i)$;
$(LMTDER4)$ $\mathcal{E}^d(A,B)\leq\bigvee_{C\leq B}\mathcal{E}^d(A,C)\wedge\mathcal{E}^d(C,A\vee B)$;
$(LMTDER5)$ $\mathcal{E}^d(A\vee B,C)=\mathcal{E}^d(A,C)\wedge\mathcal{E}^d(B,C)$.

It directly follows from (LMTDER3) and (LMTDER5) that $\mathcal{E}^d(C,D)\leq\mathcal{E}^d(A,B)$ for all $A,B,C,D\in L^X$ with $A\leq C$ and $D\leq B$.

Let $(X,\mathcal{E}^d_X)$ and $(Y,\mathcal{E}^d_Y)$ be (L,M)-fuzzy topological derived enclosed relation spaces. A mapping $f:X\to Y$ is called an (L,M)-fuzzy topological derived enclosed relation preserving mapping if for all $A,B\in L^Y$,

$$\mathcal{E}^d_Y(A,B)\leq\mathcal{E}^d_X(f^\leftarrow_L(A),f^\leftarrow_L(B)\vee f^\leftarrow_L(A)).$$

The category of (L,M)-fuzzy topological derived enclosed relation spaces and (L,M)-fuzzy topological derived enclosed relation preserving mappings is denoted by **LMTDER**.

Next, we consider relations between **LMTDER** and **LMTER**.

Theorem 4.3. *Let $(X,\mathcal{E}^d)$ be an (L,M)-fuzzy topological derived enclosed relation space. Define a binary relation $\mathcal{E}_{\mathcal{E}^d}:L^X\times L^X\to M$ by*

$$\forall A,B\in L^X,\quad \mathcal{E}_{\mathcal{E}^d}(A,B)=\bigvee_{A\vee C=B}\mathcal{E}^d(A,C).$$

Then $\mathcal{E}_{\mathcal{E}^d}$ is an (L,M)-fuzzy topological enclosed relation.

Proof. Let $A, B \in L^X$. If $A \not\leq B$, then $\mathcal{E}_{\mathcal{E}^d}(A,B) = \bigvee \emptyset = \bot$. If $A \leq B$, then $\mathcal{E}^d(A,C) \leq \mathcal{E}^d(A,B)$ for any $C \in L^X$ with $A \vee C = B$. Thus $\mathcal{E}_{\mathcal{E}^d}(A,B) = \mathcal{E}^d(A,B)$ in this case.

Clearly, (LMTER1)–(LMTER3) hold for $\mathcal{E}_{\mathcal{E}^d}$. Next, we check that (LMTER4) and (LMTER5) hold for $\mathcal{E}_{\mathcal{E}^d}$.

(LMTER4). Let $A, B \in L^X$. If $A \not\leq B$ then $\mathcal{E}_{\mathcal{E}^d}(A,B) = \bot$. Thus the desired result is trivial. Assume that $A \leq B$. By (LMTDER4) and (LMTDER5), we have $\mathcal{E}_{\mathcal{E}^d}(A,B) = \mathcal{E}^d(A,B)$ and

$$\begin{aligned}\mathcal{E}^d(A,B) &= \bigvee_{D \leq B} \mathcal{E}^d(A,D) \wedge \mathcal{E}^d(D, A \vee B) \wedge \mathcal{E}^d(A,B) \\ &= \bigvee_{D \leq B} \mathcal{E}^d(A,D) \wedge \mathcal{E}^d(A,B) \wedge \mathcal{E}^d(D,B) \\ &\leq \bigvee_{D \leq B} \mathcal{E}^d(A, D \vee A) \wedge \mathcal{E}^d(A \vee D, B) \\ &\leq \bigvee_{C \in L^X} \mathcal{E}_{\mathcal{E}^d}(A,C) \wedge \mathcal{E}_{\mathcal{E}^d}(C,B).\end{aligned}$$

(LMTER5). Let $A, B, C \in L^X$. If $A \vee B \not\leq C$, then $A \not\leq C$ or $B \not\leq C$. Thus $\mathcal{E}_{\mathcal{E}^d}(A \vee B, C) = \mathcal{E}_{\mathcal{E}^d}(A,C) \wedge \mathcal{E}_{\mathcal{E}^d}(B,C) = \bot$. Assume that $A \vee B \leq C$. By (LMTDER5), we have

$$\mathcal{E}_{\mathcal{E}^d}(A \vee B, C) = \mathcal{E}^d(A \vee B, C) = \mathcal{E}^d(A,C) \wedge \mathcal{E}^d(B,C) = \mathcal{E}_{\mathcal{E}^d}(A,C) \wedge \mathcal{E}_{\mathcal{E}^d}(B,C).$$

Therefore $\mathcal{E}_{\mathcal{E}^d}$ is an (L,M)-fuzzy topological enclosed relation. □

Theorem 4.4. *Let $(X, \mathcal{E}^d_X)$ and $(Y, \mathcal{E}^d_Y)$ be (L,M)-fuzzy topological derived enclosed relation spaces. If $f : X \to Y$ is an (L,M)-fuzzy topological derived enclosed relation preserving mapping, then $f : (X, \mathcal{E}_{\mathcal{E}^d_X}) \to (Y, \mathcal{E}_{\mathcal{E}^d_Y})$ is an (L,M)-fuzzy topological enclosed relation preserving mapping.*

Proof. Let $A, B \in L^X$. We have $\mathcal{E}_{\mathcal{E}^d_Y}(A,B) \leq \bigvee_{f_L^{\leftarrow}(A) \vee f_L^{\leftarrow}(C) = f_L^{\leftarrow}(B)} \mathcal{E}^d_X(f_L^{\leftarrow}(A), f_L^{\leftarrow}(A) \vee f_L^{\leftarrow}(C)) \leq \mathcal{E}_{\mathcal{E}^d_X}(f_L^{\leftarrow}(A), f_L^{\leftarrow}(B))$. So f is an (L,M)-fuzzy topological enclosed relation preserving mapping. □

Theorem 4.5. *Let $(X, \mathcal{E})$ be an (L,M)-fuzzy topological enclosed relation space. Define a binary relation $\mathcal{E}^d_{\mathcal{E}} : L^X \times L^X \to M$ by*

$$\forall A, B \in L^X, \quad \mathcal{E}^d_{\mathcal{E}}(A,B) = \bigwedge_{x_\lambda \not\leq^* B} \bigwedge_{\mu \in \beta^*_\lambda(L)} \mathcal{E}(A_{x_\mu}, \underline{\top}_{x_\lambda}).$$

Then $\mathcal{E}^d_{\mathcal{E}}$ is an (L,M)-fuzzy topological derived enclosed relation.

Proof. It is easy to check that $\mathcal{E}^d_{\mathcal{E}}(C,D) \leq \mathcal{E}^d_{\mathcal{E}}(A,B)$ for any $A, B, C, D \in L^X$ with $A \leq C$ and $D \leq B$. Further, if $A \leq B$ then $\mathcal{E}^d_{\mathcal{E}}(A,B) = \mathcal{E}(A, \bigwedge_{x_\lambda \not\leq^* B} \underline{\top}_{x_\lambda}) = \mathcal{E}(A,B)$. Next, we check that $\mathcal{E}^d_{\mathcal{E}}$ satisfies (LMTDER1)–(LMTDER5).

(LMTDER1). We have $\mathcal{E}^d_{\mathcal{E}}(\underline{\bot}, \underline{\bot}) = \mathcal{E}(\underline{\bot}, \underline{\bot}) = \top$ by (LMTER1).

(LMTDER2). Let $A, B \in L^X$. If there are $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$ with $A_{x_\mu} \not\leq \underline{\top}_{x_\lambda}$, then $\mathcal{E}^d_{\mathcal{E}}(A,B) \leq \mathcal{E}(A_{x_\mu}, \underline{\top}_{x_\lambda}) = \bot$ by (LMTER2). Thus (LMTDER2) holds in this case. Assume that $\bigvee_{\mu \in \beta^*_\lambda(L)} A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\leq^* B$. We have

$$\mathcal{E}^d_{\mathcal{E}}(A,B) = \bigwedge_{x_\lambda \not\leq^* B} \bigwedge_{\mu \in \beta^*_\lambda(L)} \mathcal{E}^d_{\mathcal{E}}(A_{x_\mu}, \underline{\top}_{x_\lambda}) = \bigwedge_{x_\lambda \not\leq^* B} \bigwedge_{\mu \in \beta^*_\lambda(L)} \mathcal{E}^d_{\mathcal{E}}(A_{x_\mu}, \underline{\top}_{x_\lambda}) \wedge \chi_{\underline{\top}_{x_\lambda}}(A_{x_\mu}).$$

(LMTDER3). We have $\mathcal{E}^d_{\mathcal{E}}(A, \bigwedge_{i \in I} B_i) = \bigwedge_{i \in I} \bigwedge_{x_\lambda \not\leq^* B_i} \bigwedge_{\mu \in \beta^*_\lambda(L)} \mathcal{E}(A_{x_\mu}, \underline{\top}_{x_\lambda}) = \bigwedge_{i \in I} \mathcal{E}^d_{\mathcal{E}}(A, B_i)$.

(LMTDER4). Let $a \in \beta(\top)$ with $a \prec \mathcal{E}^d_{\mathcal{E}}(A,B)$. To find some $E \in L^X$ such that $E \leq B$ and $a \leq \mathcal{E}^d_{\mathcal{E}}(A,E) \wedge \mathcal{E}^d_{\mathcal{E}}(E, A \vee B)$, let $E = \bigwedge\{H \in L^X : a \leq \mathcal{E}^d_{\mathcal{E}}(A,H)\}$ We have $\mathcal{E}^d_{\mathcal{E}}(A,E) \geq a$ by (LMTDER3). To prove that $a \leq \mathcal{E}^d_{\mathcal{E}}(E, A \vee B)$, let $x_\lambda \not\leq^* A \vee B$ and $\mu \in \beta^*_\lambda(L)$. Then $x_\lambda \not\leq^* B$ and $A_{x_\mu} = A$. By (LMTER4) and (LMTER2), we have

$$a \prec \mathcal{E}^d_{\mathcal{E}}(A,B) \leq \mathcal{E}(A, \underline{\top}_{x_\lambda}) \leq \bigvee_{C \in L^X} \mathcal{E}(A,C) \wedge \mathcal{E}(C, \underline{\top}_{x_\lambda}) = \bigvee_{A \leq C \leq \underline{\top}_{x_\lambda}} \mathcal{E}(A,C) \wedge \mathcal{E}(C, \underline{\top}_{x_\lambda}).$$

So there is a set $C \in L^X$ such that $A \leq C \leq \underline{\top}_{x_\lambda}$ and $a \leq \mathcal{E}(A,C) \wedge \mathcal{E}(C, \underline{\top}_{x_\lambda}) \leq \mathcal{E}(A, \underline{\top}_{x_\lambda})$. Further, for all $z_\eta \not\leq C$ and $\theta \in \beta^*_\eta(L)$, we have $A = A_{z_\theta}$ and $C = C_{z_\eta} \leq \underline{\top}_{z_\eta}$. Hence $\mathcal{E}^d_{\mathcal{E}}(A,C) = \bigwedge_{z_\eta \not\leq C} \bigwedge_{\theta \in \beta^*_\eta(L)} \mathcal{E}(A_{z_\theta}, \underline{\top}_{z_\eta}) \geq \mathcal{E}(A,C) \geq a$. Thus $E \leq C$ and $a \leq \mathcal{E}(C, \underline{\top}_{x_\lambda}) \leq \mathcal{E}(E, \underline{\top}_{x_\lambda})$. Hence

$$\mathcal{E}^d_{\mathcal{E}}(E, A \vee B) = \bigwedge_{x_\lambda \not\leq^* A \vee B} \bigwedge_{\mu \in \beta^*_\lambda(L)} \mathcal{E}(E_{x_\mu}, \underline{\top}_{x_\lambda}) \geq \bigwedge_{x_\lambda \not\leq^* A \vee B} \bigwedge_{\mu \in \beta^*_\lambda(L)} \mathcal{E}(E, \underline{\top}_{x_\lambda}) \geq a.$$

As a result, $a \le \mathcal{E}^d_{\mathcal{E}}(A,E) \wedge \mathcal{E}^d_{\mathcal{E}}(E,A\vee B) \le \bigvee_{G\le A\vee B} \mathcal{E}^d_{\mathcal{E}}(A,G) \wedge \mathcal{E}^d_{\mathcal{E}}(G,A\vee B)$. By the arbitrariness of a, we conclude that $\mathcal{E}^d_{\mathcal{E}}(A,B) \le \bigvee_{G\le A\vee B} \mathcal{E}^d_{\mathcal{E}}(A,G) \wedge \mathcal{E}^d_{\mathcal{E}}(G,A\vee B)$.

(LMTDER5). By (LMTER5), we have

$$\mathcal{E}^d_{\mathcal{E}}(A\vee B,C) = \bigwedge_{x_\lambda \not\le^* B} \bigwedge_{\mu\in\beta^*_\lambda(L)} \mathcal{E}(A_{x_\mu}, \underline{\top}_{x_\lambda}) \wedge \mathcal{E}(B_{x_\mu}, \underline{\top}_{x_\lambda}) = \mathcal{E}^d_{\mathcal{E}}(A,C) \wedge \mathcal{E}^d_{\mathcal{E}}(B,C).$$

Therefore $\mathcal{E}^d_{\mathcal{E}}$ is an (L,M)-fuzzy topological derived enclosed relation. □

Theorem 4.6. *Let $(X,\mathcal{E}_X)$ and $(Y,\mathcal{E}_Y)$ be (L,M)-fuzzy topological enclosed relation spaces. If $f : X\to Y$ is an (L,M)-fuzzy topological enclosed relation preserving mapping, then $f : (X,\mathcal{E}^d_{\mathcal{E}_X}) \to (Y,\mathcal{E}^d_{\mathcal{E}_Y})$ is an (L,M)-fuzzy topological derived enclosed relation preserving mapping.*

Proof. For all $A,B\in L^Y$, we have $\mathcal{E}^d_{\mathcal{E}_Y}(A,B) \le \mathcal{E}^d_{\mathcal{E}_Y}(A,A\vee B) \le \bigwedge_{x_\lambda\not\le^* f^\leftarrow_L(A\vee B)} \mathcal{E}_X(f^\leftarrow_L(A), \underline{\top}_{x_\lambda}) = \mathcal{E}^d_{\mathcal{E}_X}(f^\leftarrow_L(A), f^\leftarrow_L(A\vee B))$. So f is an (L,M)-fuzzy topological derived enclosed relation preserving mapping. □

Theorem 4.7. *We have $\mathcal{E}^d_{\mathcal{E}_{\mathcal{E}^d}} = \mathcal{E}^d$ for any (L,M)-fuzzy topological derived enclosed relation space $(X,\mathcal{E}^d)$ and $\mathcal{E}_{\mathcal{E}^d_{\mathcal{E}}} = \mathcal{E}$ for any (L,M)-fuzzy topological enclosed relation space $(X,\mathcal{E})$.*

Proof. Let $(X,\mathcal{E})$ be an (L,M)-fuzzy topological enclosed relation space. Let $A,B\in L^X$. If $A\not\le B$, we have $\mathcal{E}_{\mathcal{E}^d_{\mathcal{E}}}(A,B) = \mathcal{E}(A,B) = \bot$. Assume that $A\le B$. By (LMTDER3), we have

$$\mathcal{E}_{\mathcal{E}^d_{\mathcal{E}}}(A,B) = \mathcal{E}^d_{\mathcal{E}}(A,B) = \bigwedge_{x_\lambda \not\le^* B} \bigwedge_{\mu\in\beta^*_\lambda(L)} \mathcal{E}(A_{x_\mu}, \underline{\top}_{x_\lambda}) = \bigwedge_{x_\lambda\not\le^* B} \mathcal{E}(A, \underline{\top}_{x_\lambda}) = \mathcal{E}(A,B).$$

Thus $\mathcal{E}_{\mathcal{E}^d_{\mathcal{E}}}(A,B) = \mathcal{E}(A,B)$. That is, $\mathcal{E}_{\mathcal{E}^d_{\mathcal{E}}} = \mathcal{E}$.

Let $(X,\mathcal{E}^d)$ be an (L,M)-fuzzy topological derived enclosed relation space. For all $A,B\in L^X$, we obtain from (LMTDER2) that

$$\mathcal{E}^d_{\mathcal{E}_{\mathcal{E}^d}}(A,B) = \bigwedge_{x_\lambda\not\le^* B} \bigwedge_{\mu\in\beta^*_\lambda(L)} \bigvee_{A_{x_\mu}\vee C = \underline{\top}_{x_\lambda}} \mathcal{E}^d(A_{x_\mu}, C) = \mathcal{E}^d(A,B).$$

Thus $\mathcal{E}^d_{\mathcal{E}_{\mathcal{E}^d}}(A,B) = \mathcal{E}^d(A,B)$. Therefore $\mathcal{E}^d_{\mathcal{E}_{\mathcal{E}^d}} = \mathcal{E}^d$. □

From Theorems 4.3 and 4.4, we define a functor $\mathbb{F}$:**LMTDER**→**LMTER** by:

$$\mathbb{F}((X,\mathcal{E}^d)) = (X,\mathcal{E}_{\mathcal{E}^d}), \qquad \mathbb{F}(f) = f.$$

Based on Theorems 4.3–4.7, $\mathbb{F}$ is an isomorphic functor. So we have the following conclusion.

Theorem 4.8. *The category* **LMTDER** *is isomorphic to the category* **LMTER**.

From Theorems 4.3–4.7, we can establish relations between **LMTDER** and **LMTOP** are as follows.

Corollary 4.9. (1) *Let $(X,\mathcal{T})$ be an (L,M)-fuzzy topological space. Define $\mathcal{E}^d_{\mathcal{T}} : L^X\times L^X \to M$ by*

$$\forall A,B\in L^X,\quad \mathcal{E}^d_{\mathcal{T}}(A,B) = \bigwedge_{x_\lambda\not\le^* B} \bigwedge_{\mu,\eta\in\beta^*_\lambda(L)} \bigvee_{x_\eta\not\le D\ge A_{x_\mu}} \mathcal{T}(D').$$

Then $\mathcal{E}^d_{\mathcal{T}}$ is an (L,M)-fuzzy topological derived enclosed relation.

(2) *Let $(X,\mathcal{E}^d)$ be an (L,M)-fuzzy topological derived enclosed relation space. Define $\mathcal{T}_{\mathcal{E}^d} : L^X\to M$ by*

$$\forall A\in L^X,\quad \mathcal{T}_{\mathcal{E}^d}(A) = \bigwedge_{x_\lambda\not\le A'} \bigvee_{x_\lambda\not\le B} \bigvee_{A'\vee C = B} \mathcal{E}^d(C,B).$$

Then $\mathcal{T}_{\mathcal{E}^d}$ is an (L,M)-fuzzy topology.

Corollary 4.10. *The category* **LMTDER** *is isomorphic to the category* **LMTOP**.

To simply characterize (L,M)-fuzzy topological derived enclosed relation space, we introduce the following notion.

Definition 4.11. *An operator $Cl^d : L^X \to M^{\beta^*(\underline{\top})}$ is called an (L,M)-fuzzy topological derived closure operator on L^X and the pair (X, Cl^d) is called an (L,M)-fuzzy topological derived closure space if for all $A, B \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$,*

(LMTDCl1) $Cl^d(\underline{\perp})(x_\lambda) = \perp$;

(LMTDCl2) $Cl^d(A)(x_\lambda) = \bigvee_{\mu,\eta\in\beta^*_\lambda(L)} Cl^d(A_{x_\mu})(x_\eta) \vee (\chi_{\underline{\top}_{x_\lambda}}(A_{x_\mu}))'$;

(LMTDCl3) $Cl^d(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq^* B} \bigvee_{y_\mu \not\leq^* B} Cl^d(A \vee B)(y_\mu)$;

(LMTDCl4) $Cl^d(A \vee B) = Cl^d(A) \vee Cl^d(B)$.

Let (X, Cl^d_X) and (Y, Cl^d_Y) be (L,M)-fuzzy topological derived closure spaces. A mapping $f : X \to Y$ is called an (L,M)-fuzzy topological derived closure preserving mapping, if for all $A \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$,

$$Cl^d_X(A)(x_\lambda) \leq (Cl^d_Y(f^\rightarrow_L(A)) \vee \chi_{f^\rightarrow_L(A)})(f^\rightarrow_L(x_\lambda)).$$

The category of (L,M)-fuzzy topological derived closure spaces and (L,M)-fuzzy topological derived closure preserving mappings is denoted by **LMTDCL**.

In the sequel, for convenience, we denote $A^-_{x_\lambda} = \bigvee_{\mu\in\beta^*_\lambda(L)} A_{x_\mu}$ for all $A \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$.

Lemma 4.12. *Let $\varphi : L^X \to M^{\beta^*(\underline{\top})}$ be a monotonic operator. For all $A \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$,*

(1) *$\varphi(A)(x_\lambda) = \varphi(A^-_{x_\lambda})(x_\lambda)$ provided that φ satisfies (LMTDCl2);*

(2) *$\varphi(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \varphi(A)(x_\mu)$ provided that φ satisfies (LMTDCl3);*

(3) *φ satisfies (LMTDCl3) iff $\varphi(A)(x_\lambda) = \bigwedge_{x_\lambda\not\leq^* B} \varphi(A\vee B)(x_\lambda) \vee \bigvee_{y_\mu \not\leq^* B} \varphi(A)(y_\mu)$ for all $A \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$.*

Proof. (1) and (2) are direct. (3) can be proved similar to Lemma 4.6 in [28]. □

Next, we discuss relations between **LMTDER** and **LMTDCL**.

Theorem 4.13. *Let (X, Cl^d) be an (L,M)-fuzzy topological derived closure space. Define a binary relation $\mathcal{E}^d_{Cl^d} : L^X \times L^X \to M$ by*

$$\forall A, B \in L^X, \quad \mathcal{E}^d_{Cl^d}(A,B) = \bigwedge_{x_\lambda \not\leq^* B} (Cl^d(A)(x_\lambda))'.$$

Then $\mathcal{E}^d_{Cl^d}$ is an (L,M)-fuzzy topological derived enclosed relation.

Proof. (LMTDER1). We have $\mathcal{E}^d_{Cl^d}(\underline{\perp},\underline{\perp})(x_\lambda) = \bigwedge_{x_\lambda\in\beta^*(\underline{\top})} (Cl^d(\underline{\perp})(x_\lambda))' = \top$ by (LMTDCl1).

(LMTDER2). By (LMTDCl2) and Proposition 4.1, we have

$$\begin{aligned}
\mathcal{E}^d_{Cl^d}(A,B) &= \bigwedge_{x_\lambda\not\leq^* B} \bigwedge_{\mu,\eta\in\beta^*_\lambda(L)} (Cl^d(A_{x_\mu})(x_\eta))' \wedge \chi_{\underline{\top}_{x_\lambda}}(A_{x_\mu}) \\
&= \bigwedge_{x_\lambda\not\leq^* B} \bigwedge_{\mu\in\beta^*_\lambda(L)} [\bigwedge_{y_\eta \not\leq^* \underline{\top}_{x_\lambda}} (Cl^d(A_{x_\mu})(y_\eta))'] \wedge \chi_{\underline{\top}_{x_\lambda}}(A_{x_\mu}) \\
&= \bigwedge_{x_\lambda\not\leq^* B} \bigwedge_{\mu\in\beta^*_\lambda(L)} \mathcal{E}^d_{Cl^d}(A_{x_\mu}, \underline{\top}_{x_\lambda}) \wedge \chi_{\underline{\top}_{x_\lambda}}(A_{x_\mu}).
\end{aligned}$$

(LMTDER3). We have $\mathcal{E}^d_{Cl^d}(A, \bigwedge_{i\in I} B_i) = \bigwedge_{i\in I}\bigwedge_{x_\lambda\not\leq^* B_i} (Cl^d(A)(x_\lambda))' = \bigwedge_{i\in I} \mathcal{E}^d_{Cl^d}(A, B_i)$.

(LMTDER4). To prove that $\mathcal{E}^d_{Cl^d}(A,B) \leq \bigvee_{C\leq B} \mathcal{E}^d_{Cl^d}(A,C) \wedge \mathcal{E}^d_{Cl^d}(C, A\vee B)$, it is sufficient to prove that

$$\bigwedge_{x_\lambda\not\leq^* B} (Cl^d(A)(x_\lambda))' \quad\leq\quad \bigvee_{C\leq B} \bigwedge_{z_\nu\not\leq^* C} (Cl^d(A)(z_\nu))' \wedge \bigwedge_{y_\mu\not\leq^* B} (Cl^d(A\vee C)(y_\mu))'.$$

Let $a \in \beta(\top)$ with $a \prec \bigwedge_{x_\lambda\not\leq^* B} (Cl^d(A)(x_\lambda))'$. Then $a \prec (Cl^d(A)(x_\lambda))'$ for any $x_\lambda \not\leq^* B$. By $a \prec (Cl^d(A)(x_\lambda))'$ and (3) of Lemma 4.12, there is a set $D_\lambda \in L^X$ with $x_\lambda \not\leq^* D_\lambda$ such that $a \leq (Cl^d(A\vee D_\lambda)(x_\lambda))'$ and $a \leq \bigwedge_{y_\mu\not\leq^* D_\lambda} (Cl^d(A)(y_\mu))'$. Let $D = \bigwedge_{x_\lambda\not\leq^* B} D_\lambda$. Then $D \leq B$. Thus

$$\begin{aligned}
a &\leq \bigwedge_{x_\lambda\not\leq^* B} (Cl^d(A\vee D_\lambda)(x_\lambda))' \wedge \bigwedge_{x_\lambda\not\leq^* B} \bigwedge_{z_\nu\not\leq^* D_\lambda} (Cl^d(A)(z_\nu))' \\
&\leq \bigwedge_{x_\lambda\not\leq^* B} (Cl^d(A\vee D)(x_\lambda))' \wedge \bigwedge_{z_\nu\not\leq^* D} (Cl^d(A)(z_\nu))' \\
&\leq \bigvee_{C\leq B} \bigwedge_{z_\nu\not\leq^* C} (Cl^d(A)(z_\nu))' \wedge \bigwedge_{y_\mu\not\leq^* B} (Cl^d(A\vee C)(y_\mu))'.
\end{aligned}$$

By the arbitrariness of a, we have

$$\bigwedge_{x_\lambda\not\leq^* B}(Cl^d(A)(x_\lambda))' \quad\leq\quad \bigvee_{C\leq B}\bigwedge_{z_\nu\not\leq^* C}(Cl^d(A)(z_\nu))'\wedge\bigwedge_{y_\mu\not\leq^* B}(Cl^d(A\vee C)(y_\mu))'.$$

(LMTDER5). By (LMTDCl4), we have

$$\mathcal{E}^d_{Cl^d}(A\vee B,C)=\bigwedge_{x_\lambda\not\leq^* C}((Cl^d(A)\vee Cl^d(B))(x_\lambda))'=\mathcal{E}^d_{Cl^d}(A,C)\wedge\mathcal{E}^d_{Cl^d}(B,C).$$

Therefore $\mathcal{E}^d_{Cl^d}$ is an (L,M)-fuzzy topological derived enclosed relation. □

Theorem 4.14. *Let (X,Cl^d_X) and (Y,Cl^d_Y) be (L,M)-fuzzy topological derived closure spaces. If $f:X\to Y$ is an (L,M)-fuzzy topological derived closure preserving mapping, then $f:(X,\mathcal{E}^d_{Cl^d_X})\to(Y,\mathcal{E}^d_{Cl^d_Y})$ is an (L,M)-fuzzy topological derived enclosed relation preserving mapping.*

Proof. Let $A,B\in L^Y$. We have $\mathcal{E}^d_{Cl^d_Y}(A,B)\leq\bigwedge_{f_L^\rightarrow(x_\lambda)\not\leq^* B}(Cl^d_Y(A)(f_L^\rightarrow(x_\lambda)))'\leq\bigwedge_{x_\lambda\not\leq^* f_L^\leftarrow(A\vee B)}(Cl^d_X(f_L^\leftarrow(A))(x_\lambda))'=\mathcal{E}^d_{Cl^d_X}(f_L^\leftarrow(A),f_L^\leftarrow(A\vee B))$. Therefore f is an (L,M)-fuzzy topological derived enclosed relation preserving mapping. □

Theorem 4.15. *Let $(X,\mathcal{E}^d)$ be an (L,M)-fuzzy topological derived enclosed relation space. Define an operator $Cl^d_{\mathcal{E}^d}:L^X\to M^{\beta^*(\top)}$ by*

$$\forall A\in L^X,\forall x_\lambda\in\beta^*(\top),\quad Cl^d_{\mathcal{E}^d}(A)(x_\lambda)=\bigwedge_{x_\lambda\not\leq^* B}(\mathcal{E}^d(A,B))'.$$

Then $Cl^d_{\mathcal{E}^d}$ is an (L,M)-fuzzy topological derived closure operator.

Proof. (LMTDCl1). By (LMTDER1), we have $Cl^d_{\mathcal{E}^d}(\underline{\perp})(x_\lambda)\leq(\mathcal{E}^d(\underline{\perp},\underline{\perp}))'=\perp$.

(LMTDCl2). By (6) of Proposition 4.1 and (LMTDER3), we have

$$\bigvee_{\mu\in\beta^*_\lambda(L)}(\mathcal{E}^d(A_{x_\mu},\underline{\top}_{x_\lambda}))'=\bigvee_{\mu,\eta\in\beta^*_\lambda(L)}(\mathcal{E}^d(A_{x_\mu},\underline{\top}_{x_\eta}))'\leq\bigvee_{\mu,\eta\in\beta^*_\lambda(L)}\bigwedge_{x_\eta\not\leq^* B}(\mathcal{E}^d(A_{x_\mu},B))'=\bigvee_{\mu,\eta\in\beta^*_\lambda(L)}Cl^d_{\mathcal{E}^d}(A_{x_\mu})(x_\eta).$$

Conversely, since $x_\eta\not\leq\underline{\top}_{x_\lambda}$ for any $\eta\in\beta^*_\lambda(L)$ by (5) of Proposition 4.1, we have

$$\bigvee_{\mu,\eta\in\beta^*_\lambda(L)}Cl^d_{\mathcal{E}^d}(A_{x_\mu})(x_\eta)=\bigvee_{\mu,\eta\in\beta^*_\lambda(L)}\bigwedge_{x_\eta\not\leq^* B}(\mathcal{E}^d(A_{x_\mu},B))'\leq\bigvee_{\mu,\eta\in\beta^*_\lambda(L)}(\mathcal{E}^d(A_{x_\mu},\underline{\top}_{x_\lambda}))'=\bigvee_{\mu\in\beta^*_\lambda(L)}(\mathcal{E}^d(A_{x_\mu},\underline{\top}_{x_\lambda}))'.$$

Thus $\bigvee_{\mu,\eta\in\beta^*_\lambda(L)}Cl^d_{\mathcal{E}^d}(A_{x_\mu})(x_\eta)=\bigvee_{\mu\in\beta^*_\lambda(L)}(\mathcal{E}^d(A_{x_\mu},\underline{\top}_{x_\lambda}))'$. By this result and (LMTDER2), we have

$$\begin{aligned}Cl^d_{\mathcal{E}^d}(A)(x_\lambda)&=\bigwedge_{x_\lambda\not\leq^* B}\bigvee_{y_\eta\not\leq^* B}\bigvee_{\mu\in\beta^*_\eta(L)}(\mathcal{E}^d(A_{y_\mu},\underline{\top}_{y_\eta}))'\vee(\chi_{\underline{\top}_{y_\eta}}(A_{y_\mu}))'\\&\geq\bigwedge_{x_\lambda\not\leq^* B}\bigvee_{\mu\in\beta^*_\lambda(L)}(\mathcal{E}^d(A_{x_\mu},\underline{\top}_{x_\lambda}))'\vee(\chi_{\underline{\top}_{x_\lambda}}(A_{x_\mu}))'\\&=\bigvee_{\mu,\eta\in\beta^*_\lambda(L)}Cl^d_{\mathcal{E}^d}(A_{x_\mu})(x_\eta)\vee(\chi_{\underline{\top}_{x_\lambda}}(A_{x_\mu}))'.\end{aligned}$$

Conversely, fix a $B\in L^X$ with $x_\lambda\not\leq^* B$. Let $D_B=B_{x_\lambda}\vee\hat{B}$, where $\hat{B}\in L^X$ is defined by $\hat{B}(x)=\perp$ and $\hat{B}(z)=\top$ otherwise. Then $x_\lambda\not\leq D_B$. For any $y_\eta\in\beta^*(\underline{\top})$, $y_\eta\not\leq^* D_B$ iff $x=y$ and $\eta\in\beta^*_\lambda(L)$. Thus

$$\begin{aligned}Cl^d_{\mathcal{E}^d}(A)(x_\lambda)&\leq\bigvee_{y_\eta\not\leq^* D_B}\bigvee_{\mu\in\beta^*_\eta(L)}(\mathcal{E}^d(A_{y_\mu},\underline{\top}_{y_\eta}))'\vee(\chi_{\underline{\top}_{y_\eta}}(A_{y_\mu}))'\\&\leq\bigvee_{\mu,\eta\in\beta^*_\lambda(L)}(\mathcal{E}^d(A_{x_\mu},\underline{\top}_{x_\eta}))'\vee(\chi_{\underline{\top}_{x_\eta}}(A_{x_\mu}))'\\&\leq\bigvee_{\mu,\eta\in\beta^*_\lambda(L)}Cl^d_{\mathcal{E}^d}(A_{x_\mu})(x_\eta)\vee(\chi_{\underline{\top}_{x_\lambda}}(A_{x_\mu}))'.\end{aligned}$$

Therefore (LMTDCl2) holds.

(LMTDCl4). By (LMTDER5), we have

$$Cl^d_{\mathcal{E}^d}(A\vee B)(x_\lambda)=\bigwedge_{x_\lambda\not\leq^* C}[(\mathcal{E}^d(A,C))'\wedge(\mathcal{E}^d(B,C))']=Cl^d_{\mathcal{E}^d}(A)(x_\lambda)\wedge Cl^d_{\mathcal{E}^d}(B)(x_\lambda).$$

(LMTDCl3). To prove that $Cl^d_{\mathcal{E}^d}(A)(x_\lambda)=\bigwedge_{x_\lambda\not\leq^* B}Cl^d_{\mathcal{E}^d}(A\vee B)(x_\lambda)\vee\bigvee_{y_\mu\not\leq^* B}Cl^d_{\mathcal{E}^d}(A)(y_\mu)$, we firstly check that

$$Cl^d_{\mathcal{E}^d}(A)(x_\lambda) = \bigwedge_{x_\lambda\not\leq^* B}Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda}\vee B)(x_\lambda)\vee\bigvee_{y_\mu\not\leq^* B}Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda})(y_\mu).$$

Notice that $Cl^d_{\mathcal{E}^d}$ satisfies (LMTDCl2) and (LMTDCl4). By (1) of Lemma 4.12, we have

$$Cl^d_{\mathcal{E}^d}(A)(x_\lambda)=Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda})(x_\lambda) \leq \bigwedge_{x_\lambda\not\leq^* B}Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda}\vee B)(x_\lambda)\vee\bigvee_{y_\mu\not\leq^* B}Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda})(y_\mu).$$

To prove the inverse inequality, we prove that

$$(Cl^d_{\mathcal{E}^d}(A)(x_\lambda))' \leq \bigvee_{x_\lambda\not\leq^* B}(Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda}\vee B)(x_\lambda))'\wedge\bigwedge_{y_\mu\not\leq^* B}(Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda})(y_\mu))'.$$

Let $a\in\beta^*(\top)$ with $a\prec(Cl^d_{\mathcal{E}^d}(A)(x_\lambda))'$. We now check that $x_\lambda\not\leq^* A^-_{x_\lambda}$. Since it is easy to check that $Cl^d_{\mathcal{E}^d}(A)(x_\lambda)=\bigwedge_{\gamma\in\beta^*(\lambda)}Cl^d_{\mathcal{E}^d}(A)(x_\gamma)$, we have $a\prec(Cl^d_{\mathcal{E}^d}(A)(x_\lambda))'=\bigvee_{\gamma\in\beta^*(\lambda)}\bigvee_{x_\gamma\not\leq^* D}\mathcal{E}^d(A,D)$. There are $\gamma\in\beta^*(\lambda)$ and $D\in L^X$ such that $x_\gamma\not\leq^* D$ and $a\leq\mathcal{E}^d(A,D)$. So $A^-_{x_\gamma}\leq\underline{\top}_{x_\gamma}$ by (LMTDER2).

Suppose that $x_\lambda\leq A^-_{x_\lambda}$. By $\gamma\in\beta^*(\lambda)$, there is a $\delta\in\beta^*(\lambda)$ such that $\gamma\in\beta^*(\delta)$ (i.e., $\delta\in\beta^*_\gamma(L)$). But $x_\delta\prec A^-_{x_\lambda}\leq A^-_{x_\gamma}\leq\underline{\top}_{x_\gamma}$ which is a contradiction. So $x_\lambda\not\leq^* A^-_{x_\lambda}$. By $a\prec(Cl^d_{\mathcal{E}^d}(A)(x_\lambda))'$ and (LMTDER4),

$$a\prec\bigvee_{x_\lambda\not\leq^* B}\mathcal{E}^d(A,B)\leq\bigvee_{x_\lambda\not\leq^* B}\mathcal{E}^d(A^-_{x_\lambda},B)\leq\bigvee_{x_\lambda\not\leq^* B}\bigvee_{C\leq B}\mathcal{E}^d(A^-_{x_\lambda},C)\wedge\mathcal{E}^d(C,A^-_{x_\lambda}\vee B).$$

Thus there are $B,C\in L^X$ such that $x_\lambda\not\leq^* B\geq C$ and $a\prec\mathcal{E}^d(A^-_{x_\lambda},C)\wedge\mathcal{E}^d(C,A^-_{x_\lambda}\vee B)$. By $a\prec\mathcal{E}^d(A^-_{x_\lambda},C)$ and (LMTDER4), there is a set $D\in L^X$ such that $D\leq C$ and $a\leq\mathcal{E}^d(A^-_{x_\lambda},D)\wedge\mathcal{E}^d(D,A^-_{x_\lambda}\vee C)$. So $x_\lambda\not\leq A^-_{x_\lambda}\vee B\geq A^-_{x_\lambda}\vee C\vee D$. Hence

$$a\leq\mathcal{E}^d(A^-_{x_\lambda},A^-_{x_\lambda}\vee B)\wedge\mathcal{E}^d(D,A^-_{x_\lambda}\vee B)\leq\bigvee_{x_\lambda\not\leq^* G}\mathcal{E}^d(A^-_{x_\lambda},G)\wedge\mathcal{E}^d(D,G)=(Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda}\vee D)(x_\lambda))'$$

and $a\leq\bigwedge_{y_\eta\not\leq^* D}\mathcal{E}^d(A^-_{x_\lambda},D)\leq\bigwedge_{y_\eta\not\leq^* D}(Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda})(y_\eta))'$. So $a\leq(Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda}\vee D)(x_\lambda))'\wedge\bigwedge_{y_\eta\not\leq^* D}(Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda})(y_\eta))'$ By the arbitrariness of a, we conclude that

$$(Cl^d_{\mathcal{E}^d}(A)(x_\lambda))' \leq \bigvee_{x_\lambda\not\leq^* B}(Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda}\vee B)(x_\lambda))'\wedge\bigwedge_{y_\mu\not\leq^* B}(Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda})(y_\mu))'.$$

Hence $Cl^d_{\mathcal{E}^d}(A)(x_\lambda)\geq\bigwedge_{x_\lambda\not\leq^* B}Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda}\vee B)(x_\lambda)\vee\bigvee_{y_\mu\not\leq B}Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda})(y_\mu)$. In conclusion, we have

$$Cl^d_{\mathcal{E}^d}(A)(x_\lambda) = \bigwedge_{x_\lambda\not\leq^* B}Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda}\vee B)(x_\lambda)\vee\bigvee_{y_\mu\not\leq B}Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda})(y_\mu).$$

Applying (1) of Lemma 4.12, $Cl^d_{\mathcal{E}^d}(A)(y_\mu)=Cl^d_{\mathcal{E}^d}(A^-_{y_\mu})(y_\mu)$ and $Cl^d_{\mathcal{E}^d}(A\vee B)(x_\lambda)=Cl^d_{\mathcal{E}^d}(A^-_{x_\lambda}\vee B)(x_\lambda)$. Therefore

$$Cl^d_{\mathcal{E}^d}(A)(x_\lambda)=\bigwedge_{x_\lambda\not\leq^* B}Cl^d_{\mathcal{E}^d}(A\vee B)(x_\lambda)\vee\bigvee_{y_\mu\not\leq^* B}Cl^d_{\mathcal{E}^d}(A)(y_\mu).$$

By (3) of Lemma 4.12, we conclude that $Cl^d_{\mathcal{E}^d}$ satisfies (LMTDCl3).

In conclusion, $Cl^d_{\mathcal{E}^d}$ is an (L,M)-fuzzy topological derived closure operator. □

Theorem 4.16. *Let $(X,\mathcal{E}^d_X)$ and $(Y,\mathcal{E}^d_Y)$ be (L,M)-fuzzy topological derived enclosed relation spaces. If $f:X\to Y$ is an (L,M)-fuzzy topological derived enclosed relation preserving mapping, then $f:(X,Cl^d_{\mathcal{E}^d_X})\to(Y,Cl^d_{\mathcal{E}^d_Y})$ is an (L,M)-fuzzy topological derived closure preserving mapping.*

Proof. Let $A \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$. If $f_L^{\rightarrow}(x_\lambda) \leq f_L^{\rightarrow}(A)$, then $Cl^d_{\mathcal{E}^d_X}(A)(x_\lambda) \leq \chi_{f_L^{\rightarrow}(A)}(f_L^{\rightarrow}(x_\lambda)) = \top$. Assume that $f_L^{\rightarrow}(x_\lambda) \not\leq^* f_L^{\rightarrow}(A)$. We have

$$Cl^d_{\mathcal{E}^d_X}(A)(x_\lambda) \leq \bigwedge_{f_L^{\rightarrow}(x_\lambda)\not\leq^* B \vee f_L^{\rightarrow}(A)} (\mathcal{E}^d_X(f_L^{\leftarrow}(f_L^{\rightarrow}(A)), f_L^{\leftarrow}(B) \vee f_L^{\leftarrow}(f_L^{\rightarrow}(A))))' = Cl^d_{\mathcal{E}^d_Y}(f_L^{\rightarrow}(A))(f_L^{\rightarrow}(x_\lambda)).$$

Therefore f is an (L,M)-fuzzy topological derived closure preserving mapping. □

Theorem 4.17. *We have $\mathcal{E}^d_{Cl^d_{\mathcal{E}^d}} = \mathcal{E}^d$ for any (L,M)-fuzzy topological derived enclosed relation space $(X,\mathcal{E}^d)$ and $Cl^d_{\mathcal{E}^d_{Cl^d}} = Cl^d$ for any (L,M)-fuzzy topological derived closure operator space (X, Cl^d).*

Proof. Let $(X,\mathcal{E}^d)$ be an (L,M)-fuzzy topological derived enclosed relation space. For all $A,B \in L^X$,

$$\mathcal{E}^d_{Cl^d_{\mathcal{E}^d}}(A,B) = \bigwedge_{x_\lambda\not\leq^* B}\ \bigvee_{x_\lambda\not\leq^* D} \mathcal{E}^d(A,D) = \bigvee_{\varphi\in\Pi_{x_\lambda\not\leq^* B}\mathcal{B}_{x_\lambda}}\ \bigwedge_{x_\lambda\not\leq^* B} \mathcal{E}^d(A,\varphi(x_\lambda)) = \mathcal{E}^d(A,B),$$

where $\Pi_{x_\lambda\not\leq^* B}\mathcal{B}_{x_\lambda} = \{D \in L^X : x_\lambda \not\leq^* D\}$. Hence $\mathcal{E}^d_{Cl^d_{\mathcal{E}^d}}(A,B) = \mathcal{E}^d(A,B)$. That is, $\mathcal{E}^d_{Cl^d_{\mathcal{E}^d}} = \mathcal{E}^d$.

Let (X,Cl^d) be an (L,M)-fuzzy topological derived closure space. Let $A \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$. Clearly, we have $Cl^d_{\mathcal{E}^d_{Cl^d}}(A)(x_\lambda) \geq Cl^d(A)(x_\lambda)$. Conversely, by (LMTDCl3), we have

$$Cl^d_{\mathcal{E}^d_{Cl^d}}(A)(x_\lambda) = \bigwedge_{x_\lambda\not\leq^* B}\ \bigvee_{y_\mu\not\leq^* B} Cl^d(A)(y_\mu) \leq \bigwedge_{x_\lambda\not\leq^* B}\ \bigvee_{y_\mu\not\leq^* B} Cl^d(A\vee B)(y_\mu) = Cl^d(A)(x_\lambda).$$

Therefore $Cl^d_{\mathcal{E}^d_{Cl^d}}(A)(x_\lambda) = Cl^d(A)(x_\lambda)$. That is, $Cl^d_{\mathcal{E}^d_{Cl^d}} = Cl^d$. □

From Theorems 4.13 and 4.14, we obtain a functor $\mathbb{G}$:**LMTDCL**→ **LMTDER** by

$$\mathbb{G}((X,Cl^d)) = (X,\mathcal{E}^d_{Cl^d}), \quad \mathbb{G}(f) = f.$$

Based on Theorems 4.13–4.17, $\mathbb{G}$ is an isomorphic functor. Thus we have the following result.

Theorem 4.18. *The category* **LMTDER** *is isomorphic to the category* **LMTDCL**.

From Corollary 4.9 and Theorems 4.13–4.17, we have the following conclusions.

Corollary 4.19. (1) *Let $(X,\mathcal{T})$ be an (L,M)-fuzzy topological space. Define $Cl^d_{\mathcal{T}} : L^X \to M^{\beta^*(\underline{\top})}$ by*

$$\forall A \in L^X, \forall x_\lambda \in \beta^*(\underline{\top}), \quad Cl^d_{\mathcal{T}}(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*_\lambda(L)}\ \bigwedge_{x_\mu\not\leq B\geq A_{x_\mu}} [\mathcal{T}(B')]'.$$

Then $Cl^d_{\mathcal{T}}$ is an (L,M)-fuzzy topological derived closure operator.

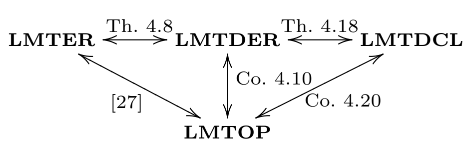
(2) *Let (X,Cl^d) be an (L,M)-fuzzy topological derived closure space. Define $\mathcal{T}_{Cl^d} : L^X \to M$ by*

$$\forall A \in L^X, \quad \mathcal{T}_{Cl^d}(A) = \bigwedge_{x_\lambda\not\leq^* A'} [Cl^d(A')(x_\lambda)]'.$$

Then $\mathcal{T}_{Cl^d}$ is an (L,M)-fuzzy topology.

Corollary 4.20. *The category* **LMTDCL** *is isomorphic to the category* **LMTOP**.

The relations of categories mentioned in this section can be shown by the following diagram.



Relations 2.

At the end of this section, we present a simple example to show that concepts of (L,M)-fuzzy derived enclosed relations and (L,M)-fuzzy derived closure operators are reasonable.

Example 4.21. *Let X, L and M be defined as in Example 3.21. Let $\mathcal{E}^d : L^X \times L^X \to M$ be defined by:*

$$\mathcal{E}^d(\underline{\bot},\underline{\bot}) = \mathcal{E}^d(\underline{\bot},x_a) = \mathcal{E}^d(\underline{\bot},x_b) = \mathcal{E}^d(\underline{\bot},\underline{\top}) = 1;$$
$$\mathcal{E}^d(x_a,\underline{\bot}) = \mathcal{E}^d(x_a,x_a) = \mathcal{E}^d(x_a,x_b) = \mathcal{E}^d(x_a,\underline{\top}) = 1;$$
$$\mathcal{E}^d(x_b,x_a) = \mathcal{E}^d(x_b,\underline{\top}) = \mathcal{E}^d(\underline{\top},x_a) = \mathcal{E}^d(\underline{\top},\underline{\top}) = 1;$$
$$\mathcal{E}^d(x_b,\underline{\bot}) = \mathcal{E}^d(x_b,x_b) = \mathcal{E}^d(\underline{\top},\underline{\bot}) = \mathcal{E}^d(\underline{\top},x_b) = 0.$$

It is routine to check that $\mathcal{E}^d$ is an (L,M)-fuzzy topological derived enclosed relation. In addition, the (L,M)-fuzzy topological enclosed relation $\mathcal{E}_{\mathcal{E}^d}$ and the (L,M)-fuzzy topological derived closure operator $cl^d_{\mathcal{E}^d}$ satisfy

$$\mathcal{E}_{\mathcal{E}^d}(\underline{\bot},\underline{\bot}) = \mathcal{E}_{\mathcal{E}^d}(\underline{\bot},x_a) = \mathcal{E}_{\mathcal{E}^d}(\underline{\bot},x_b) = \mathcal{E}_{\mathcal{E}^d}(\underline{\bot},\underline{\top}) = 1;$$
$$\mathcal{E}_{\mathcal{E}^d}(x_a,x_a) = \mathcal{E}_{\mathcal{E}^d}(x_a,\underline{\top}) = \mathcal{E}_{\mathcal{E}^d}(x_b,\underline{\top}) = \mathcal{E}_{\mathcal{E}^d}(\underline{\top},\underline{\top}) = 1;$$
$$\mathcal{E}_{\mathcal{E}^d}(x_a,\underline{\bot}) = \mathcal{E}_{\mathcal{E}^d}(x_a,x_b) = \mathcal{E}_{\mathcal{E}^d}(x_b,\underline{\bot}) = \mathcal{E}_{\mathcal{E}^d}(x_b,x_a) = 0;$$
$$\mathcal{E}_{\mathcal{E}^d}(x_b,x_b) = \mathcal{E}_{\mathcal{E}^d}(\underline{\top},\underline{\bot}) = \mathcal{E}_{\mathcal{E}^d}(\underline{\top},x_a) = \mathcal{E}_{\mathcal{E}^d}(\underline{\top},x_b) = 0$$

and

$$Cl^d_{\mathcal{E}^d}(x_b)(x_a) = Cl^d_{\mathcal{E}^d}(\underline{\top})(x_a) = 1;$$
$$Cl^d_{\mathcal{E}^d}(\underline{\bot})(x_a) = Cl^d_{\mathcal{E}^d}(\underline{\bot})(x_b) = Cl^d_{\mathcal{E}^d}(x_a)(x_a) = Cl^d_{\mathcal{E}^d}(x_a)(x_b) = Cl^d_{\mathcal{E}^d}(x_b)(x_b) = Cl^d_{\mathcal{E}^d}(\underline{\top})(x_b) = 0.$$

Also, the (L,M)-fuzzy topology $\mathcal{T}_{\mathcal{E}^d}$ satisfying $\mathcal{T}_{\mathcal{E}^d}(\underline{\bot}) = \mathcal{T}_{\mathcal{E}^d}(x_b) = \mathcal{T}_{\mathcal{E}^d}(\underline{\top}) = 1$ and $\mathcal{T}_{\mathcal{E}^d}(x_a) = 0$.

5 Conclusions

In the framework of (L,M)-fuzzy topological space, we introduced several new operators which solve the problems presented in the introduction Section. Specifically, we have the following diagrams.

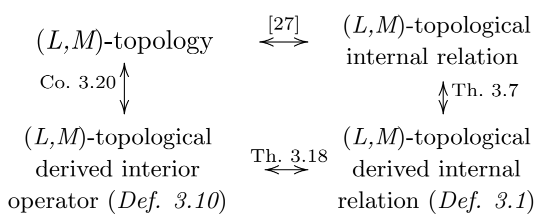


Figure 3: Solution of Problem 1

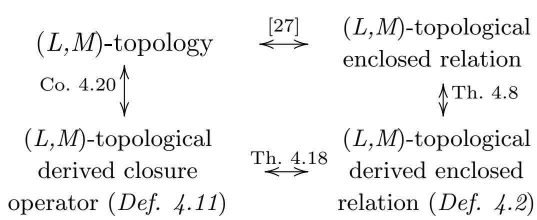


Figure 4: Solution of Problem 2

Relationships among the categories presented in Relations 1 and 2 in Section 3 and 4 may provide some alternative ways in discussing properties of (L,M)-fuzzy convex space and (L,M)-fuzzy convergence space. In addition, they may also be helpful in characterizing (L,M)-fuzzy topological-convex spaces.

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