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Original paper

Neighborhood connectivity index of a fuzzy graph and its application to human trafficking

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Abstract

Connectivity is an inevitable part of fuzzy graph theory. This article discusses about a parameter in fuzzy graph theory termed as neighborhood connectivity index. Several bounds and index values of structures like trees, cycles and complete fuzzy graphs are obtained. Generalized formula for neighborhood connectivity index of fuzzy graphs obtained by operations like union, join, composition, Cartesian product and tensor product are also developed. An algorithm for finding neighborhood connectivity index is also proposed. On practical grounds, a human trafficking problem is discussed as a real-life application.

Keywords: Fuzzy graph, connectivity, neighborhood connectivity index, human trafficking.

1 Introduction

Graph theory always has a significant role in mathematics. While discussing the relationship among objects, it concentrates on things that have an indisputable membership. When ambiguity and uncertainty established dominance in the criteria of membership, in 1965 Zadeh [27] initiated the concepts of fuzzy sets and fuzzy relations; where each point in the underlying set is associated with a value in the interval [0,1]. Later in 1973, Kauffman [11] put forward the idea of fuzzy graphs. In the following years Rosenfeld [21] and Yeh and Bang [26] laid the main foundation; they discussed the basic structure and various connectivity parameters of fuzzy graphs and their applications. Bhattacharya [2] later found that a fuzzy group can be associated with a fuzzy graph in a natural way. On the way of growth, Tuksen [25] developed interval-valued fuzzy sets. Later the properties of fuzzy bridges and fuzzy cut vertices were studied by Sunitha and Vijayakumar [22]. They obtained a characterization of fuzzy trees and it was a major breakthrough. Blocks were studied formally by Sunitha and Vijayakumar [23], which led to the invention of characterizations of blocks in fuzzy graphs. After a subtle study in 2010, Mathew and Sunitha [15], developed the characterizations to another level. In the meantime, Bhutani and Rosenfeld [4] brought up the concept of strong edges and strong paths, which later opened the way to different types of edges in fuzzy graphs through the works of Mathew and Sunitha [16]. Mordeson and Peng [20] discussed operations such as Cartesian product, union, join and composition. More related work can be found on [1, 12, 14]

Mathematicians and chemists were always interested in studying different indices of graphs. Several indices such as the Hosoya index, Wiener index, Estrada index, Hyper-Wiener index, Randic index, Padmakar-Ivan index, Szeged index, Eccentric connectivity index and Zagreb indices were deeply studied in graph theory and they were applied in vivid areas. Now, we try to study those in fuzzy graphs. Binu and Mathew [5, 6, 7] studied connectivity index, cyclic connectivity index and Wiener index and some of those are extended to bipolar graphs too.

Modeling of networks like traffic and water supply should have prior knowledge about the possible traffic strength from nodes to their neighbors. It is one among the several parameters like reachability and possible cost. The capacities of links between nodes can be adjusted in the modeling stage itself. This motivated the authors to work with

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neighborhood connectivity in normalized fuzzy networks. The new parameter gives the value of neighborhood traffic in the graph model. The authors believe that this concept is significant in modern networks and can be used in allocation problems and routing problems as well.

In this paper, we study neighborhood connectivity index of a fuzzy graph. In Section 2, we discuss the essential preliminaries related to the work. While in Section 3, neighborhood connectivity index is discussed along with its bounds and generalization to known structures, where we also see a set of vertex sets called the twinning vertex sets. In Section 4, we notice how the index works on some operations on fuzzy graphs. Section 5 puts forward an algorithm for calculating the index, whereas in Section 6, we discuss an application related to human trafficking.

2 Preliminaries

Major results of this section are based on the work by S. Mathew et al. [13, 18]. Throughout this paper, $\wedge$ represents the minimum function and $\vee$ represents the maximum function. Consider a set V. A function defined from V into the closed interval $[0,1]$ is called a *fuzzy subset* of V. The set $\{m \in V \mid \gamma(m) > 0\}$ is defined as *support of* γ, and is denoted as $Supp(\gamma)$. A triple consisting of a nonempty set V together with a pair of functions $\psi\colon V \longrightarrow [0,\ 1]$ and $\gamma\colon V \times V \longrightarrow [0,\ 1]$ is called a *fuzzy graph* $G = (V,\ \psi,\ \gamma)$, when it also satisfies the condition that for all $m,\ p \in V,\ \gamma(mp) \leq \psi(m) \wedge \psi(p)$, where $\wedge$ denotes minimum. When explicit reference to the vertex set is not needed, $G = (V, \psi, \gamma)$ will be denoted simply by $G = (\psi, \gamma)$. The *fuzzy vertex set* of G is the fuzzy set ψ and the *fuzzy edge set* of G is γ. If $\tau \subseteq \psi$ and $\nu \subseteq \gamma$, a fuzzy graph $H = (V, \tau, \nu)$ is called a *partial fuzzy subgraph* of G. Similarly, if $P \subseteq V$, $\tau(m) = \psi(m)$ for all $m \in P$ and $\nu(mp) = \gamma(mp)$ for all $m, p \in P$, then the fuzzy graph $H = (P, \tau, \nu)$ is called a *fuzzy subgraph* of G induced by P, and is denoted as $< P >$. In a fuzzy graph a sequence of distinct vertices $m_0,\ m_1, \ldots,\ m_n$ such that $\gamma(m_{i-1}m_i) > 0,\ i = 1, 2, \ldots,\ n$ is called a *path* P of length n. The strength of a path is defined as the lowest membership value of its edges. The maximum of the strengths of all paths between m and p is known as the *strength of connectedness* between two vertices m and p, it is denoted as $\gamma^{\infty}(m, p)$ or $CONN_G(m, p)$. If $m_0 = m_n$ and $n \geq 3$ we call P a cycle. If the removal of an edge causes a reduction in the connectedness between some pair of vertices, such an edge is called a *fuzzy bridge*. If the removal of a vertex causes a reduction in the connectedness between some pair of vertices, such a vertex is called a *fuzzy cutvertex*. If a fuzzy graph has no fuzzy cutvertices it is called nonseparable or a *block*. A *maximum spanning tree* of a connected fuzzy graph (ψ, γ) is a fuzzy spanning subgraph $T = (\psi, \nu)$ of G, which is a tree, such that $\gamma^{\infty}(m, p)$ is the strength of the unique strongest $m - p$ path in T for all $m, p \in G$. A fuzzy graph with $\gamma(mp) = \psi(m) \wedge \psi(p)$ for all $m, p \in \psi^*$ is a *complete fuzzy graph (CFG)*. A fuzzy graph is called a forest if the graph consisting of its nonzero edges is a forest, and it is a tree when the graph is connected. A fuzzy graph $F = (\psi, \nu)$ is a *fuzzy forest*, if it is a forest which is also a partial fuzzy spanning subgraph of a fuzzy graph $G = (\psi, \gamma)$ where for all edges mp not in F, i.e., such that $\nu(mp) = 0$, we have $\gamma(mp) < \nu^{\infty}(m, p)$. If (Supp($\psi$), Supp($\gamma$)) is a cycle then $G = (\psi, \gamma)$ is called a *cycle*. A strong edge is an edge mp for which $\gamma(mp) > 0$ and $\gamma(mp) \geq CONN_{G\backslash mp}(m, p)$, is $\alpha-$*strong* if $\gamma(mp) > CONN_{G\backslash mp}(m, p)$, is $\beta-$*strong* if $\gamma(mp) = CONN_{G\backslash mp}(m, p)$ and is a $\delta-$*edge* if $\gamma(mp) < CONN_{G\backslash mp}(m, p)$. If every vertex of a fuzzy graph has an $\alpha-$strong edge incident to it then such a graph is called $\alpha-$*saturated*. $\beta-$*saturated* graphs are also defined similarly. If every vertex has at least one $\alpha-$strong edge and at least one $\beta-$strong edge incident to it, such a graph is called *saturated*, otherwise *unsaturated*. An *isomorphism* $h : G \to G'$ is a map $h : S \to S'$ which is bijective that satisfies $\psi(m) = \psi'(h(m))$ for all $m \in S$, $\gamma(m, p) = \gamma'(h(m), h(p))$ for all $m, p \in S$ [3].The *complement of a fuzzy graph* $G = (\psi, \gamma)$ is the fuzzy graph $G^c = (\psi^c, \gamma^c)$ where $\psi^c = \psi$ and $\gamma^c(m, p) = \psi(m) \wedge \psi(p) - \gamma(m, p)$ for all $m, p \in V$ [24]. From [19] we see that the connectedness of a partial fuzzy subgraph is always less than or equal to the connectedness of the parent graph. The concepts such as union, join, composition, Cartesian product and tensor product are defined in [8, 9, 20]. For a fuzzy graph $G = (\psi, \gamma)$, the *Connectivity Index* (CI) [5] is defined as $CI(G) = \sum_{m,p \in \psi^*} \psi(m)\psi(p)CONN_G(m, p)$, where $CONN_G(m, p)$ is the strength of connectedness between m and p. For a fuzzy graph $G = (\psi, \gamma)$, the *Wiener Index* (WI) [6] is defined as $WI(G) = \sum_{m,p \in \psi^*} \psi(m)\psi(p)d_S(m, p)$, where $d_S(m, p)$ is the minimum sum of weights of geodesics from m to p.

Theorem 2.1. [10] *If $G_1 = (\psi_1, \gamma_1)$ and $G_2 = (\psi_2, \gamma_2)$ are two fuzzy graphs such that $\psi_1 \leq \gamma_2$, then $\psi_2 \geq \gamma_1$ and vice versa.*

Theorem 2.2. [6] *For a complete fuzzy graph $CI(G) = WI(G)$.*

3 Neighborhood connectivity index of a fuzzy graph

This section discusses about a new parameter related to the potential of vertices and its neighborhood. We represent a fuzzy graph by $G = (\psi, \gamma)$ in all the following sections. In examples, most cases assume $\psi(m) = 1$ for all $m \in \psi^*$, otherwise it will be mentioned explicitly. Next we put forward the idea of neighborhood connectivity index through Definition 3.1.

Definition 3.1. *The Neighborhood Connectivity Index* (NCI)*, of a fuzzy graph* G *is defined as* $NCI(G) = \sum_{m \in V(G)} d(m)e(m)$, *where* $d(m)$ *is the cardinality of* $N(m)$ *and* $e(m) = \vee\{\gamma(mp) : p \in N(m)\}$ *with* $N(m) = \{p : \gamma(mp) > 0, m, p \in \psi^*\}$. $e(m)$ *is termed as the potential of the vertex* m.

In a graph, a vertex with maximum potential is termed as *maximum potential vertex.* While discussing potential of a vertex in graph G, we denote it by $e_G(m)$. Similarly we represent $d_G(m), N_G(m)$ for $d(m), N(m)$.
Note that $e(m)$ can be defined in terms of connectivity in a different manner. For any vertex m, $e(m) = \vee\{CONN_G(m, p) : p \in V(G)\}$. For every $x \in \psi^* \setminus \{m\}$, a strongest $m - x$ path P, (say) contains an edge from $E(m)$, where $E(m) = \{mp : p \in N(m)\}$. If $\vee\{\gamma(mp) : p \in N(m)\} = \alpha$, then strength of P is less than or equal to α. In particular if $\gamma(mz) = \alpha$ then mz is a strongest path with strength $e(m)$. Therefore both the definitions of $e(m)$ are equivalent.

Example 3.2. *Consider* $G = (\psi, \gamma)$ *with* $\psi^* = \{l, a, m, b, n, c\}$; $\psi(l) = 0.5,\ \psi(a) = 0.6,\ \psi(m) = 0.2,\ \psi(b) = 0.7,\ \psi(n) = 0.4,\ \psi(c) = 0.3,$ *and* $\gamma(la) = 0.4,\ \gamma(ln) = 0.1,\ \gamma(lc) = 0.2,\ \gamma(am) = 0.1,\ \gamma(an) = 0.3,\ \gamma(mn) = 0.2,\ \gamma(bn) = 0.4,\ \gamma(nc) = 0.3.$

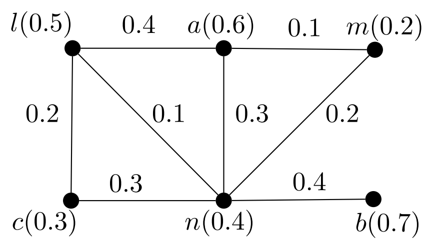

Figure 1: A fuzzy graph G with $NCI(G) = 5.8$.

Now we can find that $d(l) = 3, e(l) = \vee\{0.4, 0.2, 0.1\} = 0.4$. *Similarly we can find for the rest of the vertices also.*

Vertex	$d(x)$	$e(x)$	$d(x)e(x)$
l	3	0.4	1.2
a	3	0.4	1.2
m	2	0.2	0.4
b	1	0.4	0.4
n	5	0.4	2
c	2	0.3	0.6
		$NCI(G)$	5.8

Thus for G *in Figure 1,* $NCI(G) = 3 \times 0.4 + 3 \times 0.4 + 2 \times 0.2 + 1 \times 0.4 + 5 \times 0.4 + 2 \times 0.3 = 5.8$.

We have an obvious observation, which is given as the next remark.

Remark 3.3. *Neighborhood connectivity index of a fuzzy graph is zero if and only if the cardinality of its edge set is zero.*

Proposition 3.4. *If* $H = (\tau, \nu)$ *is a partial fuzzy subgraph of* $G = (\psi, \gamma)$*, then* $NCI(H) \leq NCI(G)$.

Proof. Suppose $H = (\tau, \nu)$ be a partial fuzzy subgraph of $G = (\psi, \gamma)$, with $\psi^* = \{m_1, m_2, \cdots, m_n\}$. Let m be an arbitrary vertex in τ^*. Then $\nu(mm_i) \leq \gamma(mm_i)$ for all other vertices m_i in τ^*. Therefore, $\vee_i\{\nu(mm_i)\} \leq \vee_i\{\gamma(mm_i)\}$. Also, $d_H(m) \leq d_G(m)$. Therefore, $NCI(H) = \sum_{m_i} d_H(m_i) \vee_i\{\nu(mm_i)\} \leq \sum_{m_i} d_G(m_i) \vee_i\{\gamma(mm_i)\} = NCI(G)$. □

Example 3.5. *Consider $H = (\tau, \nu)$ in Figure 2. Clearly H is a partial fuzzy subgraph of $G = (\psi, \gamma)$ mentioned in Example 3.2. After computing the connectedness between the vertices and cardinality of neighborhood for each vertex, neighborhood connectivity index of H can be calculated as* 4.2 *which is less than neighborhood connectivity index of G, which is* 5.8.

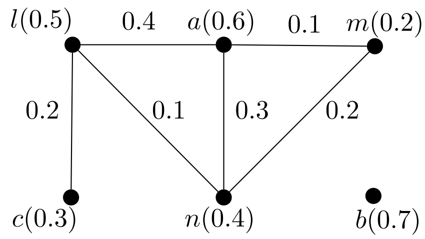

Figure 2: Subgraph H of G with $NCI(H) = 4.2$.

In the following results the paper establishes some bounds for the index. From Proposition 3.4, Corollary 3.6. follows.

Corollary 3.6. *For a fuzzy graph $G = (\psi, \gamma)$ with vertex set ψ^* and complete fuzzy super graph $G' = (\psi', \gamma')$ spanned by ψ^*, we have $0 \leq NCI(G) \leq NCI(G')$.*

Proposition 3.7. *For G with $|\psi^*| = n$, $0 \leq NCI(G) \leq n(n-1)$.*

Proof. Consider $G = (\psi, \gamma)$. If $\gamma^* = \phi$, then $d(m) = 0$, $e(m) = 0$ for all $m \in \psi^*$. Which implies $NCI(G) = 0$. If $|\gamma^*| > 0$, then $0 < d(m) \leq n-1$, $0 < e(m) \leq 1$ for at least one $m \in \psi^*$. Which implies $0 < NCI(G) \leq \sum_{m \in \psi^*} (n-1) \times 1 = n(n-1)$. The upper bound occurs when the underlying graph is a complete graph and there exist at least one edge incident to each vertex having strength 1. Therefore, $0 \leq NCI(G) \leq n(n-1)$. □

Proposition 3.8. *Let $G = (\psi, \gamma)$ be a connected fuzzy graph with n edges. Then $2nt \leq NCI(G) \leq 2ns$ where $t = \wedge\{e(m) : m \in \psi^*\}$ and $s = \vee\{e(m) : m \in \psi^*\}$.*

Proof. Suppose $G = (\psi, \gamma)$ is a fuzzy graph with n edges. Then

$$NCI(G) = \sum_{m \in \psi^*} e(m)d(m) \leq \sum_{m \in \psi^*} sd(m) = s \sum_{m \in \psi^*} d(m) = s \times 2n = 2sn.$$

Similarly,

$$NCI(G) = \sum_{m \in \psi^*} e(m)d(m) \geq \sum_{m \in \psi^*} td(m) = t \sum_{m \in \psi^*} d(m) = t \times 2n = 2tn.$$

Therefore, $2nt \leq NCI(G) \leq 2ns$. □

Remark 3.9. *Equality in the above result holds only if all vertices of G have same potential.*

The following results shows the neighborhood connectivity index of known structures such as trees, cycles and complete fuzzy graphs.

Corollary 3.10. *Consider a fuzzy graph $G = (\psi, \gamma)$ where G^* is a tree. Let $\vee\{d(m) : m \in \psi^*\} = r$ and let S_i be the set of vertices containing all vertices with degree i, $1 \leq i \leq r$. Then $NCI(G) = \sum_{m \in S_i, i=1}^{r} i \sum_{p \in N(m)} \vee\{\gamma(mp)\}$.*

Corollary 3.11. *For a cycle $G = (\psi, \gamma)$ having edges $e_1, e_2, \cdots, e_n$ with $\gamma(e_i) = t_i$ and $t_{n+1} = t_1$ we have $NCI(G) = 2\sum_{i=1}^{n} \vee\{t_i, t_{i+1}\}$.*

Corollary 3.12. *Let $G = (\psi, \gamma)$ be a CFG with $\psi^* = \{m_1, m_2, \cdots, m_n\}$ such that $t_1 \leq t_2 \leq \cdots \leq t_n$, where $t_i = \psi(m_i)$, $1 \leq i \leq n$. Then $NCI(G) = (n-1)(t_1 + t_2 + \cdots + t_{n-2} + t_{n-1} + t_{n-1})$.*

Proof. Consider the graph G. We know that, for a CFG, $\gamma(m_im_j) > 0$ for all $m_i, m_j \in \psi^*$. Therefore, $d(m_i) = n-1$ for all m_i, $1 \leq i \leq n$. Now, we can check the potential of vertices. While considering m_1 we see that it is the vertex with minimum membership value. So we can see that $CONN_G(m_1, m_i) = t_1$, $2 \leq i \leq n$. Therefore, $e(m_1) = t_1$. Next, consider the vertices m_i, $1 < i < n$. Here, $CONN_G(m_s, m_i) \leq t_i$ for all $s < i$, $CONN_G(m_r, m_i) = t_i$ for all $r > i$; therefore, $e(m_i) = t_i$, $2 \leq i \leq n-1$. At last, we consider the vertex m_n. Here we can see that $CONN_G(m_i, m_n) \leq t_{n-1}, 1 \leq i \leq n-1$, since there is no edge of membership value t_n, and there is an edge of membership value t_{n-1}. Therefore, $e(m_n) = t_{n-1}$. Summing up all those values, we get $NCI(G) = (n-1)(t_1 + t_2 + \cdots + t_{n-2} + t_{n-1} + t_{n-1})$. □

Proposition 3.13. *Neighborhood connectivity index of two isomorphic fuzzy graphs are equal.*

Proof. Let j be a bijection between the isomorphic fuzzy graphs G_1 and G_2. Since weights of the edges and vertices are preserved by an isomorphism, $N_{G_1}(m) = N_{G_2}(j(m))$, which implies $d_{G_1}(m) = d_{G_2}(j(m))$ for $m \in \psi_1^*$. Similarly, $CONN_{G_1}(m, p) = CONN_{G_2}(j(m), j(p))$ for $m, p \in \psi_1^*$. Implying $e_{G_1}(m) = e_{G_2}(j(m))$. Therefore,

$$NCI(G_1) = \sum_{m \in V(G)} d_{G_1}(m) e_{G_2}(m) = \sum_{f(m) \in V(G)} d_{G_2}(j(m)) e_{G_2}(j(m)) = NCI(G_2),$$

ie, $NCI(G_1) = NCI(G_2)$. □

Theorem 3.14. *Consider a fuzzy graph $G = (\psi, \gamma)$. If $0 \leq t_1 \leq t_2 \leq 1$, then $NCI(G^{t_2}) \leq NCI(G^{t_1})$.*

Proof. Consider a fuzzy graph $G = (\psi, \gamma)$. In G^{t_2} number of edges with non zero strength incident at a vertex is less than or equal to the number of edges with non zero strength incident at a vertex in G^{t_1}. Therefore, $d_{G^{t_2}}(m) \leq d_{G^{t_1}}(m)$. If $\gamma_G(mp) \leq t_1$, then $\gamma_{G^{t_2}}(mp) = \gamma_{G^{t_1}}(mp)$. If $t_1 < \gamma_G(mp) \leq t_2$, then $\gamma_{G^{t_2}}(mp) \leq \gamma_{G^{t_1}}(mp)$. If $\gamma_G(mp) > t_2$, then $\gamma_{G^{t_2}}(mp) = \gamma_{G^{t_1}}(mp)$. Now for $m \in \psi^*, CONN_{G^{t_2}}(m, p) \leq CONN_{G^{t_1}}(m, p)$ for all $p \in \psi^*$. Therefore, $e_{G^{t_2}}(m) \leq e_{G^{t_1}}(m)$. Therefore, $NCI(G^{t_2}) = \sum_{m \in V(G)} e_{G^{t_2}}(m) d_{G^{t_2}}(m) \leq \sum_{m \in V(G)} e_{G^{t_1}}(m) d_{G^{t_1}}(m) = NCI(G^{t_1})$. □

Next, a result about saturated fuzzy cycles is discussed.

Theorem 3.15. *Consider a saturated fuzzy cycle G with $|V(G^*)| = n$ for which every $\alpha-$ strong edge is of strength t and every $\beta-$ strong edge is of constant strength, then $NCI(G) = 2nt$.*

Proof. Suppose $G = (\psi, \gamma)$ is as in statement of the theorem. Since G^* is a saturated fuzzy cycle, $d(m) = 2$ for any $m \in \psi^*$. Also from the assumption it follows that t is greater than the constant strength of $\beta-$ strong edges, which implies, $e(m) = t$ for all $m \in \psi^*$. Therefore, $NCI(G) = \sum_{i=1}^{n} 2t = 2nt$. □

Example 3.16. *Consider the fuzzy cycle G so that $G^* = C_n$ as given in Figure 3. Clearly it is a saturated fuzzy cycle with $\psi^* = \{l, a, m, b, n, c, o, d\}$, $\gamma(la) = 0.4$, $\gamma(am) = 0.2$, $\gamma(mb) = 0.4$, $\gamma(bn) = 0.2$, $\gamma(nc) = 0.4$, $\gamma(co) = 0.2$, $\gamma(od) = 0.4$, $\gamma(dl) = 0.2$. Then neighborhood connectivity index, $NCI(G) = 2 \times 8 \times 0.4 = 6.4$.*

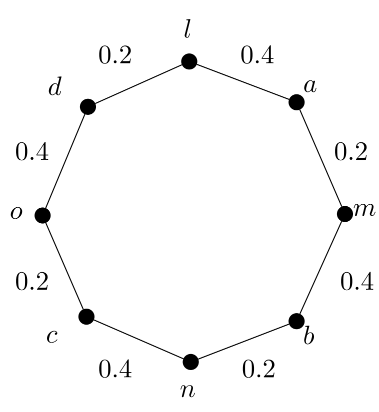

Figure 3: Saturated fuzzy cycle G with $NCI(G) = 6.4$.

Theorem 3.17. *There does not exist any kind of connected super fuzzy graph with equal neighborhood connectivity index as that of the parent graph.*

Proof. Consider a graph H which has a vertex p in addition to the parent graph G as shown in Figure 4. Let $m \in \psi^*_G$ then $d_G(m) \leq d_H(m)$, since m may or may not have an edge with p. Now consider p, since $p \notin G, 0 = d_G(p) < d_H(p)$. While considering the potential of the edges. For $m \in \psi^*_G, e_G(m) \leq e_H(m)$, since there may or may not have an edge

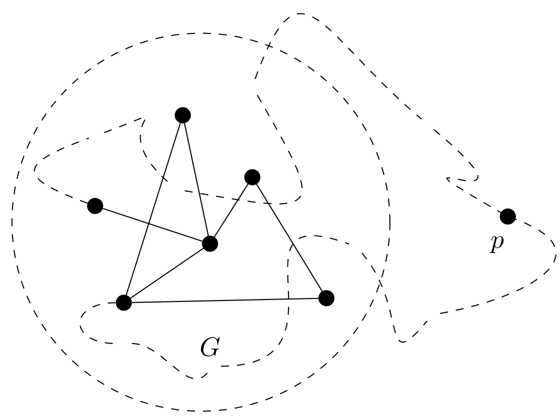

Figure 4: Model of a fuzzy graph which has a vertex in addition to the parent graph.

with strength greater than $e_G(m)$, adjacent to p. While considering p it is obvious that $0 = e_G(p) < e_H(p)$. Therefore, $NCI(G) = \sum_{m \in \psi^*_G} d_G(m)e_G(m) = \sum_{m \in \psi^*_G} d_G(m)e_G(m) + d_G(p)e_G(p) < \sum_{m \in \psi^*_G} d_H(m)e_H(m) + d_H(p)e_H(p) = NCI(H)$. Now we have shown that there does not exist a connected super graph having same neighborhood connectivity index as that of the parent graph when we add a vertex. Next consider a graph H which has an edge e in addition to the parent graph G as shown in Figure 5. There exists at least one vertex m in G to which is e is incident, then $d_G(m) < d_H(m)$. Also we can see that $e_G(m) \leq e_H(m)$. Therefore, $NCI(G) = \sum_{m \in \psi^*_G} d_G(m)e_G(m) < \sum_{m \in \psi^*_G} d_H(m)e_H(m) = NCI(H)$.

Now we have shown that there does not exist a connected super graph having same neighborhood connectivity index as that of the parent graph when we add an edge.

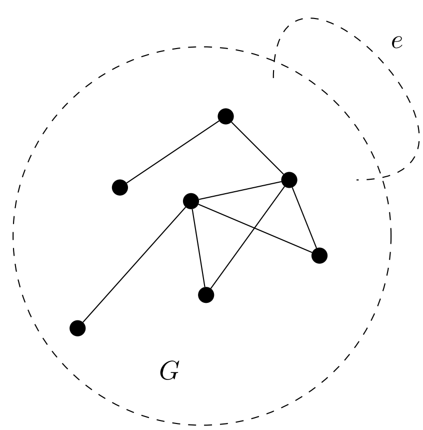

Figure 5: Model of a fuzzy graph which has an edge in addition to the parent graph.

□

The following two theorems ventilates a way for construction of fuzzy graphs with a given neighborhood connectivity index value with some predefined constrains.

Theorem 3.18. *For a given $n \in \mathbb{N}, x \in \mathbb{R}^+$ with $x \leq 2n$, there exists a fuzzy graph $G = (\psi, \gamma)$ of neighborhood connectivity index x with $|\gamma^*| = n$.*

Proof. Let $|\gamma^*| = n$. Construct a fuzzy graph $G = (\psi, \gamma)$ such that $\psi(m_i) \geq \frac{x}{2n}$ for all $m_i \in \psi^*$, $\gamma(m_i m_j) = \frac{x}{2n}$ for all $m_i m_j \in \gamma^*$. Now we can check neighborhood connectivity index of the constructed graph. Here $e(m_i) = \frac{x}{2n}$ for all $m_i \in \psi^*$. Therefore, $NCI(G) = \sum_{m_i \in V(G)} d(m_i)\frac{x}{2n} = \frac{x}{2n} \sum_{m_i \in V(G)} d(m_i) = \frac{x}{2n} \times 2n = x$. Hence our constructed graph is a fuzzy graph of NCI x with $|\gamma^*| = n$.

□

Theorem 3.19. *For a given $n \in \mathbb{N}, x \in \mathbb{R}^+$ with $x \leq n(n-1)$, there exists a fuzzy graph $G = (\psi, \gamma)$ of neighborhood connectivity index x with $|\psi^*| = n$.*

Proof. We can prove this theorem by similar construction from Theorem 3.18 by taking $|\psi^*| = n$, $\psi(m_i) \geq \frac{x}{n(n-1)}$ for all $m_i \in \psi^*$ and $\gamma(m_i m_j) = \frac{x}{n(n-1)}$ for all $m_i m_j \in \gamma$. □

Example 3.20. *Let $|\gamma^*| = 4, x = 4$. Clearly, $4 \leq 8$. Now we can find a fuzzy graph $G = (\psi, \gamma)$ given in Figure 6 such that $\psi(l) = 0.8$, $\psi(a) = 0.6$, $\psi(m) = 0.5$, $\psi(b) = 0.6$, $\gamma(la) = 0.5$, $\gamma(lm) = 0.5$, $\gamma(am) = 0.5$, $\gamma(ab) = 0.5$ with neighborhood connectivity index, $NCI(G) = 4$.*

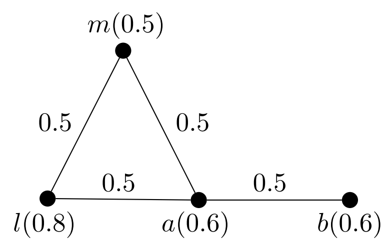

Figure 6: Fuzzy graph with $NCI(G) = 4$, given $m = 4, x = 4$.

Proposition 3.21. *Consider a fuzzy cycle $G = (\psi, \gamma)$ with $|\psi^*| = n \geq 4$ and $\psi(m_i) = t$ for all $m_i \in \psi^*$. Then $NCI(G^c) - NCI(G) \geq n^2t - 5nt$, where $G^c = (\psi^c, \gamma^c)$ is the fuzzy complement of the fuzzy graph $G = (\psi, \gamma)$.*

Proof. Suppose $G = (\psi, \gamma)$ is a fuzzy cycle. The neighborhood of each vertex in the fuzzy cycle has two vertices. Therefore, $d(m) = 2$. The potential of each vertex will always be less than t, since each vertex has strength t. Therefore, $e(m) \leq t$. Therefore,

$$NCI(G) = \sum_{m \in V(G)} d(m)e(m) = 2 \sum_{m \in V(G)} e(m) \leq 2nt \cdots \tag{1}$$

Now consider the complement $G^c = (\psi^c, \gamma^c)$ of the graph $G = (\psi, \gamma)$. Clearly G^c will have all edges which are not present on the cycle. In addition to that some edges of the cycles can also appear. Therefore, each vertex can have a neighborhood of cardinality greater than $n - 3$. ie, $d(m) \geq n - 3$ for all $m \in \psi$. Since all the edges other than those lying on the cycle has strength t, and all others has strength less than t, we can say $e(m) = t$. Therefore,

$$NCI(G^c) = \sum_{m \in V(G)} d(m)e(m) = t \sum_{m \in V(G)} d(m) \geq nt(n-3) = n^2t - 3nt \cdots \tag{2}$$

From equation (1) and (2), $NCI(G^c) - NCI(G) \geq n^2t - 3nt - 2nt = n^2t - 5nt$. □

Theorem 3.22. *For a fuzzy tree G, which is not a tree, with $F = (\psi, \nu)$ as its maximum spanning tree, $NCI(F) < NCI(G)$.*

Proof. Suppose $G = (\psi, \gamma)$ is a fuzzy tree, which is not a tree. Let $F = (\psi, \nu)$ be the maximum spanning tree of G.

Claim: For each vertex p in G, the edge with maximum strength incident at p will also lie on the maximum spanning tree F of G.

Proof of claim: Suppose not, let p be a vertex in G and pm be the edge with maximum strength incident at p. Suppose pm does not lie on the maximum spanning tree. Then $CONN_F(p,m) < CONN_G(p,m)$, a contradiction. Hence the claim. Now consider an arbitrary vertex m, then $e(m)$ is the maximum of the weight of edges starting from m. Hence by the claim we proved that $e_F(m) = e_G(m)$. Now we will show that $d_F(m) < d_G(m)$. Since our given fuzzy graph is not a tree, the maximum spanning tree of G will be different from G. There will be at least one edge removed from G. Let mp be such an edge. Then clearly, $d_F(m) < d_G(m)$ and $d_F(p) < d_G(p)$. Therefore, $NCI(F) = \sum_{m\in V(G)} d_F(m)e_F(m) = \sum_{m\in V(G)} d_F(m)e_G(m) < \sum_{m\in V(G)} d_G(m)e_G(m) = NCI(G)$. ie, $NCI(F) < NCI(G)$. □

Definition 3.23. *Two sets of vertices are called a twinning vertex sets of cardinality r, if each set has cardinality r and neighborhood connectivity index of the graph obtained after removing each set is same.*

Theorem 3.24. *Consider a fuzzy graph G. Let A be the set of pendant vertices with potential a. B be the set of supporting vertices of vertices from A with degree c and potential b. Then all the sub graphs obtained after removing any one vertex from the set A will have equal neighborhood connectivity index.*

ie, any two subsets of cardinality one of set A are examples of twinning vertex sets of cardinality one.

Proof. Consider a fuzzy graph G. Let A and B be as defined in the theorem statement. We show that for $u, v \in A$, $NCI(G \setminus u) = NCI(G \setminus v)$. First, we consider vertices which does not belong to B or A. Let f be such a vertex. Then clearly $d_{G\setminus u}(f) = d_{G\setminus v}(f)$ and $e_{G\setminus u}(f) = e_{G\setminus v}(f)$. Let a be the supporting vertex of u and b be the supporting vertex of v. Then $d_{G\setminus u}(a) = d_{G\setminus v}(b)$, since $d_G(a)$ and $d_G(b)$ are equal and removing a pendant vertex reduces it by one and $e_{G\setminus u}(a) = e_{G\setminus v}(b)$, by condition. For those vertices g which belong to A, but they are not u and v and those vertices belong to B but they are not a and b, $d_{G\setminus u}(g) = d_{G\setminus v}(g)$ and $e_{G\setminus u}(g) = e_{G\setminus v}(g)$. Now consider $u, v \in A$. For them we have $d_{G\setminus v}(u) = d_{G\setminus u}(v)$ and $e_{G\setminus v}(u) = e_{G\setminus u}(v)$.

Now,

$$\begin{aligned}
NCI(G \setminus v) &= \sum_{t\notin B, t\notin A} d_{G\setminus v}(t)e_{G\setminus v}(t) + \sum_{t\in A, t\neq u, t\neq v} d_{G\setminus v}(t)e_{G\setminus v}(t) \\
&\quad + \sum_{t\in B, t\neq a, t\neq b} d_{G\setminus v}(t)e_{G\setminus v}(t) + d_{G\setminus v}(u)e_{G\setminus v}(u) + d_{G\setminus v}(b)e_{G\setminus v}(b) \\
&= \sum_{t\notin B, t\notin A} d_{G\setminus u}(t)e_{G\setminus u}(t) + \sum_{t\in A, t\neq u, t\neq v} d_{G\setminus u}(t)e_{G\setminus u}(t) \\
&\quad + \sum_{t\in B, t\neq a, t\neq b} d_{G\setminus u}(t)e_{G\setminus u}(t) + d_{G\setminus u}(v)e_{G\setminus u}(v) + d_{G\setminus u}(b)e_{G\setminus u}(b) \\
&= NCI(G \setminus u).
\end{aligned}$$

□

The following corollary follows from Theorem 3.24.

Corollary 3.25. *Consider a fuzzy graph G. Let*

(1) *A be the set of pendant vertices with potential a.*

(2) *B be the set of supporting vertices of vertices from A with degree c and potential b.*

(3) *A_i be the set of vertices of A having same supporting vertex from B.*

Then the neighborhood connectivity index of the sub graph obtained after removing s number of vertices from any A_i will be same.

ie, such sets are twinning vertex sets of cardinality s.

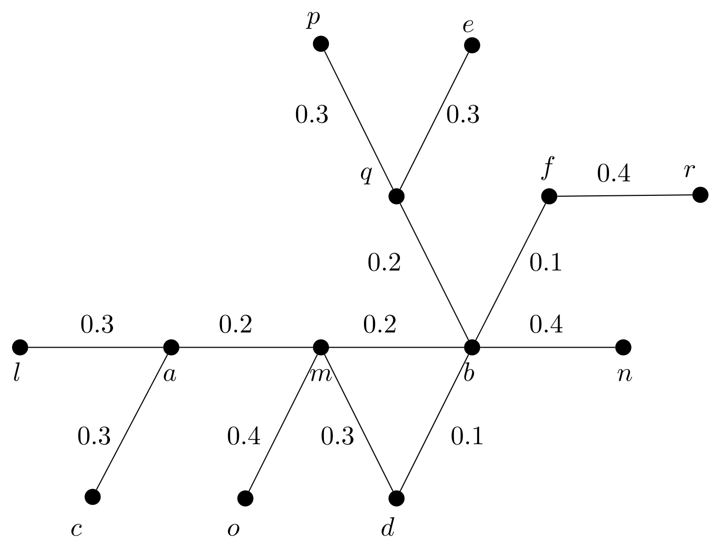

Figure 7: Fuzzy graph having twinning vertex sets.

Example 3.26. *Consider the fuzzy graph G as in Figure 7 with $\psi^* = \{l, a, m, b, n, c, o, d, p, e, q, f, r\}$ and $\gamma(la) = 0.3$, $\gamma(am) = 0.2$, $\gamma(ac) = 0.3$, $\gamma(mb) = 0.2$, $\gamma(md) = 0.3$, $\gamma(mo) = 0.4$, $\gamma(bn) = 0.4$, $\gamma(bd) = 0.1$, $\gamma(bq) = 0.2$, $\gamma(bf) = 0.1$, $\gamma(pq) = 0.3$, $\gamma(eq) = 0.3$, $\gamma(fr) = 0.4$.*
Here $NCI(G) = 8.9, NCI(G \setminus l) = 7.9, NCI(G \setminus p) = 7.9, NCI(G \setminus \{l, c\}) = 7.2, NCI(G \setminus \{p, e\}) = 7.2$. It shows that $\{l\}$ and $\{p\}$ are twinning vertex sets of cardinality one and $\{l, c\}$ and $\{p, e\}$ are twinning vertex sets of cardinality two.

The remaining of the section compares neighborhood connectivity index with connectivity index and Wiener index.

Theorem 3.27. *Let $G = (\psi, \gamma)$ be a complete fuzzy graph, $CI(G)$ be the connectivity index of G and $NCI(G)$ be the neighborhood connectivity index of G. Then $2CI(G) \leq NCI(G)$.*

Proof. Let $G = (\psi, \gamma)$ be a complete fuzzy graph. Then

$$\begin{aligned} 2CI(G) &= 2 \sum_{m,p \in \psi^*} \psi(m)\psi(p) CONN_G(m, p) \\ &\leq 2 \sum_{m,p \in \psi^*} CONN_G(m, p), \qquad (\text{since } 0 < \psi(m), \psi(p) \leq 1) \\ &\leq \sum_{m,p \in \psi^*} e(m) + \sum_{m,p \in \psi^*} e(p), \qquad (\text{replace one } CONN_G(m, p) \text{ with } e(m) \text{ and another with } e(p)) \\ &= NCI(G), \qquad \text{since each } e(m) \text{ repeats } d(m) \text{ times altogether.} \end{aligned}$$

□

Remark 3.28. *For a complete fuzzy graph we have $2WI(G) \leq NCI(G)$ since $CI(G) = WI(G)$ by Theorem 2.2.*

The above result is not always true. Consider the fuzzy graph $G = (\psi, \gamma)$ given in Figure 8 such that $\psi^* = \{l, a, m, b, n\}$, $\gamma(la) = 0.1, \gamma(am) = 0.2, \gamma(an) = 0.3, \gamma(mn) = 0.1, \gamma(bn) = 0.3$. The neighborhood connectivity index, $NCI(G) = 2.6$ and the connectivity index, $CI(G) = 1.9$. Here $2CI(G) = 3.8 \nleq 2.6 = NCI(G)$. Also note that Wiener index, $WI(G) = 4.2$ which is greater than 2.6.

4 Fuzzy graph operations and neighborhood connectivity index

There are several fuzzy graph operations in fuzzy graph theory. This section deals with the neighborhood connectivity index of graphs obtained by some of these operations. As defined earlier, in this section $G_1 \cup G_2$ represents union, $G_1 + G_2$ represents join, $G_1[G_2]$ represents composition, $G_1 \times G_2$ represents Cartesian product and $G_1 \otimes G_2$ represents tensor product of two fuzzy graphs G_1 and G_2.

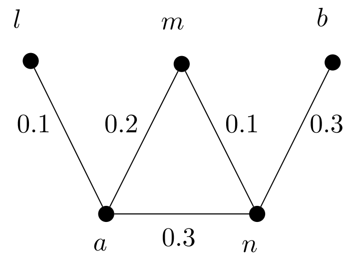

Figure 8: Fuzzy graph with $2CI(G) > NCI(G)$.

Theorem 4.1. *Let $G_i = (\psi_i, \gamma_i)$ be fuzzy graphs where $i = 1, 2$. Then*

$$NCI(G_1 \cup G_2) = \sum_m [(\vee\{e_{G_1}(m), e_{G_2}(m)\})(d_{G_1}(m) + d_{G_2}(m) - |E_1 \cap E_2(m)|)],$$

where E_1 and E_2 are the edge sets of G_1 and G_2 and $|E_1 \cap E_2(m)|$ is the number of edges arising from the vertex m which lies in both G_1 and G_2.

Proof. Consider $G_i = (\psi_i, \gamma_i), i = 1, 2$. We prove this theorem by considering three cases. As the first case we take $m \in V_1$ or $m \in V_2$, but not both. If $m \in V_1$ then $d_{G_1 \cup G_2}(m) = d_{G_1}(m)$ (since there is no new neighbor by construction.)$=$ $d_{G_1}(m) + d_{G_2}(m) - |E_1 \cap E_2(m)|$ (since in this case $d_{G_2}(m) = |E_1 \cap E_2(m)| = 0$). Similar case arises when $m \in V_2$ also. Now consider the potential of the vertex in $G_1 \cup G_2$. For $m \in G_1, e_{G_1 \cup G_2}(m) = e_{G_1}(m)$ (since there is no new edge originating from m and there is no change in weight for the existing edges) $= \vee\{e_{G_1}(m), e_{G_2}(m)\}$ (since in this case $e_{G_2}(m) = 0$). Similarly for $m \in G_2$ also. As the second case we take $m \in V_1 \cap V_2$, but no edge incident at m lies in $E_1 \cap E_2$. Here for $m \in V_1 \cap V_2, d_{G_1 \cup G_2}(m) = d_{G_1}(m) + d_{G_2}(m) = d_{G_1}(m) + d_{G_2}(m) - |E_1 \cap E_2(m)|$. While considering the potential of the vertex, $e_{G_1 \cup G_2}(m) = \vee\{e_{G_1}(m), e_{G_2}(m)\}$, since no edge incident at m lies in $E_1 \cap E_2$. As the third case we take $m \in V_1 \cap V_2$, but some edges incident at m are in $E_1 \cap E_2$. Here for $m \in V_1 \cap V_2, d_{G_1 \cup G_2}(m) =$ $d_{G_1}(m) + d_{G_2}(m) - |E_1 \cap E_2(m)|$. The potential of the vertex m is $e_{G_1 \cup G_2}(m) = \vee\{e_{G_1}(m), e_{G_2}(m)\}$, in this case, since the edges are taking the maximum weight and the maximum will be any of the $e_{G_i}(m), i = 1, 2$. From the three cases,

$$NCI(G_1 \cup G_2) = \sum_m [(\vee\{e_{G_1}(m), e_{G_2}(m)\})(d_{G_1}(m) + d_{G_2}(m) - |E_1 \cap E_2(m)|)].$$

□

Theorem 4.2. *Let $G_i = (\psi_i, \gamma_i)$ be fuzzy graphs with $|\psi_i^*| = n_i$ where $i = 1, 2$. Assuming $V_1 \cap V_2 = \phi$ we have*

$$NCI(G_1 + G_2) = \sum_{m \in G_i, i \neq j} [(d_{G_i}(m) + n_j)(\vee_{p \in G_j}\{\psi(m) \wedge \psi(p)\})].$$

Proof. Let $G_i = (\psi_i, \gamma_i)$ be fuzzy graphs with $|\psi_i^*| = n_i$ where $i = 1, 2$. Suppose $m \in G_1$, then the neighborhood of m has all elements in G_2 in addition to its neighborhood in G_1 itself. Therefore, $d_{G_1+G_2}(m) = d_{G_1}(m)+n_2$. Similarly if $m \in G_2$, $d_{G_1+G_2}(m) = d_{G_2}(m)+n_1$. Now we can check the potential of the vertex $m \in G_1$. Since $V_1 \cap V_2 = \phi$ there are two types of edges arising from m. One is those edges whose other end point is in G_1 and other is those edges whose other endpoint is in G_2. Edges of the first case has maximum connectedness $e_{G_1}(m)$. Edges of the second case has connectedness the minimum of the weight of its adjacent vertices. The maximum among them is greater than or equal to $e_{G_1}(m)$. Therefore, $e_{G_1+G_2}(m) = \vee_{p \in G_2}\{\psi(m) \wedge \psi(p)\}$. Similarly if $m \in G_2$, Therefore, $e_{G_1+G_2}(m) = \vee_{p \in G_1}\{\psi(m) \wedge \psi(p)\}$. Therefore,

$$\begin{aligned} NCI(G_1 + G_2) &= \sum_{m \in G_1 + G_2} d_{G_1+G_2}(m) e_{G_1+G_2}(m) \\ &= \sum_{m \in G_1} [(d_{G_1}(m) + n_2)(\vee_{p \in G_2}\{\psi(m) \wedge \psi(p)\})] + \sum_{m \in G_2} [(d_{G_2}(m) + n_1)(\vee_{p \in G_1}\{\psi(m) \wedge \psi(p)\})] \\ &= \sum_{m \in G_i, i \neq j} [(d_{G_i}(m) + n_j)(\vee_{p \in G_j}\{\psi(m) \wedge \psi(p)\})]. \end{aligned}$$

□

Example 4.3. *Consider the fuzzy graphs G_1 and G_2 given in Figure 9 with $\psi_1^* = \{l, a, m\}$ and $\psi_2^* = \{b, n\}$ where $\psi(l) = 0.9$, $\psi(a) = 0.4$, $\psi(m) = 1$, $\psi(b) = 0.8$, $\psi(n) = 0.7$ and $\gamma(la) = 0.3$, $\gamma(am) = 0.1$, $\gamma(lm) = 0.8$, $\gamma(bn) = 0.6$. After finding $G_1 + G_2$ we calculate $NCI(G_1 + G_2) = 4 \times 0.8 + 4 \times 0.4 + 4 \times 0.8 + 4 \times 0.8 + 4 \times 0.7 = 14$. Now using Theorem 4.2 we can find this without actually finding $G_1 + G_2$, $NCI(G_1 + G_2) = (2+2) \vee \{0.8, 0.7\} + (2+2) \vee \{0.4, 0.4\}$ $+(2+2) \vee \{0.8, 0.7\}$ $+(1+3) \vee \{0.8, 0.4, 0.8\} + (1+3) \vee \{0.7, 0.4, 0.7\} = 4 \times 0.8 + 4 \times 0.4 + 4 \times 0.8 + 4 \times 0.8 + 4 \times 0.7 = 14$.*

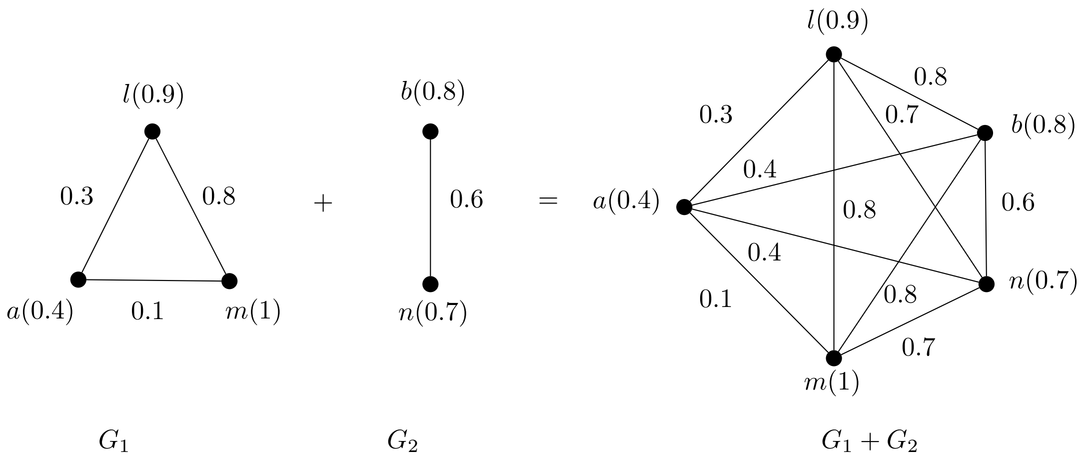

Figure 9: Join of two fuzzy graphs.

Theorem 4.4. *Let $G_i = (\psi_i, \gamma_i)$ be fuzzy graphs with $|\psi_i^*| = n_i$ where $i = 1, 2$.*

(i) if $\psi_1 \leq \gamma_2$, then $NCI(G_1[G_2]) = \sum_{(m,p)\in V_1\times V_2} [n_2 d_{G_1}(m) + d_{G_2}(p)]\psi_1(m)$.

(ii) if $\psi_1 \geq \gamma_2, \psi_2 \geq \gamma_1$, then $NCI(G_1[G_2]) = \sum_{(m,p)\in V_1\times V_2} [n_2 d_{G_1}(m) + d_{G_2}(p)][\vee\{e_{G_1}(m), e_{G_2}(p)\}]$.

Proof. Let G_1 and G_2 be two fuzzy graphs. $NCI(G_1[G_2]) = \sum_{(m,p)\in V_1\times V_2} d_{G_1[G_2]}(m,p) e_{G_1[G_2]}(m,p)$. We can calculate $d_{G_1[G_2]}(m,p)$ and $e_{G_1[G_2]}(m,p)$ separately. The neighborhood of (m,p) consists of three types of vertices. (1) $\{(x,y) : x = m, py \in E_2\}$, the cardinality of this set is $d_{G_2}(p)$. (2) $\{(x,y) : y = p, mx \in E_1\}$, the cardinality of this set is $d_{G_1}(m)$. (3) $\{(x,y) : y \neq p, mx \in E_1\}$, the cardinality of this set is $(n_2 - 1)d_{G_1}(m)$. Therefore, $d_{G_1[G_2]}(m,p) = d_{G_2}(p) + d_{G_1}(m) + (n_2 - 1)d_{G_1}(m) = n_2 d_{G_1}(m) + d_{G_2}(p)$. Now we can look into the potential of the vertex (m,p).

$$\begin{aligned} e_{G_1[G_2]}(m,p) &= \vee_{(m,p)\in V_1\times V_2}\{\gamma_{G_1[G_2]}((m,p)(x,y)) : (x,y) \in V_1 \times V_2\} \\ &= \vee_{(m,p)\in V_1\times V_2} \begin{cases} \psi_1(m) \wedge \gamma_2(py) & : \text{if} \quad x = m, py \in E_2 \\ \psi_2(p) \wedge \gamma_1(mx) & : \text{if} \quad y = p, mx \in E_1 \\ \psi_2(p) \wedge \psi_2(y) \wedge \gamma_1(mx) & : \text{if} \quad y \neq p, mx \in E_1 \end{cases} \cdots (*) \end{aligned}$$

Now we can analyse the two parts of the theorem.

Part i: $\psi_1 \leq \gamma_2$. By Theorem 2.1 equation $(*)$ becomes

$$e_{G_1[G_2]}(m,p) = \vee_{(m,p)\in V_1\times V_2} \begin{cases} \psi_1(m) & : \text{if} \quad x = m, py \in E_2 \\ \gamma_1(mx) & : \text{if} \quad y = p, mx \in E_1 \\ \gamma_1(mx) & : \text{if} \quad y \neq p, mx \in E_1 \end{cases} = \psi_1(m).$$

Therefore, $NCI(G_1[G_2]) = \sum_{(m,p)\in V_1\times V_2} [n_2 d_{G_1}(m) + d_{G_2}(p)]\psi_1(m)$.

Part ii: $\psi_1 \geq \gamma_2, \psi_2 \geq \gamma_1$. Here equation $(*)$ becomes

$$e_{G_1[G_2]}(m,p) = \vee_{(m,p)\in V_1\times V_2} \begin{cases} \gamma_2(py) & : \text{if} \quad x = m, py \in E_2 \\ \gamma_1(mx) & : \text{if} \quad y = p, mx \in E_1 \\ \gamma_1(mx) & : \text{if} \quad y \neq p, mx \in E_1 \end{cases} = \vee\{e_{G_1}(m), e_{G_2}(p)\}.$$

Therefore, $NCI(G_1[G_2]) = \sum_{(m,p)\in V_1\times V_2} [n_2 d_{G_1}(m) + d_{G_2}(p)][\vee\{e_{G_1}(m), e_{G_2}(p)\}]$. □

Corollary 4.5. *Let $G_i = (\psi_i, \gamma_i)$ be fuzzy graphs with $|\psi_i^*| = n_i$ where $i = 1, 2$.*

(i) if $\psi_1 \le \gamma_2$, then $NCI(G_1 \times G_2) = \sum_{(m,p)\in V_1\times V_2} [d_{G_1}(m) + d_{G_2}(p)]\psi_1(m)$.

(ii) if $\psi_1 \ge \gamma_2, \psi_2 \ge \gamma_1$, then $NCI(G_1 \times G_2) = \sum_{(m,p)\in V_1\times V_2} [d_{G_1}(m) + d_{G_2}(p)][\vee\{e_{G_1}(m), e_{G_2}(p)\}]$.

Proof. Since by construction, the Cartesian product of two fuzzy graphs differ from composition only by the set of edges $\{xy : y \ne p, mx \in E_1\}$. There is no change for $e_{G_1\times G_2}(m,p)$ from $e_{G_1[G_2]}(m,p)$ which can be observed from Equation $(*)$ in the Theorem 4.4. While considering the neighborhood of the vertex (m,p), third type mentioned in the above proof is missing. Therefore, $d_{G_1\times G_2}(m,p) = d_{G_1}(m) + d_{G_2}(p)$. Hence if $\psi_1 \le \gamma_2$, then $NCI(G_1 \times G_2) = \sum_{(m,p)\in V_1\times V_2} [d_{G_1}(m) + d_{G_2}(p)]\psi_1(m)$ and if $\psi_1 \ge \gamma_2, \psi_2 \ge \gamma_1$, then $NCI(G_1 \times G_2) = \sum_{(m,p)\in V_1\times V_2} [d_{G_1}(m) + d_{G_2}(p)]$ $[\vee\{e_{G_1}(m), e_{G_2}(p)\}]$.

□

Theorem 4.6. *Let $G_i = (\psi_i, \gamma_i)$ be fuzzy graphs with $|\psi_i^*| = n_i$ where $i = 1, 2$. Then*

$$NCI(G_1 \otimes G_2) = \sum_{(m,p)\in V_1\times V_2} (d(m)d(p))(\wedge\{e(m), e(p)\}).$$

Proof. Let $G_i = (\psi_i, \gamma_i)$ be fuzzy graphs with $|\psi_i^*| = n_i$ where $i = 1, 2$. First, we find $d_{G_1\otimes G_2}(m,p)$ and then $e_{G_1\otimes G_2}(m,p)$. Consider the vertex $(m,p) \in V_1 \times V_2$. In the vertex set of $V_1 \times V_2$ we can find n_2 number of vertices with same first coordinate. Among the n_2 vertices there exists $d(p)$ vertices which has a neighborhood with (m,p). And this case repeats $d(m)$ times. Therefore, $d_{G_1\otimes G_2}(m,p) = d(m)d(p)$. Now $e_{G_1\otimes G_2}(m,p) = \vee\{\gamma_{G_1\otimes G_2}((m,p)(x,y)); (x,y) \in V_1 \times V_2\} = \vee\{\gamma_{G_1}(mx) \wedge \gamma_{G_2}(py); mx \in E_1 \text{and } py \in E_2\} = \wedge\{[\vee\gamma_{G_1}(mx), \vee\gamma_{G_2}(py)]; mx \in E_1 \text{and } py \in E_2\}$ $= \wedge\{e(m), e(p)\}$. Therefore, $NCI(G_1 \otimes G_2) = \sum_{(m,p)\in V_1\times V_2} (d(m)d(p))(\wedge\{e(m), e(p)\})$. □

5 Algorithm

This section discusses an algorithm to find the neighborhood connectivity index of a fuzzy graph.

Algorithm 5.1. Let $G = (\psi, \gamma)$ be a fuzzy graph with n vertices.

1. Construct the matrix $A = [a_{ij}]$ with $a_{ij} = \gamma(m_i m_j)$.
2. Find the largest membership value in each row of the matrix. Let it be t_i.
3. Find the number of non-zero entries in each row of the matrix. Let it be s_i.
4. Then $NCI(G) = \sum_{i=1}^{n} t_i \times s_i$.

Illustration of Algorithm: Let $A = (\psi, \gamma)$ be a fuzzy graph in Figure 10 with $\psi^* = \{l, a, m, b, n, c, o, d\}$ such that $\gamma(la) = 0.5$, $\gamma(lm) = 0.3$, $\gamma(am) = 0.4$, $\gamma(mb) = 0.1$, $\gamma(bn) = 0.2$, $\gamma(nc) = 0.3$, $\gamma(no) = 0.6$, $\gamma(nd) = 0.6$, $\gamma(co) = 0.7$, $\gamma(od) = 0.5$.
The matrix representation of the given fuzzy graph is

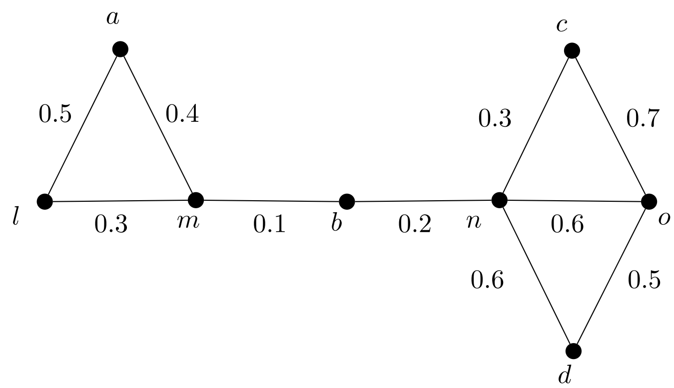

Figure 10: Illustration for Algorithm.

$$A = \begin{array}{c} \\ l \\ a \\ m \\ b \\ n \\ c \\ o \\ d \end{array} \begin{array}{c} \begin{array}{cccccccc} l & a & m & b & n & c & o & d \end{array} \\ \begin{pmatrix} 0 & 0.5 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.3 & 0.6 & 0.6 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.7 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.6 & 0 & 0.5 & 0 \end{pmatrix} \end{array}$$

Now neighborhood connectivity index can be calculated by summing the product of highest value of each row and number of non zero entries in each row. Here neighborhood connectivity index, $NCI(G) = 0.5 \times 2 + 0.5 \times 2 + 0.4 \times 3 + 0.2 \times 2 + 0.6 \times 4 + 0.7 \times 2 + 0.7 \times 3 + 0.6 \times 2 = 10.7$.

6 Application

Human trafficking has always been widely studied since its impact on human race is huge. In 2017, Mathew and Mordeson discussed about this in [17]. Directed graph technique was used by them to analyze the given data in the Table. 1. The ✓ in the data represents extremely small flow between regions and thus we neglect that further. The flow within a region is also not considered.

First of all, we construct the directed fuzzy graph S from this data. The vertices represent the regions and the directed edges represent the direction of the transition. Since the adjacency matrix of S is similar to the analyzed data, it is not mentioned again. Now by using the algorithm which we mentioned previously, we calculate neighborhood connectivity index of S and it is 6.22. After several calculations we came to an assumption that there does not exist a twinning vertex set of cardinality one. Consider the vertex set $\{l, c\}$. Now we construct the adjacency matrix of $S \setminus \{l, c\}$.

$$S \setminus \{l, c\} = \begin{array}{c} \\ a \\ m \\ b \\ n \\ o \\ d \\ p \\ e \end{array} \begin{array}{c} \begin{array}{cccccccc} a & m & b & n & o & d & p & e \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.27 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 \\ 0.04 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0.06 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.07 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0.33 \\ 0 & 0 & 0 & 0.07 & 0 & 0 & 0 & 0.18 \\ 0.16 & 0 & 0 & 0 & 0 & 1.0 & 0 & 0.10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

	WC Eur (l)	WS Eur (a)	C Eur & Bal(m)	E Eur & C Asia(b)	N Am & C Am & Car(n)	S Am (c)	E Asia & Pac(o)	S Asia (d)	S S Afr (p)	Mid East (e)
WC Eur(l)	0.62	0.13								
WS Eur(a)		0.16								
C Eur & Bal(m)		0.27	0.79		0.05					
E Eur & C Asia(b)	0.04	0.04	0.05	0.99						0.06
N Am & C Am & Car(n)	0.08	✓	✓		0.59	0.04				
S Am(c)		0.07			0.03	0.94				
E Asia & Pac(o)	0.07	0.07	✓		0.25	0.01	0.97			0.33
S Asia(d)		✓			0.07			0.96		0.18
S S Afr(p)		0.16						1.0		0.10
Mid East(e)										0.31

Table 1: Flows between different regions.

After the computation using algorithm we get the neighborhood connectivity index of $S \setminus \{l, c\}$ as 5.07. Next consider the vertex set $\{m, c\}$ and construct the adjacency matrix of $S \setminus \{m, c\}$.

$$
S \setminus \{m, c\} = \begin{array}{c} \\ l \\ a \\ b \\ n \\ o \\ d \\ p \\ e \end{array}
\begin{array}{c}
\begin{array}{cccccccc} l & a & b & n & o & d & p & e \end{array} \\
\begin{pmatrix}
0 & 0.13 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.04 & 0.04 & 0 & 0 & 0 & 0 & 0 & 0.06 \\
0.08 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.07 & 0.07 & 0 & 0.25 & 0 & 0 & 0 & 0.33 \\
0 & 0 & 0 & 0.07 & 0 & 0 & 0 & 0.18 \\
0 & 0.16 & 0 & 0 & 0 & 1.0 & 0 & 0.10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Here also after the computation we get the neighborhood connectivity index of $S \setminus \{m, c\}$ as 5.07. Therefore, $\{l, c\}$ and $\{m, c\}$ are examples of twinning vertex sets of cardinality two. Similarly, we can see that $\{l, p\}$ and $\{m, d\}$ are also twinning vertex sets of cardinality two with neighborhood connectivity index 2.58.

As this application gives the neighborhood connectivity index of a human trafficking network, the value obtained is of real concern. We can subdivide the networks into smaller portions and compare the regions with different neighborhood connectivity. Also, this examples gives the locality where we have to focuss as the removal of the twinning vertex sets $\{l, p\}$ and $\{m, d\}$ provide a much lesser index. Controlling traffic through the locations l, p, m and d can substantially reduce the traffic in the network.

7 Conclusion

Connectivity is always considered as a cornerstone in fuzzy graph theory. A new parameter related to connectivity is studied in this work. Indices of several predefined graphical structures are found and a new set of vertex sets called the twinning vertex set is also introduced. With the help of an algorithm this index is applied to human trafficking problems. Since neighborhood traffic is relevant in most of the modern networks, the concepts of this paper can be used in a wide variety of problems. This can also be used in analyzing the effectiveness of scheduling and routing in different areas.

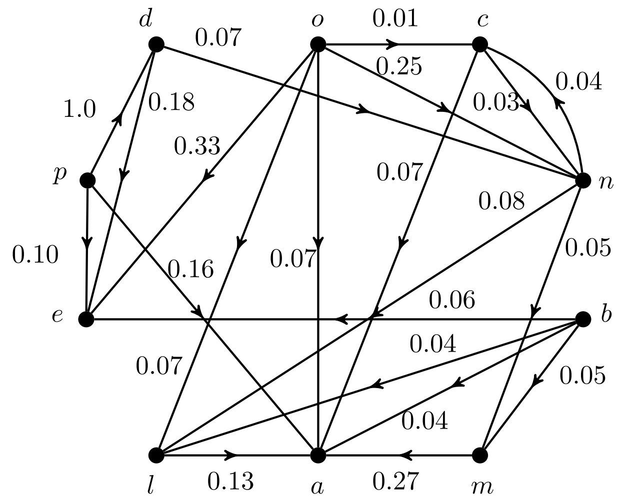

Figure 11: Directed fuzzy graph of the given data.

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