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Maximal product of two L-graph automata and some applications

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Abstract

The current study aimed to describe the notion of the L-graph, which is constructed on a residuated lattice, presenting the idea of the maximal product of two L-graphs. Moreover, the applied algorithm demonstrated its efficiency to a great extent to improve the educational system. The results showed that the maximal product of both G and H and reversely, H and G are isomorphic L-graphs. The maximal product of two L-graph automata was represented as a new notion, followed by investigating their self-sufficiency conditions in two different modes. In addition, $A(Z(G)) \star A(Z(H))$ and $A(Z(H)) \star A(Z(G))$ were proved as two isomorphic-related L-graph automata considering the residuated lattice properties. Then, correlations among $A(Z(G)) \star A(Z(H))$, $A(Z(G))$, and $A(Z(H))$ behaviors were explained. As a result, some theorems of the relationship between L-graph automata and their maximal product were introduced. Subsequently, some related theorems were proved, and several examples were provided to illustrate these new notions. Further, an application of the maximal product of two related L-graph automata was expressed in the factors affecting the spared of the coronavirus, and accordingly, some solutions were suggested to reduce the spread of the virus.

Keywords: Residuated lattice, L-graph, maximal product of two L-graphs, maximal product of two L-graph automata.

1 Introduction

Leonhard Euler (1736) formulated the concept of a graph to solve the Konigsberg Bridge [8]. After developing graph theory, many researchers have used this concept to solve real problems in various fields. This concept has also been used to determine intercity roads, prepare urban maps, and etc. Suppose G is a graph with V as its vertex set. The graph's vertices are either white or black. Assume that Z represents the (main) set of black vertices in graph G. When a black vertex has only the white vertex as a neighbor, the color of the vertex is turned to black from white according to the color-change rule. Set Z is a zero forcing set of G if all vertices of G are turned to black following finite applications of the color-change rule. The minimum of $|Z|$ overall zero forcing sets is denoted as $Z(G)$. All white vertices are transformed black using this so-called "zero forcing procedure."This is an illustration of a graph propagation process. In computer science and mathematics, these processes are among the most often discussed issues (refer to [6]). For modeling social or technological processes, such as social network analysis [13] and physics [5], several graph approaches are used. In their study, Barrat, Barthelemy and Vespignani [6] provided a thorough examination of several applications and models. The zero forcing procedure was described in [4] for constraining the graph G's minimal rank or maximum nullity. Zadeh [28] first introduced the concept of a fuzzy subset and used the set type to model ambiguous natural events. Since then, many applications have been explored for this concept by many researchers (e.g Kuzmin [14] and Bezdek [7]). In addition, fuzzy automaton was first introduced by Wee [26] and Santos [21]. Fuzzy finite automata have several applications in pattern recognition, learning systems, database theory, and neural networks. Qiu [18] and [19] investigated the fuzzy automata generalization, and Tiwari and Sharan [24] presented the concept of fuzzy automata based on lattice-ordered monoid L-fuzzy automata. According to these research, the results of such L-fuzzy automata are dependent on whether the related monoid has zero divisors. Since then, many notions have been explored for this automaton by many authors (e.g Abolpour [1, 2] and Shamsizade [3, 22, 23]). Recently, new definitions of graph

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automata based on zero-forcing sets have been presented by Zahedi and Golmohamadian [11]. Further, the L-graph notion based on the residuated lattice was introduced and its applications were explicated and discussed [20]. Moreover, a novel L-fuzzy automaton, known as L-graph automaton, was also introduced by considering a minimum zero-forcing set obtained from an L-graph [20].
The present study is an attempt to introduce and investigate a novel maximal product of two L-graphs, followed by representing one of the applications of this L-graph to improve the educational system. Additionally, the maximal product of two related L-graph automata is specified. Since the transition with label n cannot affect the behavior of these automata, the L''-graph automata are notified by the contraction of all transitions with label n in L-graph automata [20]. Besides, some examples and theorems are suggested to examine the maximal product of two L-graphs and two related L-graph automata. Further, the relationship between two structures (i.e., the maximal product of L-graphs and the maximal product of related L-graph automata) is also identified in this study. Furthermore, the applications of this automaton are investigated in various fields.

2 Preliminaries

In this section, the basic notion of graph [10, 27], fuzzy graph [16, 17], and L-graph [20] are described, followed by reviewing definitions and theorems of residuated lattice and L-graph automaton [20].

Definition 2.1. [25] *A residuated lattice is an algebra $L = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ such that*
(i) $L = (L, \vee, \wedge, 0, 1)$ is a lattice (the corresponding order will be denoted by $\leq$) with the smallest element 0 and the greatest element 1,
(ii) $L = (L, \otimes, 1)$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $x \otimes 1 = x$ holds),
(iii) $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ holds (adjointness condition).

Proposition 2.2. [9] *Let $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. Then the following properties hold:*
(R_1) $1 * x = x$, where $* \in \{\wedge, \otimes, \rightarrow\}$,
(R_2) $x \otimes 0 = 0, 1' = 0, 0' = 1$,
(R_3) $x \otimes y \leq x \wedge y \leq x, y$, and $y \leq (x \rightarrow y)$,
(R_4) $x \leq y$ implies $x * z \leq y * z$, where $* \in \wedge, \vee, \otimes$,
(R_5) $z \otimes (x \wedge y) \leq (z \otimes x) \wedge (z \otimes y)$,
(R_6) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$,
(R_7) $(x \vee y) \rightarrow z = (x \rightarrow y) \wedge (x \rightarrow z)$.

Definition 2.3. [10] *A path is a simple graph whose vertices can be arranged in a linear sequence so that two vertices are adjacent if they are consecutive in the line, and are nonadjacent otherwise.*
Likewise, a cycle on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise.
A complete graph is a simple graph where any two vertices are adjacent.
A graph is bipartite if its vertex set can be partitioned into two subsets, X and Y, so that every edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of the graph, and X and Y are considered as its parts. We denote a bipartite graph G with bipartition (X, Y) by $G[X, Y]$. If $G[X, Y]$ is simple and every vertex in X is joined to every vertex in Y, then G is called a complete bipartite graph.

Definition 2.4. [20] *$G = (\alpha, \beta)$ is called an L-graph on $G^* = (V, E)$ that is a simple graph if $\alpha : V \rightarrow L$ and $\beta : E \rightarrow L$ are functions, with $\beta(st) \leq \alpha(s) \otimes \alpha(t)$ for every $st \in E$.*
Besides, if G^* is a path (cycle, bipartite, complete, complete bipartite) graph, then G is called a path (cycle, bipartite, complete, complete bipartite) L-graph on G^*.
The degree of a vertex v in L-graph G, denoted by $d_G(v)$, is the number of edges of G incident with v.

Definition 2.5. [20] *Let $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ be two L-graphs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $c \in L \backslash \{1\}$. Then G_1 and G_2 are isomorphic with threshold c, denoted by $G_1 \cong_c G_2$ if there exists a bijection h from V_1 into V_2 such that the following conditions hold true for all $u, v \in V_1$:*
(i) $uv \in E_1$ if and only if $h(u)h(v) \in E_2$,
(ii) $\alpha_1(u) > c$ if and only if $\alpha_2(h(u)) > c$,
(iii) $\beta_1(uv) > c$ if and only if $\beta_2(h(u)h(v)) > c$.
h is an isomorphism ($\cong$) if and only if h is an isomorphism with threshold c for every $c \in L \backslash \{1\}$.

Definition 2.6. [12] *A finite automaton (FSA) is described by a five-tuple $(Q, \Sigma, I, \delta, F)$ where*
(i) Q is a finite set of states,
(ii) Σ is a finite input alphabet,
(iii) I is the initial states,
(iv) $\delta : Q \times \Sigma \to Q$ is the transition function,
(v) F is the set of accepting (final) states.

For any element q of Q and any symbol $\sigma \in \Sigma$, we interpret $\delta(q, \sigma)$ as the state to which the FSA moves, if it is in state q and receives the input σ.

Definition 2.7. [15] *Let L be a lattice. An L-fuzzy automaton A is a 5-tuple (Q, X, μ, F, σ), where*
(i) Q is a finite nonempty set (states),
(ii) X is a finite nonempty set (inputs),
(iii) μ is a function from $Q \times X \times Q$ into L (transition function),
(iv) F is a subset of Q (final states),
(v) σ is a function from Q into L (initial distribution).

Elements of X are called input symbols. Finite sequences of input symbols are called input strings, words, or tapes. Remember that the empty tape Λ has the property that $x\Lambda = x = \Lambda x$ for every tape x. The collection of all input tapes is denoted by X^*. So, the expanded function μ is defined as

$$\mu^*(q, \Lambda, q') = \begin{cases} 1 & \text{if } q = q', \\ 0 & \text{if } q \neq q', \end{cases} \qquad \mu^*(q, xu, q') = \bigvee_{q'' \in Q} \mu^*(q, x, q'') \otimes \mu(q'', u, q').$$

Definition 2.8. [20] *Let $G = (\alpha, \beta)$ be an L-graph on $G^* = (V, E)$ and $Z(G)$ be a zero forcing set of G^*. Then an L-graph automaton $A(Z(G))$ related to G is defined by five-tuple (Q, X, μ, F, σ), where*
(i) $Q = V$ is the finite nonempty set of states,
(ii) $X = \{f, n\}$ is the set of alphabet letters,
(iii) $\mu : Q \times X \times Q \longrightarrow L$ is the transition function; $\mu(q_i, u, q_j) = \begin{cases} \beta(q_i q_j) & \text{if } u = f, \\ 1 & \text{if } u = n, \end{cases}$
(iv) F is the set of final states, where $q \in F$ if and only if q does not force any vertex,
(v) $\sigma : Q \longrightarrow L$ is the initial distribution; $\sigma(q_i) = \begin{cases} 1 & \text{if } q_i \in Z(G), \\ 0 & \text{otherwise,} \end{cases}$ for every $q_i \in Q$.

Also, $Z(A(Z(G))) = \{q_i \in Q | \sigma(q_i) = 1\} = \{q_i \in Q | q_i \in Z(G)\}$ was defined.

Definition 2.9. [20] *A response function of $A(Z(G))$ is defined*

$$r_\mu(x, q) = \bigvee_{q' \in Q} \sigma(q') \otimes \mu^*(q', x, q).$$

Definition 2.10. [20] *A k-behavior of $A(Z(G))$ with threshold c is the set*

$$B_k(A(Z(G)), c) = \{x \in X^* | \bigvee_{q \in F} r_\mu(x, q) > c \text{ and } |x| \leq k\}.$$

Definition 2.11. [20] *Let $A(Z(G)) = (Q_1, X, \mu_1, F_1, \sigma_1)$ and $A(Z(H)) = (Q_2, X, \mu_2, F_2, \sigma_2)$ be two L-graph automata related to G and H, respectively, and $c \in L \setminus \{1\}$. If a function g from Q_1 to Q_2 is bijection and the following conditions hold:*
(i) $\mu_1(q, u, q') > c$ if and only if $\mu_2(g(q), u, g(q')) > c$,
(ii) $\sigma_1(q) > c$ if and only if $\sigma_2(g(q)) > c$,
(iii) $q \in F_1$ if and only if $g(q) \in F_2$,

for every $q, q' \in Q_1$ and $u \in X$, then g is called an isomorphism from $A(Z(G))$ onto $A(Z(H))$ with threshold c ($\simeq_c$). g is an isomorphism ($\simeq$) if and only if g is an isomorphism with threshold c for every $c \in L \setminus \{1\}$.

Remark 2.12. [20] *Let $A(Z(G))$ be an L-graph automaton. A contraction of transition $\mu(q_i, u, q_j)$ is to delete the transition and replace states q_i and q_j with a single state incident. All new transitions, incident in $A(Z(G))$, belong to q_i or q_j. We denote the resulting L-graph automaton by $A(Z(G)) \odot \mu(q_i, u, q_j)$.*

Definition 2.13. [20] *Let $A(Z(G)) = (Q, X, \mu, F, \sigma)$ be an L-graph automaton. An L'-graph automaton $A'(Z(G))$ is a five-tuple $(Q, X', \mu', F, \sigma')$. The transitions of $A'(Z(G))$ are contracted by transitions of $A(Z(G))$ labeled n, where*

$X' = \{f\}$ for every $q_i, q_j \in Q$, and $\mu'(q_i, f, q_j) = \mu(q_i, f, q_j)$. Furthermore, if q_i, q_j such that $i < j$ are endpoints of contraction of the transition, then $\sigma'(q_i) = \sigma'(q_j)$ is $\sigma(q_i) \vee \sigma(q_j)$; otherwise $\sigma'(q_i) = \sigma(q_i)$, and

$$q \in Z(A'(Z(G))) \Longleftrightarrow q \in Z(A(Z(G))).$$

Definition 2.14. [20] *Let $L = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. Then a new set $P_n^*(L) = \{\{x_1, x_2, \ldots, x_n\} | \ x_1, x_2, \ldots, x_n \in L\}$ is defined.*

Definition 2.15. [20] *Let $A'(Z(G)) = (Q, X', \mu', F, \sigma')$ be an L'-graph automaton. Then a label is selected with less index for every state of L'-graph automaton that have s labels, known as L''-graph automaton. We write $A''(Z(G)) = (Q'', X', \mu'', F'', \sigma'')$, where $Q'' \subseteq Q$, $F'' \subseteq F$, $\mu'' : Q \times X \times Q \longrightarrow P_n^*(L)$; $\mu''(q_i, f, q_j) = \{\mu'(q_i, f, q_j), \mu'(q_{i+k_1}, f, q_j), \ldots, \mu'(q_{i+k_l}, f, q_j), \mu'(q_i, f, q_{j+k_1}), \ldots, \mu'(q_i, f, q_{j+k_h}),$ for some q_{i+k_l}, q_{j+k_h}, where $1 \leq l \leq m, 1 \leq h \leq t$, and $l + h = s + 1\}$, $\sigma''(q_i) = \sigma'(q_i)$ and $\sigma''(q_j) = \sigma'(q_j)$.*

Definition 2.16. [20] *A chromatic number of an L''-graph automaton $A''(Z(G))$, written as $\chi(A''(Z(G)))$, is the smallest number of colors needed to label the states, with adjacent states receiving (except itself) different colors in the L''-graph automaton.*

Definition 2.17. [20] *Let G be an L-graph. Then G is called self-sufficient if $\chi(G) = \chi(A''(Z(G)))$.*

3 The maximal product of two L-graphs

Throughout this study, L denotes a residuated lattice, G represents an L-graph $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ as a simple graph, and H denotes an L-graph $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ as a simple graph. This section determines the maximal product of two L-graphs, and shows that its operation is well-defined. Moreover, the applied algorithm and an example are provided for clarification. We can use the maximal product of two L-graphs to reach a higher and more effective performance by connecting two interconnected systems.

Definition 3.1. *Let G and H be two L-graphs. Then the maximal product of two L-graphs G and H is defined by $G \star H = (\alpha, \beta)$ on $(G \star H)^* = (V, E)$, where*
(i) $V = V_1 \times V_2$,
(ii) $E = \{(q, q'_k)(q, q'_l) | \ q \in V_1, q'_k q'_l \in E_2\} \cup \{(q_i, q')(q_j, q') | \ q' \in V_2, q_i q_j \in E_1\}$,
(iii) $\alpha(q_i, q'_k) = \alpha_1(q_i) \vee \alpha_2(q'_k)$, for every $(q_i, q'_k) \in V$,
(iv) $\beta((q_i, q'_k)(q_j, q'_l)) = \begin{cases} \alpha_1(q_i) \otimes \beta_2(q'_k q'_l) \text{ if } q_i = q_j, q'_k q'_l \in E_2, \\ \alpha_2(q'_k) \otimes \beta_1(q_i q_j) \text{ if } q'_k = q'_l, q_i q_j \in E_1, \end{cases}$ *for every $(q_i, q'_k)(q_j, q'_l) \in E$.*

Algorithm 3.2. *The maximal product of two L-graphs:*
1. Input $G = (V_1, E_1, \alpha_1, \beta_1)$ and $H = (V_2, E_2, \alpha_2, \beta_2)$.
2. Set all $V'' = (q_i, q'_j)$ such that q_i in V_1 and q'_j in V_2 and $\alpha''(q_i, q'_j) = sup(\alpha_1(q_i), \alpha_2(q'_j))$.
3. Find all two (q_i, q'_j) and (q_s, q'_t) in V'' that $q_i = q_s$ and exists an edge between q'_j and q'_t in E_2 set $\beta''((q_i, q'_j)(q_s, q'_t)) = \alpha_1(q_i) \otimes \beta_2(q'_j q'_t)$ or $q'_j = q'_t$ and there exists an edge between q_i and q_s in E_1 then set $\beta''((q_i, q'_j)(q_s, q'_t)) = \beta_1(q_i q_s) \otimes \alpha_2(q'_j)$ (the edge between (q_i, q'_j) and (q_s, q'_t) in E'').
(iv) Output $G' = (V'', E'', \alpha'', \beta'')$.

Notation 3.3. *Consider two L-graphs G and H. Then GH denotes their maximal product $G \star H = (\alpha, \beta)$ on $(G \star H)^* = (V, E)$.*

Theorem 3.4. *Let G and H be two L-graphs. Then*
(i) GH is an L-graph,
(ii) If (q, q') is a vertex of GH, then $d_{GH}(q, q') = d_G(q) + d_H(q')$,
(iii) $|E| = |E_1| \times |V_2| + |V_1| \times |E_2|$.

Proof. (i) According to the definition of L-graph, this is proved immediately by showing $\beta((q_i, q'_k)(q_j, q'_l)) \leq \alpha(q_i, q'_k) \otimes \alpha(q_j, q'_l)$. We know that $\beta((q_i, q'_k)(q_j, q'_l))$ has two cases. Case 1: $q_i = q_j$, and $q'_k q'_l \in E_2$

and Case 2: $q'_k = q'_l$, and $q_iq_j \in E_1$. Assume the first case. Hence,

$$\begin{aligned}
\beta((q_i,q'_k)(q_j,q'_l)) &= \alpha_1(q_i)\otimes\beta_2(q'_kq'_l), \text{ by definition of } \beta\\
&\leq \alpha_1(q_i)\otimes(\alpha_2(q'_k)\otimes\alpha_2(q'_l)), \text{ by definition of L-graph H}\\
&\leq \alpha_2(q'_k)\otimes\alpha_2(q'_l), \text{ by } (R_3) \text{ of Proposition 2.2}\\
&\leq (\alpha_1(q_i)\otimes\alpha_1(q_j))\vee(\alpha_1(q_i)\otimes\alpha_2(q'_l))\vee(\alpha_2(q'_k)\otimes\alpha_1(q_j))\vee(\alpha_2(q'_k)\otimes\alpha_2(q'_l)), \text{ by definition of } \vee\\
&= (\alpha_1(q_i)\otimes(\alpha_1(q_j)\vee\alpha_2(q'_l)))\vee(\alpha_2(q'_k)\otimes(\alpha_1(q_j)\vee\alpha_2(q'_l))), \text{ by } (R_6) \text{ of Proposition 2.2}\\
&= (\alpha_1(q_i)\vee\alpha_2(q'_k))\otimes(\alpha_1(q_j)\vee\alpha_2(q'_l)), \text{ by } (R_6) \text{ of Proposition 2.2}\\
&= \alpha(q_i,q'_k)\otimes\alpha(q_j,q'_l), \text{ by definition of } \alpha.
\end{aligned}$$

The above results were obtained through the second case; therefore, we can say that GH is an L-graph.
(ii) Each vertex (q,q') of L-graph GH is connected to the vertices (q_i,q') that q_i are adjacent vertices of q in the L-graph G and is connected to the vertices (q,q'_j) that q'_j are vertices adjacent of q' in the L-graph H. So,
$d_{GH}(q,q') = d_G(q) + d_H(q')$.
(iii) Consider $K = \{(q,q'_k)(q,q'_l)|\ \ q\in V_1, q'_kq'_l\in E_2\}$, and $M = \{(q_i,q')(q_j,q')|\ \ q'\in V_2, q_iq_j\in E_1\}$. As $E = M\cup K$,

$$|E| = |M| + |K| = |E_1|\times|V_2| + |V_1|\times|E_2|.$$

□

Example 3.5. *Suppose* $L = ([0,1],\vee,\wedge,\otimes,\rightarrow,0,1)$, *and two L-graphs* G *in Figure 1 and* H, *as in Figure 2, where* $a\otimes b = \begin{cases}(a+b-1) & \text{if } a+b\geq 1,\\ 0 & \text{if } a+b<1,\end{cases}$ $a\rightarrow b = \begin{cases}1 & \text{if } b-a\geq 0,\\ (1-a+b) & \text{if } b-a<0,\end{cases}$ $V_1 = \{q_1,q_2,q_3,q_4\}$, $E_1 = \{q_1q_2, q_2q_3, q_1q_3, q_3q_4\}$, $\beta_1(q_iq_j) = \alpha_1(q_i)\otimes\alpha_1(q_j)$, *for every* $q_iq_j\in E_1$, $\alpha_1(q_1) = 0.6$, $\alpha_1(q_2) = 0.7$, $\alpha_1(q_3) = 0.8$, $\alpha_1(q_4) = 0.5$, $\beta_1(q_1q_2) = 0.3$, $\beta_1(q_1q_3) = 0.4$, $\beta_1(q_2q_3) = 0.5$, $\beta_1(q_3q_4) = 0.3$, $V_2 = \{q'_1,q'_2,q'_3,q'_4\}$, $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_1q'_4, q'_2q'_4\}$, $\beta_2(q'_iq'_j) = (\alpha_2(q'_i)\wedge\alpha_2(q'_j))\otimes(\alpha_2(q'_i)\wedge\alpha_2(q'_j))$, *for every* $q'_iq'_j\in E_2$, $\alpha_2(q'_1) = 0.8$, $\alpha_2(q'_2) = 0.6$, $\alpha_2(q'_3) = 0.4$, $\alpha_2(q'_4) = 0.9$, $\beta_2(q'_1q'_2) = 0.2$, $\beta_2(q'_2q'_3) = 0$, $\beta_2(q'_3q'_4) = 0$, $\beta_2(q'_1q'_4) = 0.6$, *and* $\beta_2(q'_2q'_4) = 0.2$. *Then* GH *is the maximal product of* G *and* H *(Figure 3) where* $V = \{(q_i,q'_j)|\ 1\leq i,j\leq 4\}$, $E = \{(q_i,q'_1)(q_i,q'_2),(q_i,q'_1)(q_i,q'_4),(q_i,q'_2)(q_i,q'_4),(q_i,q'_2)(q_i,q'_3),(q_i,q'_3)(q_i,q'_4),(q_1,q'_i)(q_2,q'_i),(q_1,q'_i)(q_3,q'_i),(q_2,q'_i)(q_3,q'_i),(q_3,q'_i)(q_4,q'_i)|\ 1\leq i\leq 4\}$, $\alpha(q_1,q'_2) = 0.6$, $\alpha(q_1,q'_3) = 0.6$, $\alpha(q_2,q'_2) = 0.7$, $\alpha(q_2,q'_3) = 0.7$, $\alpha(q_3,q'_2) = 0.8$, $\alpha(q_3,q'_3) = 0.8$, $\alpha(q_4,q'_2) = 0.6$, $\alpha(q_4,q'_3) = 0.5$, $\alpha(q_i,q'_4) = 0.9$, $\alpha(q_i,q'_1) = 0.8$,

$$\begin{array}{llll}
\beta((q_i,q'_1)(q_i,q'_2)) = 0 & \beta((q_i,q'_3)(q_i,q'_4)) = 0 & \beta((q_i,q'_2)(q_i,q'_3)) = 0 & \beta((q_i,q'_2)(q_i,q'_4)) = 0\\
\beta((q_1,q'_1)(q_1,q'_4)) = 0.2 & \beta((q_2,q'_1)(q_2,q'_4)) = 0.3 & \beta((q_3,q'_1)(q_3,q'_4)) = 0.4 & \beta((q_4,q'_1)(q_4,q'_4)) = 0.1\\
\beta((q_1,q'_1)(q_2,q'_1)) = 0.1 & \beta((q_1,q'_1)(q_3,q'_1)) = 0.2 & \beta((q_3,q'_1)(q_4,q'_1)) = 0.1 & \beta((q_2,q'_1)(q_3,q'_1)) = 0.3\\
\beta((q_1,q'_2)(q_2,q'_2)) = 0 & \beta((q_1,q'_2)(q_3,q'_2)) = 0 & \beta((q_3,q'_2)(q_4,q'_2)) = 0 & \beta((q_2,q'_2)(q_3,q'_2)) = 0.1\\
\beta((q_1,q'_3)(q_2,q'_3)) = 0 & \beta((q_1,q'_3)(q_3,q'_3)) = 0 & \beta((q_3,q'_3)(q_4,q'_3)) = 0 & \beta((q_2,q'_3)(q_3,q'_3)) = 0\\
\beta((q_1,q'_4)(q_2,q'_4)) = 0.2 & \beta((q_1,q'_4)(q_3,q'_4)) = 0.3 & \beta((q_3,q'_4)(q_4,q'_4)) = 0.2 & \beta((q_2,q'_4)(q_3,q'_4)) = 0.4
\end{array}$$

for every $1\leq i\leq 4$. *Also,* $d_{GH}(q_i,q'_j) = 4 = 2+2 = d_G(q_i)+d_H(q'_j)$, $d_{GH}(q_3,q'_j) = 5 = 3+2 = d_G(q_3)+d_H(q'_j)$, $d_{GH}(q_4,q'_j) = 3 = 1+2 = d_G(q_4)+d_H(q'_j)$, $d_{GH}(q_i,q'_k) = 5 = 2+3 = d_G(q_i)+d_H(q'_k)$, $d_{GH}(q_3,q'_k) = 6 = 3+3 = d_G(q_3)+d_H(q'_k)$, *and* $d_{GH}(q_4,q'_k) = 4 = 1+3 = d_G(q_4)+d_H(q'_k)$, *for every* $i = 1,2$, $j = 1,3$, *and* $k = 2,4$. *Additionally,* $|E| = 36 = 16+20$.

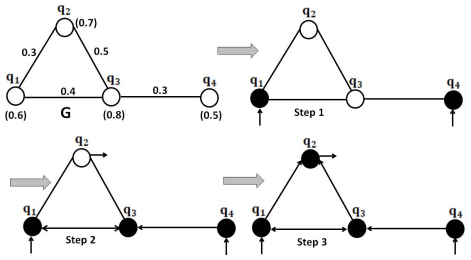


Figure 1: The L-graph G and the number steps of $Z_1(G)$

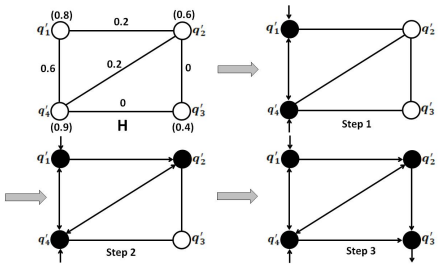


Figure 2: The L-graph H and the number steps of $Z_1(H)$

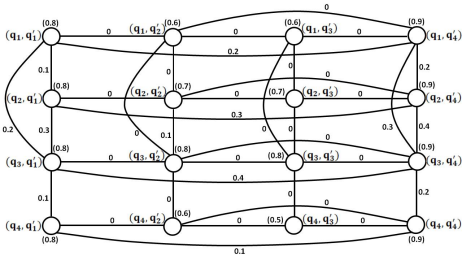


Figure 3: GH as the maximal product of two L-graphs G and H

The number steps of an L-graph with a zero-forcing set $Z(G)$ was determined when all vertices turned in black and denoted by $S(Z(G))$. Furthermore, some examples were presented to clarify this definition. In this regard, an L-graph can have the different number of steps with changing the $Z(G)$. Therefore, the optimization phase was introduced and clarified through an example. In some zero-forcing set applications of L-graph, the speed of action is very critical; thus, this definition helps optimize projects. In the following, we state the notion of a unified L-graph, which has the unique number of steps for different zero forcing sets. In addition, $S(Z(G))$ of cycle L-graph, path L-graph, complete L-graph, and complete bipartite L-graph are examined.

Remark 3.6. *Consider an L-graph G with $Z(G)$. Then the number of steps in which all vertices of G be black will be shown by $S(Z(G))$.*

Example 3.7. *Let G and H be as in Example 3.5, where $Z_1(G) = \{q_1, q_4\}$ and $Z_1(H) = \{q'_1, q'_4\}$. Then $S(Z_1(G)) = 3$, as in Figure 1, and $S(Z_1(H)) = 3$, as in Figure 2.*

Example 3.8. *Suppose $L = (\{1, 2, \ldots, 10\}, \vee, \wedge, \otimes, \rightarrow, 1, 10)$, and the L-graph G in Figure 4, where* $a \otimes b = \begin{cases} (a+b-10) & \text{if } a+b > 10, \\ 1 & \text{if } a+b \leq 10, \end{cases}$ *and* $a \rightarrow b = \begin{cases} 10 & \text{if } b-a \geq 0, \\ (10-a+b) & \text{if } b-a < 0, \end{cases}$ $V_1 = \{q_1, q_2, q_3, q_4, q_5, q_6\}$, $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_5, q_5q_6, q_1q_6, q_2q_5\}$, $\alpha_1(q_1) = 2$, $\alpha_1(q_2) = 9$, $\alpha_1(q_3) = 5$, $\alpha_1(q_4) = 6$, $\alpha_1(q_5) = 6$, $\alpha_1(q_6) = 7$, $\beta_1(q_1q_2) = 1$, $\beta_1(q_2q_3) = 3$, $\beta_1(q_3q_4) = 1$, $\beta_1(q_4q_5) = 2$, $\beta_1(q_5q_6) = 1$, $\beta_1(q_1q_6) = 1$, *and* $\beta_1(q_2q_5) = 4$. *Furthermore, if $Z_1(G) = \{q_1, q_6\}$, then $S(Z_1(G)) = 3$, as in Figure 4, and if $Z_2(G) = \{q_1, q_2\}$, then $S(Z_2(G)) = 4$, as in Figure 5.*

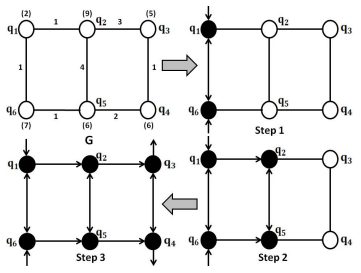


Figure 4: The L-graph G and $S(Z_1(G)) = 3$

Note 3.9. *The above example indicates that an L-graph with two $Z(G)$s has different $S(Z(G))$s.*

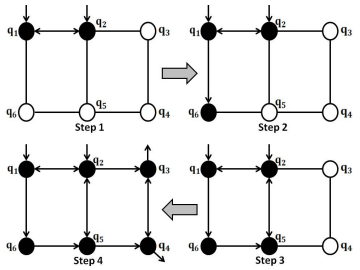


Figure 5: $S(Z_2(G)) = 4$

Definition 3.10. *Let G be the L-graph. Then the optimum of $Z(G)$s is those in which $S(Z(G))$ has the most negligible value. Also, $S(Z(G))$ that has least value is noted by $S_{min}(Z(G))$.*

Example 3.11. *Suppose the L-graph G in as Example 3.8. Then $Z_1(G)$ is the optimum $Z(G)$s of this L-graph, and $S_{min}(Z(G)) = 3$.*

Note 3.12. *Suppose the L-graph G. Then it is possible that it has more than one optimum of $Z(G)$s.*

Example 3.13. *Consider the L-graph G in Example 3.8. According to Example 3.11, $Z_1(G) = \{q_1, q_6\}$ is the optimum of $Z(G)$s. Clearly, $Z_3(G) = \{q_3, q_4\}$ is also the optimum of $Z(G)$s. Then this L-graph has more than one optimum of $Z(G)$s.*

Definition 3.14. *If G is an L-graph and all $Z(G)$s have the same $S(Z(G))$s, then G is a unified L-graph.*

Example 3.15. *Consider $L = (P(X), \cup, \cap, \otimes, \rightarrow, \emptyset, X)$, where $X = \{a, b, c, d\}$, $A \otimes B = A \cap B$, and*

$$A \rightarrow B = \begin{cases} X & \text{if } A \subseteq B, \\ B & \text{otherwise.} \end{cases}$$

Also, suppose the path L-graph G, as in Figure 6, where $V_1 = \{q_1, q_2, q_3\}$, $E_1 = \{q_1q_2, q_2q_3\}$, $\alpha_1(q_1) = \{a, b\}$, $\alpha_1(q_2) = \{c, d\}$, $\alpha_1(q_3) = \{b, c, d\}$, $\beta_1(q_1q_2) = \emptyset$, and $\beta_1(q_2q_3) = \{c\}$. Since $Z_1(G) = \{q_1\}$ with $S(Z_1(G)) = 3$, and $Z_2(G) = \{q_3\}$ with $S(Z_2(G)) = 3$, as in Figure 6, G is the unified L-graph.

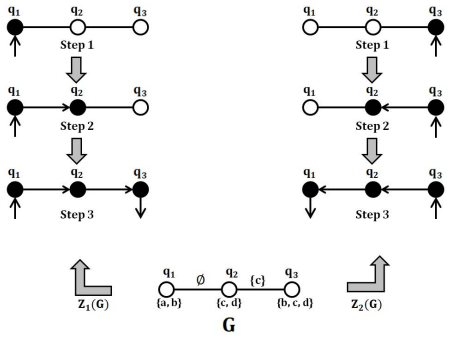


Figure 6: The path L-graph G, $S(Z_1(G)) = 3$ and $S(Z_2(G)) = 3$

Lemma 3.16. *Suppose an L-graph G. Then*
(i) If G is the cycle L-graph with $2k$ vertices, then it is the unified L-graph and $S_{min}(Z(G)) = k$.
(ii) If G is the cycle L-graph with $2k - 1$ vertices, then it is the unified L-graph and $S_{min}(Z(G)) = k$.
(iii) If G is the path L-graph with n vertices, then it is the unified L-graph and $S_{min}(Z(G)) = n$.
(iv) If G is the complete L-graph with n vertices, then it is the unified L-graph and $S_{min}(Z(G)) = 2$.
(v) If G is the complete bipartite L-graph with n vertices (as definition 2.3) such that $|X| > 1$, and $|Y| > 1$, then it is the unified L-graph and $S_{min}(Z(G)) = 2$.

Proof. (i) Suppose $Z(G)$ has two vertices; then in the first step, these two vertices are colored black. After that, these two vertices paint the other two vertices, and this process continues. Then $S_{min}(Z(G)) = k$. Since each zero forcing set of this graph has two adjacent vertices, each of these sets has the number of equal steps. Thus, G is the unified L-graph.

(ii) This proof is similar to the above with some modifications.

(iii) By using Definition 2.1 of [4], this L-graph has two zero forcing sets with one vertex, as well as an equal number of steps. Only one vertex is colored at each step of coloring the vertices. Then $S_{min}(Z(G)) = n$.

(iv) Each zero forcing set of this L-graph has $n-1$ vertices. Besides, the first step is to color these vertices, then they color a vertex. So, it is the unified L-graph and $S_{min}(Z(G)) = 2$.

(v) As every zero forcing set has $|X|-1$ vertices of X and $|Y|-1$ vertices of Y, this is proven. □

Definition 3.17. *Let two L-graphs G and H. If each of G and H have at least zero forcing sets $Z(G)$, and $Z(H)$ and $S(Z(G)) = S(Z(H))$, then these L-graphs are S-equivalent.*

Example 3.18. *Consider L and the L-graph G in Example 3.15. Also, the L-graph H, as in Figure 7, where $V_2 = \{q'_1, q'_2, q'_3, q'_4, q'_5\}$, $E_2 = \{q'_1q'_2, q'_2q'_3, q'_3q'_4, q'_4q'_5, q'_1q'_5\}$, $\alpha_2(q'_1) = \{a, c\}$, $\alpha_2(q'_2) = \{b, c\}$, $\alpha_2(q'_3) = \{d\}$, $\alpha_2(q'_4) = \{a, b, d\}$, $\alpha_2(q'_5) = \{b, d\}$, $\beta_2(q'_1q'_2) = \{c\}$, $\beta_2(q'_2q'_3) = \emptyset$, $\beta_2(q'_3q'_4) = \{d\}$, $\beta_2(q'_4q'_5) = \{b, d\}$, and $\beta_2(q'_1q'_5) = \emptyset$. Suppose $Z_1(H) = \{q'_1, q'_2\}$. Hence, $S(Z_1(H)) = 3$, as in Figure 7. As $S(Z_1(G)) = 3$ in Example 3.15, L-graphs G and H are S-equivalent.*

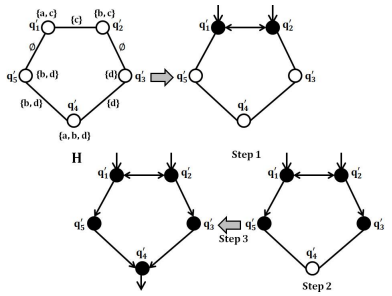


Figure 7: The cycle L-graph H and $S(Z_1(H)) = 3$

Definition 3.19. *If two unified L-graphs G and H are S-equivalent, then they are completely S-equivalent.*

Example 3.20. *Let two L-graphs G and H in Example 3.18. Then they are completely S-equivalent.*

Corollary 3.21. *Suppose two L-graphs G and H. Then*
(i) If G is the cycle L-graph with $2k$ vertices and H is the cycle L-graph with $2k-1$, then they are completely S-equivalent L-graphs.
(ii) If G is the path L-graph with n vertices and H is the cycle L-graph with $2n$ or $2n-1$, then they are completely S-equivalent L-graphs.
(iii) If G is the complete L-graph with n vertices and H is the complete bipartite L-graph with n, then they are completely S-equivalent L-graphs.

Proof. By using Lemma 3.16 and the definition of completely S-equivalent L-graphs, these are proved. □

Afterward, the theorem in which the L-graphs GH and HG are isomorphic is proved. Next, we demonstrate the correlation among the number of vertices of GH, and H and G. Consequently, the correlation between the number steps of GH, and L-graphs G and H was presented and clarified through an example. In the following, one of the applications of the maximal product of two L-graphs is presented in the quality of the education system.

Theorem 3.22. *Let G and H be two L-graphs. Then*
(i) GH and HG are isomorphic L-graphs.
(ii) GH has the copies of G^* in which their number is $|V_2|$, and it has $|V_1|$ copies of H^*.
(iii) If $|Z(G)| \times |V_2| < |Z(H)| \times |V_1|$, then $Z(GH) = \{(x, y) | x \in Z(G), y \in V_2\}$, $|Z(GH)| = |Z(G)| \times |V_2|$, and $S(Z(GH)) = S(Z(G))$.
(iv) If $|Z(H)| \times |V_1| < |Z(G)| \times |V_2|$, then $Z(GH) = \{(x, y) | x \in V_1, y \in Z(H)\}$, $|Z(GH)| = |Z(H)| \times |V_1|$, and

$S(Z(GH)) = S(Z(H))$.
(v) *If* $|Z(H)| \times |V_1| = |Z(G)| \times |V_2|$, *then the maximal product of two L-graphs has at least two zero forcing sets, where* $Z_1(GH) = \{(x,y)|\ x \in Z(G),\ y \in V_2\}$ *with* $S(Z_1(GH)) = S(Z(G))$ *and* $Z_2(GH) = \{(x,y)|\ x \in V_1,\ y \in Z(H)\}$ *with* $S(Z_2(GH)) = S(Z(H))$.

Proof. (i) Let $h : V_1 \times V_2 \to V_2 \times V_1$ such that $h(u,u') = (u',u)$ for every $u \in V_1$ and $u' \in V_2$. Then by the definition of the maximal product of two L-graphs, h is an isomorphism between GH and HG.
(ii) The vertices GH are arranged in pairs (u,u') such that $u \in V_1$ and $u' \in V_2$. If we consider all vertices with the same u', then the vertices will be an isomorphic graph with the graph G^*, which has the copies of G^* whose number is $|V_2|$. Similarly, it has the copies of H^* whose number is $|V_1|$.
The proofs of (iii), (iv), and (v) are straightforward. □

Example 3.23. *Suppose* L *and two L-graphs* G *and* H *as in Example 3.5. Then we see* $HG = (\alpha', \beta')$ *on* $(HG)^* = (V', E')$ *is the maximal product of* H *and* G, *as in Figure 8, where* $V' = \{(q'_1,q_i),(q'_2,q_i),(q'_3,q_i),(q'_4,q_i)|\ 1 \le i \le 4\}$, $E' = \{(q'_1,q_i)(q'_2,q_i),(q'_1,q_i)(q'_4,q_i),(q'_2,q_i)(q'_4,q_i),(q'_2,q_i,)(q'_3,q_i),(q'_3,q_i)(q'_4,q_i),(q'_i,q_1)(q'_i,q_2),(q'_i,q_1)(q'_i,q_3),(q'_i,q_2)(q'_i,q_3),(q'_i,q_3)(q'_i,q_4)|\ 1 \le i \le 4\}$,

$$\alpha'(q'_1,q_i) = 0.8\ ,\ \alpha'(q'_2,q_1) = 0.6\ ,\ \alpha'(q'_3,q_1) = 0.6\ ,\ \alpha'(q'_4,q_i) = 0.9\ ,\ \alpha'(q'_2,q_2) = 0.7\ ,$$
$$\alpha'(q'_3,q_2) = 0.7\ ,\ \alpha'(q'_2,q_3) = 0.8\ ,\ \alpha'(q'_3,q_3) = 0.8\ ,\ \alpha'(q'_2,q_4) = 0.6\ ,\ \alpha'(q'_3,q_4) = 0.5\ ,$$

$$\beta'((q'_1,q_i)(q'_2,q_i)) = 0\ ,\ \beta'((q'_1,q_1)(q'_4,q_1)) = 0.2\ ,\ \beta'((q'_2,q_i)(q'_4,q_i)) = 0\ ,\ \beta'((q'_2,q_i)(q'_3,q_i)) = 0\ ,$$
$$\beta'((q'_3,q_i)(q'_4,q_i)) = 0\ ,\ \beta'((q'_1,q_2)(q'_4,q_2)) = 0.3\ ,\ \beta'((q'_1,q_3)(q'_4,q_3)) = 0.4\ ,\ \beta'((q'_1,q_4)(q'_4,q_4)) = 0.1\ ,$$
$$\beta'((q'_1,q_1)(q'_1,q_2)) = 0.1\ ,\ \beta'((q'_1,q_1)(q'_1,q_3)) = 0.2\ ,\ \beta'((q'_1,q_2)(q'_1,q_3)) = 0.3\ ,\ \beta'((q'_1,q_3)(q'_1,q_4)) = 0.1\ ,$$
$$\beta'((q'_2,q_1)(q'_2,q_2)) = 0\ ,\ \beta'((q'_2,q_1)(q'_2,q_3)) = 0\ ,\ \beta'((q'_2,q_2)(q'_2,q_3)) = 0.1\ ,\ \beta'((q'_2,q_3)(q'_2,q_4)) = 0\ ,$$
$$\beta'((q'_3,q_1)(q'_3,q_2)) = 0\ ,\ \beta'((q'_3,q_1)(q'_3,q_3)) = 0\ ,\ \beta'((q'_3,q_2)(q'_3,q_3)) = 0\ ,\ \beta'((q'_3,q_3)(q'_3,q_4)) = 0\ ,$$
$$\beta'((q'_4,q_1)(q'_4,q_2)) = 0.2\ ,\ \beta'((q'_4,q_1)(q'_4,q_3)) = 0.3\ ,\ \beta'((q'_4,q_2)(q'_4,q_3)) = 0.4\ ,\ \beta'((q'_4,q_3)(q'_4,q_4)) = 0.2\ ,$$

for every $1 \le i \le 4$. *Clearly,* GH *and* HG *are isomorphic L-graphs. Additionally, this L-graph* HG *has* 4 *copies of* G *and* 4 *copies of* H. *As* $|Z_1(G)| = 2$, $|Z_1(H)| = 2$, $|V_2| = 4$, *and* $|V_1| = 4$, $|Z_1(G)| \times |V_2| = 8 = |Z_1(H)| \times |V_1|$. *Therefore,* $Z_1(HG) = \{(x,y)|\ x \in V_2,\ y \in Z_1(G)\} = \{(q'_i,q_1),(q'_i,q_4)|\ 1 \le i \le 4\}$, *and* $|Z_1(HG)| = 8$. *At the first step, it is colored* $Z_1(HG)$. *Next, the vertices,* $(q'_i,q_4)s$, *color the vertices* $(q'_i,q_3)s$, *for every* $1 \le i \le 4$. *The last step, the vertices,* $(q'_i,q_3)s$ *and* $(q'_i,q_1)s$, *color the vertices* $(q'_i,q_2)s$, *for every* $1 \le i \le 4$. *Thus,* $S(Z_1(HG)) = S(Z_1(G)) = 3$. *Also,* $Z_2(HG) = \{(x,\ y)|\ x \in Z_1(H),\ y \in V_1\} = \{(q'_1,q_i),(q'_4,q_i)|\ 1 \le i \le 4\}$, *and* $|Z_2(HG)| = 8$. *At the first step, it is colored* $Z_2(HG)$. *Next, the vertices,* $(q'_1,q_i)s$, *color the vertices* $(q'_2,q_i)s$, *for every* $1 \le i \le 4$. *The last step, the vertices,* $(q'_2,q_i)s$ *and* $(q'_4,q_i)s$, *color the vertices* $(q'_3,q_i)s$, *for every* $1 \le i \le 4$. *Besides,* $S(Z_2(HG)) = S(Z_1(H)) = 3$.

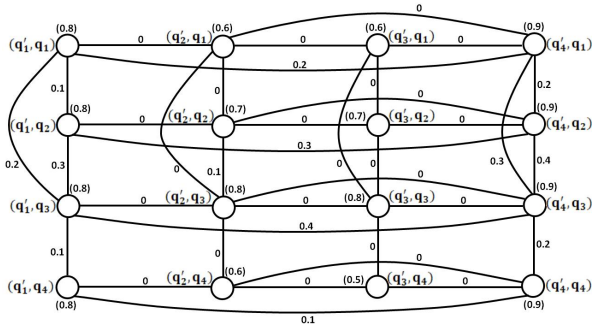


Figure 8: HG as the maximal product of two L-graphs H and G

Example 3.24. *Suppose* L, G, *and* H *in Example 3.18. Then* GH *is their maximal product, as in Figure 9, where* $V = \{(q_1,q'_i),(q_2,q'_i),(q_3,q'_i)|\ 1 \le i \le 5\}$, $E = \{(q_1,q'_i)(q_2,q'_i),(q_2,q'_i)(q_3,q'_i),(q_j,q'_1)(q_j,q'_2),(q_j,q'_2)(q_j,q'_3),(q_j,q'_3)(q_j,q'_4),(q_j,q'_4)(q_j,q'_5),(q_j,q'_1)(q_j,q'_5)|\ 1 \le i \le 5, 1 \le j \le 3\}$,

$$\alpha(q_1,q'_1) = \{a,b,c\}\ ,\ \alpha(q_1,q'_2) = \{a,b,c\}\ ,\ \alpha(q_1,q'_3) = \{a,b,d\}\ ,\ \alpha(q_1,q'_4) = \{a,b,d\}\ ,\ \alpha(q_1,q'_5) = \{a,b,d\}\ ,$$
$$\alpha(q_2,q'_1) = \{a,c,d\}\ ,\ \alpha(q_2,q'_2) = \{b,c,d\}\ ,\ \alpha(q_2,q'_3) = \{c,d\}\ ,\ \alpha(q_2,q'_4) = \{a,b,c,d\}\ ,\ \alpha(q_2,q'_5) = \{b,c,d\}\ ,$$
$$\alpha(q_3,q'_1) = \{a,b,c,d\}\ ,\ \alpha(q_3,q'_2) = \{b,c,d\}\ ,\ \alpha(q_3,q'_3) = \{b,c,d\}\ ,\ \alpha(q_3,q'_4) = \{a,b,c,d\}\ ,\ \alpha(q_3,q'_5) = \{b,c,d\}\ ,$$

$$\beta((q_1,q_i')(q_2,q_i')) = \emptyset \quad , \ \beta((q_2,q_1')(q_3,q_1')) = \{c\} \quad , \ \beta((q_2,q_2')(q_3,q_2')) = \{c\} \quad , \ \beta((q_2,q_j')(q_3,q_j')) = \emptyset \ ,$$
$$\beta((q_1,q_1')(q_1,q_2')) = \emptyset \quad , \ \beta((q_2,q_1')(q_2,q_2')) = \{c\} \quad , \ \beta((q_3,q_1')(q_3,q_2')) = \{c\} \quad , \ \beta((q_k,q_2')(q_k,q_3')) = \emptyset \ ,$$
$$\beta((q_1,q_3')(q_1,q_4')) = \emptyset \quad , \ \beta((q_2,q_3')(q_2,q_4')) = \{d\} \quad , \ \beta((q_3,q_3')(q_3,q_4')) = \{d\} \ , \ \beta((q_1,q_4')(q_1,q_5')) = \{b\} \ ,$$
$$\beta((q_2,q_4')(q_2,q_5')) = \{d\} \ , \ \beta((q_3,q_4')(q_3,q_5')) = \{b,d\} \ , \quad \beta((q_k,q_1')(q_k,q_5')) = \emptyset,$$

for every $1 \leq i \leq 5$, $3 \leq j \leq 5$, *and* $1 \leq k \leq 3$. *By using Example 3.18,* $|Z_1(G)| = 1$, $|Z_1(H)| = 2$, $|G| = 3$, *and* $|H| = 5$. *Since* $|Z_1(G)| \times |H| < |Z_1(H)| \times |G|$, $Z(GH) = \{(q_1,q_i') | \ 1 \leq i \leq 5\}$. *Also,* $|Z(GH)| = 5$. *At the first step, the vertices* $Z(GH)$ *is black. Next,* $(q_1,q_i')s$ *color* $(q_2,q_i')s$, *for every* $1 \leq i \leq 5$. *The third step,* $(q_3,q_i')s$ *are colored by* $(q_2,q_i')s$, *for every* $1 \leq i \leq 5$. *So* $S(Z(GH)) = 3 = S(Z_1(G))$.

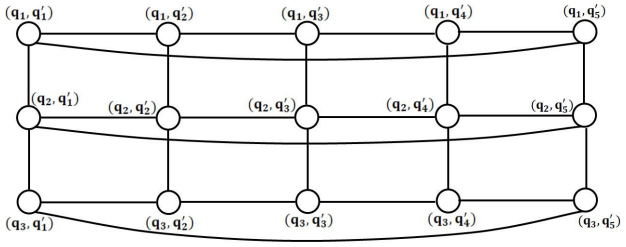


Figure 9: The graph $(GH)^*$

Application 3.25. *An educational system has two types of effects, i.e., personality traits and general conditions, which should be considered to determine the quality of an educational system. General requirements include the quality of the school, teachers, books, learning assist tools, discipline in the school, and the curriculum at school. On the other hand, personality traits include the quality of the family environment, interest rate of study, quality of the curriculum at home, and training tools at home. Therefore, we determine two L-graphs that cover the general conditions and personality traits. Each factor of these conditions was considered as the vertices of the L-graph. Also, each of these factors was connected through an edge if there is an influential relationship between them. For instance, as school and teachers are interdependent, there exists an edge between the vertex school and the vertex teachers. Using the L-graph, we can check personality traits and general conditions to upgrade the educational system, and using the maximal product of two L-graphs, we can get the effect of each personality trait on the general conditions by* α, *and using the* β, *the effect of each new condition (a combination of personality traits and general conditions) to obtain each other. As a result, using this modeling, a suitable solution can be found to improve the educational system. In the initial review, we described the fastest way for people to succeed in the education system. In the following, this case is represented in more details. Let* $L = (\{1,2,\ldots,10\}, \vee, \wedge, \otimes, \rightarrow, 1, 10)$, *where* $a \otimes b = \begin{cases} (a+b-10) & \text{if } a+b > 10, \\ 1 & \text{if } a+b \leq 10, \end{cases}$ *and*

$a \rightarrow b = \begin{cases} 10 & \text{if } b-a \geq 0, \\ (10-a+b) & \text{if } b-a < 0, \end{cases}$ *and let* G, *and* H *be two L-graphs, as in Figure 10, where* $V_1 = \{school(a_1), teachers(a_2), books(a_3), learning\ assist\ tools(a_4), discipline\ in\ the\ school(a_5), the\ curriculum\ at\ school(a_6)\}$, $E_1 = \{a_1a_2, a_1a_4, a_1a_5, a_1a_6, a_2a_4, a_2a_5, a_2a_6, a_3a_4, a_5a_6\}$, $\alpha_1(a_i) = quality\ of\ a_i$, $\beta_1(a_ia_j) = \alpha_1(a_i) \otimes \alpha_1(a_j)$, *for every* $a_ia_j \in E_1$, $Z_1(G) = \{a_1, a_2, a_6\}$, $V_2 = \{the\ family\ environment(b_1), interest\ of\ study(b_2), the\ curriculum\ at\ home(b_3), training\ tools\ at\ home(b_4)\}$, $E_2 = \{b_1b_2, b_1b_3, b_1b_4, b_2b_3, b_2b_4\}$, $\alpha_2(b_k) = quality\ of\ b_k$, *for every* $k = 1, 3, 4$, *and* $\alpha_2(b_2) = rate\ of\ b_2$; *moreover,* $\beta_2(b_ib_j) = \alpha_2(b_i) \otimes \alpha_2(b_j)$, *for every* $b_ib_j \in E_2$, *and* $Z_1(H) = \{b_2, b_3\}$. *Hence,* $S(Z_1(G)) = 4$ *in Figure 11 and* $S(Z_1(H)) = 3$, *according to Figure 12; then their maximal product is L-graph* GH, *as shown in Figure 13, where* $V = \{(a_i,b_1), (a_i,b_2), (a_i,b_3), (a_i,b_4) | \ 1 \leq i \leq 6\}$, $E = \{(a_i,b_1)(a_i,b_2), (a_i,b_1)(a_i,b_3), (a_i,b_1)(a_i,b_4), (a_i,b_2)(a_i,b_4), (a_i,b_2)(a_i,b_3), (a_1,b_j)(a_2,b_j), (a_1,b_j)(a_4,b_j), (a_1,b_j)(a_5,b_j), (a_1,b_j)(a_6,b_j), (a_2,b_j)(a_4,b_j), (a_2,b_j)(a_5,b_j), (a_2,b_j)(a_6,b_j), (a_3,b_j)(a_4,b_j), (a_5,b_j)(a_6,b_j) | \ 1 \leq i \leq 6,$ *and* $1 \leq j \leq 4\}$, $\beta((a_i,b_m)(a_i,b_n)) = \alpha_1(a_i) \otimes \beta_2(b_mb_n)$, $\beta((a_i,b_m)(a_j,b_m)) = \beta_1(a_ia_j) \otimes \alpha_2(b_m)$, *and* $\alpha(a_i,b_m) = \alpha_1(a_i) \vee \alpha_2(b_m)$, *for every* $a_ia_j \in E_1$, $b_mb_n \in E_2$, $1 \leq i \leq 6$, *and* $1 \leq m \leq 4$. *In this state, since* $|Z_1(H)| \times |V_1| = 12 = |Z_1(G)| \times |V_2|$, *according to Theorem 3.22 (v),* GH *has* $Z_1(GH) = \{(a_1,b_i), (a_2,b_i), (a_6,b_i) | \ 1 \leq i \leq 4\}$ *and* $Z_2(GH) = \{(a_i,b_2), (a_i,b_3) | \ 1 \leq i \leq 6\}$ *with* $S(Z_1(GH)) = 4$ *and* $S(Z_2(GH)) = 3$, *respectively. Since* $S(Z_2(GH)) < S(Z_1(GH))$, *the fastest way for people to progress in education is to improve the quality of* $Z_2(GH)$. *In general, there are two methods to improve the quality of the education system. The*

first method is to increase the quality of $Z(G)$, and the second method is to increase the quality of $Z(H)$. Therefore, it can be concluded that people in small communities with low welfare facilities can achieve scientific progress by improving the quality of study and curriculum.

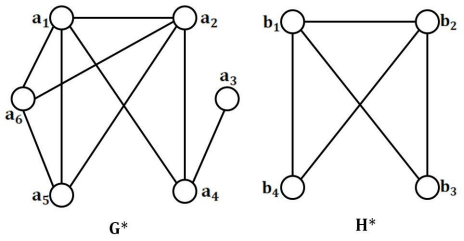


Figure 10: Two graphs G^* and H^*

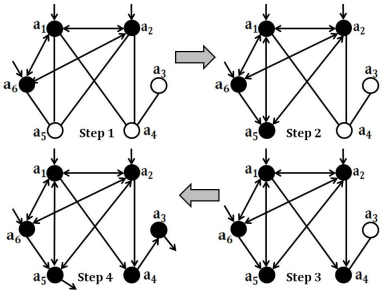


Figure 11: $S(Z(G)) = 4$

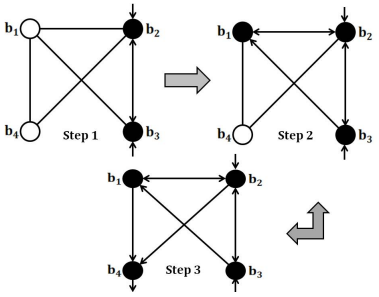


Figure 12: $S(Z(H)) = 3$

Note 3.26. *Using L-graphs, many problems that are composed of two different aspects can be modeled and using their maximal product, the effect of these conditions on each other can be obtained. We examine this model as needed and obtain useful results, and the above application was just an example of the application of these concepts. In our future research, we plan to thoroughly examine more complex issues such as improving community health and get the lowest cost of upgrading it.*

4 The maximal product of two L-graph automata

In this section, the concept of the maximal product of two L-graph automata is introduced in three cases. Moreover, the algorithm of generating the L-graph automaton from the L-graph is determined, and an example is given for more clarification.

Notation 4.1. *Suppose two L-graphs G and H with their zero forcing sets $Z(G)$ and $Z(H)$, respectively. Then their related L-graph automata $A(Z(G)) = (Q_1, X, \mu_1, F_1, \sigma_1)$ and $A(Z(H)) = (Q_1, X, \mu_2, F_2, \sigma_2)$ are denoted by $A(Z(G))$ and $A(Z(H))$, respectively.*

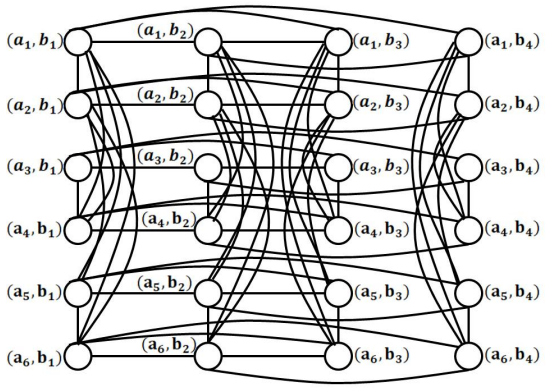


Figure 13: The graph $(GH)^*$.

Definition 4.2. *Let $A(Z(G))$ and $A(Z(H))$ be two related L-graph automata with $Z(A(Z(G)))$ and $Z(A(Z(H)))$, respectively. Then $A = A(Z(G)) \star A(Z(H)) = (Q, X, \mu, F, \sigma)$ is called a maximal product of $A(Z(G))$ and $A(Z(H))$ that has three cases:*
(i) $Q = Q_1 \times Q_2$ is the finite nonempty set of states,
(ii) $\sigma : Q \to L$ is the initial distribution; $\sigma(q_i, q'_j) = \begin{cases} 1 & \text{if } (q_i, q'_j) \in Z(A), \\ 0 & \text{otherwise}, \end{cases}$

Case 1. *If $|Z(A(Z(G)))| \times |Q_2| < |Z(A(Z(H)))| \times |Q_1|$, then*
(iii) $\mu : Q \times X \times Q \to L$ is the transition function;
If $q'_i = q'_l$, and there exists $\mu_1(q_t, u, q_k) \geq 0$, then there exists a transition with label u from (q_t, q'_i) to (q_k, q'_l) such that $\mu((q_t, q'_i), u, (q_k, q'_l)) = \begin{cases} \mu_1(q_t, f, q_k) \otimes \sigma_2(q'_i) & \text{if } u = f, \\ 1 & \text{if } u = n, \end{cases}$
If $q_t = q_k$, and H has $\mu_2(q'_i, u, q'_l) \geq 0$, then there exists a transition with label n between (q_t, q'_i) and (q_k, q'_l) such that $\mu((q_t, q'_i), n, (q_k, q'_l)) = \mu((q_k, q'_l), n, (q_t, q'_i)) = 1$,
(iv) F is the set of final states; $F = F_1 \times Q_2$,
(v) $Z(A) = Z(A(Z(G))) \times Q_2$.

Case 2. *If $|Z(A(Z(H)))| \times |Q_1| < |Z(A(Z(G)))| \times |Q_2|$, then*
(iii) $\mu : Q \times X \times Q \to L$ is the transition function;
If $q_t = q_k$, and there exists $\mu_2(q'_i, u, q'_l) \geq 0$, then there exists a transition with label u from (q_t, q'_i) to (q_k, q'_l) such that $\mu((q_t, q'_i), u, (q_k, q'_l)) = \begin{cases} \mu_2(q'_i, f, q'_l) \otimes \sigma_1(q_t) & \text{if } u = f, \\ 1 & \text{if } u = n, \end{cases}$
If $q'_i = q'_l$, and G has $\mu_1(q_t, u, q_k) \geq 0$, then there exists a transition with label n between (q_t, q'_i) and (q_k, q'_l) such that $\mu((q_t, q'_i), n, (q_k, q'_l)) = \mu((q_k, q'_l), n, (q_t, q'_i)) = 1$,
(iv) F is the set of final states; $F = Q_1 \times F_2$,
(v) $Z(A) = Q_1 \times Z(A(Z(H)))$.

Case 3. *If $|Z(A(Z(H)))| \times |Q_1| = |Z(A(Z(G)))| \times |Q_2|$, then it has two choices:*
1. If $Z(A) = Z(A(Z(G))) \times Q_2$ is selected, then the maximal product of $A(Z(G))$ and $A(Z(H))$ is similar to Case 1.
2. If $Z(A) = Q_1 \times Z(A(Z(H)))$ is selected, then the maximal product of $A(Z(G))$ and $A(Z(H))$ is similar to Case 2.

Algorithm 4.3. *Generate an L-graph automaton from an L-graph.*
1. Input $G = (V, E, \alpha, \beta)$, $Z(G) = V'$, $S = \{\}$.
2. All vertices v_is in V color them white, except v_js such that in V' and color all v_js in V' to black.
3. If all v_is in V are black, go to step 5, else find all v_is that are black and have one neighbor white and add to set S ($S = S \cup v_i$).
4. Color black all neighbors of v_js in S, and all the edges from v_js to their neighbors such that they are white and labeled f (force) and μ of these edges equals to β of these, then go to step 3.
5. All the edges that have no label, are labeled n (non-force), and μ of these edges equals to 1.
6. If v_is in $Z(G)$, σ is equals to 1 else σ equals 0.
7. Set $F = V \backslash S$, $X = \{f, n\}$ and $Q = V$.
8. Output $A(Z(G)) = (Q, X, \mu, F, \sigma)$.

Example 4.4. *Consider L, and let G and H be two L-graphs, as in Example 3.5. Then $A(Z_1(G))$ and $A(Z_1(H))$ are two L-graph automata with $Z(A(Z_1(G))) = \{q_1, q_4\}$ and $Z(A(Z_1(H))) = \{q'_1, q'_4\}$, respectively, as shown in Figure 14, where $Q_1 = \{q_1, q_2, q_3, q_4\}$, $X = \{n, f\}$, $\mu_1(q_1, n, q_3) = 1 = \mu_1(q_3, n, q_1)$, $\mu_1(q_4, f, q_3) = 0.3$, $\mu_1(q_1, f, q_2) = 0.3$, $\mu_1(q_3, f, q_2) = 0.5$, $F_1 = \{q_2\}$, $\sigma_1(q_1) = \sigma_1(q_4) = 1$, $\sigma_1(q_2) = \sigma_1(q_3) = 0$, $Q_2 = \{q'_1, q'_2, q'_3, q'_4\}$, $\mu_2(q'_1, f, q'_2) = 0.2$, $\mu_2(q'_1, n, q'_4) = 1 = \mu_2(q'_4, n, q'_1)$, $\mu_2(q'_2, n, q'_4) = 1 = \mu_2(q'_4, n, q'_2)$, $\mu_2(q'_2, f, q'_3) = 0$, $\mu_2(q'_4, f, q'_3) = 0$, $\sigma_2(q'_1) = \sigma_2(q'_4) = 1$, $\sigma_2(q'_2) = \sigma_2(q'_3) = 0$, $F_2 = \{q'_3\}$, and $|Z(A(Z_1(H)))| \times |Q_1| = 8 = |Z(A(Z_1(G)))| \times |Q_2|$. Then $A_1 = A(Z_1(G)) \star A(Z_1(H)) = (Q, X, \mu, F, \sigma)$ is their maximal product with $Z(A_1) = Z(A(Z_1(G))) \times Q_2$, as shown in Figure 15, where $Q = \{(q_1, q'_i), (q_2, q'_i), (q_3, q'_i), (q_4, q'_i) | \ 1 \le i \le 4\}$, $Z(A_1) = \{(q_1, q'_i), (q_4, q'_i) | \ 1 \le i \le 4\}$,*

$$\begin{aligned}
&\mu((q_1,q'_1),f,(q_2,q'_1)) = 0.3 = \mu((q_1,q'_4),f,(q_2,q'_4)) \ , \ \mu((q_4,q'_1),f,(q_3,q'_1)) = 0.3 = \mu((q_4,q'_4),f,(q_3,q'_4)) \ ,\\
&\mu((q_3,q'_1),f,(q_2,q'_1)) = 0.5 = \mu((q_3,q'_4),f,(q_2,q'_4)) \ , \quad \mu((q_1,q'_2),f,(q_2,q'_2)) = 0 = \mu((q_1,q'_3),f,(q_2,q'_3)) \ ,\\
&\mu((q_3,q'_2),f,(q_2,q'_2)) = 0 \quad = \mu((q_3,q'_3),f,(q_2,q'_3)) \ , \quad \mu((q_4,q'_2),f,(q_3,q'_2) = 0 = \mu((q_4,q'_3),f,(q_3,q'_3)) \ ,\\
&\mu((q_i,q'_1),n,(q_i,q'_2)) = 1 \quad = \mu((q_i,q'_2),n,(q_i,q'_1)) \ , \quad \mu((q_i,q'_1),n,(q_i,q'_4)) = 1 = \mu((q_i,q'_4),n,(q_i,q'_1)) \ ,\\
&\mu((q_i,q'_2),n,(q_i,q'_3)) = 1 \quad = \mu((q_i,q'_3),n,(q_i,q'_2)) \ , \quad \mu((q_i,q'_2),n,(q_i,q'_4)) = 1 = \mu((q_i,q'_4),n,(q_i,q'_2)) \ ,\\
&\mu((q_i,q'_3),n,(q_i,q'_4)) = 1 \quad = \mu((q_i,q'_4),n,(q_i,q'_3)) \ , \quad \mu((q_1,q'_i),n,(q_3,q'_i)) = 1 = \mu((q_3,q'_i),n,(q_1,q'_i)) \ ,
\end{aligned}$$

$\sigma(q_1, q'_i) = 1 = \sigma(q_4, q'_i)$, $\sigma(q_2, q'_i) = 0 = \sigma(q_3, q'_i)$, *for every* $1 \le i \le 4$, *and* $F = \{(q_2, q'_1), (q_2, q'_2), (q_2, q'_3), (q_2, q'_4)\}$. *Also,* $A_2 = A(Z_1(G)) \star A(Z_1(H)) = (Q', X, \mu', F', \sigma')$ *is their maximal product with* $Z(A_2) = Q_1 \times Z(A(Z_1(H)))$, *as shown in Figure 16, where* $Q' = \{(q_i, q'_j) | 1 \le i, j \le 4\}$, $Z(A_2) = \{(q_i, q'_1), (q_i, q'_4) | \ 1 \le i \le 4\}$,

$$\begin{aligned}
&\mu'((q_1,q'_1),f,(q_1,q'_2)) = 0.2 = \mu'((q_4,q'_1),f,(q_4,q'_2)) \ , \ \mu'((q_2,q'_1),f,(q_2,q'_2)) = 0 = \mu'((q_3,q'_1),f,(q_3,q'_2)) \ ,\\
&\mu'((q_i,q'_4),f,(q_i,q'_3)) = 0 = \quad \mu'((q_i,q'_2),f,(q_i,q'_3)) \ , \ \mu'((q_1,q'_i),n,(q_2,q'_i)) = 1 = \mu'((q_2,q'_i),n,(q_1,q'_i)) \ ,\\
&\mu'((q_2,q'_i),n,(q_3,q'_i)) = 1 = \quad \mu'((q_3,q'_i),n,(q_2,q'_i)) \ , \ \mu'((q_3,q'_i),n,(q_4,q'_i)) = 1 = \mu'((q_4,q'_i),n,(q_3,q'_i)) \ ,\\
&\mu'((q_1,q'_i),n,(q_3,q'_i)) = 1 = \quad \mu'((q_3,q'_i),n,(q_1,q'_i)) \ , \ \mu'((q_i,q'_1),n,(q_i,q'_4)) = 1 = \mu'((q_i,q'_4),n,(q_i,q'_1)) \ ,\\
&\mu'((q_i,q'_2),n,(q_i,q'_4)) = 1 = \quad \mu'((q_i,q'_4),n,(q_i,q'_2)) \ ,
\end{aligned}$$

$\sigma'(q_i, q'_1) = 1 = \sigma'(q_i, q'_4)$, $\sigma'(q_i, q'_2) = 0 = \sigma'(q_i, q'_3)$, *for every* $1 \le i \le 4$, *and* $F' = \{(q_i, q'_3) | 1 \le i \le 4\}$.

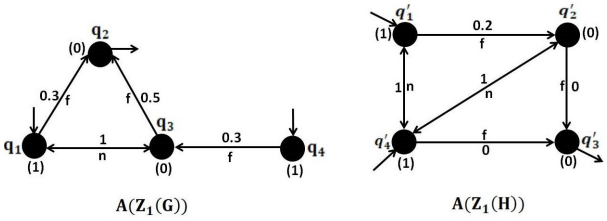


Figure 14: Two L-graph automata $A(Z_1(G))$ and $A(Z_1(H))$

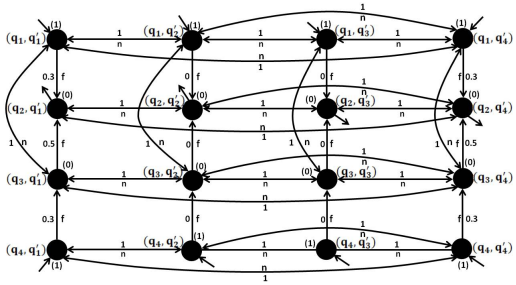


Figure 15: The maximal product of two L-graph automata A_1

In applications where the values of the edges are the same, the relationship between $S(Z(G))$ and the word of the automaton can be used to obtain its behavior. Here, it is shown that $A(Z(G)) \star A(Z(H))$ and $A(Z(H)) \star A(Z(G))$ are two isomorphic related L-graph automata.

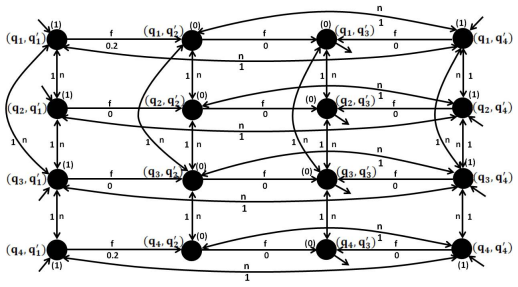


Figure 16: The maximal product of two L-graph automata A_2

Lemma 4.5. *Let $A(Z(G))$ be an L-graph automaton related to G, $x \in B(A(Z(G)), c)$, and $x^{'} = x$ without any label n. Then $|x^{'}| \leq S(Z(G)) - 1$.*

Proof. Given that at least one vertex of the graph is colored at each stage of coloring, the proof is straightforward. □

Theorem 4.6. *If G is an L-graph with $Z(G)$, where $\beta_1(q_iq_j) = k$ for every $q_iq_j \in E_1$ such that k is constant in L. Also, $A(Z(G))$ is the related L-graph automaton, then*
(i) $r_\mu(x, q) \geq k \otimes k \otimes \ldots \otimes k = (k \otimes k)^{(S(Z(G))-2)}$, for every $x \in X^*$ and $q \in F_1$.
(ii) $r_\mu(x, q) = k \otimes k \otimes \ldots \otimes k = (k \otimes k)^{(S(Z(G))-2)}$, for every $x \in X^*$ such that $x' = x$ without label n and $|x'| = |(S(Z(G)) - 1)|$ and $q \in F_1$.

Proof. (i) We know that

$$\begin{aligned} r_\mu(x, q) &= \bigvee_{q' \in Q} \sigma(q') \otimes \mu^*(q', x, q) = \bigvee_{q' \in Z(G)} \mu^*(q', x, q) = \bigvee_{q'' \in Z(G)} \mu^*(q'', x', q); x' = x \quad \text{without any label n} \\ &\geq k \otimes k \otimes \ldots k \otimes k, \text{ by Lemma 4.5 and } (R_2) \text{ of Proposition2.2} \\ &= (k \otimes k)^{(S(Z(G))-2)}. \end{aligned}$$

(ii) The proof is similar as above with some modifications. □

Theorem 4.7. *Let $A(Z(G))$ and $A(Z(H))$ be two related L-graph automata. Then $A(Z(G)) \star A(Z(H))$ and $A(Z(H)) \star A(Z(G))$ are two isomorphic related L-graph automata.*

Proof. Suppose $A(Z(G)) \star A(Z(H)) = (Q, X, \mu, \sigma, F)$, and $A(Z(H)) \star A(Z(G)) = (Q', X, \mu', \sigma', F')$. Also, $h : Q \to Q'$ is a function such that $h(q, q') = (q', q)$. Then using the definition of the maximal product of two L-graph automata, the proof is straightforward. □

We explicated that when word belongs to the behavior of $A(Z(G)) \star A(Z(H))$, then it belongs to the behavior of $A(Z(G))$ or belongs to the behavior of $A(Z(H))$. We also checked the opposite, i.e., when a word belongs to the behavior of $A(Z(G))$, then it belongs to $A(Z(G)) \star A(Z(H))$, or when a word belongs to the behavior of $A(Z(H))$ then it belongs to $A(Z(G)) \star A(Z(H))$. Accordingly, some examples are provided for these two states.

Theorem 4.8. *Let $A(Z(G))$ and $A(Z(H))$ be two related L-graph automata. Moreover, let $x \in B_k(A(Z(G)) \star A(Z(H)), c)$ such that x without label n. Then*
(i) If $|Z(A(Z(G)))| \times |Q_2| \leq |Z(A(Z(H)))| \times |Q_1|$, and $Z(A(Z(G)) \star A(Z(H))) = Z(A(Z(G))) \times Q_2$, then $x \in B_k(A(Z(G)), c)$.
(ii) If $|Z(A(Z(H)))| \times |Q_1| \leq |Z(A(Z(G)))| \times |Q_2|$, and $Z(A(Z(G)) \star A(Z(H))) = Q_1 \times Z(A(Z(H)))$, then $x \in B_k(A(Z(H)), c)$.

Proof. (i) Suppose $A = A(Z(G)) \star A(Z(H))$, and $A = (Q, X, \mu, F, \sigma)$. If $x = u_1u_2 \ldots u_m \in B_k(A, c)$, then, we have $|x| = |u_1u_2 \ldots u_m| \leq k$, $0 \leq c < 1$, and $\vee_{(q,q') \in F} r_\mu(x, (q, q')) > c$. So, there exists $(q_i, q'_j) \in F$, and $(q_t, q'_k), (q_{t_1}, q'_{k_1}), \ldots, (q_{t_{m-1}}, q'_{k_{m-1}}) \in Q$ such that $\sigma(q_t, q'_k) \otimes \mu((q_t, q'_k), u_1, (q_{t_1}, q'_{k_1})) \otimes \mu((q_{t_1}, q'_{k_1}), u_2, (q_{t_2}, q'_{k_2})) \otimes \ldots \otimes \mu((q_{t_{m-1}}, q'_{k_{m-1}}), u_m, (q_i, q'_j)) > c$. Based on the assumption, as $\sigma(q_t, q'_k) > 0$, $q_t \in Z(A(Z(G)))$, and $q'_k \in Q_2$. In addition, $q_i \in F_1$, and $q'_j \in Q_2$. On the other hand, if $q'_{k_i} = q'_{k_{i+1}}$, and $u_{i+1} = f$, then there exists $\mu_1(q_{t_i}, f, q_{t_{i+1}}) \geq 0$ such that $\mu((q_{t_i}, q'_{k_i}), f, (q_{t_{i+1}}, q'_{k_{i+1}})) = \sigma_2(q'_{k_i}) \otimes \mu_1(q_{t_i}, f, q_{t_{i+1}})$, Hence, there exists $q_t, q_{t_1}, q_{t_2} \ldots, q_{t_{m-1}}, q_i \in Q_1$ such that $\mu_1(q_t, u_1, q_{t_1}) \otimes \sigma_2(q'_{k_1}) \otimes \mu_1(q_{t_1}, u_2, q_{t_2}) \otimes \sigma_2(q'_{k_2}) \otimes \ldots \otimes \mu_1(q_{t_{m-2}}, u_{m-1}, q_{t_{m-1}}) \otimes \sigma_2(q'_{k_{m-1}}) \otimes \mu_1(q_{t_{m-1}}, u_m, q_i) \otimes \sigma_2(q'_j) > c$. Then $\mu_1^*(q_t, u_1u_2 \ldots u_m, q_i) \otimes \sigma_2(q'_{k_1}) \otimes \sigma_2(q'_{k_2}) \otimes \ldots \otimes \sigma_2(q'_{k_{m-1}}) \otimes \sigma_2(q'_j) > c$. By Proposition 2.2 (R_2), $\sigma_2(q'_{k_1}) = 1$, $\sigma_2(q'_{k_2}) = 1, \ldots, \sigma_2(q'_{k_{m-1}}) = 1$, $\sigma_2(q'_j) = 1$. So, $q'_{k_1}, q'_{k_2}, \ldots, q'_{k_m} \in Z(A(Z(H)))$, and $\mu_1^*(q_t, u_1u_2 \ldots u_m, q_i) > c$. Then, $u_1u_2 \ldots u_m = x \in B_k(A(Z(G)), c)$.
(ii)The proof is similar to the above with some modifications. □

Example 4.9. *Let $A(Z_1(G))$ and $A(Z_1(H))$ be two related L-graph automata and their maximal product A_1, as in Example 4.4. Also,*

$$\bigvee_{1\leq i\leq 4} r_\mu(f,(q_2,q_i')) = 0.3\vee 0\vee 0\vee 0.3 = 0.3.$$

Therefore, if $c=0.28$, then $x = f \in B_k(A_1, 0.28)$. Hence,

$$r_{\mu_1}(f,q_2) = \mu_1(q_1,f,q_2) = 0.3 \Rightarrow f\in B_k(A(Z_1(G)),0.28).$$

Theorem 4.10. *Let $A(Z(G)$ and $A(Z(H)$ be two related L-graph automata, and $A = A(Z(G))\star A(Z(H)) = (Q,X,\mu,F,\sigma)$ be their maximal product. Then*
(i) If $x\in B_k(A(Z(G)),c)$, $|Z(A(Z(G)))|\times|Q_2| \leq |Z(A(Z(H)))|\times|Q_1|$, and $Z(A) = Z(A(Z(G)))\times Q_2$, then $x\in B_k(A,c)$.
(ii) If $x\in B_k(A(Z(H)),c)$, $|Z(A(Z(H)))|\times|Q_1| \leq |Z(A(Z(G)))|\times|Q_2|$, and $Z(A) = Q_1\times Z(A(Z(H)))$, then $x\in B_k(A,c)$.

Proof. (i) We suppose that for every $x = u_1u_2\dots u_m \in B_k(A(Z(G)),c)$. Then $|x| = |u_1u_2\dots u_m| \leq k$, $0\leq c<1$, and $\vee_{q\in F_1} r_\mu(x,q) > c$. From the definition of r_{μ_1}, there exists $q_j\in F_1$, and $q_i, q_{t_1}, q_{t_2},\dots,q_{t_{m-1}} \in Q_1$ such that $\sigma_1(q_i)\otimes \mu_1(q_i,u_1,q_{t_1})\otimes\mu_1(q_{t_1},u_2,q_{t_2})\otimes\dots\otimes\mu_1(q_{t_{m-1}},u_m,q_j) > c$. By (R_2) of Proposition 2.2, $\sigma_1(q_i)>0$, $\mu_1(q_i,u_1,q_{t_1})>0$, $\mu_1(q_{t_1},u_1,q_{t_2})>0,\dots$, and $\mu_1(q_{t_{m-1}},u_m,q_j)>0$. By definition of $Z(A(Z(G)))$, $q_i\in Z(A(Z(G)))$. By Definition 4.2, $\mu((q_i,q_s'),u_1,(q_{t_1},q_s'))>0$, $\mu((q_{t_1},q_s'),u_2,(q_{t_2},q_s'))>0,\dots,\mu((q_{t_{m-1}},q_s'),u_m,(q_j,q_s'))>0$, and $\sigma(q_i,q_s')>0$, for every $q_s'\in Z(A(Z(H)))$. Clearly, $x\in B_k(A,c)$.
(ii) The proof is similar to the above with some slight modifications. □

Example 4.11. *Suppose $A(Z_1(G))$ and $A(Z_1(H))$ are two related L-graph automata and A_1 be the maximal product of them, as in Example 4.4. Let $x = n^{2k}f$, and $c=0.2$, then we can see that*

$$r_{\mu_1}(n^{2k}f,q_2) = 1\otimes\mu_1(q_1,f,q_2) = 0.3.$$

So, $x = n^{2k}f\in B_k(A(Z_1(G)),0.2)$. Since

$$r_\mu(n^{2k}f,(q_2,q_1')) = 1\otimes\mu((q_3,q_1'),f,(q_2,q_1')) = 0.5,$$

for every $k\geq 1$, and $r_\mu(f,(q_2,q_1')) = 0.3$. So, $x = n^*f\in B_k(A,0.2)$.

In the following, we express the relationship between the language of $A(Z(G))\star A(Z(H))$ and the language of $A(Z(GH))$ and provide an example for that.

Theorem 4.12. *Let G and H be two L-graphs, and let $Z(GH) = Z(A(Z(G))\star A(Z(H)))$, and $x\in B_k(A(Z(GH)),c)$, then $x\in B_k(A(Z(G))\star A(Z(H)),c)$.*

Proof. Suppose $GH=(\alpha,\beta)$, and let $A(Z(G)$ and $A(Z(H)$ be two related L-graph automata. Then there are two cases.
Case. 1. Let $|Z(G)|\times|V_2|\leq|Z(H)|\times|V_1|$. Then $A(Z(GH)) = (Q,X,\mu,F,\sigma)$ is a related L-graph automaton, where
(i) $Q = V_1\times V_2$,
(ii) $\mu: Q\times X\times Q\to L$ is the transition function;
If $q_t' = q_s'$, and there exists $\mu_1(q_k,f,q_l)\geq 0$, then $\mu((q_k,q_t'),f,(q_l,q_s')) = \mu_1(q_k,f,q_l)\otimes\alpha_2(q_t')$,
If $q_t' = q_s'$, and there exists $\mu_1(q_k,n,q_l) = 1$, then $\mu((q_k,q_t'),n,(q_l,q_s')) = 1$,
If $q_k = q_l$, and there exists $\mu_2(q_t',u',q_s')\geq 0$, then $\mu((q_k,q_t),u,(q_l,q_s')) = \mu((q_l,q_s'),u,(q_k,q_t')) = 1$, and $u=n$,
(iii) $F = F_1\times V_2$,
(iv) $\sigma: Q\to L$ is the initial distribution; $\sigma(q_i,q_j') = \begin{cases} 1 & (q_i,q_j')\in Z(G)\times V_2,\\ 0 & \text{otherwise,}\end{cases}$
(v) $Z(A(Z(GH))) = \{(x,y)|\ x\in Z(G), y\in V_2\} = Z(A(Z(G)))\times V_2$,
and $A(Z(G))\star A(Z(H)) = (Q,X,\mu',F',\sigma')$ is the maximal product of two related L-graph automata. Let $x\in B_k(A(Z(GH)),c)$. Then according to the definition of r_μ, there exists $(q_i,q_j')\in F$, and $(q_s,q_t')\in Z(A(Z(GH)))$ such that $\mu^*((q_s,q_t'),x',(q_i,q_j'))>c$. If $x' = u_1u_2\dots u_m$ such that $u_h = f$ for every $1\leq h\leq m\leq n$, then according to the definition of μ^*, there exists $(q_i,q_j')\in F$, $(q_s,q_t')\in Z(G_1)\times Q_2$, and $(q_{s_1},q_{k_1}'),(q_{s_2},q_{k_2}'),\dots,(q_{s_{m-1}},q_{k_{m-1}}')\in Q$ such that $q_t' = q_{k_1}' = q_{k_2}' = \dots = q_{k_{m-1}}' = q_j'$, and $\mu^*((q_s,q_t'),x',(q_i,q_j')) = \mu_1(q_s,u_1,q_{s_1})\otimes\alpha_2(q_t')\otimes\dots\otimes\mu_1(q_{s_{m-1}},u_m,q_i)\otimes\alpha_2(q_{k_{m-1}}')>c$. By definition μ'^*, $\mu'^*((q_s,q_t'),x',(q_i,q_j'))>c$. Therefore, $x\in B_k(A(Z(G))\star A(Z(H)),c)$.
Case. 2. Let $|Z(H)|\times|Q_1|\leq|Z(G)|\times|Q_2|$. Then this proof is the same as above. □

Example 4.13. *Let $A(Z_1(G))$ and $A(Z_1(H))$ be two related L-graph automata and A_1 be the maximal product of them, as in Example 4.4. Moreover, let GH be the maximal product of two L-graphs, as in Example 3.5, and $Z(GH) = \{(q_1, q'_i), (q_4, q'_i) | \ 1 \leq i \leq 4\}$. Thus, $A(Z(GH)) = (Q', X, \mu', F', \sigma')$ is the related L-graph automaton, as in Figure 17, where $Q' = Q$,*

$$\mu((q_1,q'_1),f,(q_2,q'_1)) = 0.1\ ,\ \mu((q_1,q'_2),f,(q_2,q'_2)) = 0\ \ ,\ \mu((q_1,q'_3),f,(q_2,q'_3)) = 0\ ,\ \mu((q_1,q'_4),f,(q_2,q'_4)) = 0.2\ ,$$
$$\mu((q_4,q'_1),f,(q_3,q'_1)) = 0.1\ ,\ \mu((q_4,q'_2),f,(q_3,q'_2) = 0\ \ ,\ \mu((q_4,q'_3),f,(q_3,q'_3)) = 0\ ,\ \ \mu((q_4,q'_4),f,(q_3,q'_4) = 0.2\ ,$$
$$\mu((q_3,q'_1),f,(q_2,q'_1)) = 0.3\ ,\ \mu((q_3,q'_2),f,(q_2,q'_2)) = 0.1\ ,\ \mu((q_3,q'_3),f,(q_2,q'_3)) = 0\ ,\ \mu((q_3,q'_4),f,(q_2,q'_4)) = 0.4\ ,$$

$$\mu((q_i,q'_1),n,(q_i,q'_2)) = 1 = \mu((q_i,q'_2),n,(q_i,q'_1))\ ,\ \mu((q_i,q'_1),n,(q_i,q'_4)) = 1 = \mu((q_i,q'_4),n,(q_i,q'_1))\ ,$$
$$\mu((q_i,q'_2),n,(q_i,q'_3)) = 1 = \mu((q_i,q'_3),n,(q_i,q'_2))\ ,\ \mu((q_i,q'_2),n,(q_i,q'_4)) = 1 = \mu((q_i,q'_4),n,(q_i,q'_2))\ ,$$
$$\mu((q_i,q'_3),n,(q_i,q'_4)) = 1 = \mu((q_i,q'_4),n,(q_i,q'_3))\ ,\ \mu((q_1,q'_i),n,(q_3,q'_i)) = 1 = \mu((q_3,q'_i),n,(q_1,q'_i))\ ,$$

$\sigma(q_1,q'_i) = 1 = \sigma(q_4,q'_i)$, $\sigma(q_2,q'_i) = 0 = \sigma(q_3,q'_i)$, for every $1 \leq i \leq 4$, and $F = \{(q_2,q'_1),(q_2,q'_2),(q_2,q'_3),(q_2,q'_4)\}$. Then, we can see if $c = 0.18$ and $x = f$, then $\bigvee_{(q,q')\in F'} r_{\mu'}(f,(q,q')) = 0.2$. So, $x = f \in B_k(A(Z(GH)), 0.18)$. Also, $\bigvee_{(q,q')\in F} r_{\mu}(f,(q,q')) = 0.3$. Hence, $x = f \in B_k(A_1, 0.18)$. However, this is shown that if $c = 0.2$, then the reverse is not true.

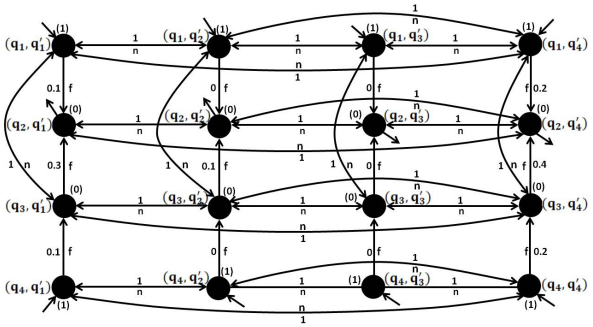


Figure 17: The L-graph automaton $A(Z(GH))$

Note 4.14. *The above example illustrates that it is possible to have c such that there exists $x \in B_k(A(Z(G)) \star A(Z(H)), c)$ but $x \notin B_k(A(Z(GH)), c)$.*

We also extracted some theorems of the relationship between the maximal product of two L-graph automata and each L-graph automaton, followed by providing an example for its clarification.

Theorem 4.15. *Let $A(Z(G))$ and $A(Z(H))$ be two related L-graph automata, and let $A''(Z(G))$ and $A''(Z(H))$ be two L''-graph automata of them, respectively. Moreover, suppose that $A = (Q, X, \mu, F, \sigma)$ is the maximal product of $A(Z(G))$ and $A(Z(H)$, and let A'' be its L''-graph automaton. Then*

(i) If $|Z(A(Z(G)))| \times |Q_2| \leq |Q_1| \times |Z(A(Z(H)))|$, and $Z(A) = Z(A(Z(G))) \times Q_2$, then
- *(a) A'' has $|Z(H)|$ copies of $A''(Z(G))$,*
- *(b) $\chi(A'') = \chi(A''(Z(G)))$,*
- *(c) If G is self-sufficient, and $\chi(G) = \chi(GH)$, then GH is self-sufficient,*
- *(d) If $A''(Z(G))$ has n loops, then A'' has at least $n \times |Q_2|$ loops.*

(ii) If $|Q_1| \times |Z(A(Z(H)))| < |Z(A(Z(G)))| \times |Q_2|$, and $Z(A) = Q_1 \times Z(A(Z(H)))$, then
1. *A'' has $|Z(G)|$ copies of $A''(Z(H))$,*
2. *$\chi(A'') = \chi(A''(Z(H)))$,*
3. *If H is self-sufficient, and $\chi(H) = \chi(GH)$, then GH is self-sufficient,*
4. *If $A''(Z(H))$ has n loops, then $A''(Z(G))$ has at least $n \times |Q_1|$ loops.*

Proof. (i)(a) We know that if $q'_i \in Z(A(H)) = Z(H)$, then $\mu((q_t,q'_i),u,(q_k,q'_i)) = \mu_1(q_t,u,q_k)$. Thus, A has $|Z(H)|$ copies of $A(Z(G))$. After applying the construction action on A and create A'', it has $|Z(H)|$ copies of $A''(Z(G))$.
(i)(b) According to definition of A'', all states that have at least a transition with label n overlap. Hence, $\chi(A'') = \chi(A''(Z(G)))$.
(i)(c) By using part (b), $\chi(A(Z(GH))) = \chi(A)$ and definition self-sufficient, it is proved
(i)(d) By using part (i)(a), it is proved.
The proof of (ii) is similar to the above with some modifications. □

Example 4.16. *Suppose that two related L-graph automata $A(Z_1(G))$ and $A(Z_1(H))$ and their maximal product L-graph automata A_1 and A_2, as in Example 4.4 exist. Consider the $L''-$graph automaton $A''(Z_1(G))$, as in Figure 18, and the $L''-$graph automaton A_1'', as in Figure 19. Then A_1'' has 2 copies of $A''(Z_1(G))$, and $\chi(A_1'') = 2 = \chi(A''(Z_1(G)))$. Moreover, the number of the loop of $A''(Z_1(G))$ is 0, then A_1'' does not have any loop. In addition, A_2'', as in Figure 20, has 2 copies of $A''(Z_1(H))$, as in Figure 21, and $\chi(A_2'') = 2 = \chi(A''(Z_1(H)))$. Also, the number of the loop of $A''(Z_1(H))$ is 1, then A_2'' has 4 loops.*

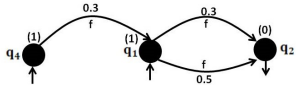


Figure 18: The L''-graph automaton $A''(Z_1(G))$

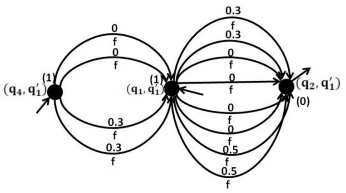


Figure 19: The L''-graph automaton A_1''

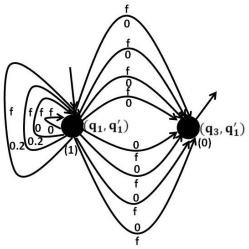


Figure 20: The L''-graph automaton A_2''

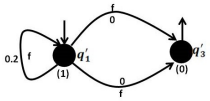


Figure 21: The L''-graph automaton $A''(Z_1(H))$

5 Application

Coronavirus disease 2019 (COVID-19) is an infectious disease caused by severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). It was first identified in December 2019 in Wuhan, China, and has since spread globally and resulted in an ongoing pandemic. As of 10 May 2020, more than 4.02 million cases have been reported across 187 countries and territories, resulting in more than 279,000 deaths, while more than 1.37 million people have been recovered. In this section, the application of the maximal product of two L-graph automata is presented. There are two critical ways in communities to get COVID-19 disease. One of these ways is related to the individual and the other is related to the community. From the individual aspect, people have to wash their hands regularly, keep (safe) distance in communities, not touch their eyes, nose, or mouth and use face masks in a public place. On the other hand, a community has to be prepared to respond to an outbreak of COVID-19, limit human-to-human transmission through the implementation of IPC recommendations suggested by the WHO, and identify, quarantine, and report suspected and confirmed cases. In

this study, we established two L-graphs that cover the individual aspect and community aspect. Each factor of these aspects was considered as the vertices of the L-graphs. Moreover, each of these factors was connected to another one through an edge if there is an influential relationship between them. We define α as the amount of performed and β as the amount of performed together with such that $\beta(q_i, q_j) = \alpha(q_i) \otimes \alpha(q_j)$. Since the resulting L-graphs are complete, only one vertex is not included in $Z(G)$. Therefore, we selected the vertex that has the highest of vertices. In the following, this case is represented by an example.

Let $L = ([0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1)$, where $a \otimes b = ab$ and $a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{if } a > b. \end{cases}$ Let G, as in Figure 22, and H, as in Figure 23, be two L-graphs, where $V_1 = \{wash\ hands\ regularly\ (q_1), \text{keep (safe)distance in communities } (q_2), \text{not to touch eyes, nose or mouth } (q_3), \text{the maskin a public place } (q_4)\} = \{q_1, q_2, q_3, q_4\}$, $E_1 = \{q_1q_2, q_1q_3, q_1q_4, q_2q_3, q_2q_4, q_3q_4\}$, $\alpha_1(q_1) = 1$, $\alpha_1(q_2) = 0.5$, $\alpha_1(q_3) = 0.8$, $\alpha_1(q_4) = 0.3$, $\beta_1(q_1q_2) = 0.5$, $\beta_1(q_1q_3) = 0.8$, $\beta_1(q_1q_4) = 0.3$, $\beta_1(q_2q_3) = 0.4$, $\beta_1(q_2q_4) = 0.15$, $\beta_1(q_3q_4) = 0.24$, $V_2 = \{\text{prepared to respond to an outbreak of COVID-19 } (q'_1), \text{limit human-to-human transmission through the implementation of IPC recommendations suggested by the WHO } (q'_2), \text{identify, quarantine and report suspected and confirmed cases } (q'_3)\} = \{q'_1, q'_2, q'_3\}$, $E_2 = \{q'_1q'_2, q'_2q'_3, q'_1q'_3\}$, $\alpha_2(q'_1) = 0.7$, $\alpha_2(q'_2) = 0.4$, $\alpha_2(q'_3) = 0.5$, $\beta_2(q'_1q'_2) = 0.28$, $\beta_2(q'_2q'_3) = 0.2$ and $\beta_2(q'_1q'_3) = 0.35$. So, $A(Z(G))$, as in Figure 22, and $A(Z(H))$ are related L-graph automata, as in Figure 23, where $Q_1 = \{q_1, q_2, q_3, q_4\}$, $Z(G) = \{q_2, q_3, q_4\}$, $\mu_1(q_2, n, q_3) = 1$, $\mu_1(q_3, n, q_2) = 1$, $\mu_1(q_3, n, q_4) = \mu_1(q_4, n, q_3) = 1$, $\mu_1(q_2, n, q_4) = \mu_1(q_4, n, q_2) = 1$, $\mu_1(q_2, f, q_1) = 0.5$, $\mu_1(q_3, f, q_1) = 0.8$, $\mu_1(q_4, f, q_1) = 0.3$, $F_1 = \{q_1\}$, $\sigma_1(q_1) = 0$, $\sigma_1(q_2) = \sigma_1(q_3) = \sigma_1(q_4) = 1$, $Q_2 = \{q'_1, q'_2, q'_3\}$, $Z(H) = \{q'_2, q'_3\}$, $\mu_2(q'_2, f, q'_1) = 0.28$, $\mu_2(q'_3, f, q'_1) = 0.35$, $\mu_2(q'_2, n, q'_3) = \mu_2(q'_3, n, q'_2) = 1$, $F_2 = \{q'_1\}$, $\sigma_2(q'_2) = \sigma_2(q'_3) = 1$, and $\sigma_2(q'_1) = 0$. Hence, $A(Z(G)) \star A(Z(H)) = (Q, X, \mu, F, \sigma)$ is the maximal product of two related L-graph automata, as in Figure 24, where $Q = \{(q_i, q'_1), (q_i, q'_2), (q_i, q'_3) |\ 1 \leq i \leq 4\}$, $|Q_1| \times |Z(A(Z(H)))| < |Z(A(Z(G)))| \times |Q_2|$,

$$\mu((q_i, q'_2), n, (q_i, q'_3)) = 1 = \mu((q_i, q'_3), n, (q_i, q'_2)) \ , \qquad \mu((q_1, q'_j), n, (q_2, q'_j)) = 1 = \mu((q_2, q'_j), n, (q_1, q'_j)) \ ,$$
$$\mu((q_1, q'_j), n, (q_3, q'_j)) = 1 = \mu((q_3, q'_j), n, (q_1, q'_j)) \ , \qquad \mu((q_1, q'_j), n, (q_4, q'_j)) = 1 = \mu((q_4, q'_j), n, (q_1, q'_j)) \ ,$$
$$\mu((q_2, q'_j), n, (q_3, q'_j)) = 1 = \mu((q_3, q'_j), n, (q_2, q'_j)) \ , \qquad \mu((q_2, q'_j), n, (q_4, q'_j)) = 1 = \mu((q_4, q'_j), n, (q_2, q'_j)) \ ,$$
$$\mu((q_3, q'_j), n, (q_4, q'_j)) = 1 = \mu((q_4, q'_j), n, (q_3, q'_j)) \ , \qquad \mu((q_1, q'_2), f, (q_1, q'_1)) = 0 = \mu((q_1, q'_3), f, (q_1, q'_1)) \ ,$$
$$\mu((q_2, q'_2), f, (q_2, q'_1)) = 0.28 = \mu((q_3, q'_2), f, (q_3, q'_1)) \ , \qquad \mu((q_2, q'_3), f, (q_2, q'_1)) = 0.35 = \mu((q_3, q'_3), f, (q_3, q'_1)) \ ,$$
$$\mu((q_4, q'_2), f, (q_4, q'_1)) = 0.28, \qquad \mu((q_4, q'_3), f, (q_4, q'_1)) = 0.35 \ ,$$

$\sigma(q_j, q'_1) = 0$, $\sigma(q_j, q'_2) = \sigma(q_j, q'_3) = 1$, for every $1 \leq i \leq 4$ and for every $1 \leq j \leq 3$, $F = \{(q_i, q'_1) |\ 1 \leq i \leq 4\}$, and $Z(A(Z(G)) \star A(Z(H))) = \{(q_i, q'_2), (q_i, q'_3) |\ 1 \leq i \leq 4\}$. Then $\bigvee_{(q_i, q'_j) \in F} r_\mu(n^* f, (q_i, q'_j)) = 0.35$. Hence, everyone catches the disease at a rate of 0.35. Always because two L-graphs G and H are unified and $|Q_1| \times |Z(A(H))| < |Z(A(G))| \times |Q_2|$. Then the model shows that the community aspect is more important than individual aspect. Besides, to reduce this rate, it is necessary to improve the community's performance. Therefore, it could be stated that 'culture creation' developed by governments to prevent the spread of this disease is of particular importance in this regard.

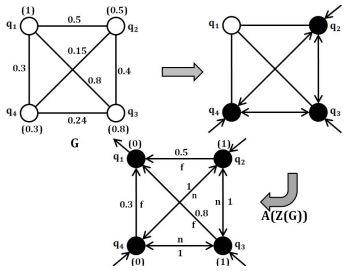


Figure 22: The L-graph G and the related L-graph automaton $A(Z(G))$

6 Conclusion

This study was an attempt to define and explicate the maximal product of two L-graphs and the maximal product of two L-graph automata. Scientists conducted much research on biological issues that resulted in facing a new disease. Thus, they need to take measures to prevent the spread of such diseases. Moreover, they are interested in treating

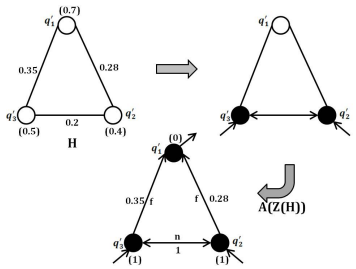


Figure 23: The L-graph H and the related L-graph automaton $A(Z(H))$

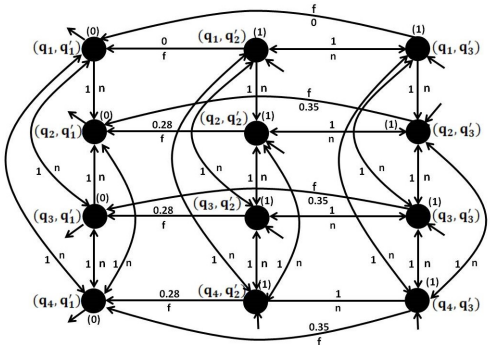


Figure 24: The maximal product of two L-graph automata $A(Z(G)) \star A(Z(H))$

these types of diseases. Since the world has been afflicted with COVID-19, we found social distancing compelling on its spread. As far as the direction of our future studies is concerned, this subject has achieved extensive results about these structures. Overall, we found the optimal solution with the highest efficiency using the available data in solving a problem, similar to what was been investigated in this study.

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